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Research article

Stability and convergence analysis for a uniform temporal high accuracy of the time-fractional diffusion equation with 1D and 2D spatial compact finite difference method

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Abstract: The 1D and 2D spatial compact finite difference schemes (CFDSs) for time-fractional diffusion equations (TFDEs) were presented in this article with uniform temporal convergence order. Based on the idea of the modified block-by-block method, the CFDSs with uniform temporal convergence order for TFDEs were given by combining the fourth-order CFDSs in space and the high order scheme in time. The stability analysis and convergence order of CFDSs with uniform convergence order in time for TFDEs strictly proved that the provided uniform accuracy time scheme is $(3 - \alpha)$ temporal order and spatial fourth-order, respectively. Ultimately, the astringency of 1D and 2D spatial CFDSs was verified by some numerical examples.

Keywords: compact finite difference method; stability analysis; time-fractional diffusion equation; convergence analysis; optimal convergence order
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1. Introduction

The fractional differential equations can more accurately describe real materials with memory and genetic properties than classical equations, simulating many physical processes, and have received increasing attention in recent decades. Due to their potential science value and engineering applications, the fractional partial differential equations have become hot research topics for scholars and are already widely applied in classical mechanics, astrophysics, quantum mechanics, and other science and engineering fields [1].

In certain special cases, analytical solutions to fractional diffusion equations can be constructed using Laplace and Fourier transforms, and these analytical solutions contain infinite series. Therefore, constructing an efficient numerical solution for fractional differential equations is a very important research topic. In recent years, many effective numerical methods have been constructed to solve weak singularity TFDEs, such as the spectral method [2–5], the L1-type scheme [6–8], and the L2-type scheme [9–14], etc.

In this article, we consider the following TFDEs in the interval $\Omega = \prod_{i=1}^{d}[a_i, b_i]$,

$$\begin{cases} {}_0D_t^{\alpha} v(\mathbf{x}, t) - \sum_{i=1}^{d} \kappa_i \partial_{x_i}^2 v(\mathbf{x}, t) = f(\mathbf{x}, t), & \mathbf{x} \in \Omega, t \in (0, T], \quad (1.1a) \\ v(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega, t \in (0, T], \quad (1.1b) \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}), & \mathbf{x} \in \Omega, \quad (1.1c) \end{cases}$$

where $T > 0$ is a bounded constant, $\mathbf{x} = (x_1, x_2, \cdots, x_d) \in R^d$, $\partial\Omega$ is the boundary of Ω, $0 < \alpha < 1$ and ${}_0D_t^{\alpha}$ is an α order temporal Caputo fractional derivative defined by

$${}_0D_t^{\alpha} v(\mathbf{x}, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial v(\mathbf{x}, \tau)}{\partial \tau} d\tau,$$

with $\Gamma(\cdot)$ being Eulers gamma function. $\kappa_i, i = 1, 2, \cdots, d$ are the generalized diffusion constants, $v_0(\mathbf{x})$ and $f(\mathbf{x}, t)$ are known smooth functions.

The application of the CFDSs offer numerical high accuracy, making it suitable for fractional differential equations. In [15], Alikhanov, Beshtokov, and Mehra gave a Crank-Nicolson type CFDS for a time-fractional Hallaire equation. The 2D multi-term fractional wave equation was solved by a fast CFDS with the temporal second-order in [16]. A numerical scheme was constructed for the TFDEs by using the temporal L1 scheme on graded meshes and the spatial CFDS in [17]. An alternating direction implicit CFDS for the 2D multi-term TFDEs were given in [18]. In [19], a high-order CFDS was proposed for solving 2D nonlinear TFDEs with apatial fourth order. Based on a proper orthogonal decomposition technique, the TFDEs were solved by a low-dimensional graded mesh CFDS in [20]. The multi-term fractional sub-diffusion equations with the Dirichlet boundary conditions were solved by a fast CFDS with graded meshes based on the sum-of-exponentials approximation method in [21]. An unequal time-steps second-order fast CFDS for sub-diffusion problems was established by the sum-of-exponentials technique in [22]. A CFDS for TFDEs with nonhomogeneous Neumann boundary conditions was given in [23]. The 2D nonlinear fractional partial integro-differential equation with a weakly singular kernel was solved by using the time two-grid finite difference algorithm in [24]. The 3D nonlocal evolution equation with a weakly singular kernel was solved by the first order fractional convolution quadrature scheme and backward Euler alternating direction implicit method in [25]. In [26], they gave an efficient numerical algorithm for the fourth-order nonlocal evolution equation with a weakly singular kernel by using second-order fractional convolution quadrature rule and the L1 method. In [27], they gave the fast CFDS for the fourth-order TFDEs by using the sum-of-exponentials.

Constructing high-order numerical schemes for TFDEs is a hot research topic. In [28], they proposed a novel numerical approximate method for the Caputo fractional derivative by using the piecewise linear and quadratic Lagrange interpolation functions with $(3-\alpha)$ order for the linear function f. In [29], they constructed the $(3-\alpha)$ order in time for the time fractional diffusion equation from the second time layer. Therefore, the main work is to propose an efficient high order time uniform numerical scheme for TFDEs (1.1a)–(1.1c) by using space fourth-order CFDS and $(3-\alpha)$ order time scheme. In this paper, we construct a novel fully discrete numerical scheme for TFDEs with uniform convergence order, which can be used for the general nonlinear function f. The modified block-by-block method is introduced to the discrete time fractional derivative, and the fourth-order CFDS is used to approximate the second derivative of space. By combining the spatial fourth-order CFDS and temporal $(3-\alpha)$ order scheme, we obtain the fully discrete numerical scheme for TFDEs with uniform convergence order in time. The numerical scheme will be established in this article to provide a paradigm for establishing high-order TFDEs and analyzing its convergence and stability of the high-order numerical scheme in time. It can also provide readers with a feasible method for constructing high order time numerical schemes and their theoretical analysis for similar TFDEs.

The following is the composition of this article. In Section 2, we introduce 1D spatial CFDS for the TFDEs. In Section 3, we provide stability and convergence analysis of 1D spatial CFDS for the TFDEs. In Section 4, we introduce 2D spatial CFDS for the TFDEs and its stability and convergence results. Some numerical results of 1D and 2D spatial CFDSs are given in Section 5. In Section 6, the work in this paper is summarized.

2. A 1D spatial CFDS for TFDEs

Now, we will give detailed instructions to construct a fourth order 1D CFDS for the spatial derivative and high order uniform accuracy scheme for the temporal derivative of (1.1a) with initial (1.1b) and boundary conditions (1.1c) for d=1. Let M, N be two positive integers and $\Delta x = \frac{b_1-a_1}{M}, \Delta t = \frac{T}{N}$, denote $x_i = a_1 + i\Delta x$ $(0 \le i \le M), t_k = k\Delta t$ $(0 \le k \le N)$, $\Omega_{\Delta x} = \{x_i | \ 0 \le i \le M\}$, $\Omega_{\Delta t} = \{t_k | \ 0 \le k \le N\}$. Let $V_{\Delta x} = \{v | v = (v_0, \cdots, v_M), v_0 = v_M = 0\}$ be grid function space defined on $\Omega_{\Delta x}$, then for any grid function $v \in V_{\Delta x}$, the following differential operators and compact differential operators are introduced,

$$\delta_x v_{j-\frac{1}{2}} = \frac{v_j - v_{j-1}}{\Delta x}, \ \ \delta_x^2 v_j = \frac{\delta_x v_{j+\frac{1}{2}} - \delta_x v_{j-\frac{1}{2}}}{\Delta x}. \tag{2.1}$$

$$H_{\Delta x} v_j = \begin{cases} (I + \frac{\Delta x^2}{12}\delta_x^2) v_j, & j = 1, 2, \cdots, M-1, \\ v_j, & j = 0, \text{ or } \ j = M, \end{cases} \tag{2.2}$$

where I is the identical operator. For $j = 1, 2, \cdots, M-1$, it is easy to know that

$$H_{\Delta x} v_j = \frac{v_{j-1} + 10 v_j + v_{j+1}}{12}.$$

For $\forall v, w \in V_{\Delta x}$, the discrete inner product and the discrete norms are defined as follows:

$$\langle v, w \rangle = \Delta x \sum_{j=1}^{M} (\delta_x v_{j-\frac{1}{2}})(\delta_x w_{j-\frac{1}{2}}) - \frac{\Delta x^2}{12} \Delta x \sum_{j=1}^{M-1} (\delta_x^2 v_j)(\delta_x^2 w_j), \tag{2.3}$$

$$||v||_\infty = \max_{1 \le j \le M-1} |v_j|, \ |v|_1 = \sqrt{\Delta x \sum_{j=1}^{M} (\delta_x v_{j-\frac{1}{2}})^2}, \ ||\delta_x^2 v|| = \sqrt{\Delta x \sum_{j=1}^{M-1} (\delta_x^2 v_j)^2}. \tag{2.4}$$

To start, in order to analyze the CFDS of TFDE convergence and stability with spatial fourth order, some useful lemmas for CFDS convergence and stability analysis will be introduced in following parts.

Lemma 2.1. *[30] For $\forall v \in V_{\Delta x}$, the $\langle v, v\rangle$ defined by (2.3) satisfies the following inequality*

$$\frac{2}{3}|v|_1^2 \leq \langle v, v\rangle \leq |v|_1^2.$$

Lemma 2.2. *[31] For $\forall v \in V_{\Delta x}$, the $\|v\|_\infty$ defined by (2.3) satisfies the following inequality*

$$\|v\|_\infty \leq \frac{1}{2}\sqrt{a_2 - a_1}|v|_1.$$

Lemma 2.3. *[32] Suppose $g(x) \in C^6[a_1, a_2]$ and denote $\zeta(\tau) = 5(1-\tau)^3 - 3(1-\tau)^5$, and then we have*

$$H_{\Delta x} g''(x_j) = \delta_x^2 g(x_j) + \frac{\Delta x^4}{360}\int_0^1 [g^{(6)}(x_j - \tau\Delta x) + g^{(6)}(x_j + \tau\Delta x)]\zeta(\tau)d\tau, 1 \leq j \leq M-1.$$

According to [11], let

$$\alpha_0 = \Gamma(3-\alpha)\Delta t^\alpha, \tag{2.5}$$

we will introduce an efficient temporal high order numerical scheme to discrete ${}_0D_t^\alpha v(x, t_k)$ with simplification by omitting the dependence of ${}_0D_t^\alpha v(x, t_k)$ on x as follows:

$${}_0D_t^\alpha v(t_k) = {}_0D_{\Delta t}^\alpha v(t_k) + r_k(t_k), \quad \forall k \geq 1, \tag{2.6}$$

where

$${}_0D_{\Delta t}^\alpha v(t_k) = \begin{cases} [\hat{D}_0 v(t_0) + \hat{D}_1 v(t_1) + \hat{D}_2 v(t_2)]/\alpha_0, & k = 1, \\ [\tilde{D}_0 v(t_0) + \tilde{D}_1 v(t_1) + \tilde{D}_2 v(t_2)]/\alpha_0, & k = 2, \\ [\bar{A}_k v(t_0) + \bar{B}_k v(t_1) + \bar{C}_k v(t_2) \\ + \sum_{i=1}^{k-1}(A_i v(t_{k-i-1}) + B_i v(t_{k-i}) + C_i v(t_{k-i+1}))]/\alpha_0, & k \geq 3, \end{cases} \tag{2.7}$$

and

$$\begin{aligned} &\hat{D}_0 = \frac{3\alpha - 4}{2}, \hat{D}_1 = 2(1-\alpha), \hat{D}_2 = \frac{1}{2}\alpha, \\ &\tilde{D}_0 = \frac{1}{2^\alpha}(3\alpha - 2), \tilde{D}_1 = -\frac{4\alpha}{2^\alpha}, \tilde{D}_2 = \frac{\alpha + 2}{2^\alpha}, \\ &\bar{A}_k = -\frac{(\alpha-2)(k-1)^{1-\alpha}}{2} + \frac{3(\alpha-2)k^{1-\alpha}}{2} - (k-1)^{2-\alpha} + k^{2-\alpha}, \\ &\bar{B}_k = -2(\alpha-2)k^{1-\alpha} - 2k^{2-\alpha} + 2(-1+k)^{2-\alpha}, \\ &\bar{C}_k = \frac{1}{2}(\alpha-2)[k^{1-\alpha} + (-1+k)^{1-\alpha}] + k^{2-\alpha} - (-1+k)^{2-\alpha}, \\ &A_i = \frac{(\alpha-2)[(-1+i)^{1-\alpha} + i^{1-\alpha}]}{2} - (-1+i)^{2-\alpha} + i^{2-\alpha}, \\ &B_i = 2[(2-\alpha)(i-1)^{1-\alpha} + (i-1)^{2-\alpha} - i^{2-\alpha}], \\ &C_i = -\frac{3(2-\alpha)}{2}(i-1)^{1-\alpha} + \frac{2-\alpha}{2}i^{1-\alpha} - (i-1)^{2-\alpha} + i^{2-\alpha}. \end{aligned} \tag{2.8}$$

According to Theorem 2.1 in [11], it is easy to obtain the temporal error estimate of the above proposed high order efficient numerical method (2.7) of the temporal fractional derivative.

Lemma 2.4. *[11] Suppose $v(t) \in C^3[0,T]$ and the numerical approximation error of the time fractional derivative satisfies*

$$|r_k(t_k)| = |{}_0D_t^\alpha v(t_k) - {}_0D_{\Delta t}^\alpha v(t_k)| \leq c_v \Delta t^{3-\alpha}, 0 < \alpha < 1, \forall k \geq 1,$$

where $c_v > 0$ is an independent of Δt constant.

Considering the Eq (1.1a) at the point (x_j, t_k), we have

$$ {}_0D_t^\alpha v(x_j, t_k) - \kappa_1 \partial_x^2 v(x_j, t_k) = f(x_j, t_k). \tag{2.9}$$

Bringing operator $H_{\Delta x}$ to the Eq (2.9), the CFDS is immediately established for TFDEs in space as follows:

$$H_{\Delta x}({}_0D_t^\alpha v(x_j, t_k)) - \kappa_1 H_{\Delta x}(\partial_x^2 v(x_j, t_k)) = H_{\Delta x}(f(x_j, t_k)).$$

To provide a concise description of the high-order numerical scheme, we introduce the following grid functions:

$$v_j^k = v(x_j, t_k), \quad f_j^k = f(x_j, t_k), \quad k = 0, 1, 2, \cdots, N, \quad j = 0, 1, 2, \cdots, M.$$

Suppose $v(x,t) \in C_{6,3}^{x,t}([a_1, a_2] \times [0,T])$ and use Lemmas 2.3 and 2.4. One can immediately obtain

$$H_{\Delta x}({}_0D_{\Delta t}^\alpha v(x_j, t_k)) - \kappa_1 \delta_x^2 v_j^k = H_{\Delta x}(f_j^k) + R_j^k, \tag{2.10}$$

with

$$R_j^k = \kappa_\alpha \frac{\Delta x^4}{360} \int_0^1 [\frac{\partial^6 v}{\partial x^6}(x_j - \tau\Delta x, t_k) + \frac{\partial^6 v}{\partial x^6}(x_j + \tau\Delta x, t_k)]\zeta(\tau) d\tau - H_{\Delta x}(r_k(x_j, t_k)),$$

and R_j^k satisfies the following:

$$|R_j^k| \leq C(\Delta t^{3-\alpha} + \Delta x^4), \tag{2.11}$$

where $C > 0$ is a constant and independent on Δt and Δx.

Eliminating infinitesimal quantities R_j^k, one can immediately obtain the high order efficient CFDS for the Eqs (1.1a)–(1.1c) as follows:

$$\begin{cases} H_{\Delta x}(\hat{D}_0 v_j^0 + \hat{D}_1 v_j^1 + \hat{D}_2 v_j^2) - \alpha_0 \kappa_1 \delta_x^2 v_j^1 = \alpha_0 H_{\Delta x}(f_j^1), & k = 1, \\ H_{\Delta x}(\tilde{D}_0 v_j^0 + \tilde{D}_1 v_j^1 + \tilde{D}_2 v_j^2) - \alpha_0 \kappa_1 \delta_x^2 v_j^2 = \alpha_0 H_{\Delta x}(f_j^2), & k = 2, \\ H_{\Delta x}[\bar{A}_k v_j^0 + \bar{B}_k v_j^1 + \bar{C}_k v_j^2 & \\ + \sum_{i=1}^{k-1} (A_i v_j^{k-i-1} + B_i v_j^{k-i} + C_i v_j^{k-i+1})] - \alpha_0 \kappa_1 \delta_x^2 v_j^k = \alpha_0 H_{\Delta x}(f_j^k), \ k \geq 3. & \end{cases} \tag{2.12}$$

For the sake of CFDS (2.12)'s stability and convergence analysis, we first rewrite it for $k \geq 4$ into the following equivalent form:

$$H_{\Delta x}(v_j^k - \sum_{i=1}^{k} d_{k-i}^k v_j^{k-i}) - \alpha_0 C_1^{-1} \kappa_1 \delta_x^2 v_j^k = \alpha_0 C_1^{-1} H_{\Delta x}(f_j^k), \ \ 4 \leq k \leq N, \tag{2.13}$$

where

$$d_0^k=-\frac{\bar{A}_k+A_{k-1}}{C_1},\ d_1^k=-\frac{A_{k-2}+B_{k-1}+\bar{B}_k}{C_1},d_2^k=-\frac{\bar{C}_k+A_{k-3}+B_{k-2}+C_{k-1}}{C_1},$$
$$d_{k-1}^k=-\frac{B_1+C_2}{C_1},d_{k-2}^k=-\frac{A_1+B_2+C_3}{C_1},\ d_{k-i}^k=-\frac{B_i+C_{i+1}+A_{i-1}}{C_1},\ 3\le i\le k-3.$$

For $k=3$, we similarly rewrite (2.12) into the following form:

$$H_{\Delta x}(v_j^3-d_2^3v_j^2-d_1^3v_j^1-d_0^3v_j^0)-\alpha_0C_1^{-1}\kappa_1\delta_x^2v_j^3=\alpha_0C_1^{-1}H_{\Delta x}(f_j^3),\ k=3, \tag{2.14}$$

where

$$d_2^3=-\frac{\bar{C}_3+B_1+C_2}{C_1},\ \ d_1^3=-\frac{\bar{B}_3+A_1+B_2}{C_1},\ \ d_0^3=-\frac{\bar{A}_3+A_2}{C_1}.$$

According to (2.13) and (2.14), the full discrete high order efficient CFDS of (2.12) is equivalently rewritten as follows:

$$\begin{cases} H_{\Delta x}(\hat{D}_0v_j^0+\hat{D}_1v_j^1+\hat{D}_2v_j^2)-\alpha_0\kappa_1\delta_x^2v_j^1=\alpha_0H_{\Delta x}(f_j^1),\ \ k=1, & \text{(2.15a)}\\ H_{\Delta x}(\tilde{D}_0v_j^0+\tilde{D}_1v_j^1+\tilde{D}_2v_j^2)-\alpha_0\kappa_1\delta_x^2v_j^2=\alpha_0H_{\Delta x}(f_j^2),\ \ k=2, & \text{(2.15b)}\\ H_{\Delta x}(v_j^3-d_2^3v_j^2-d_1^3v_j^1-d_0^3v_j^0)-\alpha_0C_1^{-1}\kappa_1\delta_x^2v_j^3=\alpha_0C_1^{-1}H_{\Delta x}(f_j^3),\ \ k=3, & \text{(2.15c)}\\ H_{\Delta x}(v_j^k-\sum\limits_{i=1}^{k}d_{k-i}^kv_j^{k-i})-\alpha_0C_1^{-1}\kappa_1\delta_x^2v_j^k=\alpha_0C_1^{-1}H_{\Delta x}(f_j^k),\ 4\le k\le N. & \text{(2.15d)} \end{cases}$$

In order to analyze the stability and convergence analysis of (2.15), we will first analyze the properties for the coefficients d_{k-i}^k of (2.15) in the following Lemma 2.5.

Lemma 2.5. *[11] For $\forall\alpha\in(0,1),k\ge4$, the scheme (2.15d)'s coefficients satisfy the following inequalities:*

$(1)\frac{3}{2}<C_1=\frac{4-\alpha}{2}<2$; $(2)\sum\limits_{i=1}^{k}d_{k-i}^k=1$; $(3)d_{k-i}^k>0,3\le i\le k$;

$(4)0<d_{k-1}^k<\frac{4}{3}$; $(5)d_{k-2}^k$ *has positive and negative values;* $(6)d_{k-2}^k+\frac{1}{4}(d_{k-1}^k)^2>0$.

From Lemma 2.5, we know that the symbol of coefficient d_{k-2}^k is uncertain when $\alpha\in(0,1)$. Due to the uncertainly symbol of coefficient d_{k-2}^k, it will be very difficult to analyze the unconditional stability and convergence of the high order efficient CFDS by using the classical analysis method. Therefore, we will use a novel technique for the provided scheme's stability and convergence analysis for all $\alpha\in(0,1)$ as follows.

In the following, we will use a new method for the unconditional stability and convergence analysis of the numerical scheme (2.15) and denote

$$\theta=\frac{1}{2}d_{k-1}^k. \tag{2.16}$$

By introducing parameters into the Eq (2.15d) and rewriting Eq (2.15d), one can get

$$v_j^k-\sum_{i=1}^{k}d_{k-i}^kv_j^{k-i}=v_j^k-d_{k-1}^kv_j^{k-1}-d_{k-2}^kv_j^{k-2}-\cdots-d_1^kv_j^1-d_0^kv_j^0$$

$$= (v_j^k - \theta v_j^{k-1}) - \theta(v_j^{k-1} - \theta v_j^{k-2}) - (\theta^2 + d_{k-2}^k)v_j^{k-2} - d_{k-3}^k v_j^{k-3} - \cdots - d_1^k v_j^1 - d_0^k v_j^0$$
$$= (v_j^k - \theta v_j^{k-1}) - \theta(v_j^{k-1} - \theta v_j^{k-2}) - (\theta^2 + d_{k-2}^k)(v_j^{k-2} - \theta v_j^{k-3}) - (\theta^3 + \theta d_{k-2}^k + d_{k-3}^k)(v_j^{k-3} - \theta v_j^{k-4})$$
$$- \cdots - (\theta^{k-2} + \theta^{k-4} d_{k-2}^k + \cdots + \theta d_3^k + d_2^k)(v_j^2 - \theta v_j^1) - (\theta^{k-1} + \theta^{k-3} d_{k-2}^k + \cdots + \theta d_2^k + d_1^k)(v_j^1 - \theta v_j^0)$$
$$-(\theta^k + \theta^{k-2} d_{k-2}^k + \cdots + \theta d_1^k + d_0^k)v_j^0.$$

For the sake of conciseness, we also introduce the following notations

$$\bar{d}_{k-i}^k = \theta^i + \sum_{j=2}^{i} \theta^{i-j} d_{k-j}^k, i = 2, 3, \cdots, k, \tag{2.17}$$

$$\bar{v}_j^i = v_j^i - \theta v_j^{i-1}, i = 1, 2, \cdots, k, \tag{2.18}$$

$$v_j^k - \sum_{i=1}^{k} d_{k-i}^k v_j^{k-i} = \bar{v}_j^k - \theta \bar{v}_j^{k-1} - \sum_{i=2}^{k-1} \bar{d}_{k-i}^k \bar{v}_j^{k-i} - \bar{d}_0^k v_j^0 = \bar{v}_j^k - \sum_{i=1}^{k} \bar{d}_{k-i}^k \bar{v}_j^{k-i}. \tag{2.19}$$

Using the above Eqs (2.17)–(2.19), Eq (2.15d) can be equivalent as follows:

$$H_{\Delta x}(\bar{v}_j^k - \sum_{i=1}^{k} \bar{d}_{k-i}^k \bar{v}_j^{k-i}) - \alpha_0 C_1^{-1} \kappa_1 \delta_x^2 v_j^k = \alpha_0 C_1^{-1} H_{\Delta x}(f_j^k). \tag{2.20}$$

For $k = 3$, using the same method for (2.20), the equivalent of (2.15c) is as follows:

$$H_{\Delta x}(\bar{v}_j^3 - \bar{d}_2^3 \bar{v}_j^2 - \bar{d}_1^3 \bar{v}_j^1 - \bar{d}_0^3 v_j^0) - \alpha_0 C_1^{-1} \kappa_\alpha \delta_x^2 v_j^3 = \alpha_0 C_1^{-1} H_{\Delta x}(f_j^3). \tag{2.21}$$

Combining (2.20) and (2.21), the equivalent form for the high-order uniform convergence accuracy numerical scheme (2.15) is as follows:

$$\begin{cases} H_{\Delta x}(\hat{D}_0 v_j^0 + \hat{D}_1 v_j^1 + \hat{D}_2 v_j^2) - \alpha_0 \kappa_1 \delta_x^2 v_j^1 = \alpha_0 H_{\Delta x}(f_j^1),\ k = 1, & (2.22a) \\ H_{\Delta x}(\tilde{D}_0 v_j^0 + \tilde{D}_1 v_j^1 + \tilde{D}_2 v_j^2) - \alpha_0 \kappa_1 \delta_x^2 v_j^2 = \alpha_0 H_{\Delta x}(f_j^2),\ k = 2, & (2.22b) \\ H_{\Delta x}(\bar{v}_j^3 - \bar{d}_2^3 \bar{v}_j^2 - \bar{d}_1^3 \bar{v}_j^1 - \bar{d}_0^3 v_j^0) - \alpha_0 C_1^{-1} \kappa_1 \delta_x^2 v_j^3 = \alpha_0 C_1^{-1} H_{\Delta x}(f_j^3),\ k = 3, & (2.22c) \\ H_{\Delta x}(\bar{v}_j^k - \sum_{i=1}^{k} \bar{d}_{k-i}^k \bar{v}_j^{k-i}) - \alpha_0 C_1^{-1} \kappa_1 \delta_x^2 v_j^k = \alpha_0 C_1^{-1} H_{\Delta x}(f_j^k),\ 4 \le k \le N. & (2.22d) \end{cases}$$

According to the Lemma 3.2 in [11], one can immediately obtain that the coefficients in (2.22d) satisfy the following Lemma 2.6.

Lemma 2.6. *When $k \geq 4$ for all $\alpha \in (0, 1)$, the coefficients in the numerical scheme* (2.22d) *satisfy*

$$(1)\theta \in (0, \frac{2}{3});\quad (2)\bar{d}_{k-i}^k > 0, i = 2, 3, \cdots, k;\quad (3)0 < \theta + \sum_{i=2}^{k-1} \bar{d}_{k-i}^k + \bar{d}_0^k \leq 1.$$

Because of $\theta \neq d_2^3 - \theta$, the numerical scheme (2.22c) and (2.22d) cannot write the unifying form, where

$$\bar{d}_2^3 = d_2^3 - \theta, \quad \bar{d}_1^3 = \bar{d}_2^3 \theta + d_1^3, \quad \bar{d}_0^3 = \bar{d}_1^3 \theta + d_0^3.$$

According to the Lemma 3.3 in [11], we can immediately get the coefficients in (2.22c) for $k = 3$, satisfying the following Lemma 2.7.

Lemma 2.7. *For $k = 3$, $\forall 0 < \alpha < 1$, the* (2.22c)*'s coefficients have properties as follows:*
$(1)\bar{d}_2^3 > 0,\ \bar{d}_1^3 > 0,\ \bar{d}_0^3 > 0$; $\quad (2)\bar{d}_2^3 - \theta < 0$; $\quad (3) 0 < \bar{d}_0^3 + \bar{d}_1^3 + \bar{d}_2^3 \le 1$.

To prove convergence, we will provide a lower bound estimate for $\bar{d}_0^k$ in the following Lemma 2.8.

Lemma 2.8. $\forall 0 < \alpha < 1$ *and $k \ge 3$, the coefficient $\bar{d}_0^k$ satisfies*

$$\bar{d}_0^k \ge \frac{2}{3}(2-\alpha)(1-\alpha)k^{-\alpha}C_1^{-1}, \tag{2.23}$$

where C_1 is defined by (1) in Lemma 2.5.

Proof. First, according to (2.17), it is obvious that $\bar{d}_0^k \ge d_0^k = -\frac{\bar{A}_k + A_{k-1}}{C_1}$. Next, we will estimate $-(\bar{A}_k + A_{k-1}) = \frac{2-\alpha}{2}[(k-2)^{1-\alpha} + 3k^{1-\alpha}] + (k-2)^{2-\alpha} - k^{2-\alpha}$. Let $k - 1 = \tilde{x}$, then $x \ge 2$. We obtain

$$\begin{aligned}
&-(\bar{A}_k + A_{k-1}) = \frac{1}{2}(2-\alpha)[(-1+\tilde{x})^{1-\alpha} + 3(\tilde{x}+1)^{1-\alpha}] + (\tilde{x}-1)^{2-\alpha} - (1+\tilde{x})^{2-\alpha} \\
= \ &\frac{2-\alpha}{2}\tilde{x}^{1-\alpha}[(1-\frac{1}{\tilde{x}})^{1-\alpha} + 3(1+\frac{1}{\tilde{x}})^{1-\alpha}] + \tilde{x}^{2-\alpha}[(1-\frac{1}{\tilde{x}})^{2-\alpha} - (1+\frac{1}{\tilde{x}})^{2-\alpha}] \\
= \ &\frac{2-\alpha}{2}\tilde{x}^{1-\alpha}\{1 + \frac{1}{1!}(1-\alpha)(-\frac{1}{\tilde{x}}) + \frac{1}{2!}(1-\alpha)(-\alpha)(-\frac{1}{\tilde{x}})^2 + \cdots \\
&+3[1 + \frac{1}{1!}(1-\alpha)(\frac{1}{\tilde{x}}) + \frac{1}{2!}(1-\alpha)(-\alpha)(\frac{1}{\tilde{x}})^2 + \cdots]\} \\
&+\tilde{x}^{2-\alpha}\{1 + \frac{2-\alpha}{1!}(-\frac{1}{\tilde{x}}) + \frac{(2-\alpha)(1-\alpha)}{2!}(-\frac{1}{\tilde{x}})^2 + \cdots \\
&-[1 + \frac{2-\alpha}{1!}(\frac{1}{\tilde{x}}) + \frac{(2-\alpha)(1-\alpha)}{2!}(\frac{1}{\tilde{x}})^2 + \cdots]\} \\
= \ &\frac{2-\alpha}{2}\tilde{x}^{1-\alpha}[\sum_{i=0}^{+\infty}\frac{\prod_{n=0}^{i}(1-\alpha-n)}{(i+1)!}(-\frac{1}{\tilde{x}})^{i+1} + 3\sum_{i=0}^{+\infty}\frac{\prod_{n=0}^{i}(1-\alpha-n)}{(i+1)!}(\frac{1}{\tilde{x}})^{i+1}] \\
&+(2-\alpha)\tilde{x}^{2-\alpha}[\sum_{i=0}^{+\infty}\frac{\prod_{n=0}^{i}(1-\alpha-n)}{(i+2)!}(-\frac{1}{\tilde{x}})^{i+2} - \sum_{i=0}^{+\infty}\frac{\prod_{n=0}^{i}(1-\alpha-n)}{(i+2)!}(\frac{1}{\tilde{x}})^{i+2}] \\
\doteq \ &\frac{1}{2}(2-\alpha)\tilde{x}^{-\alpha}\sum_{i=0}^{+\infty}\hat{b}_i,
\end{aligned}$$

where $\hat{b}_i = \prod_{n=0}^{i}(1-\alpha-n)\frac{(-1)^{i+1}i+3i+4}{(i+2)!}(\frac{1}{\tilde{x}})^i$. By careful calculation, we know $\sum_{i=2}^{+\infty}\hat{b}_i$ is a convergent alternating series. Furthermore, $\hat{b}_2 > 0$. Therefore, we have $0 \le \sum_{i=2}^{+\infty}\hat{b}_i \le \hat{b}_2$, and

$$\begin{aligned}
&-(\bar{A}_k + A_{k-1}) = \frac{2-\alpha}{2}\tilde{x}^{-\alpha}(\hat{b}_0 + \hat{b}_1 + \sum_{i=2}^{+\infty}\hat{b}_i) \ge \frac{2-\alpha}{2}\tilde{x}^{-\alpha}(\hat{b}_0 + \hat{b}_1) \\
&= \frac{2-\alpha}{2}\tilde{x}^{-\alpha}[2(1-\alpha) + (1-\alpha)(-\alpha)\cdot\frac{8}{3!\cdot\tilde{x}}] = (2-\alpha)(1-\alpha)\tilde{x}^{-\alpha}(1-\frac{2\alpha}{3\tilde{x}}) \ge \frac{2}{3}(2-\alpha)(1-\alpha)k^{-\alpha}.
\end{aligned}$$

Combining the expression of $\bar{d}_0^k$, we have already proved (2.23). □

3. Stability and convergence analysis

In order to analyze the stability, we take the right function $f(x,t) \equiv 0$. Therefore, scheme (2.22) becomes

$$\begin{cases} H_{\Delta x}(\hat{D}_0 v_j^0 + \hat{D}_1 v_j^1 + \hat{D}_2 v_j^2) - \alpha_0 \kappa_1 \delta_x^2 v_j^1 = 0, \quad k = 1, & (3.1a) \\ H_{\Delta x}(\tilde{D}_0 v_j^0 + \tilde{D}_1 v_j^1 + \tilde{D}_2 v_j^2) - \alpha_0 \kappa_1 \delta_x^2 v_j^2 = 0, \quad k = 2, & (3.1b) \\ H_{\Delta x}(\bar{v}_j^3 - \bar{d}_2^3 \bar{v}_j^2 - \bar{d}_1^3 \bar{v}_j^1 - \bar{d}_0^3 v_j^0) - \alpha_0 C_1^{-1} \kappa_1 \delta_x^2 v_j^3 = 0, \quad k = 3, & (3.1c) \\ H_{\Delta x}(\bar{v}_j^k - \sum_{i=1}^{k} \bar{d}_{k-i}^k \bar{v}_j^{k-i}) - \alpha_0 C_1^{-1} \kappa_1 \delta_x^2 v_j^k = 0, \quad 4 \le k \le N. & (3.1d) \end{cases}$$

First, we will give the estimation of $< \bar{v}^1, \bar{v}^1 > +\alpha_0 c_1^{-1} \kappa_1 \|\delta_x^2 v^1\|^2$ and $< \bar{v}^2, \bar{v}^2 > +\alpha_0 c_1^{-1} \kappa_1 \|\delta_x^2 v^2\|^2$ in the following Lemma 3.1.

Lemma 3.1. *Let*

$$\alpha_1 = \min\{-\tilde{D}_1 \hat{D}_1, \hat{D}_2 \tilde{D}_2, -\tilde{D}_1, \hat{D}_2\}, \quad \alpha_2 = \max\{\tilde{D}_1 \hat{D}_0, | - \hat{D}_2 \tilde{D}_0|\}, \tag{3.2}$$

and we have

$$\begin{cases} \langle \bar{v}^1, \bar{v}^1 \rangle + \alpha_0 C_1^{-1} \kappa_1 \|\delta_x^2 v^1\|^2 \le \beta \langle v^0, v^0 \rangle, & (3.3) \\ \langle \bar{v}^2, \bar{v}^2 \rangle + \alpha_0 C_1^{-1} \kappa_1 \|\delta_x^2 v^2\|^2 \le \beta \langle v^0, v^0 \rangle, & (3.4) \end{cases}$$

where β satisfies

$$\beta = \max\{8(\frac{\alpha_2}{\alpha_1})^2 + 2\theta^2, 8(\frac{\alpha_2}{\alpha_1})^2(1+\theta^2)\}. \tag{3.5}$$

Proof. Let's multiply $\tilde{D}_1 \Delta x \delta_x^2 v_j^1$ on both sides of (3.1a) for $k = 1$, and summing up for j from 1 to $M-1$,

$$\tilde{D}_1 \Delta x \sum_{j=1}^{M-1} (\delta_x^2 v_j^1)[(1 + \frac{\Delta x^2}{12}\delta_x^2)(\hat{D}_0 v_j^0 + \hat{D}_1 v_j^1 + \hat{D}_2 v_j^2)] - \tilde{D}_1 \alpha_0 \kappa_1 \Delta x \sum_{j=1}^{M-1} (\delta_x^2 v_j^1)^2 = 0. \tag{3.6}$$

We rewrite Eq (3.6) and obtain

$$\begin{aligned} &\tilde{D}_1 \Delta x \sum_{j=1}^{M-1} (\delta_x^2 v_j^1)(\hat{D}_0 v_j^0 + \hat{D}_1 v_j^1 + \hat{D}_2 v_j^2) \\ &+\tilde{D}_1 \frac{\Delta x^2}{12} \Delta x \sum_{j=1}^{M-1} (\delta_x^2 v_j^1)(\hat{D}_0 \delta_x^2 v_j^0 + \hat{D}_1 \delta_x^2 v_j^1 + \hat{D}_2 \delta_x^2 v_j^2) - \tilde{D}_1 \alpha_0 \kappa_1 \Delta x \sum_{j=1}^{M-1} (\delta_x^2 v_j^1)^2 = 0. \end{aligned} \tag{3.7}$$

We use the summation formula by parts and notice $v_0^k = v_M^k = 0$ for (3.7), then we obtain

$$\begin{aligned} &- \tilde{D}_1 \Delta x \sum_{j=1}^{M} (\delta_x v_{j-\frac{1}{2}}^1)(\hat{D}_0 \delta_x v_{j-\frac{1}{2}}^0 + \hat{D}_1 \delta_x v_{j-\frac{1}{2}}^1 + \hat{D}_2 \delta_x v_{j-\frac{1}{2}}^2) \\ &+ \tilde{D}_1 \frac{\Delta x^2}{12} \Delta x \sum_{j=1}^{M-1} (\delta_x^2 v_j^1)(\hat{D}_0 \delta_x^2 v_j^0 + \hat{D}_1 \delta_x^2 v_j^1 + \hat{D}_2 \delta_x^2 v_j^2) - \tilde{D}_0 \alpha_0 \kappa_1 \Delta x \sum_{j=1}^{M-1} (\delta_x^2 v_j^1)^2 = 0. \end{aligned} \tag{3.8}$$

By rearranging the left side of Eq (3.8), we can immediately obtain

$$
\begin{aligned}
&-\tilde{D}_1\hat{D}_0[\Delta x\sum_{j=1}^{M}(\delta_x v^1_{j-\frac{1}{2}})(\delta_x v^0_{j-\frac{1}{2}})-\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2 v^1_j)(\delta_x^2 v^0_j)]\\
&-\tilde{D}_1\hat{D}_1[\Delta x\sum_{j=1}^{M}(\delta_x v^1_{j-\frac{1}{2}})(\delta_x v^1_{j-\frac{1}{2}})-\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2 v^1_j)(\delta_x^2 v^1_j)]\\
&-\tilde{D}_1\hat{D}_2[\Delta x\sum_{j=1}^{M}(\delta_x v^1_{j-\frac{1}{2}})(\delta_x v^2_{j-\frac{1}{2}})-\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2 v^1_j)(\delta_x^2 v^2_j)]-\tilde{D}_1\alpha_0\kappa_1\Delta x\sum_{j=1}^{M-1}(\delta_x^2 v^1_j)^2=0. \quad (3.9)
\end{aligned}
$$

Using (2.3) and (2.4) for (3.9), we obtain

$$-\tilde{D}_1\hat{D}_0\langle v^1,v^0\rangle-\tilde{D}_1\hat{D}_1\langle v^1,v^1\rangle-\tilde{D}_1\hat{D}_2\langle v^1,v^2\rangle-\tilde{D}_1\alpha_0\kappa_1\|\delta_x^2 v^1\|^2=0. \quad (3.10)$$

Using the same method, multiplying $-\hat{D}_2\Delta x\delta_x^2 v_j^2$ on both sides of (3.1b) for $k=2$, and by summing variable j from 1 to $M-1$, one can obtain

$$\hat{D}_2\tilde{D}_0\langle v^2,v^0\rangle+\hat{D}_2\tilde{D}_1\langle v^2,v^1\rangle+\hat{D}_2\tilde{D}_2\langle v^2,v^2\rangle+\hat{D}_2\alpha_0\kappa_1\|\delta_x^2 v^2\|^2=0. \quad (3.11)$$

Taking (3.10) plus (3.11), one can obtain the following term:

$$-\tilde{D}_1\hat{D}_0\langle v^1,v^0\rangle+\hat{D}_2\tilde{D}_0\langle v^2,v^0\rangle-\tilde{D}_1\hat{D}_1\langle v^1,v^1\rangle+\hat{D}_2\tilde{D}_2\langle v^2,v^2\rangle-\tilde{D}_1\alpha_0\kappa_1\|\delta_x^2 v^1\|^2+\hat{D}_2\alpha_0\kappa_1\|\delta_x^2 v^2\|^2=0.$$

That is,

$$-\tilde{D}_1\hat{D}_1\langle v^1,v^1\rangle+\hat{D}_2\tilde{D}_2\langle v^2,v^2\rangle-\tilde{D}_1\alpha_0\kappa_1\|\delta_x^2 v^1\|^2+\hat{D}_2\alpha_0\kappa_1\|\delta_x^2 v^2\|^2=\langle\tilde{D}_1\hat{D}_0 v^1-\hat{D}_2\tilde{D}_0 v^2,v^0\rangle.$$

Because $-\tilde{D}_1\hat{D}_1$, $\hat{D}_2\tilde{D}_2$, $-\tilde{D}_1$, $\hat{D}_2$, and $\tilde{D}_1\hat{D}_0$ are all positive numbers depending on α, according to the definition of α_1 and α_2 in (3.2), we know $\alpha_1>0,\alpha_2>0$, and we get

$$\langle v^1,v^1\rangle+\langle v^2,v^2\rangle+\alpha_0\kappa_1\|\delta_x^2 v^1\|^2+\alpha_0\kappa_1\|\delta_x^2 v^2\|^2\le\frac{\alpha_2}{\alpha_1}[\langle v^1,v^0\rangle+\langle v^2,v^0\rangle]. \quad (3.12)$$

According to $\langle u,v\rangle\le\sqrt{\langle u,u\rangle}\cdot\sqrt{\langle v,v\rangle}$, (3.12) becomes

$$
\begin{aligned}
&\langle v^1,v^1\rangle+\langle v^2,v^2\rangle+\alpha_0\kappa_1\|\delta_x^2 v^1\|^2+\alpha_0\kappa_1\|\delta_x^2 v^2\|^2\\
&\le\frac{\alpha_2}{\alpha_1}\sqrt{\langle v^0,v^0\rangle}\cdot(\sqrt{\langle v^1,v^1\rangle}+\sqrt{\langle v^2,v^2\rangle})\\
&\le\frac{\alpha_2}{\alpha_1}\sqrt{\langle v^0,v^0\rangle}\cdot(\sqrt{\langle v^1,v^1\rangle}+\sqrt{\alpha_0\kappa_1}\|\delta_x^2 v^1\|+\sqrt{\langle v^2,v^2\rangle}+\sqrt{\alpha_0\kappa_1}\|\delta_x^2 v^2\|)\\
&=\frac{\alpha_2}{\alpha_1}\sqrt{\langle v^0,v^0\rangle}\cdot\sqrt{\langle v^1,v^1\rangle}+\frac{\alpha_2}{\alpha_1}\sqrt{\langle v^0,v^0\rangle}\cdot\sqrt{\alpha_0\kappa_1}\|\delta_x^2 v^1\|\\
&+\frac{\alpha_2}{\alpha_1}\sqrt{\langle v^0,v^0\rangle}\cdot\sqrt{\langle v^2,v^2\rangle}+\frac{\alpha_2}{\alpha_1}\sqrt{\langle v^0,v^0\rangle}\cdot\sqrt{\alpha_0\kappa_1}\|\delta_x^2 v^2\|\\
&\le\frac{1}{2}[(\frac{\alpha_2}{\alpha_1})^2\langle v^0,v^0\rangle+\langle v^1,v^1\rangle+(\frac{\alpha_2}{\alpha_1})^2\langle v^0,v^0\rangle+\alpha_0\kappa_1\|\delta_x^2 v^1\|^2
\end{aligned}
$$

$$+(\frac{\alpha_2}{\alpha_1})^2\langle v^0, v^0\rangle + \langle v^2, v^2\rangle + (\frac{\alpha_2}{\alpha_1})^2\langle v^0, v^0\rangle + \alpha_0\kappa_1\|\delta_x^2 v^2\|^2]$$
$$= \frac{1}{2}[4(\frac{\alpha_2}{\alpha_1})^2\langle v^0, v^0\rangle + \langle v^1, v^1\rangle + \langle v^2, v^2\rangle + \alpha_0\kappa_1\|\delta_x^2 v^1\|^2 + \alpha_0\kappa_1\|\delta_x^2 v^2\|^2].$$

Therefore, we have

$$\langle v^1, v^1\rangle + \langle v^2, v^2\rangle + \alpha_0\kappa_1\|\delta_x^2 v^1\|^2 + \alpha_0\kappa_1\|\delta_x^2 v^2\|^2 \le 4(\frac{\alpha_2}{\alpha_1})^2\langle v^0, v^0\rangle. \tag{3.13}$$

Because of $C_1 = \frac{4-\alpha}{2}$, we have $C_1^{-1} = \frac{2}{4-\alpha} \in (\frac{1}{2}, \frac{2}{3})$. Therefore, $C_1^{-1} \le 1$, $\alpha_0 C_1^{-1}\kappa_1 < \alpha_0\kappa_1$. From (3.13), we can get

$$\langle v^1, v^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^1\|^2 \le 4(\frac{\alpha_2}{\alpha_1})^2\langle v^0, v^0\rangle, \tag{3.14}$$
$$\langle v^2, v^2\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^2\|^2 \le 4(\frac{\alpha_2}{\alpha_1})^2\langle v^0, v^0\rangle. \tag{3.15}$$

According to (2.18): $\bar{v}^1 = v^1 - \theta v^0$ and $\sqrt{\langle a+b, a+b\rangle} \le \sqrt{\langle a, a\rangle} + \sqrt{\langle b, b\rangle}$, and one can get

$$\langle \bar{v}^1, \bar{v}^1\rangle = \langle v^1 - \theta v^0, v^1 - \theta v^0\rangle \le (\sqrt{\langle v^1, v^1\rangle} + \sqrt{\langle -\theta v^0, -\theta v^0\rangle})^2 \le 2\langle v^1, v^1\rangle + 2\theta^2\langle v^0, v^0\rangle. \tag{3.16}$$

Using (3.14) and (3.16), one can immediately obtain that

$$\begin{aligned}
&\langle \bar{v}^1, \bar{v}^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^1\|^2 \le 2\langle v^1, v^1\rangle + 2\theta^2\langle v^0, v^0\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^1\|^2\\
&\le 2[\langle v^1, v^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^1\|^2] + 2\theta^2\langle v^0, v^0\rangle\\
&\le 8(\frac{\alpha_2}{\alpha_1})^2\langle v^0, v^0\rangle + 2\theta^2\langle v^0, v^0\rangle = [8(\frac{\alpha_2}{\alpha_1})^2 + 2\theta^2]\langle v^0, v^0\rangle.
\end{aligned} \tag{3.17}$$

Similarly, we will estimate $\langle \bar{v}^2, \bar{v}^2\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^2\|^2$. Using (3.14) and (3.15), we have

$$\begin{aligned}
&\langle \bar{v}^2, \bar{v}^2\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^2\|^2 \le 2\langle v^2, v^2\rangle + 2\theta^2\langle v^1, v^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^2\|^2\\
&\le 2[\langle v^2, v^2\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^2\|^2] + 2\theta^2[\langle v^1, v^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^1\|^2]\\
&\le 2\cdot 4(\frac{\alpha_2}{\alpha_1})^2\langle v^0, v^0\rangle + 2\theta^2\cdot 4(\frac{\alpha_2}{\alpha_1})^2\langle v^0, v^0\rangle = 8(\frac{\alpha_2}{\alpha_1})^2(1+\theta^2)\langle v^0, v^0\rangle.
\end{aligned} \tag{3.18}$$

In summary, by using (3.5) and combining (3.17) and (3.18),

$$\langle \bar{v}^1, \bar{v}^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^1\|^2 \le \beta\langle v^0, v^0\rangle, \ \ \langle \bar{v}^2, \bar{v}^2\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^2\|^2 \le \beta\langle v^0, v^0\rangle.$$

Thus, we already proved Lemma 3.1. □

Next, we will give the estimate of $\langle \bar{v}^k, \bar{v}^k\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^k\|$ for $k \ge 3$ in Lemma 3.2, which is a very important result to analyze the scheme (3.1)'s stability.

Lemma 3.2. $\langle \bar{v}^k, \bar{v}^k\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^k\| \le \beta\langle v^0, v^0\rangle$, $3 \le k \le N$, *where* β *is defined in* (3.6).

Proof. First, using (2.18), we deduce the general formula,

$$
\begin{aligned}
&2\Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{v}_j^k)(\delta_x^2 v_j^k) = \Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{v}_j^k)(\delta_x^2 v_j^k + \delta_x^2 v_j^k)\\
&= \Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{v}_j^k)(\delta_x^2 v_j^k + \delta_x^2\bar{v}_j^k + \theta\delta_x^2 v_j^{k+1}) = \Delta x\sum_{j=1}^{M-1}[(\delta_x^2\bar{v}_j^k)^2 + (\delta_x^2\bar{v}_j^k)(\delta_x^2 v_j^k + \theta\delta_x^2 v_j^{k-1})]\\
&= \Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{v}_j^k)^2 + \Delta x\sum_{j=1}^{M-1}(\delta_x^2 v_j^k - \theta\delta_x^2 v_j^{k-1})(\delta_x^2 v_j^k + \theta\delta_x^2 v_j^{k-1})\\
&= \Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{v}_j^k)^2 + \Delta x\sum_{j=1}^{M-1}(\delta_x^2 v_j^k)^2 - \Delta x\theta^2\sum_{j=1}^{M-1}(\delta_x^2 v_j^{k-1})^2.
\end{aligned}
$$

By using (2.4), we get

$$
2\Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{v}_j^k)(\delta_x^2 v_j^k) = ||\delta_x^2\bar{v}^k||^2 + ||\delta_x^2 v^k||^2 - \theta^2||\delta_x^2 v^{k-1}||^2. \tag{3.19}
$$

Next, we prove for $k = 3$. Multiply on both sides of (3.1c) by $2\Delta x(-\delta_x^2\bar{v}_j^3)$, and by summing variable j from 1 to $M-1$, one can obtain

$$
2\Delta x\sum_{j=1}^{M-1}(-\delta_x^2\bar{v}_j^3)[(1+\frac{\Delta x^2}{12}\delta_x^2)(\bar{v}_j^3 - \bar{d}_2^3\bar{v}_j^2 - \bar{d}_1^3\bar{v}_j^1 - \bar{d}_0^3 v_j^0)] - 2\Delta x\sum_{j=1}^{M-1}(-\delta_x^2\bar{v}_j^3)(\alpha_0 C_1^{-1}\kappa_1\delta_x^2 v_j^3) = 0.
$$

That is, using the formula (3.19),

$$
\begin{aligned}
&2\Delta x\sum_{j=1}^{M-1}(-\delta_x^2\bar{v}_j^3)(\bar{v}_j^3 - \bar{d}_2^3\bar{v}_j^2 - \bar{d}_1^3\bar{v}_j^1 - \bar{d}_0^3 v_j^0) - \frac{2\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{v}_j^3)(\delta_x^2\bar{v}_j^3 - \bar{d}_2^3\delta_x^2\bar{v}_j^2 - \bar{d}_1^3\delta_x^2\bar{v}_j^1 - \bar{d}_0^3\delta_x^2 v_j^0)\\
&+\alpha_0 C_1^{-1}\kappa_1[||\delta_x^2\bar{v}^3||^2 + ||\delta_x^2 v^3||^2 - \theta^2||\delta_x^2 v^2||^2] = 0.
\end{aligned} \tag{3.20}
$$

By using $\bar{v}_0^k = \bar{v}_M^k = 0$, $1 \le k \le N$, and the summation scheme by parts for (3.20), it is easy to obtain

$$
\begin{aligned}
&2\Delta x\sum_{j=1}^{M-1}(\delta_x\bar{v}_{j-\frac{1}{2}}^3)(\delta_x\bar{v}_{j-\frac{1}{2}}^3 - \bar{d}_2^3\delta_x\bar{v}_{j-\frac{1}{2}}^2 - \bar{d}_1^3\delta_x\bar{v}_{j-\frac{1}{2}}^1 - \bar{d}_0^3\delta_x v_{j-\frac{1}{2}}^0)\\
&-\frac{2\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{v}_j^3)(\delta_x^2\bar{v}_j^3 - \bar{d}_2^3\delta_x^2\bar{v}_j^2 - \bar{d}_1^3\delta_x^2\bar{v}_j^1 - \bar{d}_0^3\delta_x^2 v_j^0)\\
&+\alpha_0 C_1^{-1}\kappa_1[||\delta_x^2\bar{v}^3||^2 + ||\delta_x^2 v^3||^2 - \theta^2||\delta_x^2 v^2||^2] = 0.
\end{aligned} \tag{3.21}
$$

For the purpose of theoretical analysis, (3.21) can be rewritten into the following equivalent form:

$$
2[\Delta x\sum_{j=1}^{M}(\delta_x\bar{v}_{j-\frac{1}{2}}^3)^2 - \frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{v}_j^3)^2] + \alpha_0 C_1^{-1}\kappa_1[||\delta_x^2\bar{v}^3||^2 + ||\delta_x^2 v^3||^2 - \theta^2||\delta_x^2 v^2||^2]
$$

$$= 2\bar{d}_2^3[\Delta x \sum_{j=1}^{M}(\delta_x \bar{v}_{j-\frac{1}{2}}^3)(\delta_x \bar{v}_{j-\frac{1}{2}}^2) - \frac{\Delta x^2}{12}\Delta x \sum_{j=1}^{M-1}(\delta_x^2 \bar{v}_j^3)(\delta_x^2 \bar{v}_j^2)]$$
$$+2\bar{d}_1^3[\Delta x \sum_{j=1}^{M}(\delta_x \bar{v}_{j-\frac{1}{2}}^3)(\delta_x \bar{v}_{j-\frac{1}{2}}^1) - \frac{\Delta x^2}{12}\Delta x \sum_{j=1}^{M-1}(\delta_x^2 \bar{v}_j^3)(\delta_x^2 \bar{v}_j^1)]$$
$$+2\bar{d}_0^3[\Delta x \sum_{j=1}^{M}(\delta_x \bar{v}_{j-\frac{1}{2}}^3)(\delta_x v_{j-\frac{1}{2}}^0) - \frac{\Delta x^2}{12}\Delta x \sum_{j=1}^{M-1}(\delta_x^2 \bar{v}_j^3)(\delta_x^2 v_j^0)].$$

Using (2.3), we have

$$\begin{aligned}
&2\langle \bar{v}^3, \bar{v}^3\rangle + \alpha_0 C_1^{-1}\kappa_1[\|\delta_x^2 \bar{v}^3\|^2 + \|\delta_x^2 v^3\|^2 - \theta^2\|\delta_x^2 v^2\|^2] \\
&= 2\bar{d}_2^3\langle \bar{v}^3, \bar{v}^2\rangle + 2\bar{d}_1^3\langle \bar{v}^3, \bar{v}^1\rangle + 2\bar{d}_0^3\langle \bar{v}^3, v^0\rangle \\
&\leqslant \bar{d}_2^3[\langle \bar{v}^3, \bar{v}^3\rangle + \langle \bar{v}^2, \bar{v}^2\rangle] + \bar{d}_1^3[\langle \bar{v}^3, \bar{v}^3\rangle + \langle \bar{v}^1, \bar{v}^1\rangle] + \bar{d}_0^3[\langle \bar{v}^3, \bar{v}^3\rangle + \langle v^0, v^0\rangle].
\end{aligned}$$

By using (2) and (3) in Lemma 2.7, we have

$$\begin{aligned}
&2\langle \bar{v}^3, \bar{v}^3\rangle + \alpha_0 C_1^{-1}\kappa_1[\|\delta_x^2 \bar{v}^3\|^2 + \|\delta_x^2 v^3\|^2 - \theta^2\|\delta_x^2 v^2\|^2] \\
&\leq (\bar{d}_2^3 + \bar{d}_1^3 + \bar{d}_0^3)\langle \bar{v}^3, \bar{v}^3\rangle + \bar{d}_2^3\langle \bar{v}^2, \bar{v}^2\rangle + \bar{d}_1^3\langle \bar{v}^1, \bar{v}^1\rangle + \bar{d}_0^3\langle v^0, v^0\rangle \\
&\leq \langle \bar{v}^3, \bar{v}^3\rangle + \bar{d}_2^3\langle \bar{v}^2, \bar{v}^2\rangle + \bar{d}_1^3\langle \bar{v}^1, \bar{v}^1\rangle + \bar{d}_0^3\langle v^0, v^0\rangle.
\end{aligned} \tag{3.22}$$

Rearrange (3.22) and $\alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 \bar{v}^3\|^2 \geq 0$, we have

$$\begin{aligned}
&\langle \bar{v}^3, \bar{v}^3\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^3\|^2 \\
&\leq \bar{d}_2^3\langle \bar{v}^2, \bar{v}^2\rangle + \bar{d}_1^3\langle \bar{v}^1, \bar{v}^1\rangle + \bar{d}_0^3\langle v^0, v^0\rangle + \alpha_0 C_1^{-1}\kappa_1\theta^2\|\delta_x^2 v^2\|^2 \\
&\leq \theta(\langle \bar{v}^2, \bar{v}^2\rangle + \alpha_0 C_1^{-1}\kappa_1\theta\|\delta_x^2 v^2\|^2) + \bar{d}_1^3\langle \bar{v}^1, \bar{v}^1\rangle + \bar{d}_0^3\langle v^0, v^0\rangle \\
&\leq \theta(\langle \bar{v}^2, \bar{v}^2\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^2\|^2) + \bar{d}_1^3(\langle \bar{v}^1, \bar{v}^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^1\|^2) + \bar{d}_0^3\langle v^0, v^0\rangle.
\end{aligned} \tag{3.23}$$

By using Lemma 3.1 and (4) in Lemma 2.7, (3.23) becomes:

$$\langle \bar{v}^3, \bar{v}^3\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^3\|^2 \leq \beta(\theta + \bar{d}_1^3 + \bar{d}_0^3)\langle v^0, v^0\rangle \leq \beta(1+\theta)\langle v^0, v^0\rangle. \tag{3.24}$$

For $k \geq 4$, multiply both sides of (3.1d) by $2\Delta x(-\delta_x^2 \bar{v}_j^k)$, and we sum up for j from 1 to $M-1$. Using the similar method for $k = 3$, we get

$$\begin{aligned}
&2\langle \bar{v}^k, \bar{v}^k\rangle + \alpha_0 C_1^{-1}\kappa_1[\|\delta_x^2 \bar{v}^k\|^2 + \|\delta_x^2 v^k\|^2 - \theta^2\|\delta_x^2 v^{k-1}\|^2] \\
&= 2\theta\langle \bar{v}^k, \bar{v}^{k-1}\rangle + 2\sum_{i=2}^{k-1}\bar{d}_{k-i}^k\langle \bar{v}^k, \bar{v}^{k-i}\rangle + 2\bar{d}_0^k\langle \bar{v}^k, v^0\rangle \\
&\leqslant \theta[\langle \bar{v}^k, \bar{v}^k\rangle + \langle \bar{v}^{k-1}, \bar{v}^{k-1}\rangle] + \sum_{i=2}^{k-1}\bar{d}_{k-i}^k[\langle \bar{v}^k, \bar{v}^k\rangle + \langle \bar{v}^{k-i}, \bar{v}^{k-i}\rangle] + \bar{d}_0^k[\langle \bar{v}^k, \bar{v}^k\rangle + \langle v^0, v^0\rangle] \\
&= (\theta + \sum_{i=2}^{k-1}\bar{d}_{k-i}^k + \bar{d}_0^k)\langle \bar{v}^k, \bar{v}^k\rangle + \theta\langle \bar{v}^{k-1}, \bar{v}^{k-1}\rangle + \sum_{i=2}^{k-1}\bar{d}_{k-i}^k\langle \bar{v}^{k-i}, \bar{v}^{k-i}\rangle + \bar{d}_0^k\langle v^0, v^0\rangle.
\end{aligned} \tag{3.25}$$

According to (1) and (3) in Lemma 2.6, the above inequality (3.25) becomes

$$\langle \bar{v}^k, \bar{v}^k\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^k\|^2 \le \theta\langle \bar{v}^{k-1}, \bar{v}^{k-1}\rangle + \sum_{i=2}^{k-1} \bar{d}_{k-i}^k\langle \bar{v}^{k-i}, \bar{v}^{k-i}\rangle + \bar{d}_0^k\langle v^0, v^0\rangle + \alpha_0 C_1^{-1}\kappa_1\theta^2\|\delta_x^2 v^{k-1}\|^2$$

$$\le \theta[\langle \bar{v}^{k-1}, \bar{v}^{k-1}\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^{k-1}\|^2] + \sum_{i=2}^{k-1} \bar{d}_{k-i}^k\langle \bar{v}^{k-i}, \bar{v}^{k-i}\rangle + \bar{d}_0^k\langle v^0, v^0\rangle$$

$$\le \theta[\langle \bar{v}^{k-1}, \bar{v}^{k-1}\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^{k-1}\|^2] + \sum_{i=2}^{k-1} \bar{d}_{k-i}^k[\langle \bar{v}^{k-i}, \bar{v}^{k-i}\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^{k-i}\|^2] + \bar{d}_0^k\langle v^0, v^0\rangle. \tag{3.26}$$

One can immediately prove the following inequality by using the mathematics induction

$$\langle \bar{v}^k, \bar{v}^k\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^k\|^2 \le \beta(1+\theta)\langle v^0, v^0\rangle, \quad 4 \le k \le N. \tag{3.27}$$

As $k = 4$, by (3.3), (3.4), (3.24), and (3) in Lemma 2.6, from (3.26), we can obtain

$$\langle \bar{v}^4, \bar{v}^4\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^4\|^2 \le \theta[\langle \bar{v}^3, \bar{v}^3\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^3\|^2]$$

$$+ \sum_{i=2}^{3} \bar{d}_{4-i}^4[\langle \bar{v}^{4-i}, \bar{v}^{4-i}\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^{4-i}\|^2] + \bar{d}_0^4\langle v^0, v^0\rangle$$

$$\le \beta(1+\theta)(\theta + \sum_{i=2}^{3} \bar{d}_{4-i}^4 + \bar{d}_0^4)\langle v^0, v^0\rangle \le \beta(1+\theta)\langle v^0, v^0\rangle. \tag{3.28}$$

According to (3.28), we have proven that the special case of (3.27) as $k = 4$ is correct. We assume that (3.27) is correct for $k = 5, 6, \cdots, N-1$, and one can immediately obtain that

$$\langle \bar{v}^N, \bar{v}^N\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^N\|^2 \le \beta(1+\theta)(\theta + \sum_{i=2}^{N-1} \bar{d}_{N-i}^N + \bar{d}_0^N)\langle v^0, v^0\rangle \le \beta(1+\theta)\langle v^0, v^0\rangle.$$

The proof of Lemma 3.2 is completed. □

Theorem 3.1. *The full discrete numerical scheme* (3.1) *is unconditionally stable for* $0 < \alpha < 1$*, and its numerical solution satisfies the following the following bounded estimates for all* $\Delta x > 0, \Delta t > 0$*,*

$$\|v^k\|_\infty + \sqrt{\alpha_0 C_1^{-1}\kappa_1}\|\delta_x^2 v^k\| \le \left[\sqrt{\frac{3(b_1-a_1)}{8}}(3\sqrt{\beta(1+\theta)}+1)+1\right]|v^0|_1, \quad 1 \le k \le N.$$

Proof. By using the Lemmas 3.1 and 3.2, one can immediately get the following result

$$\langle \bar{v}^k, \bar{v}^k\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 v^k\|^2 \le \beta(1+\theta)\langle v^0, v^0\rangle. \tag{3.29}$$

From (3.29), we have

$$\sqrt{\langle \bar{v}^k, \bar{v}^k\rangle} \le \sqrt{\beta(1+\theta)}\sqrt{\langle v^0, v^0\rangle}. \tag{3.30}$$

According to (2.18) and (1) in Lemma 2.6, we have

$$\sqrt{\langle v^k, v^k\rangle} = \sqrt{\langle \bar{v}^k + \theta v^{k-1}, \bar{v}^k + \theta v^{k-1}\rangle} \leqslant \sqrt{\langle \bar{v}^k, \bar{v}^k\rangle} + \sqrt{\langle \theta v^{k-1}, \theta v^{k-1}\rangle}$$

$$
\begin{aligned}
&= \sqrt{\langle \bar{v}^k, \bar{v}^k\rangle} + \theta\sqrt{\langle v^{k-1}, v^{k-1}\rangle} \leqslant \sqrt{\langle \bar{v}^k, \bar{v}^k\rangle} + \theta\sqrt{\langle \bar{v}^{k-1}, \bar{v}^{k-1}\rangle} + \theta^2\sqrt{\langle v^{k-2}, v^{k-2}\rangle} \leq \cdots \\
&\leqslant \sqrt{\langle \bar{v}^k, \bar{v}^k\rangle} + \theta\sqrt{\langle \bar{v}^{k-1}, \bar{v}^{k-1}\rangle} + \theta^2\sqrt{\langle \bar{v}^{k-2}, \bar{v}^{k-2}\rangle} + \cdots + \theta^{k-1}\sqrt{\langle \bar{v}^1, \bar{v}^1\rangle} + \theta^k\sqrt{\langle v^0, v^0\rangle} \\
&\leqslant \sqrt{\beta(1+\theta)}\left(1 + \theta + \theta^2 + \cdots + \theta^{k-1}\right)\sqrt{\langle v^0, v^0\rangle} + \theta^k\sqrt{\langle v^0, v^0\rangle} \\
&\leqslant \sqrt{\beta(1+\theta)}\cdot\frac{1}{1-\theta}\sqrt{\langle v^0, v^0\rangle} + \sqrt{\langle v^0, v^0\rangle} \leqslant (3\sqrt{\beta(1+\theta)} + 1)\sqrt{\langle v^0, v^0\rangle}.
\end{aligned} \tag{3.31}
$$

Combining (3.31) with Lemmas 2.1 and 2.2, we have

$$
\begin{aligned}
\|v^k\|_\infty^2 &\leq \frac{b_1 - a_1}{4}|v^k|_1^2 \leq \frac{b_1 - a_1}{4}\cdot\frac{3}{2}\langle v^k, v^k\rangle = \frac{3(b_1 - a_1)}{8}\langle v^k, v^k\rangle \\
&\leq \frac{3(b_1 - a_1)}{8}(3\sqrt{\beta(1+\theta)} + 1)^2\langle v^0, v^0\rangle \leq \frac{3(b_1 - a_1)}{8}(3\sqrt{\beta(1+\theta)} + 1)^2|v^0|_1^2.
\end{aligned} \tag{3.32}
$$

Again, using (3.29), we have

$$
\sqrt{\alpha_0 C_1^{-1}\kappa_1}\|\delta_x^2 v^k\| \leq \sqrt{(1+\theta)\beta}\sqrt{\langle v^0, v^0\rangle} \leq \sqrt{(1+\theta)\beta}|v^0|_1. \tag{3.33}
$$

Combining (3.32) and (3.33), we obtain

$$
\begin{aligned}
\|v^k\|_\infty + \sqrt{\alpha_0 C_1^{-1}\kappa_1}\|\delta_x^2 v^k\| &\leq \sqrt{\frac{3(b_1 - a_1)}{8}}(3\sqrt{(1+\theta)\beta} + 1)|v^0|_1 + \sqrt{(1+\theta)\beta}|v^0|_1 \\
&= \left[\sqrt{\frac{3(b_1 - a_1)}{8}}(3\sqrt{(1+\theta)\beta} + 1) + 1\right]|v^0|_1.
\end{aligned}
$$

Hence, the unconditional stability of the full discrete numerical scheme (3.1) for $0 < \alpha < 1$ with regard to the initial values has been proved. Theorem 3.1 has been proved completely. □

Let

$$
e_j^k = v(x_j, t_k) - v_j^k,\ j = 0, 1, 2, \cdots, M,\ \ k = 0, 1, 2, \cdots, N.
$$

In the following Theorem 3.2, we will establish the numerical scheme (3.1)'s convergence analysis.

Theorem 3.2. *Let $v(x,t)$ be the exact solution of Eq* (1.1a) *and v_j^k be the numerical scheme defined by* (3.1). *If $\max\limits_{t\in(0,T]}|\partial_t^3 v| \leq M$, then*

$$
\|v(x, t_k) - v^k\|_\infty \leqslant \sqrt{\frac{27}{8}(b_1 - a_1)C(\Delta t^{3-\alpha} + \Delta x^4)}, k = 1, 2, \cdots, N \tag{3.34}
$$

where $\alpha \in (0, 1)$ and C is a positive independence of $\Delta t, \Delta x$ constant. The definition is as follows:

$$
C = (1+\theta)\Gamma(1-\alpha)T^\alpha\kappa_1^{-1}\hat{C}^2(b_1 - a_1)\Big[\frac{4(1+\theta^2)\alpha_4(2-\alpha)(1-\alpha)}{\alpha_3} + \frac{3}{2}\Big]. \tag{3.35}
$$

Proof. By (2.10) and (2.12), e_j^k satisfies the following equations:

$$
\begin{cases}
H_{\Delta x}(\hat{D}_0 e_j^0 + \hat{D}_1 e_j^1 + \hat{D}_2 e_j^2) - \alpha_0\kappa_1\delta_x^2 e_j^1 = \alpha_0 R_j^1, & k = 1, \\
H_{\Delta x}(\tilde{D}_0 e_j^0 + \tilde{D}_1 e_j^1 + \tilde{D}_2 e_j^2) - \alpha_0\kappa_1\delta_x^2 e_j^2 = \alpha_0 R_j^2, & k = 2, \\
H_{\Delta x}[\bar{A}_k e_j^0 + \bar{B}_k e_j^1 + \bar{C}_k e_j^2 + \sum\limits_{i=1}^{k-1}(A_i e_j^{k-i-1} + B_i e_j^{k-i} + C_i e_j^{k-i+1})] - \alpha_0\kappa_1\delta_x^2 e_j^k = \alpha_0 R_j^k, & k \geq 3.
\end{cases} \tag{3.36}
$$

Similar to the derived equivalent form (2.15) from the full discrete method of (2.12), we rewrite (3.36) as the following equivalent form:

$$\begin{cases} H_{\Delta x}(\hat{D}_0 e_j^0 + \hat{D}_1 e_j^1 + \hat{D}_2 e_j^2) - \alpha_0\kappa_1\delta_x^2 e_j^1 = \alpha_0 R_j^1, \ k = 1, \\ H_{\Delta x}(\tilde{D}_0 e_j^0 + \tilde{D}_1 e_j^1 + \tilde{D}_2 e_j^2) - \alpha_0\kappa_1\delta_x^2 e_j^2 = \alpha_0 R_j^2, \ k = 2, \\ H_{\Delta x}(e_j^3 - d_2^3 e_j^2 - d_1^3 e_j^1 - d_0^3 e_j^0) - \alpha_0 C_1^{-1}\kappa_1\delta_x^2 e_j^3 = \alpha_0 C_1^{-1} R_j^3, \ k = 3, \\ H_{\Delta x}(e_j^k - \sum_{i=1}^{k} d_{k-i}^k e_j^{k-i}) - \alpha_0 C_1^{-1}\kappa_1\delta_x^2 e_j^k = \alpha_0 C_1^{-1} R_j^k, \ 4 \le k \le N. \end{cases} \tag{3.37}$$

Similar to the derived equivalent form (2.22) from the Eq (2.15), the error Eq (3.37) can be rewritten as the following equivalent form:

$$\begin{cases} H_{\Delta x}(\hat{D}_0 e_j^0 + \hat{D}_1 e_j^1 + \hat{D}_2 e_j^2) - \alpha_0\kappa_1\delta_x^2 e_j^1 = \alpha_0 R_j^1, \ k = 1, & \text{(3.38a)} \\ H_{\Delta x}(\tilde{D}_0 e_j^0 + \tilde{D}_1 e_j^1 + \tilde{D}_2 e_j^2) - \alpha_0\kappa_1\delta_x^2 e_j^2 = \alpha_0 R_j^2, \ k = 2, & \text{(3.38b)} \\ H_{\Delta x}(\bar{e}_j^3 - \bar{d}_2^3 \bar{e}_j^2 - \bar{d}_1^3 \bar{e}_j^1 - \bar{d}_0^3 e_j^0) - \alpha_0 C_1^{-1}\kappa_1\delta_x^2 e_j^3 = \alpha_0 C_1^{-1} R_j^3, \ k = 3, & \text{(3.38c)} \\ H_{\Delta x}(\bar{e}_j^k - \sum_{i=1}^{k} \bar{d}_{k-i}^k \bar{e}_j^{k-i}) - \alpha_0 C_1^{-1}\kappa_1\delta_x^2 e_j^k = \alpha_0 C_1^{-1} R_j^k, \ 4 \le k \le N. & \text{(3.38d)} \end{cases}$$

Next, we will divide into three parts to prove (3.34).

- (1) First, we will prove

$$\begin{cases} \langle \bar{e}^1, \bar{e}^1 \rangle + \alpha_0 C_1^{-1}\kappa_1 \|\delta_x^2 e^1\|^2 \le 2(1+\theta^2)\dfrac{\alpha_4\alpha_0}{\alpha_3\kappa_\alpha}(\|R^1\|^2 + \|R^2\|^2), & \text{(3.39)} \\ \langle \bar{e}^2, \bar{e}^2 \rangle + \alpha_0 C_1^{-1}\kappa_1 \|\delta_x^2 e^2\|^2 \le 2(1+\theta^2)\dfrac{\alpha_4\alpha_0}{\alpha_3\kappa_\alpha}(\|R^1\|^2 + \|R^2\|^2), & \text{(3.40)} \end{cases}$$

where α_0 is defined by (2.5), and α_3, α_4 are defined as follows:

$$\alpha_3 = \min\{-2\tilde{D}_1\hat{D}_1, 2\hat{D}_2\tilde{D}_2, -\tilde{D}_1, \hat{D}_2\}, \quad \alpha_4 = \max\{-\tilde{D}_1, \hat{D}_2\}. \tag{3.41}$$

Let's multiply $\tilde{D}_1\Delta x\delta_x^2 e_j^1$ on both sides of (3.38a) for $k = 1$, and summing up for j from 1 to $M-1$,

$$\begin{aligned} &\tilde{D}_1\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e_j^1)[(1+\frac{\Delta x^2}{12}\delta_x^2)(\hat{D}_0 e_j^0 + \hat{D}_1 e_j^1 + \hat{D}_2 e_j^2)] \\ &-\tilde{D}_1\alpha_0\kappa_\alpha\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e_j^1)^2 = \tilde{D}_1\alpha_0\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e_j^1)(R_j^1). \end{aligned} \tag{3.42}$$

We rewrite Eq (3.42) and obtain

$$\begin{aligned} &\tilde{D}_1\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e_j^1)(\hat{D}_0 e_j^0 + \hat{D}_1 e_j^1 + \hat{D}_2 e_j^2) + \tilde{D}_1\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e_j^1)(\hat{D}_0\delta_x^2 e_j^0 + \hat{D}_1\delta_x^2 e_j^1 + \hat{D}_2\delta_x^2 e_j^2) \\ &-\tilde{D}_1\alpha_0\kappa_1\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e_j^1)^2 = \tilde{D}_1\alpha_0\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e_j^1)(R_j^1). \end{aligned} \tag{3.43}$$

We use the summation formula by parts, notice $e_0^k = e_M^k = 0$ for (3.43), and obtain

$$\begin{aligned}
&-\tilde{D}_1\Delta x\sum_{j=1}^{M}(\delta_x e^1_{j-\frac{1}{2}})(\hat{D}_0\delta_x e^0_{j-\frac{1}{2}}+\hat{D}_1\delta_x e^1_{j-\frac{1}{2}}+\hat{D}_2\delta_x e^2_{j-\frac{1}{2}})\\
&+\tilde{D}_1\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^1_j)(\hat{D}_0\delta_x^2 e^0_j+\hat{D}_1\delta_x^2 e^1_j+\hat{D}_2\delta_x^2 e^2_j)\\
&-\tilde{D}_0\alpha_0\kappa_1\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^1_j)^2=\tilde{D}_1\alpha_0\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^1_j)(R^1_j).
\end{aligned}\tag{3.44}$$

Rewriting the left-end item of (3.44), one gets

$$\begin{aligned}
&-\tilde{D}_1\hat{D}_0[\Delta x\sum_{j=1}^{M}(\delta_x e^1_{j-\frac{1}{2}})(\delta_x e^0_{j-\frac{1}{2}})-\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^1_j)(\delta_x^2 e^0_j)]\\
&-\tilde{D}_1\hat{D}_1[\Delta x\sum_{j=1}^{M}(\delta_x e^1_{j-\frac{1}{2}})(\delta_x e^1_{j-\frac{1}{2}})-\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^1_j)(\delta_x^2 e^1_j)]\\
&-\tilde{D}_1\hat{D}_2[\Delta x\sum_{j=1}^{M}(\delta_x e^1_{j-\frac{1}{2}})(\delta_x e^2_{j-\frac{1}{2}})-\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^1_j)(\delta_x^2 e^2_j)]\\
&-\tilde{D}_1\alpha_0\kappa_1\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^1_j)^2=\tilde{D}_1\alpha_0\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^1_j)(R^1_j).
\end{aligned}\tag{3.45}$$

Using (2.3) and (2.4) for (3.45), we obtain

$$-\tilde{D}_1\hat{D}_0\langle e^1,e^0\rangle-\tilde{D}_1\hat{D}_1\langle e^1,e^1\rangle-\tilde{D}_1\hat{D}_2\langle e^1,e^2\rangle-\tilde{D}_1\alpha_0\kappa_1\|\delta_x^2 e^1\|^2=\tilde{D}_1\alpha_0\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^1_j)(R^1_j).\tag{3.46}$$

By multiplying $-\hat{D}_2\Delta x\delta_x^2 e_j^2$ on both sides of (3.38b) for $k = 2$ and summing j from 1 to $M-1$, in the same way we get

$$\hat{D}_2\tilde{D}_0\langle e^2,e^0\rangle+\hat{D}_2\tilde{D}_1\langle e^2,e^1\rangle+\hat{D}_2\tilde{D}_2\langle e^2,e^2\rangle+\hat{D}_2\alpha_0\kappa_1\|\delta_x^2 e^2\|^2=-\hat{D}_2\alpha_0\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^2_j)(R^2_j).\tag{3.47}$$

With (3.46) plus (3.47), one can obtain:

$$\begin{aligned}
&-\tilde{D}_1\hat{D}_0\langle e^1,e^0\rangle+\hat{D}_2\tilde{D}_0\langle e^2,e^0\rangle-\tilde{D}_1\hat{D}_1\langle e^1,e^1\rangle+\hat{D}_2\tilde{D}_2\langle e^2,e^2\rangle-\tilde{D}_1\alpha_0\kappa_1\|\delta_x^2 e^1\|^2+\hat{D}_2\alpha_0\kappa_1\|\delta_x^2 e^2\|^2\\
&=\tilde{D}_1\alpha_0\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^1_j)(R^1_j)-\hat{D}_2\alpha_0\Delta x\sum_{j=1}^{M-1}(\delta_x^2 e^2_j)(R^2_j).
\end{aligned}$$

That is,

$$-\tilde{D}_1\hat{D}_1\langle e^1,e^1\rangle+\hat{D}_2\tilde{D}_2\langle e^2,e^2\rangle-\tilde{D}_1\alpha_0\kappa_1\|\delta_x^2 e^1\|^2+\hat{D}_2\alpha_0\kappa_1\|\delta_x^2 e^2\|^2$$

$$= -(-\tilde{D}_1\alpha_0)\Delta x \sum_{j=1}^{M-1}(\delta_x^2 e_j^1)(R_j^1) - (\hat{D}_2\alpha_0)\Delta x \sum_{j=1}^{M-1}(\delta_x^2 e_j^2)(R_j^2)$$
$$\leq -\tilde{D}_1\alpha_0\Big(\frac{\kappa_1}{2}\|\delta_x^2 e^1\|^2 + \frac{1}{2\kappa_1}\|R^1\|^2\Big) + \hat{D}_2\alpha_0\Big(\frac{\kappa_1}{2}\|\delta_x^2 e^2\|^2 + \frac{1}{2\kappa_1}\|R^2\|^2\Big).$$

Therefore, we have

$$\begin{aligned} & -2\tilde{D}_1\hat{D}_1\langle e^1, e^1\rangle + 2\hat{D}_2\tilde{D}_2\langle e^2, e^2\rangle - \tilde{D}_1\alpha_0\kappa_1\|\delta_x^2 e^1\|^2 + \hat{D}_2\alpha_0\kappa_1\|\delta_x^2 e^2\|^2 \\ \leq \; & -\frac{1}{\kappa_1}\tilde{D}_1\alpha_0\|R^1\|^2 + \frac{1}{\kappa_1}\hat{D}_2\alpha_0\|R^2\|^2. \end{aligned} \tag{3.48}$$

Because $-2\tilde{D}_1\hat{D}_1$, $2\hat{D}_2\tilde{D}_2$, $-\tilde{D}_1$, and $\hat{D}_2$ are all positive numbers depending on α, according to the definition for α_3 and α_4 in (3.41), we know $\alpha_3 > 0, \alpha_4 > 0$. From (3.48), we obtain

$$\langle e^1, e^1\rangle + \langle e^2, e^2\rangle + \alpha_0\kappa_1\|\delta_x^2 e^1\|^2 + \alpha_0\kappa_1\|\delta_x^2 e^2\|^2 \leq \frac{\alpha_4}{\alpha_3}\Big(\frac{1}{\kappa_1}\alpha_0\|R^1\|^2 + \frac{1}{\kappa_1}\alpha_0\|R^2\|^2\Big). \tag{3.49}$$

According to the definition of C_1 in (2.8), we know $0 < C_1^{-1} \leq 1$, so $\alpha_0 C_1^{-1}\kappa_1 \leq \alpha_0\kappa_1$. From (3.49), we can get

$$\langle e^1, e^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^1\|^2 \leq \frac{\alpha_4\alpha_0}{\alpha_3\kappa_1}(\|R^1\|^2 + \|R^2\|^2), \tag{3.50}$$
$$\langle e^2, e^2\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^2\|^2 \leq \frac{\alpha_4\alpha_0}{\alpha_3\kappa_1}(\|R^1\|^2 + \|R^2\|^2). \tag{3.51}$$

According to (2.18): $\bar{e}^k = e^k - \theta e^{k-1}, k = 1, 2$. We have

$$\begin{aligned} \langle \bar{e}^1, \bar{e}^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^1\|^2 &= \langle e^1 - \theta e^0, e^1 - \theta e^0\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^1\|^2 \\ &= \langle e^1, e^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^1\|^2 \leq \frac{\alpha_4\alpha_0}{\alpha_3\kappa_1}(\|R^1\|^2 + \|R^2\|^2). \end{aligned} \tag{3.52}$$

Similarly, we will estimate $\langle \bar{e}^2, \bar{e}^2\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^2\|^2$. Using (3.50) and (3.51), we have:

$$\begin{aligned} & \langle \bar{e}^2, \bar{e}^2\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^2\|^2 = \langle e^2 - \theta e^1, e^2 - \theta e^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^2\|^2 \\ & \leq \big(\sqrt{\langle e^2, e^2\rangle} + \sqrt{\langle -\theta e^1, -\theta e^1\rangle}\big)^2 + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^2\|^2 \\ & \leq 2\langle e^2, e^2\rangle + 2\theta^2\langle e^1, e^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^2\|^2 \\ & \leq 2[\langle e^2, e^2\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^2\|^2] + 2\theta^2[\langle e^1, e^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^1\|^2] \\ & \leq 2(1 + \theta^2)\frac{\alpha_4\alpha_0}{\alpha_3\kappa_1}(\|R^1\|^2 + \|R^2\|^2). \end{aligned} \tag{3.53}$$

In summary, combining (3.52) and (3.53),

$$\langle \bar{e}^1, \bar{e}^1\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^1\|^2 \leq 2(1 + \theta^2)\frac{\alpha_4\alpha_0}{\alpha_3\kappa_\alpha}(\|R^1\|^2 + \|R^2\|^2),$$
$$\langle \bar{e}^2, \bar{e}^2\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^2\|^2 \leq 2(1 + \theta^2)\frac{\alpha_4\alpha_0}{\alpha_3\kappa_\alpha}(\|R^1\|^2 + \|R^2\|^2).$$

Thus, we already proved (3.39) and (3.40).

- (2) Second, we will prove

$$\langle \bar{e}^k, \bar{e}^k \rangle + \alpha_0 C_1^{-1} \kappa_1 \|\delta_x^2 e^k\|^2 \le 2(1+\theta)(1+\theta^2) \frac{\alpha_4 \Gamma(3-\alpha)\Delta t^\alpha}{\alpha_3 \kappa_1} (\|R^1\|^2 + \|R^2\|^2)$$
$$+ \frac{3(1+\theta)\Gamma(1-\alpha)T^\alpha}{2\kappa_\alpha} \max_{3\le i\le k} \|R^i\|^2, \ k \ge 1. \tag{3.54}$$

First, according to the expression of α_0 in (2.5), it is easy to find that (3.54) is correct for $k = 1, 2$.

Next, we prove for $k = 3$. Using (2.18), (2.4), and a similar procedure as (3.19), we deduce the general formula,

$$2\Delta x \sum_{j=1}^{M-1} (\delta_x^2 \bar{e}_j^k)(\delta_x^2 e_j^k) = \|\delta_x^2 \bar{e}^k\|^2 + \|\delta_x^2 e^k\|^2 - \theta^2 \|\delta_x^2 e^{k-1}\|^2. \tag{3.55}$$

Multiplying both sides of (3.38c) by $2\Delta x(-\delta_x^2 \bar{e}_j^3)$ and by summing j from 1 to $M-1$, we obtain:

$$2\Delta x \sum_{j=1}^{M-1} (-\delta_x^2 \bar{e}_j^3)[(1 + \frac{\Delta x^2}{12}\delta_x^2)(\bar{e}_j^3 - \bar{d}_2^3 \bar{e}_j^2 - \bar{d}_1^3 \bar{e}_j^1 - \bar{d}_0^3 e_j^0)]$$
$$-2\Delta x \sum_{j=1}^{M-1} (-\delta_x^2 \bar{e}_j^3)(\alpha_0 C_1^{-1} \kappa_1 \delta_x^2 e_j^3) = \alpha_0 C_1^{-1} 2\Delta x \sum_{j=1}^{M-1} (-\delta_x^2 \bar{e}_j^3)(R_j^3).$$

That is, using the formula (3.55),

$$2\Delta x \sum_{j=1}^{M-1} (-\delta_x^2 \bar{e}_j^3)(\bar{e}_j^3 - \bar{d}_2^3 \bar{e}_j^2 - \bar{d}_1^3 \bar{e}_j^1 - \bar{d}_0^3 e_j^0)$$
$$-\frac{2\Delta x^2}{12}\Delta x \sum_{j=1}^{M-1} (\delta_x^2 \bar{e}_j^3)(\delta_x^2 \bar{e}_j^3 - \bar{d}_2^3 \delta_x^2 \bar{e}_j^2 - \bar{d}_1^3 \delta_x^2 \bar{e}_j^1 - \bar{d}_0^3 \delta_x^2 e_j^0)$$
$$+\alpha_0 C_1^{-1} \kappa_1 [\|\delta_x^2 \bar{e}^3\|^2 + \|\delta_x^2 e^3\|^2 - \theta^2 \|\delta_x^2 e^2\|^2] = \alpha_0 C_1^{-1} 2\Delta x \sum_{j=1}^{M-1} (-\delta_x^2 \bar{e}_j^3)(R_j^3). \tag{3.56}$$

By using $\bar{e}_0^k = \bar{e}_M^k = 0$, $1 \le k \le N$, and the summation scheme by parts for (3.56), it is easy to obtain

$$2\Delta x \sum_{j=1}^{M-1} (\delta_x \bar{e}_{j-\frac{1}{2}}^3)(\delta_x \bar{e}_{j-\frac{1}{2}}^3 - \bar{d}_2^3 \delta_x \bar{e}_{j-\frac{1}{2}}^2 - \bar{d}_1^3 \delta_x \bar{e}_{j-\frac{1}{2}}^1 - \bar{d}_0^3 \delta_x e_{j-\frac{1}{2}}^0)$$
$$-\frac{2\Delta x^2}{12}\Delta x \sum_{j=1}^{M-1} (\delta_x^2 \bar{e}_j^3)(\delta_x^2 \bar{e}_j^3 - \bar{d}_2^3 \delta_x^2 \bar{e}_j^2 - \bar{d}_1^3 \delta_x^2 \bar{e}_j^1 - \bar{d}_0^3 \delta_x^2 e_j^0)$$
$$+\alpha_0 C_1^{-1} \kappa_1 [\|\delta_x^2 \bar{e}^3\|^2 + \|\delta_x^2 e^3\|^2 - \theta^2 \|\delta_x^2 e^2\|^2] = \alpha_0 C_1^{-1} 2\Delta x \sum_{j=1}^{M-1} (-\delta_x^2 \bar{e}_j^3)(R_j^3). \tag{3.57}$$

For the purpose of theoretical analysis, (3.57) can be rewritten into the following equivalent form:

$$
\begin{aligned}
&2[\Delta x\sum_{j=1}^{M}(\delta_x\bar{e}^3_{j-\frac{1}{2}})^2-\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{e}^3_j)^2]+\alpha_0C_1^{-1}\kappa_1[\|\delta_x^2\bar{e}^3\|^2+\|\delta_x^2e^3\|^2-\theta^2\|\delta_x^2e^2\|^2]\\
=\ &2\bar{d}^3_2[\Delta x\sum_{j=1}^{M}(\delta_x\bar{e}^3_{j-\frac{1}{2}})(\delta_x\bar{e}^2_{j-\frac{1}{2}})-\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{e}^3_j)(\delta_x^2\bar{e}^2_j)]\\
&+2\bar{d}^3_1[\Delta x\sum_{j=1}^{M}(\delta_x\bar{e}^3_{j-\frac{1}{2}})(\delta_x\bar{e}^1_{j-\frac{1}{2}})-\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{e}^3_j)(\delta_x^2\bar{e}^1_j)]\\
&+2\bar{d}^3_0[\Delta x\sum_{j=1}^{M}(\delta_x\bar{e}^3_{j-\frac{1}{2}})(\delta_xe^0_{j-\frac{1}{2}})-\frac{\Delta x^2}{12}\Delta x\sum_{j=1}^{M-1}(\delta_x^2\bar{e}^3_j)(\delta_x^2e^0_j)]+\alpha_0C_1^{-1}2\Delta x\sum_{j=1}^{M-1}(-\delta_x^2\bar{e}^3_j)(R^3_j).
\end{aligned}
$$

Using (2.3), we have

$$
\begin{aligned}
&2\langle\bar{e}^3,\bar{e}^3\rangle+\alpha_0C_1^{-1}\kappa_1[\|\delta_x^2\bar{e}^3\|^2+\|\delta_x^2e^3\|^2-\theta^2\|\delta_x^2e^2\|^2]\\
=\ &2\bar{d}^3_2\langle\bar{e}^3,\bar{e}^2\rangle+2\bar{d}^3_1\langle\bar{e}^3,\bar{e}^1\rangle+2\bar{d}^3_0\langle\bar{e}^3,e^0\rangle+\alpha_0C_1^{-1}2\Delta x\sum_{j=1}^{M-1}(-\delta_x^2\bar{e}^3_j)(R^3_j)\\
\leqslant\ &\bar{d}^3_2[\langle\bar{e}^3,\bar{e}^3\rangle+\langle\bar{e}^2,\bar{e}^2\rangle]+\bar{d}^3_1[\langle\bar{e}^3,\bar{e}^3\rangle+\langle\bar{e}^1,\bar{e}^1\rangle]\\
&+\bar{d}^3_0[\langle\bar{e}^3,\bar{e}^3\rangle+\langle e^0,e^0\rangle]+\alpha_0C_1^{-1}2\Delta x\sum_{j=1}^{M-1}(-\delta_x^2\bar{e}^3_j)(R^3_j).
\end{aligned}
$$

By using (2), (3) in Lemma 2.7, (3.39), and (3.40), we have

$$
\begin{aligned}
&2\langle\bar{e}^3,\bar{e}^3\rangle+\alpha_0C_1^{-1}\kappa_1[\|\delta_x^2\bar{e}^3\|^2+\|\delta_x^2e^3\|^2-\theta^2\|\delta_x^2e^2\|^2]\\
&\le(\bar{d}^3_2+\bar{d}^3_1+\bar{d}^3_0)\langle\bar{e}^3,\bar{e}^3\rangle+\bar{d}^3_2\langle\bar{e}^2,\bar{e}^2\rangle+\bar{d}^3_1\langle\bar{e}^1,\bar{e}^1\rangle+\bar{d}^3_0\langle e^0,e^0\rangle+\alpha_0C_1^{-1}2\Delta x\sum_{j=1}^{M-1}(-\delta_x^2\bar{e}^3_j)(R^3_j)\\
&\le\langle\bar{e}^3,\bar{e}^3\rangle+\bar{d}^3_2\langle\bar{e}^2,\bar{e}^2\rangle+\bar{d}^3_1\langle\bar{e}^1,\bar{e}^1\rangle+\alpha_0C_1^{-1}2\Delta x\sum_{j=1}^{M-1}(-\delta_x^2\bar{e}^3_j)(R^3_j)\\
&\le\langle\bar{e}^3,\bar{e}^3\rangle+\bar{d}^3_2\langle\bar{e}^2,\bar{e}^2\rangle+\bar{d}^3_1\langle\bar{e}^1,\bar{e}^1\rangle+\alpha_0C_1^{-1}2\Big(\frac{\kappa_1}{2}\|\delta_x^2\bar{e}^3\|^2+\frac{1}{2\kappa_1}\|R^3\|^2\Big).
\end{aligned}
\tag{3.58}
$$

Rearrange (3.58) and use (2) in Lemma 2.7, and we have

$$
\begin{aligned}
&\langle\bar{e}^3,\bar{e}^3\rangle+\alpha_0C_1^{-1}\kappa_1\|\delta_x^2e^3\|^2\\
&\le\bar{d}^3_2\langle\bar{e}^2,\bar{e}^2\rangle+\bar{d}^3_1\langle\bar{e}^1,\bar{e}^1\rangle+\alpha_0C_1^{-1}\kappa_1^{-1}\|R^3\|^2+\alpha_0C_1^{-1}\kappa_1\theta^2\|\delta_x^2e^2\|^2\\
&\le\theta(\langle\bar{e}^2,\bar{e}^2\rangle+\alpha_0C_1^{-1}\kappa_1\theta\|\delta_x^2e^2\|^2)+\bar{d}^3_1\langle\bar{e}^1,\bar{e}^1\rangle+\alpha_0C_1^{-1}\kappa_1^{-1}\|R^3\|^2\\
&\le\theta(\langle\bar{e}^2,\bar{e}^2\rangle+\alpha_0C_1^{-1}\kappa_1\|\delta_x^2e^2\|^2)+\bar{d}^3_1(\langle\bar{e}^1,\bar{e}^1\rangle+\alpha_0C_1^{-1}\kappa_1\|\delta_x^2e^1\|^2)+\bar{d}^3_0\Big(\frac{\alpha_0C_1^{-1}}{\kappa_1\bar{d}^3_0}\|R^3\|^2\Big)\\
&\le(\theta+\bar{d}^3_1+\bar{d}^3_0)\Big[2(1+\theta^2)\frac{\alpha_4\alpha_0}{\alpha_3\kappa_1}(\|R^1\|^2+\|R^2\|^2)+\frac{\alpha_0C_1^{-1}}{\kappa_1\bar{d}^3_0}\max_{3\le i\le k}\|R^i\|^2\Big].
\end{aligned}
\tag{3.59}
$$

According to $\theta + \bar{d}_1^3 + \bar{d}_0^3 \leq 1 + \theta$, by using (2.5) and (2.23), (3.54) is correct for $k = 3$.

For $k \geq 4$, multiply both sides of (3.38d) by $2\Delta x(-\delta_x^2 \bar{e}_j^k)$, and we sum up for j from 1 to $M - 1$. Using the similar method for $k = 3$, we get

$$\begin{aligned}
& 2\langle \bar{e}^k, \bar{e}^k\rangle + \alpha_0 C_1^{-1}\kappa_1[\|\delta_x^2 \bar{e}^k\|^2 + \|\delta_x^2 e^k\|^2 - \theta^2\|\delta_x^2 e^{k-1}\|^2] \\
= \;& 2\theta\langle \bar{e}^k, \bar{e}^{k-1}\rangle + 2\sum_{i=2}^{k-1} \bar{d}_{k-i}^k \langle \bar{e}^k, \bar{e}^{k-i}\rangle + 2\bar{d}_0^k\langle \bar{e}^k, e^0\rangle + \alpha_0 C_1^{-1} 2\Delta x \sum_{j=1}^{M-1}(-\delta_x^2 \bar{e}_j^k)(R_j^k) \\
& \leqslant \theta[\langle \bar{e}^k, \bar{e}^k\rangle + \langle \bar{e}^{k-1}, \bar{e}^{k-1}\rangle] + \sum_{i=2}^{k-1} \bar{d}_{k-i}^k[\langle \bar{e}^k, \bar{e}^k\rangle + \langle \bar{e}^{k-i}, \bar{e}^{k-i}\rangle] \\
& + \bar{d}_0^k[\langle \bar{e}^k, \bar{e}^k\rangle + \langle e^0, e^0\rangle] + \alpha_0 C_1^{-1} 2\Delta x \sum_{j=1}^{M-1}(-\delta_x^2 \bar{e}_j^k)(R_j^k) \\
= \;& (\theta + \sum_{i=2}^{k-1} \bar{d}_{k-i}^k + \bar{d}_0^k)\langle \bar{e}^k, \bar{e}^k\rangle + \theta\langle \bar{e}^{k-1}, \bar{e}^{k-1}\rangle + \sum_{i=2}^{k-1} \bar{d}_{k-i}^k\langle \bar{e}^{k-i}, \bar{e}^{k-i}\rangle \\
& + \alpha_0 C_1^{-1} 2\Delta x \sum_{j=1}^{M-1}(-\delta_x^2 \bar{e}_j^k)(R_j^k).
\end{aligned} \tag{3.60}$$

According to (1) and (3) in Lemma 2.6, the above inequality (3.60) becomes

$$\begin{aligned}
& \langle \bar{e}^k, \bar{e}^k\rangle + \alpha_0 C_1^{-1}\kappa_1[\|\delta_x^2 \bar{e}^k\|^2 + \|\delta_x^2 e^k\|^2 - \theta^2\|\delta_x^2 e^{k-1}\|^2] \\
& \leq \theta\langle \bar{e}^{k-1}, \bar{e}^{k-1}\rangle + \sum_{i=2}^{k-1} \bar{d}_{k-i}^k\langle \bar{e}^{k-i}, \bar{e}^{k-i}\rangle + \alpha_0 C_1^{-1} 2\Big(\frac{\kappa_1}{2}\|\delta_x^2 \bar{e}^k\|^2 + \frac{1}{2\kappa_1}\|R^k\|^2\Big).
\end{aligned} \tag{3.61}$$

Reorganizing (3.61), we have

$$\begin{aligned}
& \langle \bar{e}^k, \bar{e}^k\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^k\|^2 \\
\leq \;& \theta\langle \bar{e}^{k-1}, \bar{e}^{k-1}\rangle + \sum_{i=2}^{k-1} \bar{d}_{k-i}^k\langle \bar{e}^{k-i}, \bar{e}^{k-i}\rangle + \alpha_0 C_1^{-1}\kappa_1^{-1}\|R^k\|^2 + \alpha_0 C_1^{-1}\kappa_1\theta^2\|\delta_x^2 e^{k-1}\|^2 \\
\leq \;& \theta[\langle \bar{e}^{k-1}, \bar{e}^{k-1}\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^{k-1}\|^2] + \sum_{i=2}^{k-1} \bar{d}_{k-i}^k[\langle \bar{e}^{k-i}, \bar{e}^{k-i}\rangle \\
& + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^{k-i}\|^2] + \alpha_0 C_1^{-1}\kappa_1^{-1}\|R^k\|^2.
\end{aligned} \tag{3.62}$$

Next, we will prove (3.54) for $k \geq 4$ by using the mathematics induction.

As $k = 4$, from (3.62), we can obtain

$$\begin{aligned}
& \langle \bar{e}^4, \bar{e}^4\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^4\|^2 \leq \theta[\langle \bar{e}^3, \bar{e}^3\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^3\|^2] \\
& + \sum_{i=2}^{3} \bar{d}_{4-i}^4[\langle \bar{e}^{4-i}, \bar{e}^{4-i}\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^{4-i}\|^2] + \bar{d}_0^4\Big(\frac{\alpha_0 C_1^{-1}}{\kappa_1 \bar{d}_0^4}\max_{3\leq i\leq k}\|R^i\|^2\Big) \\
\leq \;& (\theta + \bar{d}_2^4 + \bar{d}_1^4 + \bar{d}_0^4)\Big[2(1+\theta)(1+\theta^2)\frac{\alpha_4\Gamma(3-\alpha)\Delta t^\alpha}{\alpha_3\kappa_1}(\|R^1\|^2 + \|R^2\|^2)
\end{aligned}$$

$$+\frac{3(1+\theta)\Gamma(1-\alpha)T^{\alpha}}{2\kappa_1}\max_{3\le i\le k}\|R^i\|^2\Big]. \tag{3.63}$$

According to (3) in Lemma 2.6, we can obtain (3.54) for $k = 4$. We assume that (3.54) is correct for $k = 5, 6, \cdots, N-1$, and one can immediately obtain that

$$\begin{aligned}
&\langle \bar{e}^N, \bar{e}^N\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^N\|^2 \\
\le\ & \theta\langle \bar{e}^{N-1}, \bar{e}^{N-1}\rangle + \sum_{i=2}^{N-1}\bar{d}_{N-i}^N\langle \bar{e}^{N-i}, \bar{e}^{N-i}\rangle + \alpha_0 C_1^{-1}\kappa_1^{-1}\|R^N\|^2 + \alpha_0 C_1^{-1}\kappa_1\theta^2\|\delta_x^2 e^{N-1}\|^2 \\
\le\ & \theta[\langle \bar{e}^{N-1}, \bar{e}^{N-1}\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^{N-1}\|^2] \\
& + \sum_{i=2}^{N-1}\bar{d}_{N-i}^N[\langle \bar{e}^{N-i}, \bar{e}^{N-i}\rangle + \alpha_0 C_1^{-1}\kappa_1\|\delta_x^2 e^{N-i}\|^2] + \bar{d}_0^N\Big(\frac{\alpha_0 C_1^{-1}}{\kappa_1\bar{d}_0^N}\max_{3\le i\le N}\|R^i\|^2\Big) \\
\le\ & (\theta + \sum_{i=2}^{N-1}\bar{d}_{N-i}^N + \bar{d}_0^N)\Big[2(1+\theta)(1+\theta^2)\frac{\alpha_4\Gamma(3-\alpha)\Delta t^{\alpha}}{\alpha_3\kappa_1}(\|R^1\|^2 + \|R^2\|^2) \\
& + \frac{3(1+\theta)\Gamma(1-\alpha)T^{\alpha}}{2\kappa_1}\max_{3\le i\le k}\|R^i\|^2\Big].
\end{aligned}$$

According to (3) in Lemma 2.6, we can obtain (3.54) for $k = N$, so the proof of (3.54) is completed.

- (3) Finally, we will prove

$$\|v(x, t_k) - v^k\|_\infty \leqslant \sqrt{\frac{27}{8}(b_1 - a_1)C}(\Delta t^{3-\alpha} + \Delta x^4), \quad k \ge 1, \tag{3.64}$$

where C is defined in (3.35).

From (3.54), $\forall k \ge 1$, and (2.11), we can get

$$\begin{aligned}
\langle \bar{e}^k, \bar{e}^k\rangle &\le 2(1+\theta)(1+\theta^2)\frac{\alpha_4\Gamma(3-\alpha)\Delta t^{\alpha}}{\alpha_3\kappa_\alpha}(\|R^1\|^2 + \|R^2\|^2) + \frac{3(1+\theta)\Gamma(1-\alpha)T^{\alpha}}{2\kappa_1}\max_{3\le i\le k}\|R^i\|^2 \\
&\le 2(1+\theta)(1+\theta^2)\frac{\alpha_4\Gamma(3-\alpha)\Delta t^{\alpha}}{\alpha_3\kappa_1}\cdot 2\hat{C}^2(b_1 - a_1)(\Delta t^{3-\alpha} + \Delta x^4)^2 \\
&\quad + \frac{3(1+\theta)\Gamma(1-\alpha)T^{\alpha}}{2\kappa_1}\cdot \hat{C}^2(b_1 - a_1)(\Delta t^{3-\alpha} + \Delta x^4)^2 \le C(\Delta t^{3-\alpha} + \Delta x^4)^2,
\end{aligned} \tag{3.65}$$

where C is defined in (3.35).

According to (2.18) and (1) in Lemma 2.6, we have

$$\begin{aligned}
\sqrt{\langle e^k, e^k\rangle} &= \sqrt{\langle \bar{e}^k + \theta e^{k-1}, \bar{e}^k + \theta e^{k-1}\rangle} \leqslant \sqrt{\langle \bar{e}^k, \bar{e}^k\rangle} + \theta\sqrt{\langle \bar{e}^{k-1}, \bar{e}^{k-1}\rangle} + \theta^2\sqrt{\langle e^{k-2}, e^{k-2}\rangle} \le \cdots \\
&\leqslant \sqrt{\langle \bar{e}^k, \bar{e}^k\rangle} + \theta\sqrt{\langle \bar{e}^{k-1}, \bar{e}^{k-1}\rangle} + \theta^2\sqrt{\langle \bar{e}^{k-2}, \bar{e}^{k-2}\rangle} + \cdots + \theta^{k-1}\sqrt{\langle \bar{e}^1, \bar{e}^1\rangle} + \theta^k\sqrt{\langle e^0, e^0\rangle} \\
&\leqslant \Big(1 + \theta + \theta^2 + \cdots + \theta^{k-1}\Big)\sqrt{C}(\Delta t^{3-\alpha} + \Delta x^4) \leqslant 3\sqrt{C}(\Delta t^{3-\alpha} + \Delta x^4).
\end{aligned}$$

That is, we have

$$\langle e^k, e^k\rangle \le 9C(\Delta t^{3-\alpha} + \Delta x^4)^2. \tag{3.66}$$

Combining (3.66) with Lemmas 2.1 and 2.2, we have

$$\|e^k\|_\infty^2 \le \frac{3(b_1 - a_1)}{8}\langle e^k, e^k\rangle \le \frac{27}{8}(b_1 - a_1)C(\Delta t^{3-\alpha} + \Delta x^4)^2.$$

Theorem 3.2 has been proved completely. □

4. Application: A 2D spatial CFDS for TFDEs

Next, we construct a high order numerical scheme for (1.1) as $d = 2$. Take three positive integers M_1, M_2, N, let $\Delta x_1 = \frac{b_1-a_1}{M_1}, \Delta x_2 = \frac{b_2-a_2}{M_2}, \Delta t = \frac{T}{N}$, and denote $x_{1,j} = a_1 + j\Delta x_1$ $(0 \le j \le M_1), x_{2,l} = a_2 + l\Delta x_2$ $(0 \le l \le M_2), t_k = k\Delta t$ $(0 \le k \le N)$, $\Omega_{\Delta x_1,\Delta x_2} = \{(x_{1,j}, x_{2,l})| \, 0 \le j \le M_1, 0 \le l \le M_2\}$, $V_{\Delta x_1,\Delta x_2} = \{v|v = (v_{0,0}, \cdots, v_{0,M_2}, v_{1,0}, \cdots, v_{1,M_2}, \cdots, v_{M_1,0}, \cdots, v_{M_1,M_2}), v_{0,l} = v_{M_1,l} = 0, l = 0, 1, \cdots, M_2, v_{j,0} = v_{j,M_2} = 0, j = 0, 1, \cdots, M_1\}$. For any grid function $v = \{v_{j,l}|0 \le j \le M_1, 0 \le l \le M_2\}$, denote

$$\delta_{x_1} v_{j-\frac{1}{2},l} = \frac{v_{j,l} - v_{j-1,l}}{\Delta x_1}, \ \ \delta_{x_1}^2 v_{j,l} = \frac{\delta_{x_1} v_{j+\frac{1}{2},l} - \delta_{x_1} v_{j-\frac{1}{2},l}}{\Delta x_1}.$$
$$\delta_{x_2}\delta_{x_1} v_{j-\frac{1}{2},l-\frac{1}{2}} = \frac{\delta_{x_1} v_{j-\frac{1}{2},l} - \delta_{x_1} v_{j-\frac{1}{2},l-1}}{\Delta x_2}, \delta_{x_2}\delta_{x_1}^2 v_{j,l-\frac{1}{2}} = \frac{\delta_{x_1}^2 v_{j,l} - \delta_{x_1}^2 v_{j,l-1}}{\Delta x_2}.$$

The compact difference operators $H_{\Delta x_1} v_{j,l}, H_{\Delta x_2} v_{j,l}$ for the spatial are defined as

$$H_{\Delta x_1} v_{j,l} = \begin{cases} (I + \frac{\Delta x_1{}^2}{12}\delta_{x_1}^2) v_{j,l}, & 1 \le j \le M_1 - 1, 0 \le l \le M_2, \\ v_{j,l}, & j = 0, M_1, 0 \le l \le M_2, \end{cases}$$

and

$$H_{\Delta x_2} v_{j,l} = \begin{cases} (I + \frac{\Delta x_2{}^2}{12}\delta_{x_2}^2) v_{j,l}, & 1 \le l \le M_2 - 1, 0 \le j \le M_1, \\ v_{j,l}, & l = 0, M_2, 0 \le j \le M_1. \end{cases}$$

For $\forall v, w \in V_{\Delta x_1,\Delta x_2}$, the inner product and the norms are defined by

$$\|v\|_\infty = \max_{\substack{1\le j\le M_1-1 \\ 1\le l\le M_2-1}} |v_{j,l}|, \ \|\delta_{x_1} v\| = \sqrt{\Delta x_1 \Delta x_2 \sum_{j=1}^{M_1} \sum_{l=1}^{M_2-1} (\delta_{x_1} v_{j-\frac{1}{2},l})^2}, \ \|\delta_x^2 v\| = \sqrt{\Delta x \sum_{j=1}^{M-1} (\delta_x^2 v_j)^2}.$$

Similar to the method for (2.15), we can obtain the high order CFDS for (1.1) as $d = 2$ as follows:

$$\begin{cases} H(\hat{D}_0 v_{j,l}^0 + \hat{D}_1 v_{j,l}^1 + \hat{D}_2 v_{j,l}^2) - \alpha_0(\kappa_1 H_{\Delta x_2}\delta_{x_1}^2 v_{j,l}^1 + \kappa_2 H_{\Delta x_1}\delta_{x_2}^2 v_{j,l}^1) = \alpha_0 H(f_{j,l}^1), \ k = 1, & (4.1a) \\ H(\tilde{D}_0 v_{j,l}^0 + \tilde{D}_1 v_{j,l}^1 + \tilde{D}_2 v_{j,l}^2) - \alpha_0(\kappa_1 H_{\Delta x_2}\delta_{x_1}^2 v_{j,l}^2 + \kappa_2 H_{\Delta x_1}\delta_{x_2}^2 v_{j,l}^2) = \alpha_0 H(f_{j,l}^2), \ k = 2, & (4.1b) \\ H(v_{j,l}^3 - \sum_{i=1}^{3} d_{3-i}^3 v_{j,l}^{3-i}) - \alpha_0 C_1^{-1}(\kappa_1 H_{\Delta x_2}\delta_{x_1}^2 v_{j,l}^3 + \kappa_2 H_{\Delta x_1}\delta_{x_2}^2 v_{j,l}^3) = \alpha_0 C_1^{-1} H(f_{j,l}^3), \ k = 3, & (4.1c) \\ H(v_{j,l}^k - \sum_{i=1}^{k} d_{k-i}^k v_{j,l}^{k-i}) - \alpha_0 C_1^{-1}[\kappa_1 H_{\Delta x_2}\delta_{x_1}^2 v_{j,l}^k + \kappa_2 H_{\Delta x_1}\delta_{x_2}^2 v_{j,l}^k] = \alpha_0 C_1^{-1} H(f_{j,l}^k), \ k \ge 4, & (4.1d) \end{cases}$$

where $H = H_{\Delta x_1} H_{\Delta x_2}$. Similar to the proof of Theorems 3.1 and 3.2, we can obtain the stability and convergence analysis for (4.1) as the following Theorems 4.1 and 4.2.

Theorem 4.1. *The full discrete numerical scheme* (4.1) *is unconditionally stable for* $0 < \alpha < 1$, *and its numerical solution satisfies the following bounded estimates for all* $\Delta x_1 > 0, \Delta x_2 > 0, \Delta t > 0$,

$$\|v^k\|_\infty \le C|v^0|_1, \quad 1 \le k \le N,$$

where C is a constant independent of $\Delta x_1, \Delta x_2, \Delta t$ and $|\cdot|_1$ is defined as

$$|v|_1 = \sqrt{\Delta x_1 \Delta x_2 \sum_{j=1}^{M_1} \sum_{l=1}^{M_2} (|\delta_{x_1} v_{j-\frac{1}{2},l}|^2 + |\delta_{x_2} v_{j,l-\frac{1}{2}}|^2)}.$$

Theorem 4.2. *Let $v(\mathbf{x}, t)$ be the exact solution of Eq (1.1a) and $v_{j,l}^k$ be the numerical scheme defined by (4.1). If $\max_{t\in(0,T]} |\partial_t^3 v| \leq M$, then*

$$\|v(\mathbf{x}, t_k) - v^k\|_\infty \leqslant C(\Delta t^{3-\alpha} + \Delta x_1{}^4 + \Delta x_2{}^4), k = 1, 2, \cdots, N,$$

where $0 < \alpha < 1$, and C is a positive independence of $\Delta t, \Delta x_1, \Delta x_2$.

5. Numerical validation

5.1. Algorithm implementation of 1D CFDS for TFDEs

In order to make it easy for readers to understand the algorithm (2.22), we describe the implementation process of the algorithm (2.22) as follows. In order to facilitate the implementation of the algorithm, we have changed the description of the algorithm to the form of matrix vector product.

(1) Denote $\mathcal{D} = \frac{1}{\Delta x^2}\text{tridiag}(1, -2, 1)$ as the second order space derivative's difference matrix and $\mathcal{H} = \text{tridiag}(1/12, 10/12, 1/12)$ as the compact finite difference matrix. $\mathcal{D}, \mathcal{H}$ are two tridiagonal matrices of $(M-1) \times (M-1)$. Denote

$$\mathbf{v}^k = (v_1^k, v_2^k, \cdots, v_{M-1}^k), \mathbf{f}^k = (f_1^k, f_2^k, \cdots, f_{M-1}^k), \quad k = 1, \cdots, N.$$

(2) Based on (2.22a) and (2.22b), one can immediately obtain the form of the matrix vector product for (2.22a) and (2.22b) for $k = 1, 2$ as follows:

$$\begin{pmatrix} \hat{D}_1\mathcal{H} - \alpha_0\kappa_1\mathcal{D} & \hat{D}_2\mathcal{H} \\ \tilde{D}_1\mathcal{H} & \tilde{D}_2\mathcal{H} - \alpha_0\kappa_1\mathcal{D} \end{pmatrix} \begin{pmatrix} (\mathbf{v}^1)^\mathsf{T} \\ (\mathbf{v}^2)^\mathsf{T} \end{pmatrix} = I_{2\times 2} \otimes \mathcal{H} \begin{pmatrix} (\mathbf{f}^1)^\mathsf{T} \\ (\mathbf{f}^2)^\mathsf{T} \end{pmatrix}, \tag{5.1}$$

where $I_{2\times 2}$ as an identity matrix of 2×2, $\otimes$ is the Kronecker product, and $(\mathbf{v}^1)^\mathsf{T}$ denotes the transpose of $(\mathbf{v}^1)$. Solve the above Eq (5.1) and obtain $\mathbf{v}^1, \mathbf{v}^2$.

(3) For $k = 3$, based on (2.22c), we can obtain the form of the matrix vector product for (2.22c) as follows:

$$(\mathcal{H} - \alpha_0 C_1^{-1}\kappa_1\mathcal{D})(\mathbf{v}^3) = \alpha_0 C_1^{-1}\mathcal{H}(\mathbf{f}^3) + d_2^3\mathcal{H}(\mathbf{v}^2) + d_1^3\mathcal{H}(\mathbf{v}^1) + d_0^3\mathcal{H}(\mathbf{v}^0). \tag{5.2}$$

One can immediately get $\mathbf{v}^3$ by solving the Eq (5.2) directly.

(4) For $k \geq 4$, by (2.22d), one has the form of the matrix vector product for (2.22d) as follows:

$$(\mathcal{H} - \alpha_0 C_1^{-1}\kappa_1\mathcal{D})(\mathbf{v}^k) = \alpha_0 C_1^{-1}\mathcal{H}(\mathbf{f}^k) + \mathcal{H}(\sum_{i=1}^{k} d_{k-i}^k(v^{k-i})). \tag{5.3}$$

Solving the Eq (5.3), one can immediately get $\mathbf{v}^k, k = 4, 5, \cdots, N$.

5.2. Algorithm implementation of 2D CFDS for TFDEs

In order to make it easy for readers to understand the algorithm (4.1), we describe the implementation process of the algorithm (4.1) as follows.

(1) Let $M_1 = M_2 = M$. Denote

$$\mathcal{D}_1 = \text{tridiag}(\frac{-1}{6}(\frac{5}{\Delta x_1{}^2} - \frac{1}{\Delta x_2{}^2}), \frac{5}{3}(\frac{1}{\Delta x_1{}^2} + \frac{1}{\Delta x_2{}^2}), \frac{-1}{6}(\frac{5}{\Delta x_1{}^2} - \frac{1}{\Delta x_2{}^2}))$$

and

$$\mathcal{D}_2 = \text{tridiag}(\frac{-1}{12}(\frac{1}{\Delta x_1{}^2} + \frac{1}{\Delta x_2{}^2}), \frac{-1}{6}(\frac{5}{\Delta x_1{}^2} - \frac{1}{\Delta x_2{}^2}), \frac{-1}{12}(\frac{1}{\Delta x_1{}^2} + \frac{1}{\Delta x_2{}^2})),$$

which are two $(M-1)\times(M-1)$ tridiagonal matrices. $B = tridiag(\mathcal{D}_2, \mathcal{D}_1, \mathcal{D}_2)$ as a $(M-1)\times(M-1)$ tridiagonal block matrix, which is the second order space derivative's difference matrix. Denote $Cc = \text{tridiag}(\frac{10}{144}, \frac{100}{144}, \frac{10}{144})$ and $Dd = \text{tridiag}(\frac{1}{144}, \frac{10}{144}, \frac{1}{144})$ as two $(M-1)\times(M-1)$ triagonal matrices, and $Bb = \text{tridiag}(Dd, Cc, Dd)$ as a $(M-1)\times(M-1)$ tridiagonal block matrix, which is the spatial compact finite difference matrix. Let $\kappa_1 = \kappa_2 = \kappa_\alpha$, and denote

$$\mathbf{v}_l^k = (v_{1,l}^k, v_{2,l}^k, \cdots, v_{M-1,l}^k)^T, \mathbf{V}^k = (\mathbf{v}_1^k, \mathbf{v}_2^k, \cdots, \mathbf{v}_{M-1}^k)^T,$$
$$\mathbf{f}_l^k = (f_{1,l}^k, f_{2,l}^k, \cdots, f_{M-1,l}^k)^T, \mathbf{F}^k = (\mathbf{f}_1^k, \mathbf{f}_2^k, \cdots, \mathbf{f}_{M-1}^k)^T, \quad k = 1, \cdots, N.$$

(2) Based on (4.1a) and (4.1b), one can immediately obtain the form of the matrix vector product for (4.1a) and (4.1b) for $k = 1, 2$ as follows:

$$\begin{pmatrix} \hat{D}_1 Bb + \alpha_0\kappa_\alpha B & \hat{D}_2 Bb \\ \tilde{D}_1 Bb & \tilde{D}_2 B + \alpha_0\kappa_\alpha B \end{pmatrix} \begin{pmatrix} (\mathbf{V}^1) \\ (\mathbf{V}^2) \end{pmatrix} = I_{2\times 2} \otimes Bb \begin{pmatrix} (\mathbf{F}^1) \\ (\mathbf{F}^2) \end{pmatrix}. \tag{5.4}$$

Solve the above Eq (5.4), and obtain $\mathbf{V}^1, \mathbf{V}^2$.

(3) For $k = 3$, based on (4.1c), we can obtain the form of the matrix vector product for (2.22c) as follows:

$$(Bb + \alpha_0 C_1^{-1}\kappa_\alpha B)(\mathbf{V}^3) = \alpha_0 C_1^{-1} Bb(\mathbf{F}^3) + d_2^3 Bb(\mathbf{V}^2) + d_1^3 Bb(\mathbf{V}^1) + d_0^3 Bb(\mathbf{V}^0). \tag{5.5}$$

One can immediately get $\mathbf{v}^3$ by solving the Eq (5.5) directly.

(4) For $k \geq 4$, by (4.1d), one has the form of the matrix vector product for (2.22d) as follows:

$$(Bb + \alpha_0 C_1^{-1}\kappa_\alpha B)(\mathbf{V}^k) = \alpha_0 C_1^{-1} Bb(\mathbf{F}^k) + Bb(\sum_{i=1}^{k} d_{k-i}^k(\mathbf{V}^{k-i})). \tag{5.6}$$

Solving the Eq (5.6), one can immediately get $\mathbf{V}^k, k = 4, 5, \cdots, N$.

5.3. Numerical results

Now, the numerical scheme (2.22) will be used to solve the TFDEs (1.1a)–(1.1c) based on the Eqs (5.4)–(5.6) in this section. We present two numerical examples to demonstrate its effectiveness and define the numerical solution's maximum norm errors $e_\infty(h, \Delta t), h = \Delta x$ for 1D spatial TFDEs and $h = \Delta x_1 = \Delta x_2$ for 2D spatial TFDEs as follows:

$$e_\infty(h, \Delta t) = \max_{1 \le k \le N} \|v(x, t_k) - v^k\|_\infty.$$

Denote the temporal convergence order $Rate(\Delta t)$ as

$$Rate(\Delta t) = \log_2\left(\frac{e_\infty(h, 2\Delta t)}{e_\infty(h, \Delta t)}\right),$$

if h is sufficiently small, and the spatial convergence order $Rate(h)$ as the following

$$Rate(h) = \log_2\left(\frac{e_\infty(2h, \Delta t)}{e_\infty(h, \Delta t)}\right),$$

when Δt is sufficiently small.

In this section, we choose $a_1 = a_2 = 0, b_1 = b_2 = 1, T = 1, \kappa_1 = \kappa_2 = 1$. In order to test the convergence order of the numerical scheme and their dependence on α, we take $\alpha = 0.3, 0.5, 0.7$ and $\Delta t = \frac{1}{N}, h = \frac{1}{M}$ as two groups of parameters. That is, choosing $N = 2^{11}, M = 2^k, k = 3, 4, 5, 6$ to verify the spatial convergence order and $N = 2^k, k = 3, 4, 5, 6$ with $M = 2^{11}$ to verify the convergence order in time.

Example 5.1. Case 1. *The exact solution is smooth.*

We choose the exact solution of (1.1) as $v(x, t) = t^4 \sin(2\pi x)$. *It is easy to obtain by directly calculating that* $f(x, t)$, $v_0(x)$ *in (1.1) have the form as follows:*

$$f(x, t) = \Big(\frac{24}{\Gamma(5 - \alpha)} t^{4-\alpha} + 4\pi^2 t^4\Big) \sin(2\pi x), \quad v_0(x) = 0,$$

where $\alpha \in (0, 1)$ *is the order of the fractional derivative of Eq (1.1).*

First, we choose $N = 2^{11}$ *and* $M = 2^k, k = 3, 4, 5, 6$ *to check the spatial accuracy. In Table 1, it presents the errors* $e_\infty(h, \Delta t)$ *for* $\alpha = 0.3, 0.5,$ *and* 0.7 *three different values. From the Table 1, one can immediately see that the* $Rate(h)$ *of the proposed high order numerical scheme is very close to fourth-order in space and not dependent on* α. *The numerical results of Table 1 indicate that the theoretical result of the Theorem 3.2 is correct.*

Table 1. The convergence orders of the space $Rate(h)$ under $\alpha = 0.3, 0.5, 0.7$ in Example 5.1.

Δx	$\alpha = 0.3$	$Rate(h)$	$\alpha = 0.5$	$Rate(h)$	$\alpha = 0.7$	$Rate(h)$
$\frac{1}{8}$	1.563359e-3	-	1.544025e-3	-	1.519775e-3	-
$\frac{1}{16}$	9.591726e-5	4.026714	9.473317e-5	4.026683	9.324978e-5	4.026613
$\frac{1}{32}$	5.967359e-6	4.006625	5.894044e-6	4.006539	5.804087e-6	4.005959
$\frac{1}{64}$	3.725729e-7	4.001497	3.683399e-7	4.000148	3.650455e-7	3.990921

Second, we will investigate the Rate(Δt) *by choosing the spatial divisions number M as a big number to satisfy that the error* $e_\infty(h, \Delta t)$ *stemming from the spatial approximation is negligible. We choose* $M = 2^{11}$ *and* $N = 2^k, k = 3, 4, 5, 6$. *From Table 2, it shows that the temporal convergence order is close to* $2.7, 2.5, 2.3$ *for three different constants* $\alpha = 0.3, 0.5, 0.7$, *respectively, i.e., the temporal convergence order is* $(3 - \alpha)$. *The numerical results of Table 2 indicate that the theoretical result of Theorem 3.2 is also correct for the temporal convergence order.*

Table 2. The convergence orders of the time $Rate(\Delta t)$ under $\alpha = 0.3, 0.5, 0.7$ in Example 5.1.

Δt	$\alpha = 0.3$	$Rate(\Delta t)$	$\alpha = 0.5$	$Rate(\Delta t)$	$\alpha = 0.7$	$Rate(\Delta t)$
$\frac{1}{16}$	2.191101e-5	-	6.921581e-5	-	1.875620e-4	-
$\frac{1}{32}$	3.626378e-6	2.595054	1.282670e-5	2.431951	3.938543e-5	2.251633
$\frac{1}{64}$	5.870045e-7	2.627086	2.332808e-6	2.459010	8.142194e-6	2.274172
$\frac{1}{128}$	9.368901e-8	2.647419	4.198113e-7	2.474254	1.669603e-6	2.285912

Lastly, we choose $\alpha = 0.5, N = 2^5, M = 2^5$ *in Figures 1–3. First, the Figure 1 shows that the numerical solution's curves are very close to the exact solution for* $x_j = \frac{7}{8}$ *and* $t_N = 1$, *respectively. This indicates that the numerical solution* v_j^k *of the proposed algorithm is a good approximation to the exact solution* $v(x_j, t_k)$ *of Eq (1.1a). Second, the Figure 2 provides the comparison three-dimensional surface between the numerical solution* v_j^k *of the proposed algorithm and the exact solution* $v(x_j, t_k)$ *of Eq (1.1a). It is easy to see from the Figure 2 that the numerical solution* v_j^k *is very close to the exact solution* $v(x_j, t_k)$. *Third, the Figure 3 shows the three-dimensional surface plot of* $e_\infty(h, \Delta t)$ *of the numerical solution to approximate the exact solution. It can be seen that* $e_\infty(h, \Delta t)$ *is very small in the Figure 3. Finally, it is clear from these graphs that the numerical solution of the proposed algorithm is a good approximation to the exact solution of Eq (1.1a).*

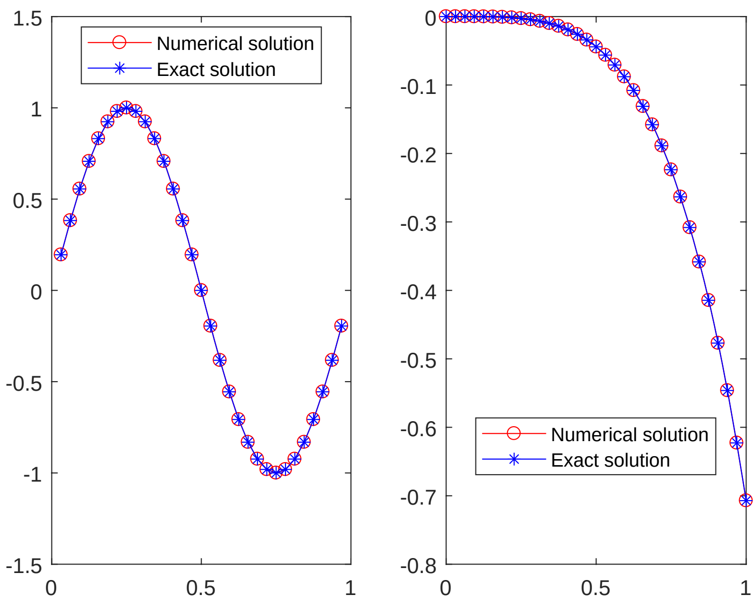

Figure 1. Images of numerical and exact solutions of $\alpha = 0.5$ for $t_N = 1$ (**left**) and $x_j = \frac{7}{8}$ (**right**).

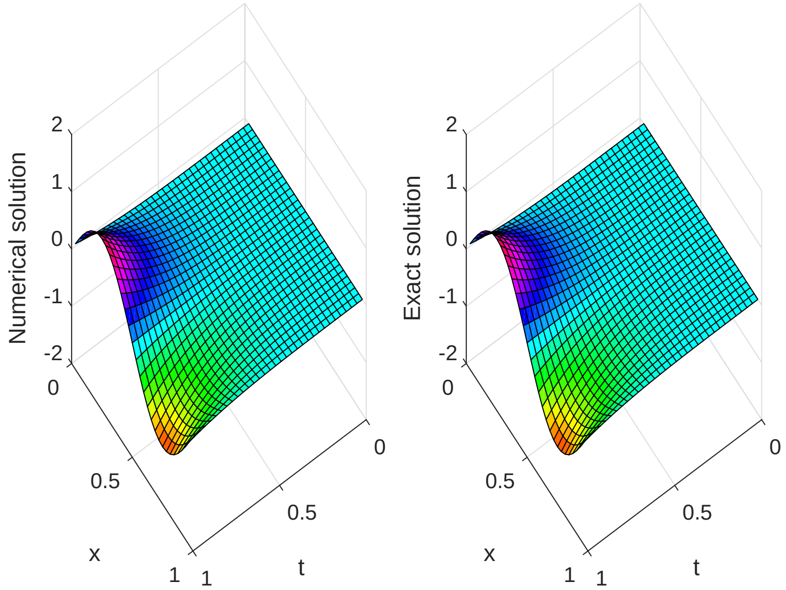

Figure 2. Images of $\alpha = 0.5$ for the numerical solution (**left**) and the exact solution (**right**).

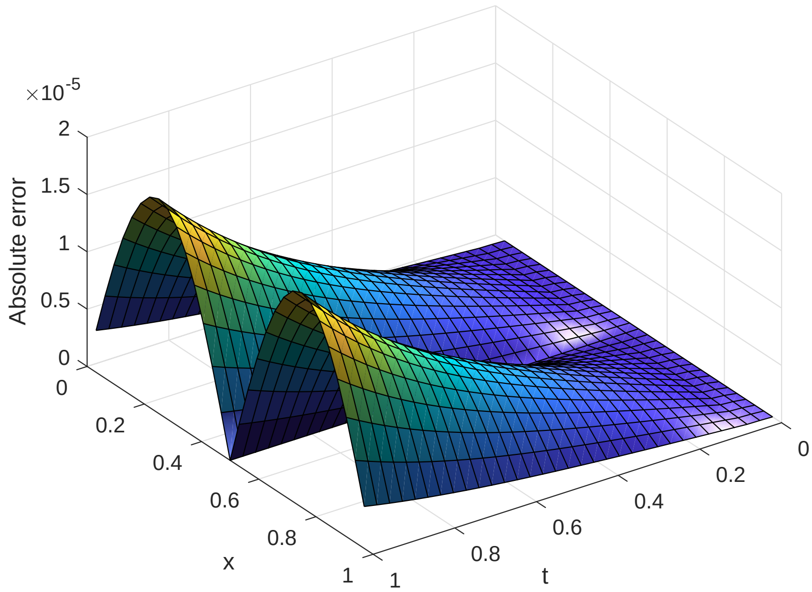

Figure 3. The absolute error distribution of $\alpha = 0.5, M = 2^5, N = 2^5$.

Case 2. *The initial singularity of the solution and the source terms* $f(x,t) = 0$.

We choose $v_0(x) = \sin(x)$ *in (1.1). It is easy to obtain that the exact solution is* $v(x,t) = E_\alpha(-t^\alpha)\sin(x)$, *where* $0 < t < 1, 0 < x < \pi$, *and* $E_\alpha(\cdot)$ *is the Mittag-Leffler function defined by*

$$E_\alpha(z) := \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1 + k\alpha)}.$$

Next, we test time accuracy. The choices of the fractional order are now taken as $\alpha = 0.3, 0.5,$ *and* 0.7. *Results are given in Table 3, from which we can observe that when* $\alpha < 1$, *the convergence order is close to* α. *The reason lies in the singularity of the Mittag-Leffler function at* $t = 0$.

Table 3. The convergence orders of the time $Rate(\Delta t)$ under $\alpha = 0.3, 0.5, 0.7$ in Example 5.1.

Δt	$\alpha = 0.3$	$Rate(\Delta t)$	$\alpha = 0.5$	$Rate(\Delta t)$	$\alpha = 0.7$	$Rate(\Delta t)$
$\frac{1}{16}$	1.415497e-3	-	7.786699e-4	-	1.496562e-3	-
$\frac{1}{32}$	1.255327e-3	0.173244	6.0949104e-4	0.353406	9.285725e-4	0.688566
$\frac{1}{64}$	1.097411e-3	0.193959	4.5933819e-4	0.408048	5.736219e-4	0.694914
$\frac{1}{128}$	9.476369e-4	0.211698	3.3821820e-4	0.441602	3.530184e-4	0.700356

Example 5.2. *We choose the exact solution of (1.1) as* $v(x_1, x_2, t) = t^4 \sin(2\pi x_1)\sin(2\pi x_2)$ *for* $\kappa_1 = \kappa_2 = \kappa_\alpha = 1$. *It is easy to obtain by directly calculating that* $f(x_1, x_2, t)$, $v_0(x_1, x_2)$ *in (1.1) have the form as follows:*

$$f(x_1, x_2, t) = \Big(\frac{\Gamma(5)}{\Gamma(5-\alpha)} t^{4-\alpha} + 8\pi^2 t^4\Big)\sin(2\pi x_1)\sin(2\pi x_2), \quad v_0(x_1, x_2) = 0.$$

We choose $N = 2^{12}, M = 2^k, k = 3, 4, 5, 6, 7$ *and* $N = 2^k, k = 3, 4, 5, 6, 7, M = 2^{10}$ *to check the spatial accuracy and temporal convergence order, respectively. As seen from the Table 4, one can get that* $Rate(\Delta t)$ *is almost* 2.3 *with respect to* $\alpha = 0.7$. *It indicates that* $Rate(\Delta t)$ *is* $(3 - \alpha)$, *which satisfies the conclusion of Theorem 4.2's theoretical analysis in space convergence order. In the same way, one can get that* $Rate(h)$ *is almost* 4 *with respect to* $\alpha = 0.3$. *It indicates that* $Rate(h)$ *is* 4, *which also satisfies the conclusion of Theorem 4.2's theoretical analysis in space convergence order.*

Table 4. The maximum errors $e_\infty(h, \Delta t)$ of $Rate(h)$ for $\alpha = 0.7$ and $Rate(\Delta t)$ for $\alpha = 0.3$ in Example 5.2.

Δt	$\alpha = 0.7$	$Rate(\Delta t)$	h	$\alpha = 0.3$	$Rate(h)$
$\frac{1}{8}$	4.394821e-4	-	$\frac{1}{8}$	9.598866e-4	-
$\frac{1}{16}$	9.517067e-5	2.207215	$\frac{1}{16}$	7.601212e-5	3.658562
$\frac{1}{32}$	1.997848e-5	2.252069	$\frac{1}{32}$	5.365581e-6	3.824422
$\frac{1}{64}$	4.129540e-6	2.274394	$\frac{1}{64}$	3.566689e-7	3.911077
$\frac{1}{128}$	8.467270e-7	2.286012	$\frac{1}{128}$	2.297780e-8	3.956272

We choose $\alpha = 0.5, N = 2^9, M = [N^{\frac{3-\alpha}{4}}]$ *in Figures 4 and 5, where* $[\cdot]$ *represents rounding. The Figure 4 shows that the numerical solution's curves are very close to the exact solution for* $t = 1$. *The Figure 5 provides the comparison three-dimensional surface between the numerical solution of the proposed algorithm and the exact solution of Eq (1.1a) for* $t = 1$. *It is easy to see from the Figures 4 and 5 that the numerical solution is very close to the exact solution.*

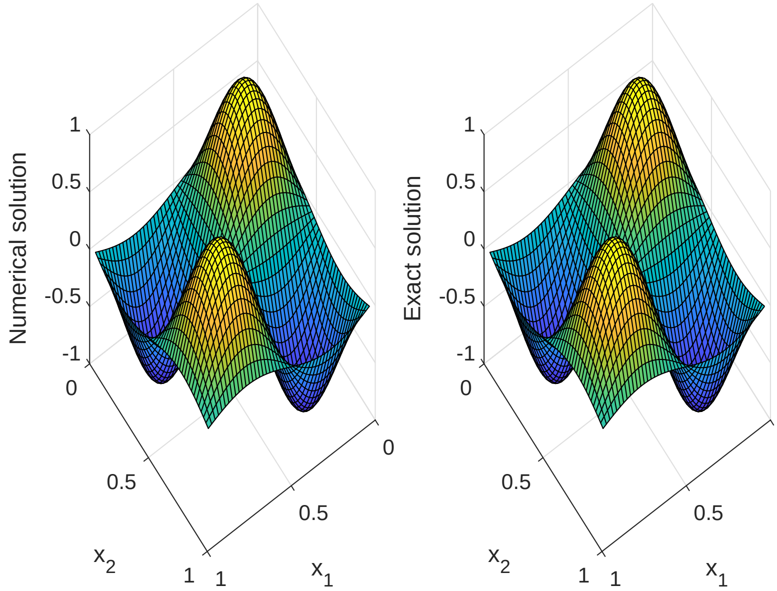

Figure 4. Numerical solution for $\alpha = 0.5$ (**left**) and exact solution for $\alpha = 0.5$ (**right**) when $t = 1$.

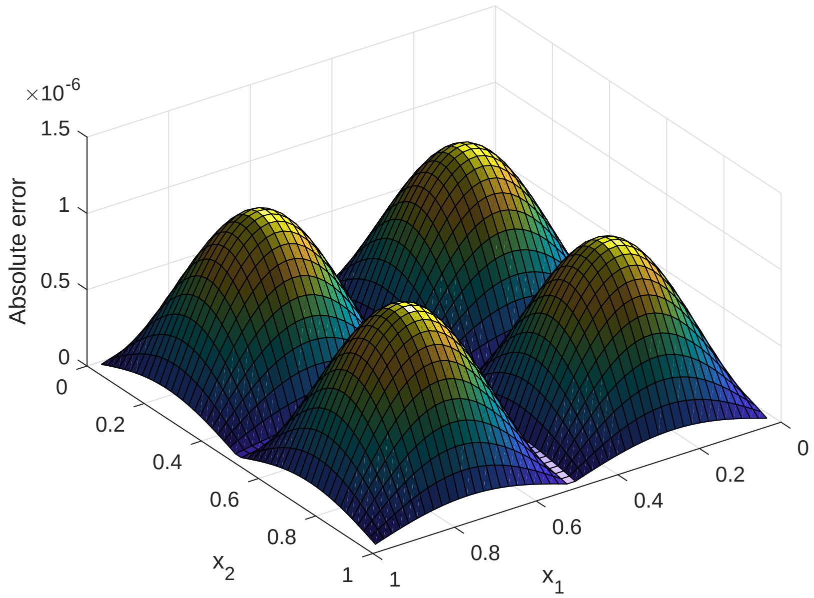

Figure 5. Error distribution of $\alpha = 0.5, t = 1$.

6. Conclusions

In this paper, we propose an efficient full discrete uniform convergence order scheme for TFDEs by using the 1D and 2D spatial fourth-order CFDS in space and $(3 - \alpha)$ order scheme in time. The proposed scheme is constructed by the modified block-by-block method and the 1D and 2D spatial fourth-order CFDSs to approximate the temporal fractional derivative and the spatial second derivative, respectively. We prove that the proposed efficient uniform convergence order schemes are stable. Numerical examples of 1D and 2D spatial fourth-order CFDS further verified the astringency of the proposed method. The numerical scheme established in this article provides a paradigm for establishing high-order TFDEs and a paradigm for analyzing its convergence and stability of high-order time numerical scheme. It can provide readers with a method reference for constructing high-order time numerical schemes and their theoretical analysis for similar TFDEs.

In the next step of research work, we will consider the following topics. First, we will construct a fast CFDS for this numerical scheme with the good idea of [27, 33, 34] by considering the high computational cost of fractional derivatives, the special matrix structure of the discrete matrix of the fourth-order CFDS, and sum-of-exponentials (SOE) technique. Second, we will investigate high-dimensional CFDS for TFDEs and CFDS for the Neumann boundary of TFDEs. We will use the above CFDS to establish the numerical solution of the optimal control problem by TFDEs and establish a CFDS with high order spatial accuracy and consistent convergence in time, and provide strict numerical analysis theory for the established numerical scheme. Third, we will investigate CFDS for time-varying fractional differential equations or spatial fourth-order differential equations and the finite volume scheme preserving maximum principle for two-dimensional time-fractional Fokker-Planck equations on distorted meshes.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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