AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 14594–14617.
DOI:10.3934/math.2024710
Received: 20 December 2023
Revised: 19 March 2024
Accepted: 16 April 2024
Published: 23 April 2024

Research article

Analyzing the normalized Laplacian spectrum and spanning tree of the cross of the derivative of linear networks

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Abstract: In this paper, we focus on the strong product of the pentagonal networks. Let R_n be a hexagonal network composed of $2n$ pentagons and n quadrilaterals. Let P_n^2 denote the graph formed by the strong product of R_n and its copy R_n'. By utilizing the decomposition theorem of the normalized Laplacian characteristics polynomial, we characterize the explicit formula of the multiplicative degree-Kirchhoff index completely. Moreover, the complexity of P_n^2 is determined.

Keywords: (multiplicative degree) Kirchhoff index; Wiener index; Gutman index; spanning trees
Mathematics Subject Classification: 05C50, 05C90

1. Introduction

Throughout this article, we only consider simple, undirected, and finite graphs and assume that all graphs are connected. Suppose $\mathscr{G}$ is a graph with the vertex set $V_(\mathscr{G}) = \{v_1, v_2, \cdots, v_n\}$ and the edge set $E_(\mathscr{G}) = \{e_1, e_2, \cdots, e_m\}$. The adjacency matrix $A(\mathscr{G})$ is a $0-1 n \times n$ matrix indexed by the vertices of $\mathscr{G}$ and defined by $a_i j = 1$ if and only if $v_s v_t \in E_{\mathscr{G}}$. For more notation, one refer to [1].

The Laplacian matrix of graph $\mathscr{G}$ is defined as $L(\mathscr{G}) = D(\mathscr{G}) - A(\mathscr{G})$, and assume that the eigenvalues of $L(\mathscr{G})$ are labeled $0 \leq \mu_1 < \mu_2 < \cdots < \mu_n$.

$$(\mathcal{L}(\mathscr{G}))_{st} = \begin{cases} 1, & s = t; \\ -1, & s \neq t\ ,\ v_s \sim v_t; \\ 0, & otherwise. \end{cases} \tag{1.1}$$

The normalized Laplacian matrix is given by

$$(\mathcal{L}(\mathscr{G}))_{st} = \begin{cases} 1, & s = t; \\ -\frac{1}{\sqrt{d_s d_t}}, & s \neq t\ ,\ v_s \sim v_t; \\ 0, & otherwise. \end{cases} \tag{1.2}$$

The distance, $d_{ij} = d_{\mathscr{G}}(v_s, v_t)$, between vertices us and ut of $\mathscr{G}$ is the length of the shortest u_s, u_t-path in $\mathscr{G}$. The Wiener index [2, 3] is the sum of the distances of any two vertices in the graph $\mathscr{G}$, that is

$$W(G) = \sum_{s<t} d_{st}.$$

In 1947, the distance-based invariant first appeared in chemistry [2, 3] and began to the applied to mathematics 30 years later [5]. Nowadays, the Wiener index is widely used in mathematics [6–8] and chemistry [9–11].

In a simple graph $\mathscr{G}$, the degree, $d_{ij} = d_{\mathscr{G}}(v_i)$, of a vertex v_i is the number of edges at v_i. The Gutman index [12] of the simple graph $\mathscr{G}$ is expressed by

$$Gut(\mathscr{G}) = \sum_{s<t} d_s d_t d_{st}.$$

Klein and Randic initially outlined the concepts associated with the resistance distance of the graph. Assume that each edge is replaced by a unit resistor, and we use rst to denote the resistance distance between two vertices s and t. Similar to the Wiener index, the Kirchhoff index [13, 14] of graph $\mathscr{G}$ is expressed as the sum of the resistance distances between each two vertices, that is

$$Kf(\mathscr{G}) = \sum_{s<t} r_{st}.$$

In 2007, Chen and Zhang [15] defined the multiplicative degree-Kirchhoff index [16, 17], that is

$$Kf^*(\mathscr{G}) = \sum_{s<t} d_s d_t r_{st}.$$

Phenyl is a conjugated hydrocarbon, and $L_n^{6,4,4}$ denote a linear chain, containing n hexagons and $2n - 1$ squares, as shown in Figure 1.

With the rapid changes of the times, organic chemistry has also developed rapidly, which has led to a growing interest in polycyclic aromatic compounds.

In 1985, the computational method and procedure of the matrix decomposition theorem were proposed by Yang [18]. This led to the solution of some problems in graph networks and allowed the unprecedented development of self-homogeneous linear hydrocarbon chains. For example, in 2021, X.L. Ma [20] got the normalized Laplacian spectrum of linear phenylene, and the linear phenylene containing it has n hexagons and $n - 1$ squares. L. Lan [21] explored linear phenylene with n hexagons and n squares. Umar Ali [22] analyzed the strong prism of a graph G, which is the strong product of the complete graph of order 2, and X.Y. Geng [23] obtained the Laplacian spectrum of $L_n^{6,4,4}$, which contains n hexagons and $2n - 1$ squares. J.B. Liu [24] derived the Kirchhoff index and complexity of O_n, which denoting linear octagonal-quadrilateral networks. C. Liu [25] got the Laplacian spectrum and Kirchhoff index of L_n, and L_n has t hexagons and $3t + 1$ quadrangles.

Inspired by these recent works, we try to investigate the Laplacians and the normalized Laplaceians for graph transformations on phenyl dicyclobutadieno derivatives.

The various sections of this article are as follows: In Section 2, we proposed some concepts and lemmas and used them in subsequent articles. In Section 3 and Section 4, we acquired the Laplacian matrix and the normalized Laplacian matrix, then we made sure the Kirchoff index, the multiplicative degree-Kirchoff index, and the complexity of L_n. In Section 5, we obtained conclusions based on the calculations in this paper.

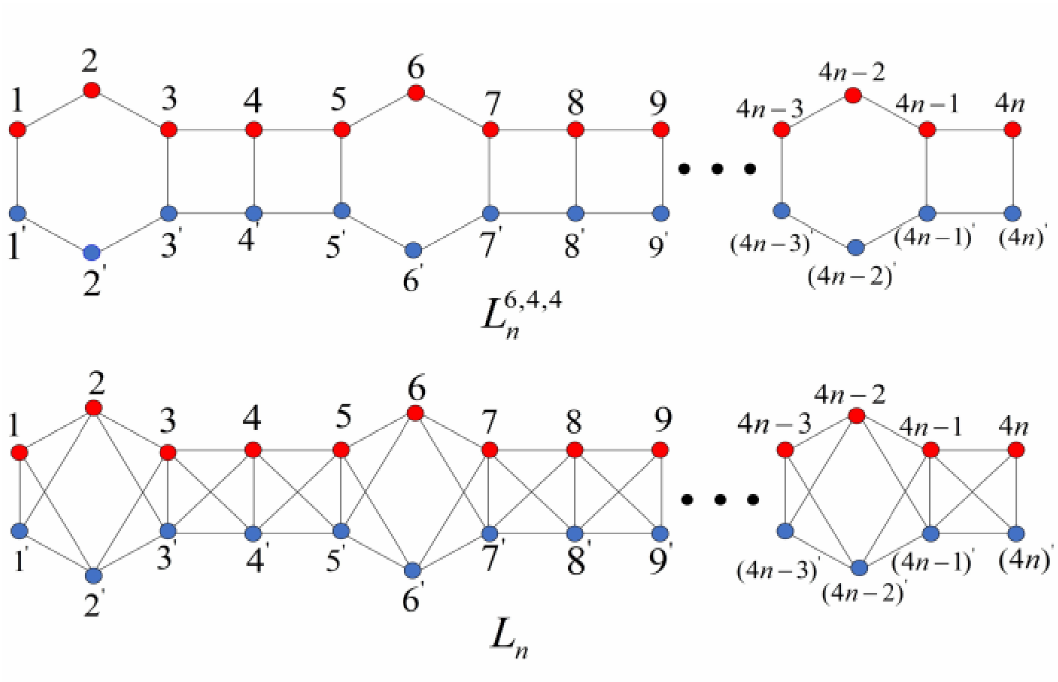

Figure 1. Graphs of $L_n^{6,4,4}$ and L_n.

2. Preliminary works

In this article, graph L_n and graph $L_n^{6,4,4}$ are portrayed in Figure 1. Define the characteristic polynomial of matrix U of order n as $P_U(x) = det(xI - U)$.

It is easy to understand that $\pi = (1, 1')(2, 2') \cdots ((4n), (4n)')$ is an automorphism. Let $V_1 = \{u_1, u_2, \cdots, u_{4n+1}, v_1, \cdots, v_{4n}\}$, $V_2 = \{u'_1, u'_2, \cdots, u'_{4n}, v'_1, \cdots, v'_{4n+1}\}$, $|V(L_n)| = 8n$ and $|E(L_n)| = 19n - 4$. Thus, the (normalized) Laplacians matrix can be expressed in the form of a block matrix, that is

$$\mathcal{L}(L_n) = \begin{pmatrix} \mathcal{L}_{V_0V_0} & \mathcal{L}_{V_0V_1} & \mathcal{L}_{V_0V_2} \\ \mathcal{L}_{V_1V_0} & \mathcal{L}_{V_1V_1} & \mathcal{L}_{V_1V_2} \\ \mathcal{L}_{V_2V_0} & \mathcal{L}_{V_2V_1} & \mathcal{L}_{V_2V_2} \end{pmatrix},$$

where $L_{V_sV_t}$ and $\mathcal{L}_{V_sV_t}$ are a submatrix consisting of rows corresponding to the vertices in V_s and columns corresponding to the vertices in V_t, $s, t = 0, 1, 2$. Let

$$Q = \begin{pmatrix} I_t & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} I_{4n} & \frac{1}{\sqrt{2}} I_{4n} \\ 0 & \frac{1}{\sqrt{2}} I_{4n} & -\frac{1}{\sqrt{2}} I_{4n} \end{pmatrix},$$

then

$$QL(L_{\mathscr{G}})Q' = \begin{pmatrix} L_A(\mathscr{G}) & 0 \\ 0 & L_S(\mathscr{G}) \end{pmatrix}, QL(\mathcal{L}_{\mathscr{G}})Q' = \begin{pmatrix} \mathcal{L}_A(\mathscr{G}) & 0 \\ 0 & \mathcal{L}_S(\mathscr{G}) \end{pmatrix},$$

and Q' is the transposition of Q.

$L_A = L_{V_1V_1} + L_{V_1V_2}$, $L_S = L_{V_1V_1} - L_{V_1V_2}$, $\mathcal{L}_A = \mathcal{L}_{V_1V_1} + \mathcal{L}_{V_1V_2}$, $\mathcal{L}_S = \mathcal{L}_{V_1V_1} - \mathcal{L}_{V_1V_2}$

Lemma 2.1. [20] *If $\mathscr{G}$ is a graph and suppose that $L_A(\mathscr{G}), L_S(\mathscr{G}), L_A(\mathscr{G})$, and $L_S(\mathscr{G})$ are determined as above, then*

$$\vartheta_{L(L_n)}(y) = \theta_{L_A(\mathscr{G})}(y)\theta_{L_S(\mathscr{G})}(y), \vartheta_{\mathcal{L}(L_n)}(y) = \theta_{\mathcal{L}_A(\mathscr{G})}(y)\theta_{\mathcal{L}_S(\mathscr{G})}(y).$$

Lemma 2.2. [26] *With the extensive study of the Kirchhoff index, Gutman and Mohar proposed an algorithm based on the relation between the Kirchhoff index and the Laplacian eigenvalues, namely*

$$Kf(\mathscr{G}) = n \sum_{t=2}^{n} \frac{1}{\xi_t},$$

and the eigenvalues of $L(\mathscr{G})$ are $0 = \xi_1 < \xi_2 \geq \cdots \geq \xi_n (n \geq 2)$.

Lemma 2.3. [14] *Suppose that the eigenvalues of $L(\mathscr{G})$ are $\xi_1 \leq \xi_2 \leq \cdots \leq \xi_n$, then its multiplicative degree-Kirchhoff index can be denoted by*

$$Kf^*(\mathscr{G}) = 2m \sum_{t=2}^{n} \frac{1}{\xi_t}.$$

Lemma 2.4. [1] *The number of spanning trees in $\mathscr{G}$ can also be called the complexity of $\mathscr{G}$. If $\mathscr{G}$ is a graph with $|V_{\mathscr{G}}| = n$ and $|E_{\mathscr{G}}| = m$. Let $\lambda_i (i = 2, 3, \cdots, n)$ be the eigenvalues of $L(\mathscr{G})$. Then the complexity of $\mathscr{G}$ is*

$$2m\tau(\mathscr{G}) = \prod_{i=1}^{n} d_i \cdot \prod_{i=2}^{n} \lambda_i.$$

3. Kirchhoff index of L_n

In this section, the main objective is to find out the Kirchhoff index of L_n. Then, combining the definition of the Laplacian matrix and Eq (1.1), we can write these block matrices as follows:

$$L_{V_1V_1} = \begin{pmatrix} 3 & -1 & & & & & & & & & \\ -1 & 4 & -1 & & & & & & & & \\ & -1 & 5 & -1 & & & & & & & \\ & & -1 & 5 & -1 & & & & & & \\ & & & -1 & 5 & -1 & & & & & \\ & & & & -1 & 4 & -1 & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & -1 & 5 & -1 & & \\ & & & & & & & -1 & 4 & -1 & \\ & & & & & & & & -1 & 5 & -1 \\ & & & & & & & & & -1 & 3 \end{pmatrix}_{(4n)\times(4n)},$$

and

$$L_{V_1V_2} = \begin{pmatrix} -1 & -1 & & & & & & & & & \\ -1 & 0 & -1 & & & & & & & & \\ & -1 & -1 & -1 & & & & & & & \\ & & -1 & -1 & -1 & & & & & & \\ & & & -1 & -1 & -1 & & & & & \\ & & & & -1 & 0 & -1 & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & -1 & -1 & -1 & & \\ & & & & & & & -1 & 0 & -1 & \\ & & & & & & & & -1 & -1 & -1 \\ & & & & & & & & & -1 & -1 \end{pmatrix}_{(4n)\times(4n)}.$$

Hence,

$$L_A = \begin{pmatrix} 2 & -2 & & & & & & & & & \\ -2 & 4 & -2 & & & & & & & & \\ & -2 & 4 & -2 & & & & & & & \\ & & -2 & 4 & -2 & & & & & & \\ & & & -2 & 4 & -2 & & & & & \\ & & & & -2 & 4 & -2 & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & -2 & 4 & -2 & & \\ & & & & & & & -2 & 4 & -2 & \\ & & & & & & & & -2 & 4 & -2 \\ & & & & & & & & & -2 & 2 \end{pmatrix}_{(4n)\times(4n)},$$

and

$$L_S = diag(4,4,6,6,6,4,\cdots,6,4,6,4).$$

Assume that $0 \le \alpha_1 < \alpha_2 \le \alpha_3 \le \cdots \le \alpha_{4n}$ are the roots of $P_{L_A}(x) = 0$, and $0 \le \beta_1 \le \beta_2 \le \beta_3 \le \cdots \le \beta_{4n}$ are the roots of $P_{L_S}(x) = 0$. By Lemma 2.2, we immediately have

$$Kf(L_n^2) = 2(4n)\left(\sum_{i=2}^{4n}\frac{1}{\alpha_i} + \sum_{j=1}^{4n}\frac{1}{\beta_j}\right). \tag{3.1}$$

Obviously, $\sum_{j=1}^{4n}\frac{1}{\beta_j}$ can be obtained according to L_S.

$$\sum_{j=1}^{4n}\frac{1}{\beta_j} = \frac{1}{6}\times(3n-2) + \frac{1}{4}\times(n+2) = \frac{9n+2}{12}. \tag{3.2}$$

Next, we focus on computing $\sum_{i=1}^{4n}\frac{1}{\alpha_i}$. Let

$$P_{L_A}(x) = det(xI - L_A) = x(x^{4n-1} + a_1x^{4n-2} + \cdots + a_{4n-2}x + a_{4n-1}), a_{4n-1} \neq 0.$$

Based on Vieta's theorem of $P_{L_A}(x)$, we can exactly get the following equation:

$$\sum_{i=2}^{4n}\frac{1}{\alpha_i} = \frac{(-1)^{4n-2}a_{4n-2}}{(-1)^{4n-1}a_{4n-1}}. \tag{3.3}$$

For the sake of convenience, let M_s be used to express the $s-th$ order principal minors of matrix A, and $m_s = detM_s$ is recorded. We can get $m_1 = 2, m_2 = 4, m_3 = 8$.

And

$$m_s = 4m_{s-1} - 4m_{s-2}, 4 \le s \le 4n,$$

by further induction, we have

$$m_s = 2^s.$$

In this way, we can get two theorems.

Theorem 3.1. $(-1)^{4n-1}a_{4n-1} = (4n)2^{4n-1}$.

Proof. Due to the sum of all the principal minors of order $4n-1$ of L_A is $(-1)^{4n-1}a_{4n-1}$,

$$\begin{aligned}(-1)^{4n-1}a_{4n-1} &= \sum_{s=1}^{4n} detL_A[s] \\ &= \sum_{s=1}^{4n} det\begin{pmatrix} M_{s-1} & 0 \\ 0 & U_{4n-s}\end{pmatrix} \\ &= \sum_{s=1}^{4n} detM_{s-1}\cdot detU_{4n-s},\end{aligned}$$

where

$$M_{s-1} = \begin{pmatrix} l_{11} & -2 & \cdots & 0 \\ -2 & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l_{s-1,s-1} \end{pmatrix}_{(s-1)\times(s-1)},$$

$$U_{4n-s} = \begin{pmatrix} l_{s+1,s+1} & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & l_{4n-1,4n-1} & \vdots \\ 0 & \cdots & -2 & l_{4n,4n} \end{pmatrix}_{(4n-s)\times(4n-s)}.$$

Let $m_0 = 1, detU_0 = 1$, because of the symmetry of matrix L_A, then $detU_{4n-s} = detM_{4n-s}$. Hence

$$\begin{aligned} (-1)^{4n-1}a_{4n-1} &= \sum_{s=1}^{4n} detm_{s-1} \cdot detm_{4n-s} \\ &= (4n)2^{4n-1}, \end{aligned}$$

as desired. □

Theorem 3.2. $(-1)^{4n-2}a_{4n-2} = \frac{(4n-1)(4n)(4n+1)2^{4n-1}}{3}$.

Proof. Since the $(-1)^{4n-2}a_{4n-2}$ is the tatal of all the principal minors of order $4n-2$ of L_A, we have

$$\begin{aligned} (-1)^{4n-1}a_{4n-1} &= \sum_{1\le s<t\le 4n} detL_A[s,t] \\ &= \sum_{1\le s<t\le 4n} detM_{s-1}detN_{t-s-1}detU_{4n-t}, \end{aligned}$$

where

$$L_A[s,t] = \begin{pmatrix} M_{p-1} & 0 & 0 \\ 0 & N_{t-s-1} & 0 \\ 0 & 0 & U_{4n-t} \end{pmatrix}, 1 \le s < t \le 4n$$

and

$$\begin{aligned} N_{t-s-1} &= \begin{vmatrix} 4 & -2 & & & & & \\ -2 & 4 & -2 & & & & \\ & -2 & 4 & -2 & & & \\ & & & \ddots & & & \\ & & & -2 & 4 & -2 & \\ & & & & -2 & 4 & -2 \\ & & & & & -2 & 4 \end{vmatrix}_{(t-s-1)} \\ &= (t-s)2^{t-s-1}. \end{aligned}$$

Therefore, we can have

$$
\begin{aligned}
(-1)^{4n-2}a_{4n-2} &= \sum_{1\le s<t\le 4n} detM_{s-1}\cdot detN_{t-s-1}\cdot detU_{4n-t}\\
&= \sum_{1\le s<t\le 4n} (t-s)2^{t-s-1}\cdot detm_{s-1}\cdot m_{4n-t}\\
&= \frac{(4n-1)(4n)(4n+1)2^{4n-1}}{3}.
\end{aligned}
$$

The proof is over. □

From the results of Theorem 3.1 and Theorem 3.2, we can get

$$\sum_{i=2}^{4n}\frac{1}{\alpha_i}=\frac{(-1)^{4n-2}a_{4n-2}}{(-1)^{4n-1}a_{4n-1}}=\frac{16n^2-1}{12}, \tag{3.4}$$

where the eigenvalues of L_A are $0\le\alpha_1<\alpha_2\le\alpha_3\le\cdots\le\alpha_{4n}$.

Lemma 3.3. Suppose $L_n^{6,4,4}$ be the dicyclobutadieno derivative of phenylenes and the graph L_n be obtained from the transformation of the graph $L_n^{6,4,4}$.

$$Kf(L_n)=\frac{32n^3+18n^2+2n}{3}.$$

Proof. Substituting Eqs (3.5) and (3.6) into (3.4), the Kirchhoff index of L_n can be expressed

$$
\begin{aligned}
Kf(L_n) &= 2(4n)\left(\sum_{i=2}^{4n}\frac{1}{\alpha_i}+\sum_{j=1}^{4n}\frac{1}{\beta_j}\right)\\
&= (8n)(\frac{9n+2}{12}+\frac{(4n+1)(4n-1)}{12})\\
&= \frac{32n^3+18n^2+2n}{3}.
\end{aligned}
$$

The result is as desired. □

The Kirchhoff index of L_n from L_1 to L_{12} is shown in Table 1.

Table 1. The Kirchhoff indices of $L_1,\ L_2, ...,\ L_{12}$.

$\mathscr{G}$	$Kf(\mathscr{G})$	$\mathscr{G}$	$Kf(\mathscr{G})$	$\mathscr{G}$	$Kf(\mathscr{G})$
L_1	17.3	L_5	1486.7	L_9	8268.0
L_2	110.7	L_6	2524.0	L_{10}	24457.3
L_3	344.0	L_7	3957.3	L_{11}	30454.7
L_4	781.3	L_8	5850.7	L_{12}	37360.0

Next, we will further consider the Wiener index of L_n.

Lemma 3.4. Let $L_n^{6,4,4}$ be the dicyclobutadieno derivative of [n]phenylenes, and the graph L_n be obtained from the transformation of the graph $L_n^{6,4,4}$, then

$$\lim_{n\to\infty}\frac{Kf(L_n)}{W(L_n)}=\frac{1}{4}.$$

Proof. Consider d_{st} for all vertices. For the sake of convenience, we divide the vertices of the graph into the following five categories:

Case 1. Vertex 1 of L_n:

$$g_1(i) = 1 + 2(\sum_{k=1}^{4n-1} k).$$

Case 2. Vertex $4j-3(j = 1, 2, \cdots, n)$ of L_n, $i = 4j-3$:

$$g_2(i) = 1 + 2(\sum_{k=1}^{4n-1} k + \sum_{k=1}^{4n-i} k).$$

Case 3. Vertex $4j-2(j = 1, 2, \cdots, n)$ of L_n, $i = 4j-2$:

$$g_3(i) = 1 + 2(\sum_{k=1}^{4n-1} k + \sum_{k=1}^{4n-i} k).$$

Case 4. Vertex $4j-1(j = 1, 2, \cdots, n)$ of L_n, $i = 4j-1$:

$$g_4(i) = 1 + 2(\sum_{k=1}^{4n-1} k + \sum_{k=1}^{4n-i} k).$$

Case 5. Vertex $4j(j = 1, 2, \cdots, n)$ of L_n, $i = 4j$:

$$g_5(i) = 1 + 2(\sum_{k=1}^{4n-1} k + \sum_{k=1}^{4n-i} k).$$

Hence, we have

$$\begin{aligned}
W(L_n) &= \frac{4g_1(i) + 2\sum_{i=4j-3} g_2(i) + 2\sum_{i=4j-2} g_3(i) + 2\sum_{i=4j-1} g_4(i) + 2\sum_{i=4j} g_5(i)}{2} \\
&= \frac{4(1 + 2\sum_{k=1}^{4n-1} k) + 2\sum_{j=1}^{n}[1 + 2(\sum_{k=1}^{4j-k} k + \sum_{k=1}^{4n-4j+2} k)]}{2} \\
&+ \frac{2\sum_{j=1}^{n}[2 + 2(\sum_{k=1}^{4j-3} k + \sum_{k=1}^{4n-4j+2} k)] + 2\sum_{j=1}^{n}[1 + 2(\sum_{k=1}^{4j-2} k + \sum_{k=1}^{4n-4j+1} k)]}{2} \\
&+ \frac{2\sum_{j=1}^{n-1}[1 + 2(\sum_{k=1}^{4j-1} k + \sum_{k=1}^{4n-4j} k)]}{2} \\
&= \frac{128n^3 + 48n^2 - 5n + 3}{3}.
\end{aligned}$$

Consider the above results of the Kirchhoff index and the Wiener index. We can get following equation when n tends to infinity:

$$\lim_{n\to\infty} \frac{Kf(L_n)}{W(L_n)} = \frac{1}{4}.$$

The result is as desired. □

4. Multiplicative degree-Kirchhoff index and complexity of L_n

In this section, we use the eigenvalues of the normalized Laplacian matrix to determine the multiplicative degree-Kirchhoff index of L_n. Besides, we calculate the complexity of L_n. Then

$$
\mathcal{L}_{V_1V_1} = \begin{pmatrix}
1 & \frac{-1}{\sqrt{12}} & & & & & & & & & \\
\frac{-1}{\sqrt{12}} & 1 & \frac{-1}{\sqrt{20}} & & & & & & & & \\
 & \frac{-1}{\sqrt{20}} & 1 & \frac{-1}{5} & & & & & & & \\
 & & \frac{-1}{5} & 1 & \frac{-1}{5} & & & & & & \\
 & & & \frac{-1}{5} & 1 & \frac{-1}{\sqrt{20}} & & & & & \\
 & & & & \frac{-1}{\sqrt{20}} & 1 & \frac{-1}{\sqrt{20}} & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & \frac{-1}{5} & 1 & \frac{-1}{\sqrt{20}} & & \\
 & & & & & & & \frac{-1}{\sqrt{20}} & 1 & \frac{-1}{\sqrt{20}} & \\
 & & & & & & & & \frac{-1}{\sqrt{20}} & 1 & \frac{-1}{\sqrt{15}} \\
 & & & & & & & & & \frac{-1}{\sqrt{15}} & 1
\end{pmatrix}_{(4n)\times(4n)},
$$

and

$$
\mathcal{L}_{V_1V_2} = \begin{pmatrix}
\frac{-1}{3} & \frac{-1}{\sqrt{12}} & & & & & & & & & \\
\frac{-1}{\sqrt{12}} & 0 & \frac{-1}{\sqrt{20}} & & & & & & & & \\
 & \frac{-1}{\sqrt{20}} & \frac{-1}{5} & \frac{-1}{5} & & & & & & & \\
 & & \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} & & & & & & \\
 & & & \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{\sqrt{20}} & & & & & \\
 & & & & \frac{-1}{\sqrt{20}} & 0 & \frac{-1}{\sqrt{20}} & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{\sqrt{20}} & & \\
 & & & & & & & \frac{-1}{\sqrt{20}} & 0 & \frac{-1}{\sqrt{20}} & \\
 & & & & & & & & \frac{-1}{\sqrt{20}} & \frac{-1}{5} & \frac{-1}{\sqrt{15}} \\
 & & & & & & & & & \frac{-1}{\sqrt{15}} & \frac{-1}{3}
\end{pmatrix}_{(4n)\times(4n)}.
$$

Therefore,

$$\mathcal{L}_A = \begin{pmatrix} \frac{2}{3} & \frac{-1}{\sqrt{3}} & & & & & & & & & & \\ \frac{-1}{\sqrt{3}} & 1 & \frac{-1}{\sqrt{5}} & & & & & & & & & \\ & \frac{-1}{\sqrt{5}} & \frac{4}{5} & \frac{-2}{5} & & & & & & & & \\ & & \frac{-2}{5} & \frac{4}{5} & \frac{-2}{5} & & & & & & & \\ & & & \frac{-2}{5} & \frac{4}{5} & \frac{-1}{\sqrt{5}} & & & & & & \\ & & & & \frac{-1}{\sqrt{5}} & 1 & \frac{-1}{\sqrt{5}} & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & \frac{-2}{5} & \frac{4}{5} & \frac{-1}{\sqrt{5}} & & & \\ & & & & & & & \frac{-1}{\sqrt{5}} & 1 & \frac{-1}{\sqrt{5}} & & \\ & & & & & & & & \frac{-1}{\sqrt{5}} & \frac{4}{5} & \frac{-2}{\sqrt{15}} \\ & & & & & & & & & \frac{-2}{\sqrt{15}} & \frac{2}{3} \end{pmatrix}_{(4n)\times(4n)},$$

and

$$\mathcal{L}_S = diag(\frac{4}{3},\ 1,\ \frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \cdots, \frac{6}{5},\ 1,\ \frac{6}{5}, \frac{4}{3}).$$

Assume that the roots of $P_{L_A}(x) = 0$ are $0 \le \xi_1 < \xi_2 \le \xi_3 \le \cdots \le \alpha_{4n}$, and $0 \le \gamma_1 \le \gamma_2 \le \gamma_3 \le \cdots \le \gamma_{4n}$ are the roots of $P_{L_S}(x) = 0$. By Lemma 2.3, we can get

$$Kf^*(L_n^2) = 2(19n-4)\left(\sum_{i=2}^{4n}\frac{1}{\xi_i} + \sum_{j=1}^{4n}\frac{1}{\gamma_j}\right).$$

Since $\mathcal{L}_s$ is a diagonal matrix. Obviously, its diagonal elements $1, \frac{4}{3}$ and $\frac{6}{5}$ correspond to the eigenvalues of $\mathcal{L}_s$, respectively. Then, it can be clearly obtained as

$$\sum_{i=1}^{4n}\frac{1}{\gamma_i} = \frac{21n-1}{6}. \tag{4.1}$$

Let

$$P_{\mathcal{L}_A}(x) = det(xI - \mathcal{L}_A) = x^{4n} + b_1x^{4n-1} + \cdots + b_{4n-1}x, b_{4n-1} \neq 0,$$

$\frac{1}{\xi_2}, \frac{1}{\xi_3}, \cdots, \frac{1}{\xi_{4n+1}}$are the roots of the following equation

$$b_{4n-1}x^{4n-1} + b_{4n-2}x^{4n-2} + \cdots + b_1x + 1 = 0.$$

Based on the Vieta' s theorem of $P_{L_A}(x)$, we can get

$$\sum_{i=2}^{4n}\frac{1}{\xi_i} = \frac{(-1)^{4n-2}b_{4n-2}}{(-1)^{4n-1}b_{4n-1}}.$$

Similarly, we can get another two interesting facts.

Theorem 4.1. $(-1)^{4n-1}b_{4n-1} = \frac{25}{9}(38n-8)(\frac{4}{125})^n$.

Proof. Let $s_p = detF_p$, then we have $s_1 = \frac{2}{3}, s_2 = \frac{1}{3}, s_3 = \frac{2}{15}, s_4 = \frac{4}{75}$, and

$$\begin{cases} s_{4p} = \frac{4}{5}s_{4p-1} - \frac{4}{25}s_{4p-2}; \\ s_{4p+1} = \frac{4}{5}s_{4p} - \frac{4}{25}s_{4p-1}; \\ s_{4p+2} = s_{4p+1} - \frac{1}{5}s_{4p}; \\ s_{4p+3} = \frac{4}{5}s_{4p+2} - \frac{1}{5}s_{4p+1}. \end{cases}$$

After further simplification, the transformation form of the above formula is obtained.

$$\begin{cases} s_{4p} = \frac{5}{3} \cdot (\frac{4}{125})^p, 1 \leq p \leq n; \\ s_{4p+1} = \frac{2}{3} \cdot (\frac{4}{125})^i, 0 \leq p \leq n-1; \\ s_{4p+2} = \frac{1}{3} \cdot (\frac{4}{125})^i, 0 \leq p \leq n-1; \\ s_{4p+3} = \frac{1}{15} \cdot (\frac{4}{125})^i, 0 \leq p \leq n-1. \end{cases}$$

Similarly, we have $t_1 = \frac{2}{3}, t_2 = \frac{4}{15}, t_3 = \frac{2}{15}, t_4 = \frac{4}{75}$, and

$$\begin{cases} t_{4p} = \frac{2}{5}t_{4p-1} - \frac{2}{5}t_{4p-2}; \\ t_{4p+1} = \frac{4}{5}t_{4p} - \frac{4}{25}t_{4p-1}; \\ t_{4p+2} = \frac{4}{5}t_{4p+1} - \frac{4}{25}t_{4p}; \\ t_{4p+3} = t_{4p+2} - \frac{1}{5}t_{4p+1}. \end{cases}$$

Therefore, the transformation form of the above formula is obtained.

$$\begin{cases} t_{4p-4} = \frac{5}{3} \cdot (\frac{4}{125})^p, 1 \leq p \leq n; \\ t_{4p-3} = \frac{2}{3} \cdot (\frac{4}{125})^p, 0 \leq p \leq n-1; \\ t_{4p-2} = \frac{4}{15} \cdot (\frac{4}{125})^p, 0 \leq p \leq n-1; \\ t_{4p-1} = \frac{2}{15} \cdot (\frac{4}{125})^p, 0 \leq p \leq n-1. \end{cases}$$

Since the $(-1)^{4n-1}b_{4n-1}$ is the total of all the principal minors of order $4n-1$ of L_A, we have

$$\begin{aligned} (-1)^{4n-1}b_{4n-1} &= \sum_{i=2}^{4n} detNL_A[i] + s_{4n} + t_{4n} \\ &= \frac{1}{45}(38n-8)(\frac{4}{125})^n. \end{aligned}$$

The proof of Theorem 4.1 completed. □

Theorem 4.2. $(-1)^{4n-2}b_{4n-2} = \frac{1}{3240}(14520n^3 + 4599n^2 - 1496n + 3)(\frac{4}{125})^n$.

Proof. We observe that the sum of all the principal minors of order $4n$ in $\mathcal{L}_A$ is $(-1)^{4n-2}b_{4n-2}$, then

$$(-1)^{4n-2}b_{4n-2} = \sum_{1\leq s<t\leq 4n} det\mathcal{L}_A[s,t] \cdot f_{s-1} \cdot f'_{4n-t}. \quad (4.2)$$

By Eq (4.8), we know that the result of $det\mathcal{L}_A[s,t]$ will change with the values of s and t. Then we can get the following twenty cases:

Case 1. $i = 4s,\ j = 4t,\ 1 \le s < t \le n,$

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{1}{\sqrt{5}} & & & & & & \\
-\frac{2}{\sqrt{5}} & 1 & -\frac{1}{5} & & & & & \\
 & -\frac{1}{5} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & \\
 & & & \ddots & & & & \\
 & & & \frac{-2}{5} & \frac{4}{5} & \frac{-2}{5} & & \\
 & & & & \frac{-2}{5} & -\frac{4}{5} & -\frac{1}{\sqrt{5}} & \\
 & & & & & -\frac{1}{5} & 1 & -\frac{1}{\sqrt{5}} \\
 & & & & & & \frac{-2}{5} & -\frac{4}{5}
\end{vmatrix}_{(4t-4s-1)} \\
&= 10(t-s)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 2. $i = 4s,\ j = 4t+1,\ 1 \le s \le t \le n-1,$

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{1}{\sqrt{5}} & & & & & & \\
-\frac{1}{\sqrt{5}} & 1 & -\frac{1}{5} & & & & & \\
 & -\frac{1}{5} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & \\
 & & & \ddots & & & & \\
 & & & \frac{-2}{5} & \frac{4}{5} & \frac{-1}{5} & & \\
 & & & & \frac{-1}{5} & 1 & -\frac{1}{\sqrt{5}} & \\
 & & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{\sqrt{5}} \\
 & & & & & & \frac{-2}{5} & -\frac{4}{5}
\end{vmatrix}_{(4t-4s)} \\
&= (4t-4s+1)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 3. $i = 4s,\ j = 4t+2,\ 1 \le s \le t \le n-1,$

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{1}{\sqrt{5}} & & & & & & \\
-\frac{1}{\sqrt{5}} & 1 & -\frac{1}{5} & & & & & \\
 & -\frac{1}{5} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & & \\
 & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & \frac{-2}{5} & \\
 & & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} \\
 & & & & & & \frac{-2}{5} & -\frac{4}{5}
\end{vmatrix}_{(4t-4s+1)} \\
&= \frac{4}{5}(2t-2s+1)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 4. $i = 4s,\ j = 4t + 3,\ 1 \le s \le t \le n - 1,$

$$
\begin{aligned}
det\,\varphi &= \left|\begin{array}{cccccccc}
\frac{4}{5} & -\frac{1}{\sqrt{5}} & & & & & & \\
-\frac{1}{\sqrt{5}} & 1 & -\frac{1}{5} & & & & & \\
 & -\frac{1}{5} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{5} & & \\
 & & & & -\frac{2}{5} & \frac{4}{5} & \frac{-2}{5} & \\
 & & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} \\
 & & & & & & -\frac{1}{\sqrt{5}} & 1
\end{array}\right|_{(4t-4s+2)} \\
&= \frac{1}{5}(4t - 4s + 3)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 5. $i \equiv 0,\ j = 4n,\ 1 \le s \le t,$

$$
\begin{aligned}
det\,\varphi &= \left|\begin{array}{cccccccc}
\frac{4}{5} & -\frac{1}{\sqrt{5}} & & & & & & \\
-\frac{2}{\sqrt{5}} & 1 & -\frac{1}{5} & & & & & \\
 & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{5} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & \\
 & & & & -\frac{2}{5} & \frac{4}{5} & \frac{-1}{\sqrt{5}} & \\
 & & & & & \frac{-1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} \\
 & & & & & & -\frac{2}{\sqrt{5}} & \frac{4}{5}
\end{array}\right|_{(4n-4s-1)} \\
&= 10(n - s)\Big(\frac{4}{125}\Big)^{n-s}.
\end{aligned}
$$

Case 6. $i = 4s + 1,\ j = 4t,\ 1 \le s < t \le n,$

$$
\begin{aligned}
det\,\varphi &= \left|\begin{array}{cccccccc}
1 & -\frac{1}{\sqrt{5}} & & & & & & \\
-\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & & \\
 & -\frac{2}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & \\
 & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & \\
 & & & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} \\
 & & & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5}
\end{array}\right|_{(4t-4s-2)} \\
&= \frac{25}{4}(4t - 4s - 1)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 7. $i = 4s + 1,\ j = 4t + 1,\ 1 \le s < t \le n - 1,$

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
1 & -\frac{1}{\sqrt{5}} & & & & & & \\
-\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & & \\
 & -\frac{2}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & & \\
 & & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & \\
 & & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{5} \\
 & & & & & & -\frac{2}{5} & \frac{4}{5}
\end{vmatrix}_{(4t-4s-1)} \\
&= 10(t-s)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 8. $i = 4s + 1,\ j = 4t + 2,\ 1 \le s < t \le n - 1,$

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
1 & -\frac{1}{\sqrt{5}} & & & & & & \\
-\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & & \\
 & -\frac{2}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{2}{5} & 1 & -\frac{1}{\sqrt{5}} & & \\
 & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{5} & \\
 & & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} \\
 & & & & & & -\frac{2}{5} & \frac{4}{5}
\end{vmatrix}_{(4t-4s)} \\
&= (4t-4s+1)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 9. $i = 4s + 1,\ j = 4t + 3,\ 0 \le s \le t \le n - 1,$

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
1 & -\frac{1}{\sqrt{5}} & & & & & & \\
-\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & & \\
 & -\frac{2}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{5} & & \\
 & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & \\
 & & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} \\
 & & & & & & -\frac{1}{\sqrt{5}} & 1
\end{vmatrix}_{(4t-4s+1)} \\
&= (2t-2s+1)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 10. $i \equiv 1,\ j = 4n+1,\ 0 \le s \le n,$

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
1 & -\frac{1}{\sqrt{5}} & & & & & & & \\
-\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & & & \\
 & -\frac{2}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{\sqrt{5}} & & & & & \\
 & & & \ddots & & & & & \\
 & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & \\
 & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & & \\
 & & & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & \\
 & & & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} &
\end{vmatrix}_{(4n-4s-2)} \\
&= \frac{25}{4}(4n-4s-1)\Big(\frac{4}{125}\Big)^{n-s}.
\end{aligned}
$$

Case 11. $i = 4s+2,\ j = 4t,\ 0 \le s < t \le n,$

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{2}{5} & & & & & & \\
-\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & & & \\
 & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & \\
 & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & \\
 & & & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} \\
 & & & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5}
\end{vmatrix}_{(4t-4s-3)} \\
&= 25(2t-2s-1)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 12. $i = 4s+2,\ j = 4t+1,\ 0 \le s < t \le n-1,$

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{2}{5} & & & & & & \\
-\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & & & \\
 & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & & \\
 & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{5} & \\
 & & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} \\
 & & & & & & -\frac{2}{5} & \frac{4}{5}
\end{vmatrix}_{(4t-4s-2)} \\
&= 5(4t-4s-1)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 13. $i = 4s + 2,\ j = 4t + 2, 0 \leq s < t \leq n - 1$,

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{2}{5} & & & & & & \\
-\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & & & \\
 & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & & \\
 & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{5} & \\
 & & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} \\
 & & & & & & -\frac{2}{5} & \frac{4}{5}
\end{vmatrix}_{(4t-4s-1)} \\
&= 8(t-s)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 14. $i = 4s + 2,\ j = 4t + 3, 0 \leq s \leq t \leq n - 1$,

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{2}{5} & & & & & & \\
-\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & & & \\
 & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{5} & & \\
 & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & \\
 & & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} \\
 & & & & & & -\frac{1}{\sqrt{5}} & 1
\end{vmatrix}_{(4t-4s)} \\
&= (4t-4s+1)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 15. $i \equiv 2,\ j = 4t + 3, 0 \leq s \leq n - 1$,

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{2}{5} & & & & & & \\
-\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & & & \\
 & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & \\
 & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & \\
 & & & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} \\
 & & & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5}
\end{vmatrix}_{(4n-4s-3)} \\
&= 25(2n-2s-1)\Big(\frac{4}{125}\Big)^{n-s}.
\end{aligned}
$$

Case 16. $i = 4s+3,\ j = 4t,\ 0 \le s < t \le n$,

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{2}{5} & & & & & & & \\
-\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & & & & & & \\
 & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & & & & & \\
 & & & \ddots & & & & & \\
 & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & & \\
 & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & & \\
 & & & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & \\
 & & & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} &
\end{vmatrix}_{(4t-4s-4)} \\
&= \frac{125}{4}(4t-4s-3)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 17. $i = 4s+3,\ j = 4t+1,\ 0 \le s < t \le n-1$,

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{2}{5} & & & & & & & \\
-\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & & & & & & \\
 & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & & & & & \\
 & & & \ddots & & & & & \\
 & & & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & & & \\
 & & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & & \\
 & & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & \\
 & & & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} &
\end{vmatrix}_{(4t-4s-3)} \\
&= 25(2t-2s-1)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 18. $i = 4s+3,\ j = 4t+2,\ 0 \le s < t \le n-1$,

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{2}{5} & & & & & & & \\
-\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & & & & & & \\
 & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & & & & & \\
 & & & \ddots & & & & & \\
 & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & & & \\
 & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{5} & & \\
 & & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & \\
 & & & & & & -\frac{2}{5} & \frac{4}{5} &
\end{vmatrix}_{(4t-4s-3)} \\
&= \frac{25}{3}(4t-4s-1)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 19. $i = 4s+3,\ j = 4t+3,\ 0 \le s < t \le n-1$,

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{2}{5} & & & & & & \\
-\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & & & & & \\
 & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{1}{\sqrt{5}} & \frac{4}{5} & -\frac{2}{5} & & \\
 & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & \\
 & & & & & -\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} \\
 & & & & & & -\frac{1}{\sqrt{5}} & 1
\end{vmatrix}_{(4t-4s-1)} \\
&= 10(t-s)\Big(\frac{4}{125}\Big)^{t-s}.
\end{aligned}
$$

Case 20. $i \equiv 3,\ j = 4t,\ 0 \le s \le n-1$,

$$
\begin{aligned}
det\,\varphi &= \begin{vmatrix}
\frac{4}{5} & -\frac{2}{5} & & & & & & \\
-\frac{2}{5} & \frac{4}{5} & -\frac{1}{\sqrt{5}} & & & & & \\
 & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & & & & \\
 & & & \ddots & & & & \\
 & & & -\frac{2}{5} & \frac{4}{5} & -\frac{2}{5} & & \\
 & & & & -\frac{2}{5} & 1 & -\frac{1}{\sqrt{5}} & \\
 & & & & & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} \\
 & & & & & & -\frac{1}{\sqrt{5}} & \frac{4}{5}
\end{vmatrix}_{(4n-4s-4)} \\
&= \frac{125}{4}(4n-4s-3)\Big(\frac{4}{125}\Big)^{n-s}.
\end{aligned}
$$

Therefore, we can get

$$(-1)^{4n-2} b_{4n-2} = E_1 + E_2 + E_3 + E_4,$$

where

$$E_1 = \frac{1}{18}(227n^3 + 347n^2 - 574n + 4)\Big(\frac{4}{125}\Big)^{n-1}.$$

$$E_2 = \frac{1}{72}(908n^3 + 3431n^2 + 523n)\Big(\frac{4}{125}\Big)^{n}.$$

$$E_3 = \frac{1}{45}(454n^3 + 1375n^2 - 1079n)\Big(\frac{4}{125}\Big)^{n}.$$

$$E_4 = \frac{1}{81}(92n^3 + 561n^2 - 611n)\Big(\frac{4}{125}\Big)^{n-1}.$$

Hence

$$(-1)^{4n-2}b_{4n-2} = E_1 + E_2 + E_3 + E_4 = \frac{1}{3240}(14520n^3 + 4599n^2 - 1496n + 4)\Big(\frac{4}{125}\Big)^n.$$

The proof of Theorem 4.2 completed. □

Let $0 \le \xi_1 \le \xi_2 \le \xi_3 \le \cdots \le \xi_{3n+2}$ are the eigenvalues of $\mathscr{L}_A$. We can get the following exact equation:

$$\sum_{i=2}^{4n}\frac{1}{\xi_i} = \frac{(-1)^{4n-2}b_{4n-2}}{(-1)^{4n-1}b_{4n-1}} = \frac{1}{72}\Big(\frac{14520n^3 + 4599n^2 - 1496n + 8}{38n - 8}\Big).$$

Theorem 4.3. Set $L_n^{6,4,4}$ be the derivative [n]pheylenes, and the expression of the multiplicative degree-Kirchhoff index is

$$Kf^*(L_n) = \Big(\frac{29040n^3 + 8996n^2 - 3198n + 8}{144}\Big).$$

Proof. Together with Eq (4.7) and Theorems 4.1 and 4.2, one can get

$$\begin{aligned} Kf^*(L_n) &= 2(19n-4)\Big(\sum_{i=2}^{4n}\frac{1}{\xi_i} + \sum_{i=1}^{4n}\frac{1}{\gamma_i}\Big) \\ &= 2(19n-4)\Big[\frac{1}{72}\Big(\frac{14520n^3 + 4599n^2 - 1496n + 8}{38n - 8}\Big) + \frac{21n-1}{6}\Big] \\ &= \Big(\frac{29040n^3 + 8996n^2 - 3198n + 8}{144}\Big). \end{aligned}$$

The result as desired. □

The multiplicative degree-Kirchhoff indices of L_n from L_1 to L_{12}, see Table 2.

Then we want to calculate the Gutman index of L_n.

Table 2. The multiplicative degree-Kirchhoff indices of $L_1,\ L_2, ...,\ L_{12}$.

$\mathscr{G}$	$Kf^*(\mathscr{G})$	$\mathscr{G}$	$Kf^*(\mathscr{G})$	$\mathscr{G}$	$Kf^*(\mathscr{G})$
L_1	241.98	L_5	26659.15	L_9	151875.4
L_2	1818.86	L_6	45675.81	L_{10}	207691.9
L_3	5940.68	L_7	72077.4	L_{11}	275733.2
L_4	13817.44	L_8	107073.9	L_{12}	357209.6

Theorem 4.4. Suppose that $L_n^{6,4,4}$ is the dicyclobutadieno derivative of [n]phenylenes and the graph L_n is obtained from the transformation of the graph $L_n^{6,4,4}$, then

$$\lim_{n\to\infty}\frac{Kf^*(L_n)}{Gut(L_n)} = \frac{1}{4}.$$

Proof. Consider d_{ij} for all vertices. We divide the vertices of L_n into the following four categories.

Case 1. Vertex $4i-2(i = 1, 2, \cdots, n)$ of L_n:

$$f_{4i-2} = \frac{10}{3}n(56n^2 - 24n + 37).$$

Case 2. Vertex $4i-1(i=1,2,\cdots,n)$ of L_n:

$$f_{4i-1}=\frac{10}{3}n(152n^2-48n+29).$$

Case 3. Vertex $4i(i=1,2,\cdots,n)$ of L_n:

$$f_{4i}=\frac{10}{3}n(140n^2-48n+43).$$

Case 4. Vertex $4i-3(i=1,2,\cdots,n)$ of L_n:

$$f_{4i-3}=\frac{10}{3}n(136n^2-6n+71).$$

According to Eq (1.3), the Gutman index of L_n is

$$\begin{aligned} Gut(L_n) &= \frac{f_{4i}+f_{4i-1}+f_{4i-2}+f_{4i-3}}{2} \\ &= \frac{10n}{3}(242n^2-63n+61). \end{aligned}$$

Therefore, combining with $Kf^*(L_n)$ and $Gut(L_n)$, we have

$$\lim_{n\to\infty}\frac{Kf^*(L_n)}{Gut(L_n)}=\frac{1}{4}.$$

The result as desired. □

Finally, we want to know the complexity of L_n.

Theorem 4.5. For the graph L_n, we have

$$\tau(L_n)=2^{3n+2}\cdot 3^{3n-2}.$$

Proof. Based on Lemma 2.4, we can get

$$\prod_{i=1}^{8n}d_i\prod_{i=2}^{4n}\alpha_i\prod_{j=1}^{4n}\beta_i=2(19n-4)\cdot\tau(L_n).$$

Note that

$$\prod_{i=1}^{8n}d_i=3^4\cdot 4^{2n}\cdot 5^{6n-4}.$$

$$\prod_{i=2}^{4n}\alpha_i=\frac{25}{9}\cdot(38n-8)\cdot(\frac{4}{125})^n.$$

$$\prod_{j=1}^{4n}\beta_i=(\frac{4}{3})^2\cdot(\frac{6}{5})^{3n-2}.$$

Hence,

$$\tau(L_n)=2^{3n+2}\cdot 3^{3n-2}.$$

The proof is over. □

Thus, we can get the complexity of L_n from W_1 to W_8 which are listed in Table 3.

Table 3. The complexity of $W_1, W_2 \cdots W_8$.

$\mathscr{G}$	$\tau(\mathscr{G})$	$\mathscr{G}$	$\tau(\mathscr{G})$
W_1	96	W_5	208971104256
W_2	20736	W_6	45137758519296
W_3	4478976	W_7	9749755840167936
W_4	967458816	W_8	2105947261476274176

5. Conclusions

In this paper, the linear chain network with n hexagons and $2n-1$ squares is considered. We have devoted ourselves to calculating the (multiplicative degree) Kirchhoff index, Wiener index, Gutman index, and complexity. In the meantime, we deduced that the ratio of the (multiplicative degree) Kirchhoff index to the (Gutman) Wiener index is nearly a quarter when n tends to infinity. These rules also apply to some other graphs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was funded by the Anhui Provincial Natural Science Research Major Project (No. 2022AH040317) and the Anhui Provincial 2023 Action Project for Cultivating Young and Middle Aged Teachers in Universities (No. DTR2023095).

Conflict of interest

No potential conflicts of interest were reported by the authors.

References

1. J. A. Bondy, U. S. R. Murty, *Graph theory with applications*, Macmillan Press Ltd., 1976.
2. F. R. K. Chung, *Spectral graph theory*, American Mathematical Society, 1997.
3. H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc.*, **69** (1947), 17–20. https://doi.org/10.1021/ja01193a005
4. A. Dobrynin, Branchings in trees and the calculation of the Wiener index of a tree, *MATCH Commun. Math. Comput. Chem.*, **41** (2000), 119–134.
5. R. C. Entringer, D. E. Jackson, D. A. Snyder, Distance in graphs, *Czechoslovak Mathematical Journal*, **26** (1976), 283–296.
6. L. H. Feng, X. M. Zhu, W. J. Liu, Wiener index, Harary index and graph properties, *Discrete Appl. Math.*, **223** (2017), 72–83. https://doi.org/72-83.10.1016/j.dam.2017.01.028

7. A. Abiad, B. Brimkov, A. Erey, On the Wiener index, distance cospectrality and transmission-regular graphs, *Discrete Appl. Math.*, **230** (2017), 1–10. https://doi.org/10.1016/j.dam.2017.07.010

8. Y. P. Mao, Z. Wang, I. Gutman, Nordhaus-Gaddum-type results for the Steiner Wiener index of graphs, *Discrete Appl. Math.*, **219** (2017), 167–175. https://doi.org/167-175.10.1016/j.dam.2016.11.014

9. A. R. Ashrafi, A. Ghalavand, Ordering chemical trees by Wiener polarity index, *Appl. Math. Comput.*, **313** (2017), 301–312. https://doi.org/301-312.10.1016/j.amc.2017.06.005

10. M. Crepnjak, N. Tratnik, The Szeged index and the Wiener index of partial cubes with applications to chemical graphs, *Appl. Math. Comput.*, **309** (2017), 324–333. https://doi.org/324-333.10.1016/j.amc.2017.04.011

11. A. Mohajeri, P. Manshour, M. Mousaee, A novel topological descriptor based on the expanded Wiener index: applications to QSPR/QSAR studies, *Iran. J. Math. Chem.*, **8** (2017), 107–135.

12. I. Gutman, Selected properties of the schultz molecular topological index, *Journal of Chemical Information and Computer Sciences*, **34** (1994), 1087–1089.

13. D. J. Klein, Resistance-distance sum rules, *Croat. Chem. Acta*, **75** (2002), 633–649.

14. D. J. Klein, O. Ivanciuc, Graph cyclicity, excess conductance, and resistance deficit, *J. Math. Chem.,* **30** (2001), 271–287. https://doi.org/10.1023/A:1015119609980

15. H. Y. Chen, F. J. Zhang, Resistance distance and the normalized Laplacian spectrum, *Discrete Appl. Math.,* **155** (2007), 654–661. https://doi.org/10.1016/j.dam.2006.09.008

16. L. H. Feng, I. Gutman, G. H. Yu, Degree Kirchhoff index of unicyclic graphs, *Match* (2013).

17. J. Huang, S. H. Li, On the normalised Laplacian spectrum, degree-Kirchhoff index and spanning trees of graphs, *B. Aust. Math. Soc.*, **91** (2015), 353–367. https://doi.org/10.1017/S0004972715000027

18. Y. L. Yang, T. Y. Yu, Graph theory of viscoelasticities for polymers with starshaped, multiple-ring and cyclic multiple-ring molecules, *Macromol. Chem. Phys.*, **186** (1985), 609–631.

19. G. H. Yua, L. H. Feng, On connective eccentricity index of graphs, *MATCH Commun. Math. Comput. Chem.*, **69** (2013), 611–628.

20. X. L. Ma, B. Hong, The normalized Laplacians, degree-Kirchhoff index and the spanning trees of cylinder phenylene chain, *Polycyclic Aromatic Compounds*, **41** (2021), 1159–1179. https://doi.org/10.1080/10406638.2019.1665553

21. L. Lei, X. Y. Geng, S. Li, Y. Peng, Y. Yu, On the normalized Laplacian of Mobius phenylene chain and its applications, *Int. J. Quantum Chem.*, **119** (2019), e26044. https://doi.org/10.1002/qua.26044

22. U. Ali, Y. Ahmad, S. A. Xu, X. F. Pan, On Normalized Laplacian, Degree-Kirchhoff Index of the Strong Prism of Generalized Phenylenes, *Polycyclic Aromatic Compounds*, **42** (2022), 6215–6232. https://doi.org/10.1080/10406638.2021.1977351

23. X. Y. Geng, L. Yu, On the Kirchhoff index and the number of spanning trees of linear phenylenes chain, *Polycyclic Aromatic Compounds*, **42** (2022), 4984–4993. https://doi.org/10.1080/10406638.2021.1923536

24. J. B. Liu, Z. Y. Shi, Y. H. Pan, J. D. Cao, Computing the Laplacian spectrum of linear octagonal-quadrilateral networks and its applications, *Polycyclic Aromatic Compounds*, **42** (2020), 1–12. https://doi.org/10.1080/10406638.2020.1748666

25. C. Liu, Y. H. Pan, J. P. Li, On the Laplacian spectrum and Kirchhoff index of generalized phenylenes, *Polycyclic Aromatic Compounds*, **41** (2019), 1–10. https://doi.org/10.1080/10406638.2019.1703765

26. I. Gutman, B. Mohar, The quasi-Wiener and the Kirchhoff indices coincide, *J. Chem. Inf. Model.*, **36** (1996), 982–985.