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*Research article*

# Solving hybrid functional-fractional equations originating in biological population dynamics with an effect on infectious diseases

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**Abstract:** This paper study was designed to establish solutions for mixed functional fractional integral equations that involve the Riemann-Liouville fractional operator and the Erdélyi-Kober fractional operator to describe biological population dynamics in Banach space. The results rely on the measure of non-compactness and theoretical concepts from fractional calculus. Darbo's fixed-point theorem for Banach spaces has been utilized. Moreover, the solvability of a specific non-linear integral equation that models the spread of infectious diseases with a seasonally varying periodic contraction rate has been explored by using the Banach contraction principle. Finally, two numerical examples demonstrate the practical application of these findings in the realm of fractional integral equation theory.

**Keywords:** hybrid nonlinear fractional equation; infectious diseases; Banach space; fixed point technique; biological population dynamics; fractional derivatives
**Mathematics Subject Classification:** 26A33, 26D15, 47H08, 47H10

## Abbreviations

| | |
|---|---|
| FIE | fractional integral equation |
| RLF | Riemann-Liouville fractional |

| | |
|---|---|
| EKF | Erdélyi-Kober fractional |
| BPD | biological population dynamics |
| BS | Banach space |
| MNC | measure of non-compactness |
| FP | fixed point |
| IEs | integral equations |
| BCP | Banach contraction principle |
| FC | fractional calculus |
| TSS | tempered sequence space |
| GF | gamma function |
| RL$\kappa$-F | Riemann-Liouville $\kappa$-fractional |

## 1. Introduction

Fractional calculus (FC) serves as a widely-used mathematical tool for characterizing nonlocal diffusion in various physical studies. Its applications span across engineering, physics, and the analysis of natural occurrences where fractional analysis plays a crucial role. By leveraging the gamma function in analytical contexts, FC plays a significant role in mathematical analysis, enabling the study of integrals and derivatives for real or complex orders. Through FC, diverse phenomena and their impacts in scientific and engineering domains like the frequency dispersion of power, electromagnetics, viscoelasticity, electrochemistry, and diffusion waves can be effectively demonstrated. Furthermore, fractional relaxation oscillation and fading memory have also been explored.

Fractional operators are valuable for the effective modeling of functional-fractional integral equations (FIEs) to investigate a range of issues, such as the fractal nature of materials, porous media seepage flow, and nonlinear earthquake oscillations. Functional-FIEs are pivotal across various fields and in the analysis diverse concrete models. Various disciplines like cytotoxic activity theory, statistical mechanics theory, radioactive transmission theory, and acoustic scattering heavily rely on nonlinear FIEs in practical applications [1–10].

Fixed point (FP) theory and the measure of non-compactness (MNC) are instrumental in examining various real-world scenarios represented by FIEs [11–19]. Initially introduced by Stephen Banach, FP theory has garnered a significant amount of attention across multiple scientific fields. Its efficacy has been widely recognized in the mathematical community owing to its potential applications in neural networks, healthcare, immunology, and aerospace, as well as its connections to recrystallization theory, wave analysis, phase-transition theory, programming language analysis, and more.

For non-linear functional integral equations (IEs), Dhage [20] provided global attractivity findings by means of an FP theorem of the Krasnoselskii type. Aghajani et al. [21] examined the FP findings for Meir-Keeler condensing maps using an MNC. Javahernia et al. [22], examined common FPs for Mizoguchi-Takahashi contractive maps. Mohammadi et al. [23] employed an extension of Darbo's FP theorem to determine if a system of IEs has a solution. Darbo's theorem was proven for multiple generalizations, and Jleli et al. [24] looked into whether it applied to FIEs. Functional FIE equations can also be solved with the use of FP theory. Numerous real-world problems have been significantly solved with the help of various types of functional FIEs. For more details, see [25–29].

Recently, a number of research articles regarding FP theory and its applications have been published. The solvability of functional FIEs was studied by Arab et al. [30] in 2020 using a MNC

$$v(\tau) = \psi(\tau, v(\tau)) + \frac{Wv(\tau)}{\Gamma(\lambda)} \int_0^\tau (K(\tau) - K(s))^{\lambda-1} K'(s) g(\tau, s, v(\tau))\, ds,\ \lambda \in (0,1),\ \tau \in [0,1].$$

By introducing a new $\mu$-set contraction operator and utilizing control functions in Banach spaces (BSs), existence results were established by using MNC methods and an extended version of Darbo's FP theorem.

The existence of implicit FIEs in a tempered sequence space (TSS) was studied by Das et al. [31] by generalizing a Darbo-type theorem

$$Q_n(\zeta) = R_n\left(\zeta, Q(\zeta), \int_c^\zeta \frac{K'(s) W_n(\zeta, s, Q(s))}{(K(\zeta) - K(s))^{1-\lambda}} ds\right),\ \lambda \in (0,1),\ \zeta \in [c, S],\ S > 0,$$

where $Q(\zeta) \in \{Q_n(\zeta)\}_{n=1}^\infty \subset \Omega$ and $\Omega$ is a sequence in a BS. For the TSS $\ell_p^\zeta$ $(1 < p < \infty)$, existence results were derived by using MNC and Darbo's FP theorem approaches.

Mohiuddine et al. [32] demonstrated that non-linear IEs have solutions in a TSS by an extended Darbo-type theorem.

$$Q_n(\zeta) = R_n\left(\zeta, Q(\zeta), \int_c^\zeta G_n(\zeta, s, Q(s))\, ds\right),\ n \in \mathbb{N},\ \zeta \in [0, S],\ S > 0,$$

where $Q(\zeta) \in \{Q_n(\zeta)\}_{n=1}^\infty$ . The authors demonstrated the existence of solutions to the IEs by using the concept of MNC and Darbo's FP theorem in TSS $C\left(\left[[0, S], \ell_p^\zeta\right]\right)$.

In this article, we investigate the $(\varphi, \theta)-$generalized Riemann-Liouville fractional (RLF) integral operator ${}_\varphi^\theta I_c^{\rho,\ell}$, where $\ell \in (0,1)$, $\theta \in \mathbb{R}\backslash\{0\}$ and $\varphi, c, \rho > 0$, for a continuous function $\Phi$ as follows:

$$\left({}_\varphi^\theta I_c^{\rho,\ell}\Phi\right)(\tau) = \frac{\theta^{1-\frac{\rho}{\varphi}}}{\ell^{\frac{\rho}{\varphi}} \varphi^{\frac{\rho}{\varphi}} \Gamma\left(\frac{\rho}{\varphi}\right)} \int_c^\tau \exp\left[\frac{\ell\left(\tau^\theta - s^\theta\right)}{(\ell - 1)}\right] \left(\tau^\theta - s^\theta\right)^{\frac{\rho}{\varphi}-1} s^{\theta-1} \Phi(s) ds.$$

Moreover, we introduce the Erdélyi-Kober fractional (EKF) integral operator $\widetilde{I}_{\xi,c}^\eta$, where $\xi, c > 0$ and $\eta \in (0,1)$ for a continuous function $\Phi$ as follows:

$$\left(\widetilde{I}_{\xi,c}^\eta \Phi\right)(\tau) = \frac{\xi}{\Gamma(\eta)} \int_c^\tau \frac{s^{\xi-1}\Phi(s)}{(\tau^\xi - s^\xi)^{1-\eta}} ds. \tag{1.1}$$

A range of fractional operators that have been developed and categorized into broad groups according to their traits and behaviors can be used to assess the biological population dynamics (BPD). In our work, we express the $(\varphi, \theta)-$generalized RLF and EKF operators in terms of each other to prove an essential relationship between them, using the notion of FC with respect to (w.r.t.) the same map on the BS $C([1, S])$.

In addition, from the standpoint of implementation, the primary objective of this article is to incorporate a hybrid fractional operators into a BPD model that accounts for the detection of the birth rate increase $\Phi(\tau)$ at any time $\tau$ to enable essential planning for the future. Mixed type IEs associated with a $(\varphi, \theta)-$generalized RLF together with the EKF operator results in the following dependency of the birthrate $\Phi(\tau)$ on prior birthrates $\Phi\left(\tau^{\vartheta} - s^{\vartheta}\right)$ for women in the childbearing age range $s \in [1, S]$ given $\vartheta > 1$:

$$\Phi(\tau) = \varkappa(\tau) + r\left(\tau, P(\tau, \Phi(\tau)), \left({}_{\varphi}^{\theta}I_{1}^{\rho,\ell}\Phi\right)(\tau)\right) + R\left(\tau, Q(\tau, \Phi(\tau)), \left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau)\right), \tag{1.2}$$

where $\Phi(s)$ is the possibility that the female will live to age $s$, $\varkappa(\tau)$, $P(\tau, \Phi(\tau))$, and $Q(\tau, \Phi(\tau))$ are the variables added to account for females who were born before the oldest child-bearing women of a certain age ($s = S$). In addition, $\ell, \eta \in (0, 1)$, $\theta \in \mathbb{R}_{+}$, $\varphi, \rho, \xi > 0$, and $\tau \in J = [1, S]$, $S > 0$.

This concept has been the subject of extensive study by a number of scholars, notably Gurtin and MacCamy [33]. A thorough explanation of the application of mathematical models in physiologically structured populations has been provided by Metz and Diekmann [34]. Cushing provides an extensive analysis of the research on predator delay [35]. See [36–41] for a comprehensive examination of age-dependent Parkinson's disease.

In this paper, we discuss the motivation for the analysis of Eq (1.2) and the details of our findings. To generalize the subjects of FC approaches, we investigate FIEs in connection with BPD modeling in this paper. Our second step involves examining earlier studies conducted in this area. Secondly, we think that the proposed FP consequence has the advantage of reducing the MNC demand, which is important for some FP outcomes. Our findings support, amplify, and reinforce findings that have already been published. Furthermore, the solvability of a particular non-linear IE that models the transmission of particular infectious diseases with a seasonally variable periodic contraction rate has been investigated by using the Banach contraction principle (BCP). Finally, we present two examples to illustrate the significance of our findings.

Our paper is summarized as follows. Some preliminary definitions and theories related to the problem under study are presented in Section 2. Section 3 is concerned with finding theoretical solutions to the problem under study under appropriate conditions in BSs. In Section 4, by using the BCP, we investigate the solution of a special non-linear IE (4.1) that models the spread of particular infectious diseases with a seasonal periodic contraction rate. In Section 5, we evaluate the applicability of our findings by looking at a few examples derived from the BPD models. Finally, a conclusion and some open problems are defined the future are expressed in Section 6.

## 2. Preliminaries

To facilitate the discussion of our key findings, we offer notations, definitions, and extra details in this section. Let $\Omega = C(J)$ be the space of real-valued continuous functions defined on $J = [1, S]$. Define the norm $\|.\|$ by

$$\|\varpi\| = \sup\{|\varpi(\tau)| : \tau \in J\}, \text{ for some } \varpi \in \Omega.$$

Henceforth, the symbols $\mathfrak{R}[\omega, z_0]$, $\overline{\mho}$, $Con\ \mho$, $\mathbb{R}_{+}$, $\mathbb{N}^{*}$, $\emptyset$, $\Lambda_{\Omega}$ and $\Delta_{\Omega}$ refer to a closed ball with center $\omega$ and radius $z_0$ in $\Omega$, the closure of a subset $\mho$ of $\Omega$, the convex hull of a subset $\mho$, the set of all positive real numbers, the set of all natural numbers without zero, the empty set, the class of all non-empty bounded subsets of $\Omega$, and the subfamily of all relatively compact subsets, respectively.

**Definition 2.1.** [42] A function $\mathfrak{I} : \Lambda_{\Omega} \to \mathbb{R}_{+}$ is called a MNC in $\Omega$ if the assumptions below hold:

(1) $\mho \in \Lambda_\Omega$ and $\Im(\mho) = 0$ imply that $\mho$ is precompact;
(2) $ker\ \psi \subset \Delta_\Omega$ and $ker\ \psi = \{\mho \in \Lambda_\Omega : \Im(\mho) = 0\}$ is non-void;
(3) $\mho \subseteq \mho_1$ implies that $\Im(\mho) \leq \Im(\mho_1)$;
(4) $\Im\left(\overline{\mho}\right) = \Im(\mho)$;
(5) $\Im(con\ \mho) = \Im(\mho)$;
(6) For all $\kappa \in [0,1]$, $\Im(\kappa\mho + (1-\kappa)\mho_1) \leq \kappa\Im(\mho) + (1-\kappa)\Im(\mho_1)$;
(7) If $\mho_n \in \Lambda_\Omega$, $\Im\left(\overline{\mho}\right) = \Im(\mho)$, $\mho_{n+1} \subseteq \mho_n$, $n = 1,2,...$, and $\lim_{n\to\infty}\Im(\mho_n) = 0$, then $\mho_\infty = \cap_{m=1}^{\infty}\mho_n \neq \emptyset$.

**Definition 2.2.** [42] Assume that $F\,(\neq \emptyset)$ is a bounded subset of $C(J)$. Then, for all $\epsilon > 0$ there exists $\varpi \in F$ such that the modulus of continuity of $\varpi$ is described by

$$d(\varpi, \epsilon) = \sup\{|\varpi(\tau_2) - \varpi(\tau_1)| : \tau_1, \tau_2 \in J;\ |\tau_2 - \tau_1| \leq \epsilon\},$$

with

$$d(F, \epsilon) = \sup\{d(\varpi, \epsilon) : \varpi \in F\} \text{ and } d_0(F) = \lim_{\epsilon\to 0} d(F, \epsilon),$$

where the map $d_0(F)$ is a regular MNC in $C(J)$. Also, there is a Hausdorff MNC $\Im$, which is given by $\Im(F) = \frac{1}{2}d_0(F)$.

**Remark 2.3.** The symbol $ker\ \psi$ refers to the kernel of MNC $\Im$. In addition, $\mho_\infty \in ker\ \Im$ and $\Im(\mho_\infty) \leq \Im(\mho_n)$ for $n \geq 1$, then, we have that $\Im(\mho_\infty) = 0$. Hence $\mho_\infty \in ker\ \psi$.

**Theorem 2.4.** *[43] (Darbo's FP theorem) Let $\Im$ be an MNC, $\Omega$ be a BS and $\Upsilon \subseteq \Omega$ be non-empty, bounded, closed, and convex. Assume that $\aleph : \Upsilon \to \Upsilon$ is a continuous mapping. Then, $\aleph$ has a FP in $\Upsilon$ provided that the following inequality is true*

$$\Im(\aleph E) \leq \sigma\Im(E),\ E \subset \Upsilon,\ \textit{for a constant } \sigma \in [0,1).$$

**Definition 2.5.** [44] For a continuous mapping $\varpi$, the RLF integral of order $\eta > 0$ is described by

$$I_c^\eta \varpi(\tau) = \frac{1}{\Gamma(\eta)}\int_c^\tau \varpi(s)(\tau - s)^{\eta-1}ds,\ \tau \in (c,d],$$

where $\Gamma(.)$ is the Euler GF. The well-known Cauchy formula serves as the inspiration for the RL integral:

$$\int_c^\tau ds_1 \int_c^{s_1} ds_2 ... \int_c^{s_{n-1}} \varpi(s_n)\,ds_n = \frac{1}{(n-1)!}\int_c^\tau \varpi(s)(\tau - s)^{n-1}ds,\ n \in \mathbb{N}^*.$$

In 2012, Pagnini [45] investigated an EKF operator as follows:

**Definition 2.6.** For a sufficiently well-behaved continuous function $\varpi(v)$, the EKF operator is described by

$$I_{\xi,c}^{\eta,\beta}\varpi(v) = \frac{\xi}{\Gamma(\beta)}v^{-\xi(\eta+\beta)}\int_c^v \frac{s^{\xi(\eta+1)-1}}{(v^\xi - s^\xi)^{1-\beta}}\varpi(s)\,ds,$$

where $\beta, \xi, c > 0$, $\eta \in \mathbb{R}$.

The usual GF, first described by Diaz and Pariguan [46], has been generalized into the $\kappa$-GF as follows:

**Definition 2.7.** For a continuous function $\kappa(\rho)$, the $\kappa-$GF is defined by

$$\Gamma\kappa(\rho) = \lim_{n\to\infty} \frac{n!\kappa^n (n\kappa)^{\frac{\rho}{\kappa}-1}}{(\rho)_{n,\kappa}},\ \kappa,\rho > 0,$$

where $(\rho)_{n,\kappa}$ is the Pochhammer's $\kappa-$symbol [47] for factorial functions. Additionally, the $\kappa$-GF is denoted and described by the following integral form [48]:

$$\Gamma\kappa(\rho) = \int_0^\infty e^{-\frac{s^\kappa}{\kappa}} s^{\rho-1} ds,\ \kappa,\rho > 0.$$

Furthermore, as presented by Mubeen and Habibullah [49], the RL$\kappa$-F integral of the function $\varpi$ of order $\beta > 0$ is designated and defined by

$$I_{0,\kappa}^{\rho,\beta}\varpi(v) = \frac{1}{\kappa\Gamma\kappa(\rho)} \int_0^v (v-s)^{\frac{\beta}{\kappa}-1} \varpi(s)\, ds,\ \kappa,\rho,v > 0.$$

## 3. Main results

The main focus of this section is on whether Eq (1.2) can be solved in BS $C(J)$. Let us consider that $\mathfrak{R}_{z_0} = \{\Phi \in \Omega : \|\Phi\| \le z_0\}$. In order to demonstrate our main theorem here, we take into account the following assumptions:

$A_1$ : $\varkappa : J \to \mathbb{R}$ is a bounded and continuous function with $\gimel_1 = \sup_{\tau\in J} |\varkappa(\tau)|$.

$A_2$ : $r : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $P : J\times\mathbb{R} \to \mathbb{R}$ are continuous functions and there are constants $\gimel_2, \gimel_3, \gimel_4 \ge 0$ such that

$$\left|r(\tau, P, I_1) - r\left(\tau, \widehat{P}, \widehat{I_1}\right)\right| \le \gimel_2 \left|P - \widehat{P}\right| + \gimel_3 \left|I_1 - \widehat{I_1}\right|,\ \text{for } P, I_1, \widehat{P}, \widehat{I_1} \in \mathbb{R} \text{ and } \tau \in J.$$

In addition,

$$\left|P(\tau, w) - P(\tau, \widehat{w})\right| \le \gimel_4 \left|w - \widehat{w}\right|,\ w, \widehat{w} \in \mathbb{R}.$$

$A_3$ : $R : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $Q : J\times\mathbb{R} \to \mathbb{R}$ are continuous functions and there are constants $\gimel_5, \gimel_6, \gimel_7 \ge 0$ such that

$$\left|R\left(\tau, Q, \widetilde{I_1}\right) - R\left(\tau, \widehat{Q}, \overline{\overline{I_1}}\right)\right| \le \gimel_5 \left|Q - \widehat{Q}\right| + \gimel_6 \left|\widetilde{I_1} - \overline{\overline{I_1}}\right|,\ \text{for } Q, \widetilde{I_1}, \widehat{Q}, \overline{\overline{I_1}} \in \mathbb{R} \text{ and } \tau \in J.$$

In addition,

$$\left|Q(\tau, \varpi) - Q(\tau, \widehat{\varpi})\right| \le \gimel_7 \left|\varpi - \widehat{\varpi}\right|,\ \varpi, \widehat{\varpi} \in \mathbb{R}.$$

$A_4$ : There is a positive constant $z_0$ such that

$$\sup_{\tau\in J} \left|\varkappa(\tau) + r(\tau, P, I_1) + R\left(\tau, Q, \widetilde{I_1}\right)\right| \le z_0,$$

for $P \in [-P', P']$, $I_1 \in [-I_1', I_1']$, $Q \in [-Q', Q']$ and $\widetilde{I_1} \in [-\widetilde{I}_1', \widetilde{I}_1']$, where

$$
\begin{aligned}
P' &= \sup\{|P(\tau, \Phi(\tau))| : \tau \in J,\ \Phi(\tau) \in [-z_0, z_0]\},\\
I_1' &= \sup\left\{\left|\left({}^{\theta}_{\varphi}I_1^{\rho,\ell}\Phi\right)(\tau)\right| : \tau \in J,\ \Phi(\tau) \in [-z_0, z_0]\right\},\\
Q' &= \sup\{|Q(\tau, \Phi(\tau))| : \tau \in J,\ \Phi(\tau) \in [-z_0, z_0]\},\\
\widetilde{I}_1' &= \sup\left\{\left|\left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)\right| : \tau \in J,\ \Phi(\tau) \in [-z_0, z_0]\right\}.
\end{aligned}
$$

Additionally, $\gimel_2\gimel_4 + \gimel_5\gimel_7 < 1$.

$A_5$ : There is a constant $z_0 > 0$ such that

$$
\gimel_1 + (\gimel_2\gimel_4 + \gimel_5\gimel_7) z_0 + \gimel_3 \frac{\theta^{-\frac{\varrho}{\varphi}} e^{\frac{\ell(S^{\theta})}{(\ell-1)}} \left(S^{\theta} - 1\right)^{\frac{\varrho}{\varphi}}}{\rho \ell^{\frac{\varrho}{\varphi}} \varphi^{\frac{\varrho}{\varphi}-1} \Gamma\left(\frac{\varrho}{\varphi}\right)} z_0 + \gimel_6 \frac{S^{\xi\eta}}{\Gamma(\eta+1)} z_0 \leq z_0.
$$

**Remark 3.1.** Based on $(A_2)$ and $(A_3)$, one has

$$
|P(\tau, 0)| = 0,\ \ |r(\tau, 0, 0)| = 0,\ \ |Q(\tau, 0)| = 0,\ \ |R(\tau, 0, 0)| = 0.
$$

**Theorem 3.2.** *With the help of Remark 3.1, the problem given be Eq (1.2) has a solution in* $C(J)$ *provided that the hypotheses* $(A_1)$*–*$(A_5)$ *are true.*

*Proof.* Define an operator $\aleph : \mathfrak{R}_{z_0} \to \Omega$ as follows:

$$
(\aleph\Phi)(\tau) = \varkappa(\tau) + r\left(\tau, P(\tau, \Phi(\tau)), \left({}^{\theta}_{\varphi}I_1^{\rho,\ell}\Phi\right)(\tau)\right) + R\left(\tau, Q(\tau, \Phi(\tau)), \left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)\right).
$$

Thus, the problem turns into the problem of finding the FP of the operator $\aleph$. In other words, solving Eq (1.2) is equivalent to finding the FP of the operator $\aleph$. To achieve this, we divide the proof into the following cases:

**Case 1.** Prove that $\aleph$ maps $\mathfrak{R}_{z_0}$ into $\mathfrak{R}_{z_0}$. Then, for $\Phi \in \mathfrak{R}_{z_0}$, we have

$$
\begin{aligned}
|(\aleph\Phi)(\tau)| &= \left|\varkappa(\tau) + r\left(\tau, P(\tau, \Phi(\tau)), \left({}^{\theta}_{\varphi}I_1^{\rho,\ell}\Phi\right)(\tau)\right) + R\left(\tau, Q(\tau, \Phi(\tau)), \left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)\right)\right|\\
&\leq |\varkappa(\tau)| + \left|r\left(\tau, P(\tau, \Phi(\tau)), \left({}^{\theta}_{\varphi}I_1^{\rho,\ell}\Phi\right)(\tau)\right) - r(\tau, 0, 0)\right| + |r(\tau, 0, 0)|\\
&\quad + \left|R\left(\tau, Q(\tau, \Phi(\tau)), \left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)\right) - R(\tau, 0, 0)\right| + |R(\tau, 0, 0)|\\
&\leq \gimel_1 + \gimel_2 |P(s, \Phi(s))| + \gimel_3 \left|\left({}^{\theta}_{\varphi}I_1^{\rho,\ell}\Phi\right)(\tau)\right| + \gimel_5 |Q(\tau, \Phi(\tau))| + \gimel_6 \left|\left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)\right|\\
&\leq \gimel_1 + \gimel_2 [|P(\tau, \Phi(\tau)) - P(\tau, 0)| + |P(\tau, 0)|] + \gimel_3 \left|\left({}^{\theta}_{\varphi}I_1^{\rho,\ell}\Phi\right)(\tau)\right|\\
&\quad + \gimel_5 [|Q(\tau, \Phi(\tau)) - Q(\tau, 0)| + |Q(\tau, 0)|] + \gimel_6 \left|\left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)\right|\\
&\leq \gimel_1 + \gimel_2\gimel_4 |\Phi(\tau)| + \gimel_3 \left|\left({}^{\theta}_{\varphi}I_1^{\rho,\ell}\Phi\right)(\tau)\right| + \gimel_5\gimel_7 |\Phi(\tau)| + \gimel_6 \left|\left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)\right|,
\end{aligned}
\tag{3.1}
$$

where

$$
\left|\left({}^{\theta}_{\varphi}I_1^{\rho,\ell}\Phi\right)(\tau)\right| = \left|\frac{\theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}} \varphi^{\frac{\varrho}{\varphi}} \Gamma\left(\frac{\varrho}{\varphi}\right)} \int_1^{\tau} e^{\left(\frac{\ell\left(\tau^{\theta} - s^{\theta}\right)}{\ell-1}\right)} \left(\tau^{\theta} - s^{\theta}\right)^{\frac{\varrho}{\varphi}-1} s^{\theta-1} \Phi(s) ds\right|
$$

$$
\begin{aligned}
&\le \frac{\theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}}\Gamma\left(\frac{\varrho}{\varphi}\right)}\left|\int_1^\tau e^{\left(\frac{\ell\left(\tau^\theta-s^\theta\right)}{\ell-1}\right)}\left(\tau^\theta-s^\theta\right)^{\frac{\varrho}{\varphi}-1}s^{\theta-1}\Phi(s)ds\right| \\
&\le \frac{z_0\theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}}\Gamma\left(\frac{\varrho}{\varphi}\right)}e^{\left(\frac{\ell S^\theta}{\ell-1}\right)}\int_1^\tau\left(\tau^\theta-s^\theta\right)^{\frac{\varrho}{\varphi}-1}s^{\theta-1}ds \\
&\le \frac{z_0\theta^{-\frac{\varrho}{\varphi}}}{\rho\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}-1}\Gamma\left(\frac{\varrho}{\varphi}\right)}e^{\left(\frac{\ell S^\theta}{\ell-1}\right)}\left(S^\theta-1\right)^{\frac{\varrho}{\varphi}}, \qquad (3.2)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)\right| &= \left|\frac{\xi}{\Gamma(\eta)}\int_1^\tau\frac{s^{\xi-1}\Phi(s)}{\left(\tau^\xi-s^\xi\right)^{1-\eta}}ds\right| \\
&\le \frac{\xi}{\Gamma(\eta)}\int_1^\tau\frac{s^{\xi-1}\left|\Phi(s)\right|}{\left(\tau^\xi-s^\xi\right)^{1-\eta}}ds \\
&\le \frac{z_0\xi}{\Gamma(\eta+1)}S^{\xi\eta}. \qquad (3.3)
\end{aligned}
$$

Applying Eqs (3.2) and (3.3) in Eq (3.1) and using $A_5$ with $|\Phi(\tau)| \le z_0$, we have

$$
\|\aleph\Phi\| \le \daleth_1+\daleth_2\daleth_4\left|\Phi(\tau)\right|+\daleth_3\frac{\theta^{-\frac{\varrho}{\varphi}}}{\rho\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}-1}\Gamma\left(\frac{\varrho}{\varphi}\right)}e^{\left(\frac{\ell S^\theta}{\ell-1}\right)}\left(S^\theta-1\right)^{\frac{\varrho}{\varphi}}z_0+\daleth_5\daleth_7z_0+\daleth_6\frac{\xi S^{\xi\eta}}{\Gamma(\eta+1)}z_0<z_0.
$$

Hence, $\aleph$ maps $\mathfrak{R}_{z_0}$ into $\mathfrak{R}_{z_0}$.

**Case 2.** Show that $\aleph$ is continuous in $\mathfrak{R}_{z_0}$. For this, assume that $\epsilon>0$ and $\Phi,\overline{\Phi}\in\mathfrak{R}_{z_0}$ such that $\left\|\Phi-\overline{\Phi}\right\|<\epsilon$; we get

$$
\begin{aligned}
&\left|(\aleph\Phi)(\tau)-\left(\aleph\overline{\Phi}\right)(\tau)\right| \\
=&\ \Big|\varkappa(\tau)+r\left(\tau,P(\tau,\Phi(\tau)),\left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\Phi\right)(\tau)\right)+R\left(\tau,Q(\tau,\Phi(\tau)),\left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)\right) \\
&-\varkappa(\tau)-r\left(\tau,P\left(\tau,\overline{\Phi}(\tau)\right),\left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\overline{\Phi}\right)(\tau)\right)-R\left(\tau,Q\left(\tau,\overline{\Phi}(\tau)\right),\left(\widetilde{I}^{\eta}_{\xi,1}\overline{\Phi}\right)(\tau)\right)\Big| \\
\le&\ \daleth_2\left|P(\tau,\Phi(\tau))-P\left(\tau,\overline{\Phi}(\tau)\right)\right|+\daleth_3\left|\left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\Phi\right)(\tau)-\left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\overline{\Phi}\right)(\tau)\right| \\
&+\daleth_5\left|Q(\tau,\Phi(\tau))-Q\left(\tau,\overline{\Phi}(\tau)\right)\right|+\daleth_6\left|\left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)-\left(\widetilde{I}^{\eta}_{\xi,1}\overline{\Phi}\right)\right| \\
\le&\ \daleth_2\daleth_4\left|\Phi(\tau)-\overline{\Phi}(\tau)\right|+\daleth_5\daleth_7\left|\Phi(\tau)-\overline{\Phi}(\tau)\right|+\daleth_3\left|\left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\Phi\right)(\tau)-\left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\overline{\Phi}\right)(\tau)\right| \\
&+\daleth_6\left|\left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)-\left(\widetilde{I}^{\eta}_{\xi,1}\overline{\Phi}\right)\right|, \qquad (3.4)
\end{aligned}
$$

where

$$
\left|\left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\Phi\right)(\tau)-\left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\overline{\Phi}\right)(\tau)\right|
$$

$$
\begin{aligned}
&= \left|\frac{\theta^{1-\frac{\rho}{\varphi}}}{\ell^{\frac{\rho}{\varphi}}\varphi^{\frac{\rho}{\varphi}}\Gamma\left(\frac{\rho}{\varphi}\right)}\int_1^{\tau} e^{\left(\frac{\ell\left(\tau^{\theta}-s^{\theta}\right)}{\ell-1}\right)}\left(\tau^{\theta}-s^{\theta}\right)^{\frac{\rho}{\varphi}-1} s^{\theta-1}\left[\Phi(s)-\overline{\Phi}(s)\right]ds\right| \\
&\leq \frac{\theta^{1-\frac{\rho}{\varphi}}}{\ell^{\frac{\rho}{\varphi}}\varphi^{\frac{\rho}{\varphi}}\Gamma\left(\frac{\rho}{\varphi}\right)}\left|\int_1^{\tau} e^{\left(\frac{\ell\left(\tau^{\theta}-s^{\theta}\right)}{\ell-1}\right)}\left(\tau^{\theta}-s^{\theta}\right)^{\frac{\rho}{\varphi}-1} s^{\theta-1}\left[\Phi(s)-\overline{\Phi}(s)\right]ds\right| \\
&< \frac{\theta^{-\frac{\rho}{\varphi}}e^{\left(\frac{\ell S^{\theta}}{\ell-1}\right)}\left(S^{\theta}-1\right)^{\frac{\rho}{\varphi}}}{\rho\ell^{\frac{\rho}{\varphi}}\varphi^{\frac{\rho}{\varphi}-1}\Gamma\left(\frac{\rho}{\varphi}\right)}\left|\Phi(s)-\overline{\Phi}(s)\right| \qquad (3.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau)-\left(\widetilde{I}^{\eta}_{\xi,1}\overline{\Phi}\right)\right| &= \left|\frac{\xi}{\Gamma(\eta)}\int_1^{\tau}\frac{s^{\xi-1}}{\left(\tau^{\xi}-s^{\xi}\right)^{1-\eta}}\left[\Phi(s)-\overline{\Phi}(s)\right]ds\right| \\
&\leq \frac{\xi}{\Gamma(\eta)}\int_1^{\tau}\frac{s^{\xi-1}}{\left(\tau^{\xi}-s^{\xi}\right)^{1-\eta}}\left|\Phi(s)-\overline{\Phi}(s)\right|ds \\
&\leq \frac{\xi S^{\xi\eta}}{\Gamma(\eta+1)}\left|\Phi(s)-\overline{\Phi}(s)\right|. \qquad (3.6)
\end{aligned}
$$

By applying Eqs (3.5) and (3.6) in Eq (3.4) and taking the supremum with $\left\|\Phi-\overline{\Phi}\right\| < \epsilon$, we conclude that

$$
\left\|\aleph\Phi-\aleph\overline{\Phi}\right\| \leq \beth_2\beth_4\epsilon+\beth_5\beth_7\epsilon+\beth_3\frac{\theta^{-\frac{\rho}{\varphi}}e^{\left(\frac{\ell S^{\theta}}{\ell-1}\right)}\left(S^{\theta}-1\right)^{\frac{\rho}{\varphi}}}{\rho\ell^{\frac{\rho}{\varphi}}\varphi^{\frac{\rho}{\varphi}-1}\Gamma\left(\frac{\rho}{\varphi}\right)}\epsilon+\beth_6\frac{\xi S^{\xi\eta}}{\Gamma(\eta+1)}\epsilon. \qquad (3.7)
$$

Letting $\epsilon \to 0$ in Eq (3.7), we have

$$
\left\|\aleph\Phi-\aleph\overline{\Phi}\right\| \to 0,
$$

which implies that $\aleph$ is continuous in $\mathfrak{R}_{z_0}$.

**Case 3.** Show the estimation of $\aleph$ with respect to $d_0$. For this, consider a $F\,(\neq \emptyset) \subseteq \mathfrak{R}_{z_0}$. In addition, let $F \in \mathfrak{R}_{z_0}$ and $\tau_1, \tau_2 \in J$ with $\tau_1 \leq \tau_2$ such that $|\tau_2-\tau_1| \leq \epsilon$; then, we get

$$
\begin{aligned}
& \left|(\aleph\Phi)(\tau_2)-(\aleph\Phi)(\tau_1)\right| \\
= & \Big|\varkappa(\tau_2)+r\left(\tau_2, P(\tau_2,\Phi(\tau_2)), \left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\Phi\right)(\tau_2)\right)+R\left(\tau_2, Q(\tau_2,\Phi(\tau_2)), \left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau_2)\right) \\
& -\varkappa(\tau_1)-r\left(\tau_1, P(\tau_1,\Phi(\tau_1)), \left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\Phi\right)(\tau_1)\right)-R\left(\tau_1, Q(\tau_1,\Phi(\tau_1)), \left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau_1)\right)\Big| \\
\leq & \left|\varkappa(\tau_2)-\varkappa(\tau_1)\right| \\
& +\left|r\left(\tau_2, P(\tau_2,\Phi(\tau_2)), \left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\Phi\right)(\tau_2)\right)-r\left(\tau_1, P(\tau_1,\Phi(\tau_1)), \left({}^{\theta}_{\varphi}I^{\rho,\ell}_1\Phi\right)(\tau_1)\right)\right| \\
& +\left|R\left(\tau_2, Q(\tau_2,\Phi(\tau_2)), \left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau_2)\right)-R\left(\tau_1, Q(\tau_1,\Phi(\tau_1)), \left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau_1)\right)\right|,
\end{aligned}
$$

which implies that

$$
\left|(\aleph\Phi)(\tau_2)-(\aleph\Phi)(\tau_1)\right|
$$

$$
\begin{aligned}
\leq\ & |\varkappa(\tau_2)-\varkappa(\tau_1)| \\
&+\left|r\left(\tau_2,P(\tau_2,\Phi(\tau_2)),\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_2)\right)-r\left(\tau_2,P(\tau_2,\Phi(\tau_2)),\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1)\right)\right| \\
&+\left|r\left(\tau_2,P(\tau_2,\Phi(\tau_2)),\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1)\right)-r\left(\tau_2,P(\tau_1,\Phi(\tau_1)),\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1)\right)\right| \\
&+\left|r\left(\tau_2,P(\tau_1,\Phi(\tau_1)),\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1)\right)-r\left(\tau_1,P(\tau_1,\Phi(\tau_1)),\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1)\right)\right| \\
&+\left|R\left(\tau_2,Q(\tau_2,\Phi(\tau_2)),\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_2)\right)-R\left(\tau_2,Q(\tau_2,\Phi(\tau_2)),\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_1)\right)\right| \\
&+\left|R\left(\tau_2,Q(\tau_2,\Phi(\tau_2)),\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_1)\right)-R\left(\tau_2,Q(\tau_1,\Phi(\tau_1)),\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_1)\right)\right| \\
&+\left|R\left(\tau_2,Q(\tau_1,\Phi(\tau_1)),\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_1)\right)-R\left(\tau_1,Q(\tau_1,\Phi(\tau_1)),\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_1)\right)\right|,
\end{aligned}
$$

and

$$
\begin{aligned}
& |(\aleph\Phi)(\tau_2)-(\aleph\Phi)(\tau_1)| \\
\leq\ & d(\varkappa,\epsilon)+\gimel_3\left|\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_2)-\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1)\right| \\
& +\gimel_2|P(\tau_2,\Phi(\tau_2))-P(\tau_1,\Phi(\tau_1))|+d_r(J,\epsilon) \\
& +\gimel_6\left|\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_2)-\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_1)\right|+\gimel_5|Q(\tau_2,\Phi(\tau_2))-Q(\tau_1,\Phi(\tau_1))|+d_R(J,\epsilon).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& |(\aleph\Phi)(\tau_2)-(\aleph\Phi)(\tau_1)| \\
\leq\ & d(\varkappa,\epsilon)+\gimel_3\left|\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_2)-\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1)\right|+d_r(J,\epsilon) \\
& +\gimel_2\left[|P(\tau_2,\Phi(\tau_2))-P(\tau_2,\Phi(\tau_1))|+|P(\tau_2,\Phi(\tau_1))-P(\tau_1,\Phi(\tau_1))|\right] \\
& +\gimel_6\left|\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_2)-\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_1)\right|+d_R(J,\epsilon) \\
& +\gimel_5\left[|Q(\tau_2,\Phi(\tau_2))-Q(\tau_2,\Phi(\tau_1))|+|Q(\tau_2,\Phi(\tau_1))-Q(\tau_1,\Phi(\tau_1))|\right] \\
\leq\ & d(\varkappa,\epsilon)+\gimel_2\gimel_4|\Phi(\tau_2)-\Phi(\tau_1)|+\gimel_2 d_P(J,\epsilon)+d_r(J,\epsilon)+d_R(J,\epsilon) \\
& +\gimel_5\gimel_7|\Phi(\tau_2)-\Phi(\tau_1)|+\gimel_5 d_Q(J,\epsilon) \\
& +\gimel_3\left|\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_2)-\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1)\right|+\gimel_6\left|\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_2)-\left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_1)\right|,
\end{aligned}
$$

where

$$
\begin{aligned}
d(\varkappa,\epsilon) &= \sup\{|\varkappa(\tau_2)-\varkappa(\tau_1)|:\tau_2,\tau_1\in J;|\tau_2-\tau_1|\leq\epsilon\}, \\
d_P(J,\epsilon) &= \sup\{|P(\tau_2,\Phi)-P(\tau_1,\Phi)|:\tau_2,\tau_1\in J;|\tau_2-\tau_1|\leq\epsilon\}, \\
d_r(J,\epsilon) &= \sup\{|r(\tau_2,P,I_1)-r(\tau_1,P,I_1)|:\tau_2,\tau_1\in J;|\tau_2-\tau_1|\leq\epsilon\}, \\
d_R(J,\epsilon) &= \sup\{|R(\tau_2,Q,I_1)-R(\tau_1,Q,I_1)|:\tau_2,\tau_1\in J;|\tau_2-\tau_1|\leq\epsilon\}, \\
d_Q(J,\epsilon) &= \sup\{|Q(\tau_2,\Phi)-Q(\tau_1,\Phi)|:\tau_2,\tau_1\in J;|\tau_2-\tau_1|\leq\epsilon\}.
\end{aligned}
$$

In addition,

$$
\left|\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_2)-\left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1)\right| = \left|\frac{\theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}}\Gamma\left(\frac{\varrho}{\varphi}\right)}\int_1^{\tau_2} e^{\left(\frac{\ell\left(\tau_2^{\theta}-s^{\theta}\right)}{\ell-1}\right)}\left(\tau_2^{\theta}-s^{\theta}\right)^{\frac{\varrho}{\varphi}-1}s^{\theta-1}\Phi(s)ds\right.
$$

$$
\begin{aligned}
&- \frac{\theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}}\Gamma\left(\frac{\varrho}{\varphi}\right)} \int_{1}^{\tau_1} e^{\left(\frac{\ell\left(\tau_1^{\theta}-s^{\theta}\right)}{\ell-1}\right)} \left(\tau_1^{\theta}-s^{\theta}\right)^{\frac{\varrho}{\varphi}-1} s^{\theta-1}\Phi(s)ds \Bigg| \\
= \; & \Bigg| \frac{\theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}}\Gamma\left(\frac{\varrho}{\varphi}\right)} \int_{1}^{\tau_2} e^{\left(\frac{\ell\left(\tau_2^{\theta}-s^{\theta}\right)}{\ell-1}\right)} \left(\tau_2^{\theta}-s^{\theta}\right)^{\frac{\varrho}{\varphi}-1} s^{\theta-1}\Phi(s)ds \\
&- \frac{\theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}}\Gamma\left(\frac{\varrho}{\varphi}\right)} \int_{1}^{\tau_1} e^{\left(\frac{\ell\left(\tau_2^{\theta}-s^{\theta}\right)}{\ell-1}\right)} \left(\tau_2^{\theta}-s^{\theta}\right)^{\frac{\varrho}{\varphi}-1} s^{\theta-1}\Phi(s)ds \Bigg| \\
&+ \Bigg| \frac{\theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}}\Gamma\left(\frac{\varrho}{\varphi}\right)} \int_{1}^{\tau_1} e^{\left(\frac{\ell\left(\tau_2^{\theta}-s^{\theta}\right)}{\ell-1}\right)} \left(\tau_2^{\theta}-s^{\theta}\right)^{\frac{\varrho}{\varphi}-1} s^{\theta-1}\Phi(s)ds \\
&- \frac{\theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}}\Gamma\left(\frac{\varrho}{\varphi}\right)} \int_{1}^{\tau_1} e^{\left(\frac{\ell\left(\tau_1^{\theta}-s^{\theta}\right)}{\ell-1}\right)} \left(\tau_1^{\theta}-s^{\theta}\right)^{\frac{\varrho}{\varphi}-1} s^{\theta-1}\Phi(s)ds \Bigg|,
\end{aligned}
$$

which yields

$$
\begin{aligned}
& \left| \left({}^{\theta}_{\varphi}I_1^{\rho,\ell}\Phi\right)(\tau_2) - \left({}^{\theta}_{\varphi}I_1^{\rho,\ell}\Phi\right)(\tau_1) \right| \\
\leq \; & \frac{\theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}}\Gamma\left(\frac{\varrho}{\varphi}\right)} \int_{\tau_1}^{\tau_2} e^{\left(\frac{\ell\left(\tau_2^{\theta}-s^{\theta}\right)}{\ell-1}\right)} \left(\tau_2^{\theta}-s^{\theta}\right)^{\frac{\varrho}{\varphi}-1} s^{\theta-1} \left|\Phi(s)\right| ds \\
&+ \frac{\theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}}\Gamma\left(\frac{\varrho}{\varphi}\right)} \int_{1}^{\tau_2} \left| \left( e^{\left(\frac{\ell\left(\tau_2^{\theta}-s^{\theta}\right)}{\ell-1}\right)} \left(\tau_2^{\theta}-s^{\theta}\right)^{\frac{\varrho}{\varphi}-1} - e^{\left(\frac{\ell\left(\tau_1^{\theta}-s^{\theta}\right)}{\ell-1}\right)} \left(\tau_1^{\theta}-s^{\theta}\right)^{\frac{\varrho}{\varphi}-1} \right) s^{\theta-1}\Phi(s) \right| ds \\
\leq \; & \frac{\theta^{-\frac{\varrho}{\varphi}}}{\rho\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}-1}\Gamma\left(\frac{\varrho}{\varphi}\right)} e^{\left(\frac{\ell S^{\theta}}{\ell-1}\right)} \left(S^{\theta}-1\right)^{\frac{\varrho}{\varphi}} \|\Phi\| \\
&+ \frac{\|\Phi\|\, \theta^{1-\frac{\varrho}{\varphi}}}{\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}-1}\Gamma\left(\frac{\varrho}{\varphi}\right)} \int_{1}^{\tau_2} \left| \left( e^{\left(\frac{\ell\left(\tau_2^{\theta}-s^{\theta}\right)}{\ell-1}\right)} \left(\tau_2^{\theta}-s^{\theta}\right)^{\frac{\varrho}{\varphi}-1} - e^{\left(\frac{\ell\left(\tau_1^{\theta}-s^{\theta}\right)}{\ell-1}\right)} \left(\tau_1^{\theta}-s^{\theta}\right)^{\frac{\varrho}{\varphi}-1} \right) s^{\theta-1} \right| ds,
\end{aligned}
$$

and

$$
\begin{aligned}
& \left| \left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau_2) - \left(\widetilde{I}^{\eta}_{\xi,1}\Phi\right)(\tau_1) \right| \\
= \; & \left| \frac{\xi}{\Gamma(\eta)} \int_{1}^{\tau_2} \frac{s^{\xi-1}\Phi(s)}{\left(\tau_2^{\xi}-s^{\xi}\right)^{1-\eta}} ds - \frac{\xi}{\Gamma(\eta)} \int_{1}^{\tau_1} \frac{s^{\xi-1}\Phi(s)}{\left(\tau_1^{\xi}-s^{\xi}\right)^{1-\eta}} ds \right| \\
\leq \; & \frac{\xi}{\Gamma(\eta)} \left| \int_{1}^{\tau_2} \frac{s^{\xi-1}\Phi(s)}{\left(\tau_2^{\xi}-s^{\xi}\right)^{1-\eta}} ds - \frac{\xi}{\Gamma(\eta)} \int_{1}^{\tau_1} \frac{s^{\xi-1}\Phi(s)}{\left(\tau_2^{\xi}-s^{\xi}\right)^{1-\eta}} ds \right| \\
&+ \frac{\xi}{\Gamma(\eta)} \left| \int_{1}^{\tau_1} \frac{s^{\xi-1}\Phi(s)}{\left(\tau_2^{\xi}-s^{\xi}\right)^{1-\eta}} ds - \int_{1}^{\tau_1} \frac{s^{\xi-1}\Phi(s)}{\left(\tau_1^{\xi}-s^{\xi}\right)^{1-\eta}} ds \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\xi}{\Gamma(\eta)} \int_{\tau_1}^{\tau_2} \frac{s^{\xi-1}\left|\Phi(s)\right|}{\left(\tau_2^{\xi}-s^{\xi}\right)^{1-\eta}} ds + \frac{\xi}{\Gamma(\eta)} \int_{1}^{\tau_1} \left( \frac{s^{\xi-1}}{\left(\tau_2^{\xi}-s^{\xi}\right)^{1-\eta}} - \frac{s^{\xi-1}}{\left(\tau_1^{\xi}-s^{\xi}\right)^{1-\eta}} \right) \left|\Phi(s)\right| ds \\
&\leq \frac{\|\Phi\|}{\Gamma(\eta+1)} \left[ 2\left(\tau_2^{\xi}-\tau_1^{\xi}\right)^{\eta} - \left(\tau_2^{\xi}-1\right)^{\eta} + \left(\tau_1^{\xi}-1\right)^{\eta} \right].
\end{aligned}
$$

Since $|\tau_2-\tau_1| \leq \epsilon$, one gets that $\tau_2 \to \tau_1$. Thus,

$$
\left| \left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_2) - \left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1) \right| \to 0 \text{ and } \left| \left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_2) - \left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_1) \right| \to 0. \tag{3.8}
$$

Hence,

$$
\begin{aligned}
\left| (\aleph\Phi)(\tau_2) - (\aleph\Phi)(\tau_1) \right| \leq\; & d(\varkappa,\epsilon) + \beth_2\beth_4 d(\Phi,\epsilon) + \beth_2 d_P(J,\epsilon) + d_r(J,\epsilon) + d_R(J,\epsilon) \\
& \beth_5\beth_7 d(\Phi,\epsilon) + \beth_5 d_Q(J,\epsilon) + \beth_3 \left| \left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_2) - \left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1) \right| \\
& + \beth_6 \left| \left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_2) - \left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_1) \right|,
\end{aligned}
$$

that is,

$$
\begin{aligned}
d(\aleph\Phi,\epsilon) \leq\; & d(\varkappa,\epsilon) + (\beth_2\beth_4 + \beth_5\beth_7) d(\Phi,\epsilon) + \beth_2 d_P(J,\epsilon) + d_r(J,\epsilon) + d_R(J,\epsilon) \\
& + \beth_5 d_Q(J,\epsilon) + \beth_3 \left| \left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_2) - \left({}_{\varphi}^{\theta}I_1^{\rho,\ell}\Phi\right)(\tau_1) \right| \\
& + \beth_6 \left| \left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_2) - \left(\widetilde{I}_{\xi,1}^{\eta}\Phi\right)(\tau_1) \right|.
\end{aligned} \tag{3.9}
$$

From the uniform continuity of the functions $\varkappa$, $P$, $Q$, $r$ and $R$ on $J$, $J \times [-z_0, z_0]$, $J \times [-z_0, z_0]$, $J \times [-P', P'] \times [-I_1', I_1']$ and $J \times [-Q', Q'] \times [-\widetilde{I}_1', \widetilde{I}_1']$, respectively; we have the following when $\epsilon \to 0$ :

$$
d(\varkappa,\epsilon) \to 0,\ d_P(J,\epsilon) \to 0,\ d_r(J,\epsilon) \to 0,\ d_R(J,\epsilon) \to 0 \text{ and } d_Q(J,\epsilon) \to 0. \tag{3.10}
$$

Applying Eqs (3.8) and (3.10) in Eq (3.9), and taking $\sup_{\Phi\in F}$ and $\epsilon \to 0$ in Eq (3.9), we have

$$
d_0(\aleph F) \leq (\beth_2\beth_4 + \beth_5\beth_7)\, d_0(F).
$$

Since $\beth_2\beth_4 + \beth_5\beth_7 < 1$, then all requirements of Theorem 2.4 are fulfilled. Therefore, $\aleph$ has an FP, which is a solution to our problem given be Eq (1.2) in $C(J)$. This finishes the proof.

□

## 4. A result on infectious diseases

Since most differential equations and IEs that arise in many real-world problems are known to be non-linear, FP theory offers a significant method for locating the solutions to these equations, which are otherwise difficult to solve by using other conventional methods. Here, by using the BCP, we investigate the solution of a special non-linear IE (4.1) that models the spread of particular infectious diseases with a seasonal periodic contraction rate.

If we take $\left(\widetilde{I}_{\xi,c}^{\eta}\Phi\right)(\tau) = h(\tau)$ with $\eta = \xi = 1$, $c = \tau - \alpha > 0$ and $\Phi(s) = \upsilon(s, h(s))$ in the EKF integral operator given by Eq (1.1), then, we obtain

$$
h(\tau) = \int_{\tau-\alpha}^{\tau} \upsilon(s, h(s)) ds. \tag{4.1}
$$

Equation (4.1) was presented by Leggett and Williams [50]. This equation can be used as a model for the spread of various infectious illnesses, whose periodic contraction rate exhibits seasonal change, where $h(\tau)$ is the number of people with diseases at time $\tau$, $\upsilon(\tau, h(\tau))$ is the number of infections in that time period ($f(\tau, 0) = 0$), and $\alpha$ is the amount of time that a person can still spread a disease.

Let $\Omega$ be a BS with the norm

$$\|h - \hbar\| = \sup_{\tau \in R} |h(\tau) - \hbar(\tau)|, \text{ for all } h, \hbar \in \Omega.$$

Assume that $\wp$ is a bounded subsets of $\Omega$. Define an operator $\aleph : \wp \to \wp$ by

$$\aleph h(\tau) = \int_{\tau-\alpha}^{\tau} \upsilon(s, h(s)) ds,$$

where $\upsilon$ satisfies the following axioms:

($a_1$) The function $\upsilon : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is continuous;
($a_2$) For all $h_1, \hbar_1 \in \wp$,

$$\|\upsilon(\tau, h_1(\tau)) - \upsilon(\tau, h_2(\tau))\| \leq \|h_1(\tau) - h_2(\tau)\|.$$

Now, we shall introduce a simple proof of the following theorem:

**Theorem 4.1.** *Assume that the axioms* ($a_1$) *and* ($a_2$) *are true; then, the problem given by Eq (4.1) has a unique solution provided that* $\alpha < 1$.

*Proof.* For all $h_1, \hbar_1 \in \wp$, we have

$$\begin{aligned} \|\aleph h_1(\tau) - \aleph h_2(\tau)\| &= \left\| \int_{\tau-\alpha}^{\tau} \upsilon(s, h_1(s)) ds - \int_{\tau-\alpha}^{\tau} \upsilon(s, h_2(s)) ds \right\| \\ &\leq \|h_1(\tau) - h_2(\tau)\| \left( \int_{\tau-\alpha}^{\tau} ds \right) \\ &\leq \alpha \|h_1(\tau) - h_2(\tau)\|. \end{aligned}$$

Since $\alpha < 1$, the mapping $\aleph$ is a contraction. Based on the BCP, $\aleph$ has a unique FP. It is the unique solution to the problem given by Eq (4.1). $\square$

## 5. Supportive applications

In this section, we will examine a few applications of BPD modeling in order to assess the effectiveness of our findings.

**Example 5.1.** Consider the following model:

$$\Phi(\tau) = \frac{\tau \sin \tau}{7} + \frac{\tau^3 \tan^{-1} \Phi(\tau)}{5 + 6\tau^3} + \frac{\cos \Phi(\tau)}{1 + \tau^3} + \frac{\left( {}^{\frac{4}{3}}_{\frac{1}{4}} I_1^{2, \frac{1}{2}} \Phi \right)(\tau)}{4^5} + \frac{\left( \widetilde{I}^{\frac{1}{8}}_{\frac{1}{4}, 1} \Phi \right)(\tau)}{64}, \tag{5.1}$$

where $\Phi(s)$ refers to the surge in the birthrate at any time $\tau$, $\tau \in J = [1, 2]$.

Equation (5.1), compared to Eq (1.2), yields

$$
\begin{aligned}
\varkappa(\tau) &= \frac{\tau \sin \tau}{7}, \\
r(\tau, P, I_1) &= P(\tau, \Phi(\tau)) + \frac{I_1}{4^5}, \\
P(\tau, \Phi(\tau)) &= \frac{\tau^3 \tan^{-1} \Phi(\tau)}{5 + 6\tau^3}, \\
\left({}^{\frac{4}{3}}_{\frac{1}{4}} I_1^{2, \frac{1}{2}} \Phi\right)(\tau) &= \frac{3^7 \times 4^{12}}{4^7 \Gamma(8)} \int_1^\tau e^{-\left(\tau^{\frac{4}{3}} - s^{\frac{4}{3}}\right)} \left(\tau^{\frac{4}{3}} - s^{\frac{4}{3}}\right)^7 s^{\frac{1}{3}} \Phi(s)\, ds, \\
\left(\widetilde{I}^{\frac{1}{8}}_{\frac{1}{4}, 1} \Phi\right)(\tau) &= \frac{1}{4\Gamma(\frac{1}{8})} \int_1^\tau \frac{s^{\frac{-3}{4}} \Phi(s)}{\left(\tau^{\frac{1}{4}} - s^{\frac{1}{4}}\right)^{\frac{7}{8}}} ds, \\
R\left(\tau, Q, \widetilde{I}_1\right) &= Q(\tau, \Phi(\tau)) + \frac{\widetilde{I}_1}{64}, \\
Q(\tau, \Phi(\tau)) &= \frac{\cos \Phi(\tau)}{1 + \tau^3},
\end{aligned}
$$

where $\Phi(s)$ is the possibility that the female will live to age $s$, $\varkappa(\tau)$, $P(\tau, \Phi(\tau))$, and $Q(\tau, \Phi(\tau))$ are the variables added to account for females born before the oldest child-bearing women of a certain age ($s = 2$) were born. The survival functions, $R$ and $r$, measure the proportion of people who live to age $\tau$.

Clearly, the functions $\varkappa$, $P$, $Q$, $r$ and $R$ are continuous and satisfy

$$
\begin{aligned}
\sup_{\tau \in J} |\varkappa(\tau)| &= \sup_{\tau \in J} \left|\frac{\tau \sin \tau}{7}\right| \leq \frac{2}{7}, \\
\left|r(\tau, P, I_1) - r\left(\tau, \overline{P}, \overline{I_1}\right)\right| &\leq \left|P - \overline{P}\right| + \frac{1}{4^5} \left|I_1 - \overline{I_1}\right|, \\
|P(\tau, \Phi_1(\tau)) - P(\tau, \Phi_2(\tau))| &\leq \frac{1}{6} |\Phi_1 - \Phi_2|, \\
\left|R\left(\tau, Q, \widetilde{I}_1\right) - R\left(\tau, \overline{Q}, \overline{\widetilde{I}_1}\right)\right| &\leq \left|Q - \overline{Q}\right| + \frac{1}{64} \left|\widetilde{I}_1 - \overline{\widetilde{I}_1}\right|, \\
|Q(\tau, \Phi_1(\tau)) - Q(\tau, \Phi_2(\tau))| &\leq \frac{1}{2} |\Phi_1 - \Phi_2|.
\end{aligned}
$$

Hence, $\gimel_1 = \frac{2}{7}$, $\gimel_2 = \gimel_5 = 1$, $\gimel_3 = \frac{1}{4^5}$, $\gimel_4 = \frac{1}{6}$, $\gimel_6 = \frac{1}{64}$ and $\gimel_7 = \frac{1}{8}$. If $\|Q\| \leq z_0$, one has

$$
P' = \frac{z_0}{6},\ Q' = \frac{z_0}{2},\ I_1' = \frac{z_0\left(3^8 \times 4^5\right)}{4^3 \Gamma(8)} e^{-2\left(\frac{1}{4}\right)} \left(2^{\left(\frac{4}{3}\right)} - 1\right)^8,\ \text{and } \widetilde{I}_1' = \frac{z_0 8 \left(2^{\frac{1}{4}} - 1\right)^{\frac{1}{8}}}{\Gamma\left(\frac{1}{8}\right)}.
$$

Additionally, the inequality created by Assumption $A_5$ transforms into

$$
\gimel_1 + (\gimel_2 \gimel_4 + \gimel_5 \gimel_7) z_0 + \gimel_3 \frac{\theta^{-\frac{\rho}{\varphi}} e^{\frac{\ell\left(S^\theta\right)}{(\ell - 1)}} \left(S^\theta - 1\right)^{\frac{\rho}{\varphi}}}{\rho \ell^{\frac{\rho}{\varphi}} \varphi^{\frac{\rho}{\varphi} - 1} \Gamma\left(\frac{\rho}{\varphi}\right)} z_0 + \gimel_6 \frac{S^{\xi\eta}}{\Gamma(\eta + 1)} z_0
$$

$$= \frac{2}{7} + \frac{7}{24}z_0 + \frac{z_0 3^8}{4^2\Gamma(8)} e^{-2\left(\frac{1}{4}\right)} \left(2^{\left(\frac{4}{3}\right)} - 1\right)^8 + \frac{z_0\left(2^{\frac{1}{4}} - 1\right)^{\frac{1}{8}}}{8\Gamma\left(\frac{1}{8}\right)} \leq z_0. \tag{5.2}$$

Taking $z_0 = 2$, we have

$$P' = \frac{1}{3},\ Q' = 1,\ I_1' = \frac{2\left(3^8 \times 4^5\right)}{4^3\Gamma(8)} e^{-2\left(\frac{1}{4}\right)} \left(2^{\left(\frac{4}{3}\right)} - 1\right)^8,\ \text{and}\ \widetilde{I}_1' = \frac{16\left(2^{\frac{1}{4}} - 1\right)^{\frac{1}{8}}}{\Gamma\left(\frac{1}{8}\right)}.$$

Hence, Eq (5.2) satisfies

$$\frac{2}{7} + \frac{14}{24} + \frac{2 \times 3^8}{4^3\Gamma(8)} e^{-2\left(\frac{1}{4}\right)} \left(2^{\left(\frac{4}{3}\right)} - 1\right)^8 + \frac{2\left(2^{\frac{1}{4}} - 1\right)^{\frac{1}{8}}}{8\Gamma\left(\frac{1}{8}\right)} \leq 2.$$

Therefore the assumptions $A_1$–$A_5$ of Theorem (2.4) are satisfied. Then, the problem given by Eq (5.1) has a solution on $C([1, 2])$.

**Example 5.2.** For more analysis, consider the following problem:

$$\Phi(\tau) = \frac{\tau e^{-\tau}}{4 + 2\tau} + \frac{\tau e^{-(1-\tau)^3}\Phi(\tau)}{5} + \frac{\sin\Phi(\tau)}{3 + \tau^2} + \frac{\left({}^{\frac{5}{3}}_{\frac{1}{4}}I_1^{3,\frac{1}{3}}\Phi\right)(\tau)}{3^{23}} + \frac{\left(\widetilde{I}^{\frac{-1}{3}}_{\frac{2}{3},1}\Phi\right)(\tau)}{5}, \tag{5.3}$$

where $\Phi(s)$ refers to the surge in the birthrate at any time $\tau$, $\tau \in J = [1, 2]$.

Equation (5.3) is a special case of Eq (1.2) with

$$\begin{aligned}
\varkappa(\tau) &= \frac{\tau e^{-\tau}}{4 + 2\tau}, \\
r(\tau, P, I_1) &= P(\tau, \Phi(\tau)) + \frac{I_1}{3^{23}}, \\
P(\tau, \Phi(\tau)) &= \frac{\tau e^{-(1-\tau)^3}\Phi(\tau)}{5}, \\
\left({}^{\frac{5}{3}}_{\frac{1}{4}}I_1^{3,\frac{1}{3}}\Phi\right)(\tau) &= \frac{3^{23} \times 4^{12}}{5^{11} \times \Gamma(12)} \int_1^\tau e^{-\left(\tau^{\frac{5}{3}} - s^{\frac{5}{3}}\right)} \left(\tau^{\frac{5}{3}} - s^{\frac{5}{3}}\right)^{11} s^{\frac{2}{3}}\Phi(s)\, ds, \\
\left(\widetilde{I}^{\frac{-1}{3}}_{\frac{2}{3},1}\Phi\right)(\tau) &= \frac{2}{3\Gamma(\frac{1}{3})} \int_1^\tau \frac{s^{\frac{-1}{3}}\Phi(s)}{\left(\tau^{\frac{2}{3}} - s^{\frac{2}{3}}\right)^{\frac{2}{3}}} ds, \\
R\left(\tau, Q, \widetilde{I}_1\right) &= Q(\tau, \Phi(\tau)) + \frac{\widetilde{I}_1}{5}, \\
Q(\tau, \Phi(\tau)) &= \frac{\sin\Phi(\tau)}{3 + \tau^2},
\end{aligned}$$

where $\Phi(s)$ is the possibility that the female will live to age $s$, $\varkappa(\tau)$, $P(\tau, \Phi(\tau))$, and $Q(\tau, \Phi(\tau))$ are the variables added to account for females born before the oldest child-bearing women of a certain age ($s = 2$) were born. The survival functions, $R$ and $r$, measure the proportion of people who live to age $\tau$.

Clearly, the functions $\varkappa$, $P$, $Q$, $r$ and $R$ are continuous and satisfy

$$\begin{aligned}
\sup_{\tau\in J}|\varkappa(\tau)| &= \sup_{\tau\in J}\left|\frac{\tau e^{-\tau}}{4+2\tau}\right| \leq \frac{1}{4}, \\
\left|r(\tau,P,I_1)-r\left(\tau,\overline{P},\overline{I_1}\right)\right| &\leq \left|P-\overline{P}\right|+\frac{1}{3^{23}}\left|I_1-\overline{I_1}\right|, \\
\left|P(\tau,\Phi_1(\tau))-P(\tau,\Phi_2(\tau))\right| &\leq \frac{1}{5}\left|\Phi_1-\Phi_2\right|, \\
\left|R\left(\tau,Q,\widetilde{I_1}\right)-R\left(\tau,\overline{Q},\overline{\overline{I_1}}\right)\right| &\leq \left|Q-\overline{Q}\right|+\frac{1}{5}\left|\widetilde{I_1}-\overline{\overline{I_1}}\right|, \\
\left|Q(\tau,\Phi_1(\tau))-Q(\tau,\Phi_2(\tau))\right| &\leq \frac{1}{4}\left|\Phi_1-\Phi_2\right|.
\end{aligned}$$

Thus, $\gimel_1=\frac{1}{4}$, $\gimel_2=\gimel_5=1$, $\gimel_3=\frac{1}{3^{23}}$, $\gimel_4=\frac{1}{5}$, $\gimel_6=\frac{1}{5}$ and $\gimel_7=\frac{1}{4}$. Taking $\|Q\|\leq z_0$, then

$$P'=\frac{z_0}{5},\ Q'=\frac{z_0}{4},\ I_1'=\frac{z_0\times 3^{24}\times 4^{12}}{60\times 5^{11}\times\Gamma(12)}e^{-(2)^{\frac{5}{3}}}\left(2^{\left(\frac{5}{3}\right)}-1\right)^{12},\ \text{and}\ \widetilde{I_1'}=\frac{z_0 9\left(2^{\frac{2}{3}}-1\right)^{\frac{1}{3}}}{3\Gamma(\frac{1}{3})}.$$

Furthermore, the inequality produced by $A_5$ changes into

$$\begin{aligned}
&\gimel_1+(\gimel_2\gimel_4+\gimel_5\gimel_7)z_0+\gimel_3\frac{\theta^{-\frac{\varrho}{\varphi}}e^{\frac{\ell(S^\theta)}{(\ell-1)}}\left(S^\theta-1\right)^{\frac{\varrho}{\varphi}}}{\rho\ell^{\frac{\varrho}{\varphi}}\varphi^{\frac{\varrho}{\varphi}-1}\Gamma\left(\frac{\varrho}{\varphi}\right)}z_0+\gimel_6\frac{S^{\xi\eta}}{\Gamma(\eta+1)}z_0 \\
=\ &\frac{1}{4}+\frac{9}{20}z_0+\frac{z_0\times 4^{12}}{60\times 5^{11}\times\Gamma(12)}e^{-(2)^{\frac{5}{3}}}\left(2^{\left(\frac{5}{3}\right)}-1\right)^{12}+\frac{z_0\left(2^{\frac{1}{4}}-1\right)^{\frac{1}{8}}}{40\Gamma\left(\frac{1}{8}\right)}\leq z_0. \qquad (5.4)
\end{aligned}$$

Taking $z_0=2$, we have

$$P'=\frac{2}{5},\ Q'=\frac{1}{2},\ I_1'=\frac{3^{24}\times 4^{12}}{30\times 5^{11}\times\Gamma(12)}e^{-(2)^{\frac{5}{3}}}\left(2^{\left(\frac{5}{3}\right)}-1\right)^{12},\ \text{and}\ \widetilde{I_1'}=\frac{18\left(2^{\frac{2}{3}}-1\right)^{\frac{1}{3}}}{3\Gamma(\frac{1}{3})}.$$

Hence, Eq (5.4) satisfies

$$\frac{1}{4}+\frac{9}{10}+\frac{4^{12}}{30\times 5^{11}\times\Gamma(12)}e^{-(2)^{\frac{5}{3}}}\left(2^{\left(\frac{5}{3}\right)}-1\right)^{12}+\frac{\left(2^{\frac{1}{4}}-1\right)^{\frac{1}{8}}}{20\Gamma\left(\frac{1}{8}\right)}\leq 2.$$

Therefore the assumptions $A_1$–$A_5$ of Theorem 2.4 are satisfied. Then, the problem given by (5.3) has a solution on $C([1,2])$.

## 6. Conclusions

In various research fields and engineering applications, problems involving the division of biological populations arise frequently. Models on population biology have contributed to studies on the dynamics of resistance in bacteria, viruses, and microparasites that appear in the treated host population and the individual treated host. Mixed Riemann-Liouville and Erdélyi-Kober fractional operators that arise in BPD issues constituted the subject of this report. For such challenges, we employed FP approaches to make sure that solutions existed. Additionally, a result on infectious disorders has been reported. Through practical examples, we have also provided several illustrative applications. In the future, we look forward to introducing the Caputo-Fabrizio fractional operators with non-singular kernels, the Caputo-Hadamard fractional operators under nonlocal anti-periodic integral boundary constraints, and the Langevin fractional operators with the $p$-Laplacian operators by using the EKF integral operators in the biological system given by Eq (1.2) with a different visualization of the variables $\Phi(s)$, $\varkappa(\tau)$, $P(\tau,\Phi(\tau))$, and $Q(\tau,\Phi(\tau))$. Finally, we plan to study the stability of the current system and the rest of the proposed systems, as the perception of stability for such models is still vague.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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All authors contributed equally and significantly in writing this article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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