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Research article

On the meromorphic continuation of the Mellin transform associated to the wave kernel on Riemannian symmetric spaces of the non-compact type

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Abstract: We considered the Mellin transform assigned to the convolution wave kernel associated to the Laplace-Beltrami operator on higher rank Riemannian symmetric spaces of the non-compact type. The occurrence of the analyticity strip of this transform can be deduced directly from the pointwise kernel estimates. Using the zeta function techniques, we established its meromorphic extension to the entire complex plane $\mathbb{C}$ with simple poles on the real line.

Keywords: Riemannian symmetric spaces; wave kernel; Mellin transform; asymptotic expansion; meromorphic continuation; zeta function
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1. Introduction

In various problems of mathematical analysis arises the need for evaluation of simple or multiple integrals. This stimulated the appearance of a large number of works dealing with the development and improvement of evaluation methods and the study of the analytical aspect of these integrals.

Among the fundamental methods for the evaluation of integrals is the use of an operational approach based on the Mellin transform defined as a function on the complex plane by the relation

$$f^*(z) = \frac{1}{\Gamma(z)} \int_0^\infty f(t) t^{z-1} dt, \tag{1.1}$$

for some function f defined on the positive real axis, and Γ denotes the usual gamma function.

This method is quite global; it is fundamentally linked to the theory of residues and integration on contours of the complex plane. Moreover, it is related to several integral transforms such as the Fourier and Laplace transforms. The method allows the evaluation of a wide class of integrals, containing elementary and special functions of hypergeometric type and arising in various problems of mathematical analysis and applications. Elements of the theory of the Mellin transform can be found, for example, in [2, 27].

In the literature, it is well-known that for some absolutely integrable function f on a finite interval $(t_1, t_2), 0 < t_1 < t_2 < \infty$, which satisfies the following bounds

$$\underbrace{|f(t)| < Ct^{-c_1}}_{t\to 0}, \qquad \underbrace{|f(t)| < Ct^{-c_2}}_{t\to\infty}, \tag{1.2}$$

for some positive constants $C,\ c_1$ and c_2 such that $c_1 < c_2$ and the Mellin transform f^* exists and it is an analytic function on the vertical strip $c_1 < \mathrm{Re}(z) < c_2$. In some cases, this strip may extend to a half-plane ($c_1 = -\infty$ or $c_2 = +\infty$) or to the whole complex z-plane ($c_1 = -\infty$ and $c_2 = +\infty$).

In the general context, where M is a compact d-dimensionnal Riemannian manifold equipped with a metric g and a Laplace-Beltrami operator Δ, most of the literature has concentrated on the zeta function defined on the complex plane $\mathbb{C}$ by the series absolutely convergent and holomorphic on $\mathrm{Re}(z) > \frac{d}{2}$

$$\zeta_M(z) = \sum_{j\geq 1} \lambda_j^{-z}, \tag{1.3}$$

where λ_j are the eigenvalues of Δ. It is known that, if $K(t, x, y)$ denotes the heat kernel of Δ and $Z(t)$ is the partition function defined by the formula

$$Z(t) = \int_M K(t, x, x)\sqrt{\det g}\, dx = \sum_{j\geq 1} e^{-t\lambda_j},$$

then the zeta function is associated to this partition function by a Mellin transform

$$\zeta_M(z) = \frac{1}{\Gamma(z)} \int_0^\infty (Z(t) - 1)t^{z-1} dt. \tag{1.4}$$

Using a short-time asymptotic expansion [26]

$$Z(t) \sim (4\pi t)^{-\frac{d}{2}} \sum_{j\geq 0} C_j t^j, \quad t \to 0^+, \tag{1.5}$$

for appropriate coefficients C_j (called Minakshisundaram-Pleijel coefficients), Minakshisundaram and Pleijel proved that $\zeta_M(z)$ can be continued to $\mathbb{C}$ to a meromorphic function with simple poles on the real line at $z_j = \frac{d}{2} - j,\ \ j \in \mathbb{N}_0$, and only a finite number of poles ($j = 0, 1, \cdots, \frac{d}{2} - 1$) in the even-dimensional case and residues are given by

$$\mathrm{res}(\zeta_M(z), z_j) = \frac{(4\pi)^{-\frac{d}{2}} C_j}{\Gamma(\frac{d}{2} - j)}, \quad j \in \mathbb{N}_0. \tag{1.6}$$

Moreover, the values of $\zeta_M(z)$ for z, a negative integer, or zero are related to the M-P coefficients by the formula

$$\forall k \in \mathbb{N}_0, \ \ \zeta_M(-k) = \begin{cases} (-1)^k k!(4\pi)^{-\frac{d}{2}} C_{\frac{d}{2}+k}, & d \text{ even;} \\ 0, & d \text{ odd.} \end{cases} \tag{1.7}$$

Computing the M-P coefficients remains a challenge, and has never been an easy task. However, the coefficients are explicitly calculated for a compact rank 1 locally symmetric space $\Gamma \setminus G/K$ [10] and also for a compact connected rank 1 symmetric space M' dual of G/K [9]. Here, G is a connected non-compact real semisimple Lie group with finite center, K is a maximal compact subgroup of G, and Γ is a co-compact torsion-free discrete subgroup of G. Note that, the M-P coefficients for $\Gamma \setminus G/K$ and M' are proportional as follows:

$$C_j(\Gamma \setminus G/K) = (-1)^j \frac{\mathrm{Vol}(\Gamma \setminus G/K)}{\mathrm{Vol}(M')} C_j(M'). \tag{1.8}$$

The computation of these coefficients is also carried out in a vector bundle context [7, 25, 29].

The zeta function $\zeta_{G/K}(z)$ on rank one-Riemannian symmetric spaces of the non-compact type G/K (Hyperbolic spaces) is defined as the Mellin transform of the heat kernel $h(t, x)$, $x \in G/K$

$$h^*(z, x) = \zeta_{G/K}(z, x) = \frac{1}{\Gamma(z)} \int_0^\infty h(t, x) t^{z-1} dt, \ \text{ for } \mathrm{Re}(z) > \frac{d}{2}. \tag{1.9}$$

In [14], R. Camporesi derived a meromorphic continuation to $\mathbb{C}$ at the origin $x_0 \in G/K$ of $\zeta_{G/K}(z, x_0) = \zeta_{G/K}(z)$, separating even and odd-dimensional cases, and in analogy with the compact case above, using a short time asymptotic expansion of $h(t, x_0)$, relations and formulas equivalent to (1.6) and (1.7) are obtained and, consequently, the M-P coefficients are deduced explicitly from the meromorphic extension. In this same paper, one can observe in the extension of zeta function for the compact case, the presence of a term proportional to $\zeta_{G/K}(z)$. This is a consequence of the proportionality given in (1.8) above of the M-P coefficients of the heat kernel in the compact and non-compact cases.

Godoy et al. [19] and F. L. Williams [30] extended the study of $\zeta_{G/K}(z, x)$ to higher real rank Riemannian symmetric spaces of the non-compact type. In particular, they proved that for a fixed real b, the function

$$\zeta_{G/K}(z, x, b) = \frac{1}{\Gamma(z)} \int_0^\infty e^{-b^2 t} h(t, x) t^{z-1} dt \tag{1.10}$$

can be analytically continued to an entire function of z if $x \neq x_0$, and has a meromorphic continuation to $\mathbb{C}$ with simple poles on the real line in the case $x = x_0$. As an application, the one-loop effective potential

$$W^{(1)} = \frac{1}{2} \log \det(-\Delta/\mu^2)$$

has been computed when $\mathbb{X}$ is of complex type (G is a semisimple complex Lie Group), whose relevance for quantum field theory was alluded and computed in many different situations in [3, 5, 6, 8, 11, 13, 15].

The results obtained in this paper can be considered as a first study of the Mellin transform associated to the wave kernel on general d-dimensional non-compact symmetric spaces $\mathbb{X} = G/K$ of higher real rank, where G denotes a semisimple Lie group, connected, non-compact, with finite center, and K is a maximal compact subgroup of G.

Let Δ be the Laplace-Beltrami operator on $\mathbb{X} = G/K$. Using the inverse spherical Fourier transform, the wave convolution kernel associate to Δ is a bi-K-invariant kernel on G expressed as follows:

$$w(t, x, \sigma) = \frac{1}{|W|} \int_{\mathfrak{a}^\star} e^{it\sqrt{|\lambda|^2 + |\rho|^2}} (|\lambda|^2 + |\rho|^2)^{-\sigma/2} \varphi_\lambda(x) |c(\lambda)|^{-2} d\lambda, \tag{1.11}$$

for $t > 0$, $x \in G$ and $\sigma \in \mathbb{C}$, such that $\mathrm{Re}(\sigma) > d = \dim(\mathbb{X})$, where $\mathfrak{a}^\star$ denotes a vector dual of a maximal abelian subspace $\mathfrak{a}$ in the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of the group G. Here, φ_λ denotes the spherical function on G of index $\lambda \in \mathfrak{a}^\star$ and $c(\lambda)$ is the Harish-Chandra c-function. $|W|$ denotes the cardinality of the Weyl group W associated to the root system Σ, and ρ is the half sum of positive roots counted with their multiplicities relative to $\mathfrak{a}$.

The kernel $w(t,x,\sigma)$ satisfies the wave equation $\frac{\partial^2}{\partial t^2} w(t,x,\sigma) = \Delta w(t,x,\sigma)$, and its spherical Fourier transform is given by

$$\widehat{w}(t,\lambda,\sigma) = e^{it\sqrt{|\lambda|^2+|\rho|^2}}(|\lambda|^2 + |\rho|^2)^{-\sigma/2}, \quad \lambda \in \mathfrak{a}^\star.$$

According to the formula

$$a^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-ua} u^{\sigma-1} du \quad \forall a > 0,$$

we easily write the integral expression (1.11) differently:

$$\begin{aligned} w(t,x,\sigma) &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \underbrace{\left(\frac{1}{|W|} \int_{\mathfrak{a}^\star} e^{-(s-it)\sqrt{|\lambda|^2+|\rho|^2}} \varphi_\lambda(x) |c(\lambda)|^{-2} d\lambda \right)}_{p(s-it,x)} s^{\sigma-1} ds \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty p(s-it,x) s^{\sigma-1} ds \\ &= p^*(r-it)(\sigma), \end{aligned} \tag{1.12}$$

where $p(r-it,x)$ denotes the bi-K-invariant convolution kernel of the Poisson operator $\mathcal{P}_{r-it}$. Thus, formally a passage to the limit ($t \to 0^+$) gives

$$p^*(r-it)(\sigma) \longrightarrow \frac{1}{|W|} \int_{\mathfrak{a}^\star} (|\lambda|^2 + |\rho|^2)^{-\sigma/2} \varphi_\lambda(x) |c(\lambda)|^{-2} d\lambda = \zeta_{G/K}(\sigma/2, x, b=0), \tag{1.13}$$

which makes the zeta function appear as a limit of a Mellin transform. In other words, it allows us to generalize formally and to describe a Mellin transform as a limit of another one, and the limit function $(\lim_{t\to 0^+} w(t,x,\sigma))_{\mathrm{Re}(\sigma)>d}$ can be continued to an entire function of σ when $x \neq x_0$. This motivates us mathematically to study the Mellin transform associated to the wave kernel $w(t,x,\sigma)$.

In this paper, we consider the Mellin transform associated to the kernel $w(t,x,\sigma)$ given by the formula

$$w^*(z,x,\sigma,b) = \frac{1}{\Gamma(z)} \int_0^\infty e^{-bt} w(t,x,\sigma) t^{z-1} dt, \tag{1.14}$$

for $z, \sigma \in \mathbb{C}$ with $\mathrm{Re}(\sigma) > d$, and a fixed element x in the Riemannian symmetric spaces of the non-compact type $\mathbb{X} = G/K$ and a real parameter $b > 0$. We show that the tools employed in the study of zeta functions on compact and non-compact manifolds can be generalized to $w^*(z,x,\sigma,b)$. Using pointwise kernel estimates proved in [21], we shall see that $w^*(z,x,\sigma,b)$ is a well-defined analytic function on $\mathrm{Re}(z) > \frac{d-1}{2}$, for every $x \in \mathbb{X} = G/K$. We prove that $w^*(z,x,\sigma,b)$ extends meromorphically to the entire complex plan $\mathbb{C}$ with simple poles on the real line at the points $z_j = \frac{d-1}{2} - j$, $j \in \mathbb{N}_0$, by using the short time asymptotic expansion of the wave kernel

$$w(t,x,\sigma) \sim c_d t^{-\frac{d-1}{2}} \sum_{j\geq 0} a_j(x,\sigma) t^j, \quad t \to 0^+,$$

and residues given by

$$\operatorname{res}(w^*(z,x,\sigma,b), z_j = \frac{d-1}{2} - j) = \frac{c_d}{\Gamma(\frac{d-1}{2} - j)} \sum_{l=0}^{j} \frac{(-1)^{j-l} b^{j-l}}{(j-l)!} a_l(x,\sigma), \quad \forall\, x \in \mathbb{X} = G/K.$$

Moreover, if $d = 2p+1$ is odd, we prove that $w^*(z,x,\sigma,b)$ has a finite number of poles at $z_j = \frac{d-1}{2} - j$, $j = 0, 1, \cdots \frac{d-1}{2} - 1$.

On the other hand, we establish under special regularity conditions a relation between the value of $w^*(z,x,\sigma,b)$ at $z = 0$, and at the origin $x_0 = eK$ of $\mathbb{X} = G/K$, and the main zeta function $Z_{\mathbb{X}}(s,a)$ associated to the heat kernel at $x_0 = eK$

$$h(t,x_0) = \frac{1}{|W|} \int_{\mathfrak{a}^\star} e^{-t(|\lambda|^2+|\rho|^2)} |c(\lambda)|^{-2} d\lambda,$$

and given by the following formula

$$Z_{\mathbb{X}}(s,a) = \int_{\mathfrak{a}^\star} (|\lambda|^2 + |\rho|^2 + a^2)^{-s} |c(\lambda)|^{-2} d\lambda, \tag{1.15}$$

for $\operatorname{Re}(s) > \frac{d}{2}$ and for a real parameter a (see [30]). In particular, we prove that this relation induces another one with the coefficient $a_j(x_0,\sigma)$ of the short time asymptotic expansion (see (3.24)), and, consequently, this expansion can be written in terms of the main zeta function.

It should be noted that, the meromorphic continuation to $\mathbb{C}$ of $w^*(z,x,\sigma,b)$ is uniform over the whole Riemannian symmetric space $\mathbb{X} = G/K$, contrary to the zeta function, where its continuation is only at the origin x_0 of $\mathbb{X}$, and this is due to the nature of pointwise estimates of the heat kernel used in [19,30].

2. Preliminaries

We recall the basic notations of Fourier analysis on Riemannian symmetric spaces of the non-compact type. We refer to [17,22,23] for geometric properties and more details for harmonic analysis on these spaces.

Throughout this paper, the symbol $A \lesssim B$ between two positive expressions means that there is a positive constant C such that $A \leq CB$.

Let G be a non-compact connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Let $\mathfrak{g}$ (resp., $\mathfrak{k}$) be the Lie algebra of G (resp., K) and consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$. Here, $\mathfrak{p}$ is the (K-invariant) orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form $\langle .\,,.\rangle$ of $\mathfrak{g}$. This form induces a K-invariant scalar product on $\mathfrak{p}$ and, hence, a G-invariant Riemannian metric on the symmetric homogeneous manifold $\mathbb{X} = G/K$ whose tangent space at origin eK is naturally identified with $\mathfrak{p}$. Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and denote by $\mathfrak{a}^\star$ (respectively, $\mathfrak{a}^\star_{\mathbb{C}}$) the real (respectively, complex) vector dual of $\mathfrak{a}$. The Killing form of $\mathfrak{g}$ induces a scalar product on $\mathfrak{a}^\star$ and a $\mathbb{C}$-bilinear form on $\mathfrak{a}^\star_{\mathbb{C}}$. If $\lambda,\ \mu \in \mathfrak{a}^\star_{\mathbb{C}}$, let $H_\lambda \in \mathfrak{a}_{\mathbb{C}}$ be determined by $\lambda(H) = \langle H_\lambda\,, H\rangle$, $\forall H \in \mathfrak{a}_{\mathbb{C}}$, and put $\langle \lambda\,, \mu\rangle = \langle H_\lambda\,, H_\mu\rangle$. We put $|\lambda| = \langle \lambda\,, \lambda\rangle^{1/2}$ for $\lambda \in \mathfrak{a}^\star$. Denote by Σ the root system of $\mathfrak{g}$ relatively to $\mathfrak{a}$. Let W be the Weyl group associated to Σ, and let m_λ denote the multiplicity of the root $\lambda \in \Sigma$. In particular, if $\mathfrak{a}^+$ denotes the positive Weyl chamber in $\mathfrak{a}$ corresponding to some fixed set Σ_+ of positive roots, we have the Cartan decomposition $G = K\exp(\overline{\mathfrak{a}^+})K$ of G, where $\overline{\mathfrak{a}^+}$ denotes the closure of $\mathfrak{a}^+$. Each element $x \in G$ is written uniquely as $x = k_1 A(x) k_2$. We denote by $|x| = |A(x)|$ the norm defined on G. Let $\rho = \frac{1}{2}\sum\limits_{\lambda\in\Sigma_+} m_\lambda \lambda$ and let d be the dimension of $\mathbb{X}$, and D be the dimension at infinity of $\mathbb{X}$. Consider $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$, the Iwasawa decomposition of $\mathfrak{g}$, and the corresponding Iwasawa decomposition $G = K\exp(\mathfrak{a})N$ of G, where N is the analytic subgroup of G associated to the nilpotent subalgebra $\mathfrak{n}$. Denote by $H(x)$ the Iwasawa component of $x \in G$ in $\mathfrak{a}$. There is a basic estimate for this component (see [22], p. 476)

$$|H(x)| \lesssim |x| \tag{2.1}$$

and it is called Iwasawa projection. Finally, let $\ell = \dim(\mathfrak{a})$, the real dimension of $\mathfrak{a}$. By definition, ℓ is the real rank of G.

We identify functions on $\mathbb{X} = G/K$ with functions on G, which are K-invariant on the right and, hence, bi-K-invariant functions on G with functions on $\mathbb{X}$, K-invariants on the left. If f is a sufficiently regular bi-K-invariant function on G, then its spherical-Fourier transform is a function on $\mathfrak{a}^\star_{\mathbb{C}}$ defined by

$$\widehat{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx, \quad \lambda \in \mathfrak{a}^\star_{\mathbb{C}},$$

where φ_λ denotes the spherical function on G defined by

$$\varphi_\lambda(x) = \int_K e^{<i\lambda-\rho, H(xk)>}dk, \quad x \in G, \quad \lambda \in \mathfrak{a}^\star_{\mathbb{C}}$$

and it satisfies the basic estimates

$$|\varphi_\lambda(x)| \le \varphi_0(x), \quad \forall x \in G, \quad \forall \lambda \in \mathfrak{a}^\star_{\mathbb{C}}.$$

The Plancherel and inversion formula for the spherical transform are, respectively, given by

$$\int_G |f(x)|^2 dx = \frac{1}{|W|}\int_{\mathfrak{a}^\star} |\widehat{f}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda,$$

$$f(x) = \tfrac{1}{|W|}\int_{\mathfrak{a}^\star} \varphi_\lambda(x)\widehat{f}(\lambda)|c(\lambda)|^{-2}d\lambda, \quad x \in G,$$

where $|W|$ denotes the order of the Weyl group W and $c(\lambda)$ is the Harish-Chandra c-function, and it satisfies the estimate

$$|c(\lambda)|^{-2} \lesssim (1+|\lambda|)^{d-\ell}, \tag{2.2}$$

together with all its derivatives.

3. Analysis of $w^*(z, x, \sigma, b)$

This section is devoted to the study of the analyticity of the Mellin transform $w^*(z, x, \sigma, b)$ associated to the kernel $w(t, x, \sigma)$ and given by the rule:

$$w^*(z, x, \sigma, b) = \frac{1}{\Gamma(z)}\int_0^\infty e^{-bt} w(t, x, \sigma) t^{z-1} dt, \tag{3.1}$$

for $z, \sigma \in \mathbb{C}$ with $\mathrm{Re}(\sigma) > d$, and fixed $x \in \mathbb{X} = G/K$ and a real parameter $b > 0$, where $w(t, x, \sigma)$ is the K-bi-invariant convolution kernel of the operator $\mathcal{W}(t, \sigma) = (-\Delta)^{-\sigma/2} e^{it\sqrt{-\Delta}}$ on the symmetric space $\mathbb{X}$:

$$\mathcal{W}(t, \sigma) f(x) = f \star w(t, x, \sigma) = \int_G w(t, y^{-1}x, \sigma) f(y) dy.$$

Our study focuses on the following pointwise estimates of the kernel $w(t, x, \sigma)$ for bounded ℓ-values (see [21]). To begin, for t large ($t \geq 1$), the estimate

$$|w(t, x, \sigma)| \leq C t^{-\ell/2} \varphi_0(x)(3 + |x|)^a, \tag{3.2}$$

is valid for $\mathrm{Re}(\sigma) > d$, for every $x \in G$, and for every $\ell < a \in 2\mathbb{N}$ and $C = C(\sigma, a) > 0$.
On the other hand, if t is small ($0 < t < 1$), the estimate (see [16])

$$|w(t, x, \sigma)| \lesssim t^{-\frac{d-1}{2}}, \tag{3.3}$$

holds for $\mathrm{Re}(\sigma) > \frac{d+1}{2}$.

Remark 3.1. *Notice that the previous kernel estimates are obtained under a strong smoothness assumption; Re(σ) $> d$, but largely sufficient to study the analyticity of $w^*(z, x, \sigma, b)$. The proof of these estimates for large t is essentially based on Morse's fundamental lemma, which describes completely the local structure of a map in a neighborhood of a nondegenerate critical point. It says that in a neighborhood of a nondegenerate critical point, the map can be written as a quadratic form in a suitable chart [28].*

3.1. Meromorphic continuation of $w^*(z, x, \sigma, b)$

The convergence of the integral given by Eq (3.1), where $\mathrm{Re}(z) > \frac{d-1}{2}$ is easily handled. Clearly, using (3.2) and (3.3), we have the following straightforward bound

$$\int_0^\infty |e^{-bt} w(t, x, \sigma) t^{z-1}| dt \lesssim \Big(\int_0^1 t^{-\frac{d-1}{2} + \mathrm{Re}(z) - 1} dt + \int_1^\infty e^{-bt} t^{-\frac{\ell}{2} + \mathrm{Re}(z) - 1} dt \Big),$$

and observe that, the first integral converges for $\mathrm{Re}(z) > \frac{d-1}{2}$. However, the second integral is controlled as follows:

$$\int_1^\infty e^{-bt} t^{-\frac{\ell}{2} + \mathrm{Re}(z) - 1} dt \lesssim \int_1^\infty e^{-bt/2} dt < \infty.$$

This concludes the absolute convergence for $\mathrm{Re}(z) > \frac{d-1}{2}$.

Theorem 3.2. *The function $w^*(z, x, \sigma, b)$ extends meromorphically to $\mathbb{C}$ with simple poles located at $z_j = \frac{d-1}{2} - j$; $j \in \mathbb{N}_0$, for all $x \in \mathbb{X}$ and for all $\sigma \in \mathbb{C}$, such that Re(σ) $> d$.*

Proof. To start, we will check the holomorphy of $w^*(z, x, \sigma, b)$. For the local part

$$w_0^*(z, x, \sigma, b) = \frac{1}{\Gamma(z)} \int_0^1 e^{-bt} w(t, x, \sigma) t^{z-1} dt \tag{3.4}$$

it suffices to see that for all real positive numbers $0 < \beta < \gamma < \infty$, this integral defines a holomorphic function in the vertical band

$$S_{\beta,\gamma} = \{z \in \mathbb{C}; \frac{d-1}{2} < \beta < \mathrm{Re}(z) < \gamma\}. \tag{3.5}$$

Let ε be a positive real number and introduce the function

$$w^*_{0,\varepsilon}(z, x, \sigma, b) = \frac{1}{\Gamma(z)} \int_\varepsilon^1 e^{-bt} w(t, x, \sigma) t^{z-1} dt.$$

This function is visibly holomorphic on the band $S_{\beta,\gamma}$. It suffices then to prove that the sequence $(w^*_{0,\varepsilon}(z, x, \sigma, b))_{\varepsilon>0}$ of holomorphic functions on $S_{\beta,\gamma}$ converges uniformly with limit $w^*_0(z, x, \sigma, b)$. Cauchy's theorem will then guarantee that this limit is holomorphic. To this end, let us estimate

$$\begin{aligned} |w^*_0(z, x, \sigma, b) - w^*_{0,\varepsilon}(z, x, \sigma, b)| &= \frac{1}{|\Gamma(z)|} \left| \int_0^\varepsilon e^{-bt} w(t, x, \sigma) t^{z-1} dt \right| \\ &\lesssim \int_0^\varepsilon t^{-\frac{d-1}{2}+\mathrm{Re}(z)-1} dt \\ &\leq \frac{1}{\beta - \frac{d-1}{2}} \varepsilon^{\beta-\frac{d-1}{2}} \xrightarrow[\varepsilon\to 0^+]{} 0 \text{ uniformly for every } z \in S_{\beta,\gamma}. \end{aligned}$$

Now, for $t \geq 1$, according to (3.2), we deduce that

$$|e^{-bt} w(t, x, \sigma) t^{z-1}| \leq C(\sigma, a) \varphi_0(x)(3 + |x|)^a e^{-bt} t^{-\ell/2+\mathrm{Re}(z)-1},$$

which ensures that the integral $\int_1^\infty e^{-bt} w(t, x, \sigma) t^{z-1} dt$ converges uniformly on $\mathrm{Re}(z) \leq R$, for each $R \in \mathbb{R}$. In particular, it converges uniformly on compact subsets of $\mathbb{C}$. It follows that the infinity part

$$w^*_\infty(z, x, \sigma, b) = \frac{1}{\Gamma(z)} \int_1^\infty e^{-bt} w(t, x, \sigma) t^{z-1} dt \tag{3.6}$$

is an entire function of z ([24], Ch. XII, Lemma 1.1, p. 308).

As mentioned above, despite the fact the integral that defines $w^*(z, x, \sigma, b)$ is not absolutely convergent for values of z such that $\mathrm{Re}(z) \leq \frac{d-1}{2}$, we will be able to extend it meromorphically to the entire complex plane $\mathbb{C}$ with poles only at points $z_j = \frac{d-1}{2} - j$, where $j \in \mathbb{N}_0$.

It is easy to prove that the kernel $w(t, x, \sigma)$ admits an asymptotic power series expansion in an appropriate neighborhood of $t = 0^+$. In other words, let us state

$$w(t, x, \sigma) \sim c_d t^{-\frac{d-1}{2}} \sum_{j\geq 0} a_j t^j, \quad t \to 0^+, \tag{3.7}$$

where the coefficients $a_j = a_j(x, \sigma)$ are to be calculated later. Here, c_d is a constant depending on d.

For our purposes, it is enough to see that for each nonnegative integer N, there exists a positive constant $C_N > 0$ such that

$$|w(t, x, \sigma) - c_d \sum_{j=0}^N a_j t^{j-\frac{d-1}{2}}| \leq C_N t^{N+1-\frac{d-1}{2}}, \quad \text{for } 0 < t \leq 1. \tag{3.8}$$

Let us then decompose $w^*(z,x,\sigma,b)$ for given z with $\mathrm{Re}(z) > \frac{d-1}{2}$ and given σ with $\mathrm{Re}(\sigma) > d$ in the easy form

$$\begin{aligned} w^*(z,x,\sigma,b) &= \underbrace{\frac{1}{\Gamma(z)}\int_0^1 e^{-bt}(w(t,x,\sigma) - c_d\sum_{j=0}^{N} a_j t^{j-\frac{d-1}{2}})t^{z-1}dt}_{A_N(z,x,\sigma,b)} \\ &+ \underbrace{\frac{c_d}{\Gamma(z)}\sum_{j=0}^{N} a_j b^{\frac{d-1}{2}-j-z}\Gamma(j-\frac{d-1}{2}+z)}_{B_{1,N}(z,b)} - \underbrace{\frac{c_d}{\Gamma(z)}\sum_{j=0}^{N} a_j \int_1^\infty e^{-bt}t^{j-\frac{d-1}{2}+z-1}dt}_{B_{2,N}(z,b)} \\ &+ \underbrace{\frac{1}{\Gamma(z)}\int_1^\infty e^{-bt}w(t,x,\sigma)t^{z-1}dt}_{w^*_\infty(z,x,\sigma,b)}. \end{aligned} \tag{3.9}$$

On the one hand, clearly, for $0 < t < 1$ and according to (3.8), we have

$$\begin{aligned} \mid e^{-bt}(w(t,x,\sigma) - c_d\sum_{j=0}^{N} a_j t^{j-\frac{d-1}{2}})t^{z-1} \mid &\leq e^{-bt} \mid w(t,x,\sigma) - c_d\sum_{j=0}^{N} a_j t^{j-\frac{d-1}{2}} \mid t^{\mathrm{Re}(z)-1} \\ &\leq C_N e^{-bt} t^{N+1-\frac{d-1}{2}+\mathrm{Re}(z)-1} \\ &\leq C_N e^{-bt} t^{\varepsilon-1}, \ \ \forall \varepsilon > 0, \end{aligned}$$

which shows that the integral

$$\int_0^1 e^{-bt}(w(t,x,\sigma) - c_d\sum_{j=0}^{N} a_j t^{j-\frac{d-1}{2}})t^{z-1}dt$$

is uniformly convergent on $\mathrm{Re}(z) \geq \frac{d-1}{2} - (N+1) + \varepsilon$ for every $\varepsilon > 0$. It follows that the function $A_N(z,x,\sigma,b)$ is holomorphic on $\mathrm{Re}(z) > \frac{d-1}{2} - (N+1)$.

On the other hand, by applying the same arguments above for $w^*_\infty(z,x,\sigma,b)$, one can easily check that $B_{2,N}(z,b)$ is also an entire function, and the function $B_{1,N}(z,b)$ containing the gamma function $\Gamma(j - \frac{d-1}{2} + z)$ can be extended meromorphically to $\mathbb{C}$ with simple poles on the real line, more precisely at $z_j = \frac{d-1}{2} - j$, with $j \in \mathbb{N}_0$.

From all the previous considerations and from (3.9), we assert that $w^*(z,x,\sigma,b)$ can be continued to a meromorphic function with simple poles at $z_j = \frac{d-1}{2} - j$.

In analogy with [30], a close study of the wave kernel using a new asymptotic power expansion of kernel $\tilde{w}(t,x,\sigma) = e^{-bt}w(t,x,\sigma)$ as $t \to 0^+$ reveals the set of poles of $w^*(z,x,\sigma,b)$ and determines exactly their locations and their residues. In fact, given the Maclaurin series

$$e^{-bt} \sim \sum_{j\geq 0} \frac{(-1)^j b^j}{j!} t^j, \ \ t \to 0^+$$

and the asymptotic expansion of $w(t,x,\sigma)$ formulated above by (3.7), the asymptotic expansion of the product is given by

$$\tilde{w}(t,x,\sigma) \sim c_d t^{-\frac{d-1}{2}} \sum_{j\geq 0} \tilde{a}_j t^j, \ \ t \to 0^+, \tag{3.10}$$

where

$$\tilde{a}_j = \tilde{a}_j(x,\sigma) = \sum_{l=0}^{j} \frac{(-1)^{j-l} b^{j-l}}{(j-l)!} a_l(x,\sigma), \quad j \in \mathbb{N}_0. \tag{3.11}$$

Therefore, the new version of (3.8) is written as follows:

$$\mid \tilde{w}(t,x,\sigma) - c_d \sum_{j=0}^{N} \tilde{a}_j t^{j-\frac{d-1}{2}} \mid \leq C_N t^{N+1-\frac{d-1}{2}}, \ \text{for} \ 0 < t \leq 1, \tag{3.12}$$

and for $\mathrm{Re}(z) > \frac{d-1}{2}$, the decomposition (3.9) takes the form

$$w^*(z,x,\sigma,b) = \tilde{A}_N(z,x,\sigma,b) + \tilde{B}_N(z,x,\sigma,b) + w^*_\infty(z,x,\sigma,b), \tag{3.13}$$

where

- $\tilde{A}_N(z,x,\sigma,b) = \frac{1}{\Gamma(z)} \int_0^1 (\tilde{w}(t,x,\sigma) - c_d \sum_{j=0}^{N} \tilde{a}_j t^{j-\frac{d-1}{2}}) t^{z-1} dt$ holomorphic function on $\mathrm{Re}(z) > \frac{d-1}{2} - (N+1)$,
- $\tilde{B}_N(z,x,\sigma,b) = \frac{c_d}{\Gamma(z)} \sum_{j=0}^{N} \tilde{a}_j \int_0^1 t^{j-\frac{d-1}{2}+z-1} dt = \frac{c_d}{\Gamma(z)} \sum_{j=0}^{N} \frac{\tilde{a}_j}{j - \frac{d-1}{2} + z}$,
- $w^*_\infty(z,x,\sigma,b) = \frac{1}{\Gamma(z)} \int_1^\infty \tilde{w}(t,x,\sigma) t^{z-1} dt$ entire function.

In conclusion, according to the uniqueness of the meromorphic extension, which is guaranteed by the uniqueness principle of holomorphic functions outside of the discrete set of poles, (3.13) provides the meromorphic continuation of $w^*(z,x,\sigma,b)$ with simple poles at $(z_j = \frac{d-1}{2} - j)_{j\in\mathbb{N}_0}$ appearing in the function $\tilde{B}_N(z,x,\sigma,b)$, and, clearly, the residues are given by

$$\begin{aligned} \mathrm{res}(w^*(z,x,\sigma,b), z_j = \frac{d-1}{2} - j) &= \frac{c_d}{\Gamma(\frac{d-1}{2} - j)} \tilde{a}_j(x,\sigma) \\ &= \frac{c_d}{\Gamma(\frac{d-1}{2} - j)} \sum_{l=0}^{j} \frac{(-1)^{j-l} b^{j-l}}{(j-l)!} a_l(x,\sigma), \ \ \forall\, x \in \mathbb{X} = G/K. \end{aligned} \tag{3.14}$$

This finishes the proof of Theorem 3.2. □

Remark 3.3. *Notice that, if $d = 2p+1$ is odd, clearly we have*

$$\frac{\Gamma(j - \frac{d-1}{2} + z)}{\Gamma(z)} = \frac{\Gamma(z - (p-j))}{\Gamma(z)} = \frac{1}{(z-1)(z-2)\cdots(z - \frac{d-1}{2} + j)},$$

which allows us to write the function $B_{1,N}(z,x,\sigma,b)$ in (3.9) as follows:

$$B_{1,N}(z,x,\sigma,b) = c_d \sum_{j=0}^{\frac{d-1}{2}} \frac{b^{j-z}}{(z-1)(z-2)\cdots(z-j)} a_{\frac{d-1}{2}-j} + \underbrace{c_d \sum_{j=\frac{d-1}{2}+1}^{N} b^{\frac{d-1}{2}-j-z} \prod_{l=0}^{j-\frac{d-1}{2}-1} (z-l) a_j}_{\textit{entire}},$$

and to affirm that $w^*(z,x,\sigma,b)$ has a finite number of simple poles at the points $z = 1,2,\cdots,\frac{d-1}{2}$ (clearly, that can be presented by the sequence $z_j = \frac{d-1}{2} - j,\ j = 0,1,\cdots,\frac{d-1}{2} - 1$). However, this is still visible simply using (3.14), since in the case $j > \frac{d-1}{2} - 1$, we obtain $\frac{d-1}{2} - j = p - j$ as a negative integer, and directly $\frac{1}{\Gamma(\frac{d-1}{2}-j)} = 0$, and consequently $res(w^*(z,x,\sigma,b),z_j) = 0$. These results are in agreement with general ζ-theory while permuting the parity of the space dimension.

3.2. Computation of the coefficients $\tilde{a}_j(x,\sigma)$

In order to find an explicit expression of the coefficients $\tilde{a}_j = \tilde{a}_j(x,\sigma)$, we write $w^*(z,x,\sigma,b)$ differently using Fubini's theorem. We set $\phi(\lambda) = \sqrt{|\lambda|^2 + |\rho|^2}$. Clearly, the following bound is valid for $\mathrm{Re}(\sigma) > d$,

$$\begin{aligned}\int_{\mathfrak{a}^\star}\int_0^\infty |e^{-bt}e^{it\phi(\lambda)}\phi(\lambda)^{-\sigma}\varphi_\lambda(x)|c(\lambda)|^{-2}t^{z-1}|dtd\lambda &= \int_{\mathfrak{a}^\star}\int_0^\infty e^{-bt}t^{\mathrm{Re}(z)-1}\phi(\lambda)^{-\mathrm{Re}(\sigma)}|\varphi_\lambda(x)||c(\lambda)|^{-2}dtd\lambda \\ &= b^{-\mathrm{Re}(z)}\Gamma(\mathrm{Re}(z))\int_{\mathfrak{a}^\star}\phi(\lambda)^{-\mathrm{Re}(\sigma)}|\varphi_\lambda(x)||c(\lambda)|^{-2}d\lambda \\ &\lesssim b^{-\mathrm{Re}(z)}\Gamma(\mathrm{Re}(z))\underbrace{\int_{\mathfrak{a}^\star}\phi(\lambda)^{-\mathrm{Re}(\sigma)}(1+|\lambda|)^{d-\ell}d\lambda}_{<\infty,\ \mathrm{Re}(\sigma)>d}.\end{aligned}$$

Thus, according to integral expression (3.1) and from the following formula [4, Equation 4],

$$\int_0^\infty e^{-bt}e^{it\phi(\lambda)}t^{z-1}dt = \Gamma(z)e^{iz\arctan(\frac{\phi(\lambda)}{b})}(|\lambda|^2+|\rho|^2+b^2)^{-z/2},\ \ \mathrm{Re}(z) > 0,$$

we obtain

$$w^*(z,x,\sigma,b) = \frac{1}{|W|}\int_{\mathfrak{a}^\star} e^{iz\arctan\frac{\sqrt{|\lambda|^2+|\rho|^2}}{b}}\varphi_\lambda(x)(|\lambda|^2+|\rho|^2)^{-\frac{\sigma}{2}}(|\lambda|^2+|\rho|^2+b^2)^{-\frac{z}{2}}|c(\lambda)|^{-2}d\lambda, \tag{3.15}$$

for every $x \in \mathbb{X}$, $b > 0$, $\mathrm{Re}(\sigma) > d$, and $\mathrm{Re}(z) > d - \mathrm{Re}(\sigma)$. Observe then its relation with the zeta function; since, in particular, for $z = 0$, we find

$$\begin{aligned}w^*(0,x,\sigma,b) &= \frac{1}{|W|}\int_{\mathfrak{a}^\star}(|\lambda|^2+|\rho|^2)^{-\frac{\sigma}{2}}\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda \\ &= \zeta(\frac{\sigma}{2},x,0),\end{aligned} \tag{3.16}$$

and for the origin $x = x_0 = eK$ in $\mathbb{X} = G/K$, we obtain

$$\begin{aligned}w^*(0,x_0,\sigma,b) &= \frac{1}{|W|}\int_{\mathfrak{a}^\star}(|\lambda|^2+|\rho|^2)^{-\frac{\sigma}{2}}|c(\lambda)|^{-2}d\lambda \\ &= \frac{1}{|W|}Z_{\mathbb{X}}(\frac{\sigma}{2},0),\ \ \text{for }\ \mathrm{Re}(\sigma) > d,\end{aligned} \tag{3.17}$$

where $Z_{\mathbb{X}}$ denotes the main zeta function associated to the heat kernel on $\mathbb{X} = G/K$.

Notice that if $x \neq x_0$ in $\mathbb{X} = G/K$, then for a real parameter a, the function $\zeta(\frac{\sigma}{2},x,a)$ can be analytically continued to $\mathbb{C}$ as the entire function of σ and, $Z_{\mathbb{X}}(\frac{\sigma}{2},0)$ extends meromorphically to $\mathbb{C}$ for all higher-rank symmetric space $\mathbb{X} = G/K$. This is true for $w^*(0, x_0, \sigma, b)$ by (3.17). Furthermore, its poles are simple and are located at the points $(\sigma_j = d - 2j)_{j\in\mathbb{N}_0}$, with residues given by

$$\operatorname{res}(w^*(0, x_0, \sigma, b), \sigma_j = d - 2j) = \operatorname{res}(Z_{\mathbb{X}}(\frac{\sigma}{2}, 0), \sigma_j = d - 2j) = \frac{2(4\pi)^{-d/2}}{\Gamma(\frac{d}{2} - j)} C_j, \ \ \forall\ j \in \mathbb{N}_0, \tag{3.18}$$

where C_j denotes the Minakshisundaram-Pleijel coefficients (see [18, 19, 30]).

Back to the Eq (3.17), it is possible to express $w^*(0, x_0, \sigma, b)$ using the meromorphic continuation of $Z_{\mathbb{X}}(\frac{\sigma}{2}, 0)$ given by Eq (3.14) in [30], and this is for the purpose to compute exactly $w^*(0, x_0, -2k, b)$ in terms of the coefficients C_j, for every $k \in \mathbb{N}_0$. Using a similar argument as that which led to Eq (3.16) in [30], we check quickly that

$$\forall k \in \mathbb{N}_0, \ \ w^*(0, x_0, -2k, b) = \begin{cases} \frac{(4\pi)^{-d/2}}{|W|}(-1)^k k! C_{\frac{d}{2}+k}, & d = 2p; \\ 0, & d = 2p + 1, \end{cases} \tag{3.19}$$

in particular

$$w^*(0, x_0, \sigma, b)_{/\sigma=0} = \begin{cases} \frac{(4\pi)^{-d/2}}{|W|} C_{\frac{d}{2}}, & d = 2p; \\ 0, & d = 2p + 1. \end{cases} \tag{3.20}$$

Let us continue to refer to the same arguments leading to equation Eq (3.16) in [30]. According to the meromorphic continuation given above by (3.13), we easily show that the values of $w^*(z, x, \sigma, b)$ for z a negative integer or zero are related to the coefficients $(\tilde{a}_j(x, \sigma))_{x\in\mathbb{X},\ \mathrm{Re}(\sigma)>d}$ by the formula

$$\forall k \in \mathbb{N}_0, \ \ w^*(-k, x, \sigma, b) = \begin{cases} c_d(-1)^k k! \tilde{a}_{\frac{d-1}{2}+k}(x, \sigma), & d = 2p + 1; \\ 0, & d = 2p, \end{cases} \tag{3.21}$$

in particular, for $k = 0$ and $x = x_0$, one has for $\mathrm{Re}(\sigma) > d$

$$w^*(0, x_0, \sigma, b) = \begin{cases} c_d(-1)^k k! \tilde{a}_{\frac{d-1}{2}}(x_0, \sigma), & d = 2p + 1; \\ 0, & d = 2p. \end{cases} \tag{3.22}$$

Summing up, we obtain the following result.

Theorem 3.4. *If $d = 2p + 1$ is odd, the coefficients $(\tilde{a}_j(x, \sigma))_{x\in\mathbb{X},\ Re(\sigma)>d}$ are given by*

$$\tilde{a}_{\frac{d-1}{2}}(x, \sigma) = c_d^{-1} \zeta(\frac{\sigma}{2}, x, 0). \tag{3.23}$$

In particular,

$$\tilde{a}_{\frac{d-1}{2}}(x_0, \sigma) = c_d^{-1} |W|^{-1} Z_{\mathbb{X}}(\frac{\sigma}{2}, 0). \tag{3.24}$$

Furthermore,

$$Res(w^*(z, x, \sigma, b), z_j = \frac{d-1}{2} - j) = \frac{1}{\Gamma(\frac{d-1}{2} - j)} \zeta(\frac{\sigma}{2}, x, 0). \tag{3.25}$$

Corollary 3.5. *The following formula holds, for d=2p+1 is odd:*

$$\sum_{l=0}^{\frac{d-1}{2}} \frac{(-1)^{\frac{d-1}{2}-l} b^{\frac{d-1}{2}-l}}{(\frac{d-1}{2} - l)!} a_l(x_0, \sigma) = c_d^{-1} |W|^{-1} Z_{\mathbb{X}}(\frac{\sigma}{2}, 0).$$

Proof. Immediate from Eq (3.11).

Remark 3.6. *In the particular case when $z = 0$ and $x = x_0$ in the origin of $\mathbb{X} = G/K$, the results presented here can be considered as a generalization of those obtained for the zeta function in the case of hyperbolic spaces H^d. In fact, Camporesi [14] obtained the following equality for $Re(\sigma) > d$*

$$\zeta_{H^d}(\sigma/2) = C_d' \int_0^\infty (\lambda^2 + \rho^2)^{-\sigma/2}|c(\lambda)|^{-2}d\lambda = C_d'|W|w^*(0, x_0, \sigma, b),$$

where C_d' is a certain constant depending on d. Consequently, the meromorphic continuation of $w^*(0, x_0, \sigma, b)$ will follow from that of $\zeta_{H^d}(\sigma)$.

Remark 3.7. *From the above, it is clear that it is possible to express the asymptotic expansion of the kernel $w(t, x, \sigma)$ using the zeta function, which explains in another way the integral limit* (1.13). *Furthermore, from the formula* (1.12)*, we recall that we have*

$$w(t, x, \sigma) = p^*(r - it)(\sigma), \quad \textit{for } Re(\sigma) > d,$$

therefore, a small time asymptotic expansion of the kernel $p(r - it)$ can make it possible to determine the poles of the wave kernel $w(t, x, \sigma)$ and to specify their locations.

4. Conclusions

It is natural to study the Mellin transform associated to the main evolution equations: The heat equation, the wave equation, and the Schrödinger equation, since all the Euclidean and Riemannian zeta-theories, and others are essentially based on the spatial and time behavior of the heat kernel, and are defined via the Mellin transform of this kernel.

In this paper, we considered the Mellin transform $w^*(z, x, \sigma, b)$ associated to the wave kernel on Riemannian symmetric spaces of the non-compact type. We proved that it extends to a meromorphic function on the entire complex plane $\mathbb{C}$ with a finite number of simple poles only on the real line. We showed in particular that under special regularity conditions, this transform is proportional to the main zeta function $Z_{\mathbb{X}}(\frac{\sigma}{2}, a)$. Furthermore, in the odd-dimensional case, we proved that the coefficients $\tilde{a}_j$ of the short time asymptotic expansion of the wave kernel are also proportional to this main zeta function.

Anker and Zhang [1] established sharp pointwise kernel estimates and dispersive properties for the wave equation on non-compact symmetric spaces of general rank under weaker smoothness conditions; $\mathrm{Re}(\sigma) = \frac{d+1}{2}$. In future work, we intend to reproduce the same study presented in this paper by investigating the mentioned estimates with $\mathrm{Re}(\sigma) = \frac{d+1}{2}$. Also, in the special case when G is a complex semisimple Lie group, the Harish-Chandra c-function and the spherical function φ_λ have elementary expressions, which allows us to analyze accurately the wave kernel $w(t, x, \sigma)$ and, consequently, its Mellin transform $w^*(z, x, \sigma, b)$ in order to find a more explicit expression of the coefficients $a_j(x, \sigma)$.

Certainly, a study of the Mellin transform associated to the wave kernel in higher rank symmetric spaces is now still open, and the questions at this level will be plenty and allow us to address several mathematical and physical applications.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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