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Research article

Characterization of (α,β) Jordan bi-derivations in prime rings

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Abstract: Let $\mathfrak{S}$ be a prime ring with automorphisms α,β. A bi-additive map $\mathfrak{D}$ is called an (α,β) Jordan bi-derivation if $\mathfrak{D}(k^2, s) = \mathfrak{D}(k, s)\alpha(k) + \beta(k)\mathfrak{D}(k, s)$. In this paper, we find conditions under which a symmetric (α,β) Jordan bi-derivation becomes a symmetric (α,β) bi-derivation. We also characterize the symmetric (α,β) Jordan bi-derivations.

Keywords: prime ring; Jordan derivation; symmetric (α,β) Jordan bi-derivation
Mathematics Subject Classification: 16W25, 16N60

1. Introduction

We consider $\mathfrak{S}$ as an associative ring throughout the paper, and $\mathfrak{Z}(\mathfrak{S})$ its center, unless otherwise mentioned. For all $k, l \in \mathfrak{S}$, if $k\mathfrak{S}l = \{0\}$ ($k\mathfrak{S}k = \{0\}$) implies that $k = 0$ or $l = 0$ ($k = 0$), then $\mathfrak{S}$ is called a prime ring (semiprime ring). For all $k, l \in \mathfrak{S}$, the representations $[k, l] = kl - lk$ and $k \circ l = kl + lk$ are termed as the commutator (Lie product) and the skew-commutator (Jordan product), respectively. For all $a \in \mathfrak{S}$, $\mathfrak{S}$ is called an n-torsion free ring if $na = 0$ implies $a = 0$. A map g is called a centralizing (resp. commuting) map on $\mathfrak{S}$ if $[g(k), l] \in \mathfrak{Z}(\mathfrak{S})$ (resp. $[g(k), l] = 0$) $\forall\ k, l \in \mathfrak{S}$. An additive map $f : \mathfrak{S} \to \mathfrak{S}$ satisfying $f(kl) = f(k)l + kf(l)$ (resp. $f(k^2) = f(k)k + kf(k)$) is termed as a derivation (resp. Jordan derivation) $\forall\ k, l \in \mathfrak{S}$. Clearly we can see that every derivation is a Jordan derivation, however the converse is not true in general. In 1957, Herstein [1, Theorem 4.1] showed that the Jordan derivation d of ring $\mathfrak{S}$ is a derivation if d satisfies $d(klk) = d(k)lk + kd(l)k + kld(k)$ for all $k, l \in \mathfrak{S}$, where $\mathfrak{S}$ is a prime ring which is not a commutative integral domain.

For a prime ring $\mathfrak{S}$, an additive map d from $\mathfrak{S}$ to $\mathfrak{S}$ is said to be a σ-derivation (resp. Jordan σ-derivation) with associated endomorphism σ if $d(kl) = d(k)l + \sigma(k)d(l)$ for all $k, l \in \mathfrak{S}$ (resp. $d(k^2) = d(k)k + \sigma(k)d(k)$). In 2015, Lee [2] proved that, for a non-commutative prime ring $\mathfrak{S}$, every Jordan

σ-derivation is a σ-derivation if it satisfies the condition $d(klk) = d(k)lk + \sigma(k)d(l)k + \sigma(kl)d(k)$ for all $k, l \in S$.

If a map d from $\mathfrak{S} \times \mathfrak{S}$ to $\mathfrak{S}$ is additive in both arguments, then it is called bi-additive and d is said to be symmetric if $d(k, l) = d(l, k)$. Let d be a symmetric bi-additive map satisfying the condition $d(kl, s) = d(k, s)l + kd(l, s)$ (resp. $d(k^2, s) = d(k, s)k + kd(k, s)$) for all $k, l, s \in \mathfrak{S}$. Then d is said to be a symmetric bi-derivation (resp. symmetric Jordan bi-derivation). In 1980 [3], Maksa was the first to introduce the idea of symmetric bi-derivation, and after that Vukman [4] proved some results related to symmetric bi-derivation using prime and semiprime rings. In 2017, Abdioğlu and Lee [5] studied a basic functional identity in the presence of bi-additive maps on a non-commutative prime ring $\mathfrak{S}$. Precisely, they have proved: "Let $\mathfrak{S}$ be a non-commutative prime ring and let $\mathfrak{B} : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{Q}_{ml}(\mathfrak{S})$ be a bi-additive map. Suppose that $[\mathfrak{B}(x, y), [x, y]] = 0$ for all $x, y \in \mathfrak{S}$. Then there exist $\lambda \in \mathfrak{C}$ and a bi-additive map $\beta : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{C}$ such that $\mathfrak{B}(x, y) = \lambda[x, y] + \beta(x, y)$ for all $x, y \in \mathfrak{S}$". We can also find some related results [2, Theorem 2.1] and their application in order to investigate Jordan σ-derivation.

Motivated by all these mentioned results, in this paper we defined symmetric (α,β) Jordan bi-derivation in prime rings. It is a map $\mathfrak{D}$ from $\mathfrak{S}\times\mathfrak{S}$ to $\mathfrak{S}$ defined by $\mathfrak{D}(k^2, s) = \mathfrak{D}(k, s)\alpha(k)+\beta(k)\mathfrak{D}(k, s)$ for all $k, s \in \mathfrak{S}$. Any symmetric (α,β) Jordan bi-derivation is symmetric (α,β) bi-derivation, but the converse is not true in general. In this article, we find the answer under which the map symmetric (α,β) Jordan bi-derivation becomes an (α,β) bi-derivation in prime rings $\mathfrak{S}$. Also, we characterize the symmetric (α,β) Jordan bi-derivation on any rings $\mathfrak{S}$.

2. Preliminaries

Lemma 2.1. [2, Theorem 2.1] *Let $\mathfrak{S}$ be a non-commutative prime ring and $f : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{Q}_{mr}(\mathfrak{S})$ be a bi-additive map. If $f(k, l)[k, l] = 0$ for all $k, l \in \mathfrak{S}$, then $f = 0$.*

3. Results

Theorem 3.1. *Let $\mathfrak{D}$ be a symmetric (α,β) Jordan bi-derivation on a ring $\mathfrak{S}$ with char$(\mathfrak{S}) \neq 2$. Consequently, the following claims are true for all $k, l, s \in \mathfrak{S}$.*

(1) $\mathfrak{D}(kl + lk, s) = \mathfrak{D}(k, s)\alpha(l) + \mathfrak{D}(l, s)\alpha(k) + \beta(l)\mathfrak{D}(k, s) + \beta(k)\mathfrak{D}(l, s)$.
(2) $\mathfrak{D}(klk, s) = \mathfrak{D}(k, s)\alpha(l)\alpha(k) + \beta(k)\beta(l)\mathfrak{D}(k, s) + \beta(k)\mathfrak{D}(l, s)\alpha(k)$.
(3) $\mathfrak{D}(klr + rlk, s) = \mathfrak{D}(k, s)\alpha(l)\alpha(r) + \beta(k)\mathfrak{D}(l, s)\alpha(r) + \mathfrak{D}(r, s)\alpha(l)\alpha(k) + \beta(r)\mathfrak{D}(l, s)\alpha(k) + \beta(r)\beta(l)\mathfrak{D}(k, s) + \beta(k)\beta(l)\mathfrak{D}(r, s)$.
(4) $\mathfrak{D}(klr + lrk, s) = \mathfrak{D}(lr, s)\alpha(k) + \mathfrak{D}(k, s)\alpha(l)\alpha(r) + \beta(k)\mathfrak{D}(lr, s) + \beta(l)\beta(r)\mathfrak{D}(k, s)$.
(5) $\mathfrak{D}(l^2, s)\alpha(k) + \beta(k)\mathfrak{D}(l^2, s) = \mathfrak{D}(l, s)\alpha(l)\alpha(k) + \beta(k)\mathfrak{D}(l, s)\alpha(l) + \beta(l)\mathfrak{D}(l, s)\alpha(k) + \beta(k)\mathfrak{D}(l, s)\alpha(l) + \beta(k)\beta(l)\mathfrak{D}(l, s)$.

Proof. (1) In the definition of symmetric (α,β) Jordan bi-derivation, substituting k by $k + l$, we get

$$\begin{aligned} \mathfrak{D}((k + l)^2, s) &= \mathfrak{D}(k + l, s)\alpha(k + l) + \beta(k + l)\mathfrak{D}(k + l, s) \\ &= \mathfrak{D}(k, s)\alpha(k) + \mathfrak{D}(k, s)\alpha(l) + \mathfrak{D}(l, s)\alpha(k) \\ &+ \mathfrak{D}(l, s)\alpha(l) + \beta(k)\mathfrak{D}(k, s) + \beta(l)\mathfrak{D}(k, s) \\ &+ \beta(k)\mathfrak{D}(l, s) + \beta(l)\mathfrak{D}(l, s). \end{aligned} \tag{3.1}$$

Also, we have

$$\begin{aligned}\mathfrak{D}((k+l)^2, s) &= \mathfrak{D}(k^2 + kl + lk + l^2, s)\\ &= \mathfrak{D}(k^2, s) + \mathfrak{D}(kl + lk, s) + \mathfrak{D}(l^2, s)\\ &= \mathfrak{D}(k, s)\alpha(k) + \beta(k)\mathfrak{D}(k, s) + \mathfrak{D}(kl + lk, s)\\ &+ \mathfrak{D}(l, s)\alpha(l) + \beta(l)\mathfrak{D}(l, s).\end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we conclude that

$$\mathfrak{D}(kl + lk, s) = \mathfrak{D}(k, s)\alpha(l) + \mathfrak{D}(l, s)\alpha(k) + \beta(l)\mathfrak{D}(k, s) + \beta(k)\mathfrak{D}(l, s). \tag{3.3}$$

(2) Replacing l by $kl + lk$ in (3.3), we obtain

$$\begin{aligned}\mathfrak{D}(k(kl + lk) + (kl + lk)k, s) &= \mathfrak{D}(k, s)\alpha(kl + lk) + \mathfrak{D}(kl + lk, s)\alpha(k)\\ &+ \beta(kl + lk)\mathfrak{D}(k, s) + \beta(k)\mathfrak{D}(kl + lk, s).\end{aligned} \tag{3.4}$$

Using Theorem 3.1 (1) in the right hand side of (3.4), after simplifying we get

$$\begin{aligned}&\mathfrak{D}(k, s)\alpha(k)\alpha(l) + \mathfrak{D}(k, s)\alpha(l)\alpha(k) + \mathfrak{D}(k, s)\alpha(l)\alpha(k) + \beta(k)\mathfrak{D}(l, s)\alpha(k)\\ &+\mathfrak{D}(l, s)\alpha(k)\alpha(k) + \beta(l)\mathfrak{D}(k, s)\alpha(k) + \beta(k)\beta(l)\mathfrak{D}(k, s) + \beta(l)\beta(k)\mathfrak{D}(k, s)\\ &+\beta(k)\mathfrak{D}(k, s)\alpha(l) + \beta(k)\beta(k)\mathfrak{D}(l, s) + \beta(k)\mathfrak{D}(l, s)\alpha(k) + \beta(k)\beta(l)\mathfrak{D}(k, s).\end{aligned} \tag{3.5}$$

Now, solving the left side of (3.4) and using (3.3), we have

$$\begin{aligned}\mathfrak{D}(k(kl + lk) + (kl + lk)k, s) &= \mathfrak{D}(k^2 l + klk + klk + lk^2, s)\\ &= \mathfrak{D}(k^2 l + lk^2, s) + 2\mathfrak{D}(klk, s)\\ &= \mathfrak{D}(k^2, s)\alpha(l) + \beta(k^2)\mathfrak{D}(l, s) + \mathfrak{D}(l, s)\alpha(k^2)\\ &+ \beta(l)\mathfrak{D}(k^2, s) + 2\mathfrak{D}(klk, s)\\ &= \mathfrak{D}(k, s)\alpha(k)\alpha(l) + \beta(k)\mathfrak{D}(k, s)\alpha(l) + \beta(k)\beta(k)\mathfrak{D}(l, s)\\ &+ \mathfrak{D}(l, s)\alpha(k)\alpha(k) + \beta(l)\mathfrak{D}(k, s)\alpha(k) + \beta(l)\beta(k)\mathfrak{D}(k, s)\\ &+ 2\mathfrak{D}(klk, s).\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we see that

$$2\mathfrak{D}(klk, s) = 2\mathfrak{D}(k, s)\alpha(l)\alpha(k) + 2\beta(k)\beta(l)\mathfrak{D}(k, s) + 2\beta(k)\mathfrak{D}(l, s)\alpha(k). \tag{3.7}$$

Since $\mathrm{char}(\mathfrak{S}) \neq 2$, then from the last relation we get the required result:

$$\mathfrak{D}(klk, s) = \mathfrak{D}(k, s)\alpha(l)\alpha(k) + \beta(k)\beta(l)\mathfrak{D}(k, s) + \beta(k)\mathfrak{D}(l, s)\alpha(k).$$

(3) Substituting k by $k + r$ in (2) for all $r \in \mathfrak{S}$, we have

$$\begin{aligned}\mathfrak{D}((k + r)l(k + r), s) &= \mathfrak{D}(k + r, s)\alpha(l)\alpha(k + r) + \beta(k + r)\beta(l)\mathfrak{D}(k + r, s)\\ &+ \beta(k + r)\mathfrak{D}(l, s)\alpha(k + r).\end{aligned} \tag{3.8}$$

Now, solving the left side of (3.8), we obtain

$$\begin{aligned}&\mathfrak{D}((k+r)l(k+r),s) = \mathfrak{D}(klk+klr+rlk+rlr,s) = \mathfrak{D}(klk,s)\\&+\mathfrak{D}(klr+rlk,s)+\mathfrak{D}(rlr,s) = \mathfrak{D}(k,s)\alpha(l)\alpha(k)+\beta(k)\beta(l)\mathfrak{D}(k,s)\\&+\beta(k)\mathfrak{D}(l,s)\alpha(k)+\mathfrak{D}(klr+rlk,s)+\mathfrak{D}(r,s)\alpha(l)\alpha(r)+\beta(r)\beta(l)\mathfrak{D}(r,s)\\&+\beta(r)\mathfrak{D}(l,s)\alpha(r).\end{aligned}\tag{3.9}$$

Now, solving the right side of (3.8), we obtain

$$\begin{aligned}&\mathfrak{D}(k,s)\alpha(l)\alpha(k)+\mathfrak{D}(r,s)\alpha(l)\alpha(k)+\mathfrak{D}(k,s)\alpha(l)\alpha(r)+\mathfrak{D}(r,s)\alpha(l)\alpha(r)\\&+\beta(k)\beta(l)\mathfrak{D}(k,s)+\beta(r)\beta(l)\mathfrak{D}(k,s)+\beta(k)\beta(l)\mathfrak{D}(r,s)+\beta(r)\beta(l)\mathfrak{D}(r,s)\\&+\beta(k)\mathfrak{D}(l,s)\alpha(k)+\beta(r)\mathfrak{D}(l,s)\alpha(k)+\beta(k)\mathfrak{D}(l,s)\alpha(r)+\beta(r)\mathfrak{D}(l,s)\alpha(r).\end{aligned}\tag{3.10}$$

From (3.9) and (3.10), we find that

$$\begin{aligned}\mathfrak{D}(klr+rlk,s) =&\mathfrak{D}(r,s)\alpha(l)\alpha(k)+\mathfrak{D}(k,s)\alpha(l)\alpha(r)+\beta(r)\beta(l)\mathfrak{D}(k,s)\\&+\beta(k)\beta(l)\mathfrak{D}(r,s)+\beta(r)\mathfrak{D}(l,s)\alpha(k)+\beta(k)\mathfrak{D}(l,s)\alpha(r).\end{aligned}\tag{3.11}$$

This is the required result.
(4) Substituting k by $k+lr$ in the definition of (α,β) Jordan bi-derivation, we get

$$\mathfrak{D}((k+lr)^2,s) = \mathfrak{D}(k+lr,s)\alpha(k+lr)+\beta(k+lr)\mathfrak{D}(k+lr,s).\tag{3.12}$$

Solving the left side of (3.12) and using the definition of (α,β) Jordan bi-derivation, we see that

$$\begin{aligned}\mathfrak{D}((k+lr)^2,s) &= \mathfrak{D}(k^2+klr+lrk+(lr)^2,s)\\&= \mathfrak{D}(k^2,s)+\mathfrak{D}(klr+lrk,s)+\mathfrak{D}((lr)^2,s)\\&= \mathfrak{D}(k,s)\alpha(k)+\beta(k)\mathfrak{D}(k,s)+\mathfrak{D}(klr+lrk,s)\\&+\mathfrak{D}(lr,s)\alpha(lr)+\beta(lr)\mathfrak{D}(lr,s).\end{aligned}\tag{3.13}$$

Now, solving the right side of (3.12) and using the definition of (α,β) Jordan bi-derivation, we find that

$$\begin{aligned}&\mathfrak{D}(k+lr,s)\alpha(k+lr)+\beta(k+lr)\mathfrak{D}(k+lr,s)\\=&\mathfrak{D}(k,s)\alpha(k)+\mathfrak{D}(k,s)\alpha(lr)+\mathfrak{D}(lr,s)\alpha(k)\\&+\mathfrak{D}(lr,s)\alpha(lr)+\beta(k)\mathfrak{D}(k,s)+\beta(lr)\mathfrak{D}(k,s)\\&+\beta(k)\mathfrak{D}(lr,s)+\beta(lr)\mathfrak{D}(lr,s).\end{aligned}\tag{3.14}$$

Comparing (3.13) and (3.14), we have

$$\begin{aligned}\mathfrak{D}(klr+lrk,s) = &\mathfrak{D}(k,s)\alpha(lr)+\mathfrak{D}(lr,s)\alpha(k)\\&+\beta(lr)\mathfrak{D}(k,s)+\beta(k)\mathfrak{D}(lr,s).\end{aligned}\tag{3.15}$$

We are done.
(5) Subtracting (3.15) from (3.11), we get

$$\mathfrak{D}([r,l]k,s) = \beta(k)\mathfrak{D}(l,s)\alpha(r)+\mathfrak{D}(r,s)\alpha(l)\alpha(k)+\beta(r)\mathfrak{D}(l,s)\alpha(k)$$

$$+[\beta(r),\beta(l)]\mathfrak{D}(k,s)+\beta(k)\beta(l)\mathfrak{D}(r,s)-\mathfrak{D}(lr,s)\alpha(k)-\beta(k)\mathfrak{D}(lr,s). \tag{3.16}$$

In particular, for $r = l$, (3.16) reduces to

$$\begin{aligned}0 =&\beta(k)\mathfrak{D}(l,s)\alpha(l)+\mathfrak{D}(l,s)\alpha(l)\alpha(k)+\beta(l)\mathfrak{D}(l,s)\alpha(k)\\&+\beta(k)\beta(l)\mathfrak{D}(l,s)-\mathfrak{D}(l^2,s)\alpha(k)-\beta(k)\mathfrak{D}(l^2,s),\end{aligned} \tag{3.17}$$

which implies that

$$\begin{aligned}\mathfrak{D}(l^2,s)\alpha(k)+\beta(k)\mathfrak{D}(l^2,s) =& \beta(k)\mathfrak{D}(l,s)\alpha(l)+\mathfrak{D}(l,s)\alpha(l)\alpha(k)\\&+\beta(l)\mathfrak{D}(l,s)\alpha(k)+\beta(k)\beta(l)\mathfrak{D}(l,s).\end{aligned} \tag{3.18}$$

So, the last equality is done. Hence, with this proof our theorem is completed. □

Definition 3.2. Let $\mathfrak{S}$ be a ring with char($\mathfrak{S}$) $\neq 2$. We define the symbol

$$k^l = \mathfrak{D}(l,s)\alpha(k)+\beta(l)\mathfrak{D}(k,s)-\mathfrak{D}(kl,s)$$

for all $k, l, s \in \mathfrak{S}$.

Now we present a theorem regarding k^l under the action of symmetric (α,β) Jordan bi-derivation.

Theorem 3.3. *Let $\mathfrak{D}$ be a symmetric (α,β) Jordan bi-derivation on a ring $\mathfrak{S}$. Consequently, the following claims are true for all $k, k_i, l, l_i, s \in \mathfrak{S}$, for $i \in \{1, 2\}$.*

(1) $k^l + l^k = 0$.
(2) $k^{l_1+l_2} = k^{l_1} + k^{l_2}$.
(3) $(k_1 + k_2)^l = k_1^l + k_2^l$.

Proof. (1) Let $k, l, s \in \mathfrak{S}$. Then, from Definition 3.2, we have

$$\begin{aligned}k^l + l^k =& \mathfrak{D}(l,s)\alpha(k)+\beta(l)\mathfrak{D}(k,s)-\mathfrak{D}(kl,s)\\&+\mathfrak{D}(k,s)\alpha(l)+\beta(k)\mathfrak{D}(l,s)-\mathfrak{D}(lk,s).\end{aligned} \tag{3.19}$$

The above relation can be re-written as

$$\begin{aligned}k^l + l^k =& \mathfrak{D}(l,s)\alpha(k)+\beta(l)\mathfrak{D}(k,s)+\mathfrak{D}(k,s)\alpha(l)\\&+\beta(k)\mathfrak{D}(l,s)-\mathfrak{D}(kl+lk,s).\end{aligned} \tag{3.20}$$

Using (2) of Theorem 3.1 in (3.20), we get

$$\begin{aligned}k^l + l^k =& \mathfrak{D}(l,s)\alpha(k)+\beta(l)\mathfrak{D}(k,s)+\mathfrak{D}(k,s)\alpha(l)\\&+\beta(k)\mathfrak{D}(l,s)-\mathfrak{D}(k,s)\alpha(l)-\mathfrak{D}(l,s)\alpha(k)\\&-\beta(k)\mathfrak{D}(l,s)-\beta(l)\mathfrak{D}(k,s).\end{aligned} \tag{3.21}$$

Solving (3.21), we get

$$k^l + l^k = 0.$$

(2) Let $k, l_1, l_2, s \in \mathfrak{S}$. Then, from Definition 3.2, we have

$$\begin{aligned} k^{l_1+l_2} &= \mathfrak{D}(l_1 + l_2, s)\alpha(k) + \beta(l_1 + l_2)\mathfrak{D}(k, s) - \mathfrak{D}(k(l_1 + l_2), s) \\ &= \mathfrak{D}(l_1, s)\alpha(k) + \mathfrak{D}(l_2, s)\alpha(k) + \beta(l_1)\mathfrak{D}(k, s) \\ &+ \beta(l_2)\mathfrak{D}(k, s) - \mathfrak{D}(kl_1, s) - \mathfrak{D}(kl_2, s). \end{aligned} \tag{3.22}$$

The above relation can be re-written as

$$\begin{aligned} k^{l_1+l_2} &= \mathfrak{D}(l_1, s)\alpha(k) + \beta(l_1)\mathfrak{D}(k, s) - \mathfrak{D}(kl_1, s) \\ &+ \mathfrak{D}(l_2, s)\alpha(k) + \beta(l_2)\mathfrak{D}(k, s) - \mathfrak{D}(kl_2, s). \end{aligned} \tag{3.23}$$

Using Definition 3.2 in (3.23), we find that

$$k^{l_1+l_2} = k^{l_1} + k^{l_2}. \tag{3.24}$$

(3) Let $k_1, k_2, l, s \in \mathfrak{S}$. Then, from the Definition 3.2, we have

$$\begin{aligned} (k_1 + k_2)^l &= \mathfrak{D}(l, s)\alpha(k_1 + k_2) + \beta(l)\mathfrak{D}(k_1 + k_2, s) - \mathfrak{D}((k_1 + k_2)l, s) \\ &= \mathfrak{D}(l, s)\alpha(l) + \mathfrak{D}(l, s)\alpha(k_2) + \beta(l)\mathfrak{D}(k_1, s) \\ &+ \beta(l)\mathfrak{D}(k_2, s) - \mathfrak{D}(k_1 l + k_2 l, s). \end{aligned} \tag{3.25}$$

The above relation can be re-written as

$$\begin{aligned} (k_1 + k_2)^l &= \mathfrak{D}(l, s)\alpha(k_1) + \beta(l)\mathfrak{D}(k_1, s) - \mathfrak{D}(k_1 l, s) \\ &+ \mathfrak{D}(l, s)\alpha(k_2) + \beta(l)\mathfrak{D}(k_2, s) - \mathfrak{D}(k_2 l, s). \end{aligned} \tag{3.26}$$

Using Definition 3.2 in (3.26), we see that

$$(k_1 + k_2)^l = k_1^l + k_2^l. \tag{3.27}$$

□

Since we know that every symmetric (α,β) bi-derivation of $\mathfrak{S}$ is always a symmetric (α,β) Jordan bi-derivation of $\mathfrak{S}$, the converse need not be true. So, in our next theorem we will show that a symmetric (α,β) Jordan bi-derivation of $\mathfrak{S}$ becomes a symmetric (α,β) bi-derivation of $\mathfrak{S}$ by imposing certain conditions on ring $\mathfrak{S}$.

Theorem 3.4. *Let $\mathfrak{S}$ be a non-commutative prime ring with automorphisms α and β. Any symmetric (α,β) Jordan bi-derivation $\mathfrak{D}$ of $\mathfrak{S}$ which satisfies the properties* (2) *and* (3) *of Theorem 3.1 (it is trivial if char ($\mathfrak{S}$) $\neq 2$), $\mathfrak{D}$ is a symmetric (α,β) bi-derivation of $\mathfrak{S}$.*

Proof. Substituting r by kl in (3) of Theorem 3.1, we get

$$\begin{aligned} \mathfrak{D}(klkl + kllk, s) &= \mathfrak{D}(kl, s)\alpha(l)\alpha(k) + \mathfrak{D}(k, s)\alpha(l)\alpha(kl) + \beta(kl)\beta(l)\mathfrak{D}(k, s) \\ &+ \beta(k)\beta(l)\mathfrak{D}(kl, s) + \beta(kl)\mathfrak{D}(l, s)\alpha(k) + \beta(k)\mathfrak{D}(l, s)\alpha(kl). \end{aligned} \tag{3.28}$$

We solve this equation in two parts. In the first part we solve the right side of (3.28), and in second we solve the left side of (3.28). Solving right side, we get

$$\begin{aligned}&\mathfrak{D}(kl,s)\alpha(l)\alpha(k)+\mathfrak{D}(k,s)\alpha(l)\alpha(k)\alpha(l)+\beta(k)\beta(l)\beta(l)\mathfrak{D}(k,s)\\&+\beta(k)\beta(l)\mathfrak{D}(kl,s)+\beta(k)\beta(l)\mathfrak{D}(l,s)\alpha(k)+\beta(k)\mathfrak{D}(l,s)\alpha(k)\alpha(l).\end{aligned} \tag{3.29}$$

Now, solving the left side of (3.28), we find that

$$\mathfrak{D}(klkl+kllk,s)=\mathfrak{D}((kl)^2+kl^2k,s)=\mathfrak{D}((kl)^2,s)+\mathfrak{D}(kl^2k,s). \tag{3.30}$$

Using the definition of (α,β) Jordan bi-derivation and (2) of Theorem 3.1 in (3.30), we see that

$$\begin{aligned}&\mathfrak{D}(kl,s)\alpha(k)\alpha(l)+\beta(k)\beta(l)\mathfrak{D}(kl,s)+\mathfrak{D}(k,s)\alpha(l)\alpha(l)\alpha(k)\\&+\beta(k)\beta(l)\beta(l)\mathfrak{D}(k,s)+\beta(k)(\mathfrak{D}(l^2,s))\alpha(k).\end{aligned} \tag{3.31}$$

The above relation can be re-written as

$$\begin{aligned}&\mathfrak{D}(kl,s)\alpha(k)\alpha(l)+\beta(k)\beta(l)\mathfrak{D}(kl,s)+\mathfrak{D}(k,s)\alpha(l)\alpha(l)\alpha(k)\\&+\beta(k)\beta(l)\beta(l)\mathfrak{D}(k,s)+\beta(k)\mathfrak{D}(l,s)\alpha(l)\alpha(k)+\beta(k)\beta(l)\mathfrak{D}(l,s)\alpha(k).\end{aligned} \tag{3.32}$$

Comparing (3.29) and (3.32), we find that

$$\begin{aligned}&\mathfrak{D}(k,s)\alpha(l)\alpha(k)\alpha(l)+\beta(k)\mathfrak{D}(l,s)\alpha(k)\alpha(l)+\mathfrak{D}(kl,s)\alpha(l)\alpha(k)\\&=\mathfrak{D}(kl,s)\alpha(k)\alpha(l)+\mathfrak{D}(k,s)\alpha(l)\alpha(l)\alpha(k)+\beta(k)\mathfrak{D}(l,s)\alpha(l)\alpha(k).\end{aligned}$$

The above equation can be re-arranged as

$$\mathfrak{D}(k,s)\alpha(l)[\alpha(k),\alpha(l)]+\beta(k)\mathfrak{D}(l,s)[\alpha(k),\alpha(l)]-\mathfrak{D}(kl,s)[\alpha(k),\alpha(l)]=0. \tag{3.33}$$

This implies that

$$(\mathfrak{D}(k,s)\alpha(l)+\beta(k)\mathfrak{D}(l,s)-\mathfrak{D}(kl,s))[\alpha(k),\alpha(l)]=0. \tag{3.34}$$

By using Lemma 2.1 in (3.34), we obtain

$$(\mathfrak{D}(k,s)\alpha(l)+\beta(k)\mathfrak{D}(l,s)-\mathfrak{D}(kl,s))=0. \tag{3.35}$$

That is,

$$\mathfrak{D}(kl,s)=\mathfrak{D}(k,s)\alpha(l)+\beta(k)\mathfrak{D}(l,s). \tag{3.36}$$

So, the above equation is a symmetric (α,β) bi-derivation of $\mathfrak{S}$, and we are done. □

4. Conclusions

In this paper, we focused on symmetric (α,β) Jordan biderivation $\mathfrak{D}$ on a prime ring $\mathfrak{S}$, where α and β are automorphisms of $\mathfrak{S}$. In Theorem 3.1, we have defined certain properties of (α,β) Jordan biderivation $\mathfrak{D}$ on a ring $\mathfrak{S}$. Finally, we have provided the conditions under which the symmetric (α,β) Jordan biderivation $\mathfrak{D}$ transforms into a symmetric (α,β) biderivation on a prime ring $\mathfrak{S}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no competing interests to declare that are relevant to the content of this article.

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