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Research article

Symmetry analysis of the canonical connection on Lie groups: six-dimensional case with abelian nilradical and one-dimensional center

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Abstract: In this article, the investigation into the Lie symmetry algebra of the geodesic equations of the canonical connection on a Lie group was continued. The key ideas of Lie group, Lie algebra, linear connection, and symmetry were quickly reviewed. The focus was on those Lie groups whose Lie algebra was six-dimensional solvable and indecomposable and for which the nilradical was abelian and had a one-dimensional center. Based on the list of Lie algebras compiled by Turkowski, there were eight algebras to consider that were denoted by $A_{6,20}$–$A_{6,27}$. For each Lie algebra, a comprehensive symmetry analysis of the system of geodesic equations was carried out. For each symmetry Lie algebra, the nilradical and a complement to the nilradical inside the radical, as well as a semi-simple factor, were identified.

Keywords: Lie symmetry; Lie group; canonical connection; geodesic system
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1. Introduction

Any Lie group is equipped with a natural linear connection ∇, and therefore, a canonical system of geodesic equations. This connection was introduced in 1926 by Cartan and Schouten [1]. Recently, a lot of work has been done on the symmetries of the geodesic equations of the canonical connection. Ghanam and Thompson considered the problem for all three and four-dimensional indecomposable Lie algebras [2]. They also considered six-dimensional nilpotent Lie algebras [3]. Almusawa et al. [4] considered the probelm for the five-dimensional indecomposable Lie algebras with co-dimension one abelian nilradical.

Recently, Almutiben et al. considered the problem for the case where the nilradical is of co-dimension two. In dimension four, there is only one such indecompsable Lie algebra with co-dimensional two nilradical, namely, $A_{4,12}$ in the Winternitz list [5]. In dimension five, there are three five-dimensional Lie algebras with co-dimension two abelian nilradical. These algebras are $A_{5,33}$–$A_{5,35}$ in [5]. In all these cases, a comprehensive analysis of the symmetries of the geodesic equations was performed. Almutiben et al. [6] also considered the problem for the six-dimensional solvable indecomposable Lie algebras. Following the classification given by Turkowski [7], there are forty classes of non-isomorphic six-dimensional Lie algebras. Among these forty algebras, the first nineteen $A_{6,1}$–$A_{6,19}$ have a four-dimensional, or equivalently co-dimension two, abelian nilradical and a two-dimensional abelian complement. Almutiben et al has given a comprehensive analysis of the symmetries in these nineteen cases [6] .

In this paper, we continue to study the symmetries corresponding to the eight algebras $A_{6,20}$–$A_{6,27}$ in [7]. These algebras are characterized by the property that they have a four-dimensional abelian nilradical and a one-dimensional center.

An outline of the paper is as follows: In Section 2, we provide some background material that helps to motivate our analysis. We do not give very specific details, but do provide some useful references. In Section 3, we give the definition of the canonical connection ∇ on a Lie group and a summary of its properties. In Section 4, we review the symmetries of differential equations and the Lie invariance condition. In Section 5, for each algebra $A_{6,20}$–$A_{6,27}$ in Turkowski's list, we give the geodesic equations, a basis for the symmetry algebra in terms of vector fields, and, finally, we identify the symmetry Lie algebra in terms of the nilradical and its complement. We will use $\rtimes$ to denote a semi-direct product and $\oplus$ for the direct sum of algebras.

2. Background concepts

In order to motivate some of the material, we shall sketch a few of the key ideas encountered below. We shall be considering certain systems of second order ordinary differential equations. The space of independent variables that occur serve as a system of local coordinates on a Lie group G. We shall take for granted the basic definitions and properties of Lie groups. One may think of a Lie group as being an object that is intermediate between a vector space and a differentiable manifold. In particular, on a Lie group, one may make sense of various geometric objects (vector fields and one-forms primarily) as being left or right invariant. We refer the reader to [8–10] for readable introductions to the topic, that will be helpful in understanding the present article. In addition, these references help to explain the relationship between Lie groups and Lie algebras in a pragmatic way. Although the differential equations treated here technically “live” on a Lie group, in practice, all of our calculations are done at the Lie algebra level. Another more advanced source that covers the same material is [11].

As regarding precise definitions related to Lie algebras, we refer in the first instance to [12] and also to [11]. For a *solvable* Lie algebra, one should think roughly of a subspace of upper triangular matrices and for a *nilpotent* Lie algebra, a subspace of the strictly upper triangular matrices. Nonetheless, abelian sub-algebras are nilpotent, so subspaces of diagonal matrices are also nilpotent.

An important construct that we shall make use of is the semi-direct product of Lie algebras. The idea can be understood in various ways, but perhaps the simplest is to say that an algebra is a semi-direct product of Lie algebras if it is a vector space direct sum of a sub-algebra and an ideal. Solvable, not nilpotent, Lie algebras are only semi-direct products when there is an *abelian* complement to the nilradical. In this article, we shall be concerned with the Lie algebras $A_{6,20}$–$A_{6,27}$ in [7]. Of these eight classes, only three, $A_{6,22}, A_{6,23}, A_{6,27}$ for which $\epsilon = 0$, are semi-direct products. However, we shall see the appearance of semi-direct products again when we analyze the symmetry algebras in Section 5. In general, a symmetry algebra need not be solvable, but rather will have a Levi decomposition, that is, it will be a semi-direct product of a solvable ideal (that itself may or may not be a semi-direct product) and a semi-simple sub-algebra. All of the algebras $A_{6,20}$–$A_{6,27}$ studied in Section 5, produce semi-simple sub-algebras.

Concerning the definition of a linear connection, one may refer to [11] among a host of many excellent references. In relation to the current paper, one really only needs to understand that a linear connection produces a system of second order ordinary differential equations, the geodesics. These systems are similar to equations encountered in particle mechanics; the simplest example arises from the flat connection on Euclidean space (in arbitrary dimension), and the corresponding differential equations are the equations of motion of a free particle. More general connections introduce, as well as second order terms, first order terms that are quadratic in velocities.

Finally, we come to the notion of symmetry of a differential equation. Lie's original idea was that a differential equation that could be integrated explicitly must possess an underlying symmetry. By the term "symmetry", we understand a change of variables may be both independent and dependent variables, such that after applying a finite transformation, the differential equation remains invariant. For a determined system of ordinary differential equations, and later, partial differential equations, the set of such symmetries comprises what was to become known as a Lie transformation group. Very quickly it was realized that the underlying structure need not be associated to a differential equation at all, and led to the idea of an abstract Lie group. It was also understood by Lie and his contemporaries, that it would be virtually impossible to calculate Lie transformation groups explicitly, even in some of the simplest cases. That circumstance led Lie to another great insight: that it would be far easier to work at the infinitesimal level and find not the Lie group, but rather its Lie algebra. In fact, Lie frequently uses the term "group", whereas today we would be more careful and refer to the "Lie algebra".

In this work, Lie groups and Lie algebras appear at two levels. First of all, the differential equations that we study constitute an intrinsic part of the Lie group on which they are defined. Second, the set of symmetries of the differential equations itself forms a Lie group. However, the relationship between the two Lie groups and, more importantly, their associated Lie algebras is not a simple one in general. It is only in the case where the first Lie algebra has a trivial center that one can be sure that the first Lie algebra is isomorphic to a sub-algebra of the second; the sub-algebra in question is then either the algebra of left or right-invariant vector fields. In fact, the Lie algebras studied below in Section 5 all have a one-dimensional center.

3. The canonical connection on Lie groups

On left-invariant vector fields X and Y, the canonical symmetric connection ∇ on a Lie group G is defined by

$$\nabla_X Y = \frac{1}{2}\,[X, Y], \tag{3.1}$$

and then extended to arbitrary vector fields using linearity and the Leibnitz rule. The connection ∇ is left-invariant. One could just as well use right-invariant vector fields to define ∇, but one must check that ∇ is well-defined. Properties of the canonical connection have been studied in [11], and we will summarize them in the following proposition:

Proposition 1. *For the canonical connection defined by* (3.1)*:*

(1) The torsion is zero.
(2) The curvature tensor R is given by $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$.
(3) The curvature tensor R is covariantly constant.
(4) The curvature tensor R is zero if, and only if, the Lie algebra is two-step nilpotent.
(5) The Ricci tensor is symmetric and in fact a multiple of the Killing form.
(6) The Ricci tensor is bi-invariant.

4. Symmetries of the geodesic equations

In this section, we explain the algorithm for finding the Lie symmetries of the geodesic equations. In local coordinates and in dimension n, the geodesic equations are given by

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \tag{4.1}$$

where Γ^i_{jk} are the connection components or Christoffel symbols, where $i, j, k = 1, ..., n$. In dimension six, let's take our coordinates to be t, p, q, x, y, z, w, where t is the independent variable and p, q, x, y, z, w are the dependant variables, so are functions of t. Define Γ to be

$$\Gamma = T\frac{\partial}{\partial t} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q} + X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z} + W\frac{\partial}{\partial w}, \tag{4.2}$$

where T, P, Q, X, Y, Z, and W are unknown functions of (t, p, q, x, y, z, w). The first prolongation Γ^1 and second prolongation Γ^2 of Γ are given by

$$\Gamma^1 = \Gamma + P_t\frac{\partial}{\partial \dot{p}} + Q_t\frac{\partial}{\partial \dot{q}} + X_t\frac{\partial}{\partial \dot{x}} + Y_t\frac{\partial}{\partial \dot{y}} + Z_t\frac{\partial}{\partial \dot{z}} + W_t\frac{\partial}{\partial \dot{w}}, \tag{4.3}$$

$$\Gamma^2 = \Gamma^1 + P_{tt}\frac{\partial}{\partial \ddot{p}} + Q_{tt}\frac{\partial}{\partial \ddot{q}} + X_{tt}\frac{\partial}{\partial \ddot{x}} + Y_{tt}\frac{\partial}{\partial \ddot{y}} + Z_{tt}\frac{\partial}{\partial \ddot{z}} + W_{tt}\frac{\partial}{\partial \ddot{w}}, \tag{4.4}$$

where

$$\begin{aligned} P_t &= D_t(P) - \dot{p}D_t(T), & P_{tt} &= D_t(P_t) - \ddot{p}D_t(T), \\ Q_t &= D_t(Q) - \dot{q}D_t(T), & Q_{tt} &= D_t(Q_t) - \ddot{q}D_t(T), \\ X_t &= D_t(X) - \dot{x}D_t(T), & X_{tt} &= D_t(X_t) - \ddot{x}D_t(T), \\ Y_t &= D_t(Y) - \dot{y}D_t(T), & Y_{tt} &= D_t(Y_t) - \ddot{y}D_t(T), \\ Z_t &= D_t(Z) - \dot{z}D_t(T), & Z_{tt} &= D_t(Z_t) - \ddot{z}D_t(T), \\ W_t &= D_t(W) - \dot{w}D_t(T), & W_{tt} &= D_t(W_t) - \ddot{w}D_t(T), \end{aligned} \tag{4.5}$$

and D_t is given by

$$D_t = \frac{\partial}{\partial t} + \dot{p}\frac{\partial}{\partial p} + \dot{q}\frac{\partial}{\partial q} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} + \dot{w}\frac{\partial}{\partial w} + \ddot{p}\frac{\partial}{\partial \dot{p}} + \ddot{q}\frac{\partial}{\partial \dot{q}} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \ddot{y}\frac{\partial}{\partial \dot{y}} + \ddot{z}\frac{\partial}{\partial \dot{z}} + \ddot{w}\frac{\partial}{\partial \dot{w}}. \tag{4.6}$$

Finally, Γ is said to be a Lie symmetry of the system of the geodesic equations if

$$\Gamma^2(\Delta_i^{(2)})|_{\Delta_i^{(2)}=0} = 0, \tag{4.7}$$

where

$$\Delta_i^{(2)} = \frac{d^2x^i}{dt^2} - f^i(t, x^i), \qquad i = 1, 2, ..., 6. \tag{4.8}$$

Equation (4.7) is called the Lie invariance condition. We equate the coefficients of the linearly independent derivation terms to zero, and this yields to an over-determined system of partial differential equations. For a good reference on symmetries of differential equations, we refer the reader to [13].

5. Lie symmetry algebras of $A_{6,20}$–$A_{6,27}$

In this section, we consider the eight six-dimensional Lie algebras with co-dimension two nilradical and one-dimensional center, $A_{6,20}$–$A_{6,27}$ in [7]. For each Lie algebra, we will list the nonzero brackets, the system of the geodesic equations and, the symmetry vector fields. Finally, we analyze the symmetry Lie algebra in terms of its nilradical, complement, and semi-simple sub-algebra.

5.1. Algebra $A_{6,20}^{ab}$ (ab: $a^2 + b^2 \neq 0$)

The nonzero brackets for the algebra $A_{6,20}^{ab}$ are given by

$$[e_1, e_4] = ae_4, \quad [e_1, e_6] = e_6, \quad [e_2, e_4] = be_4, \quad [e_1, e_2] = e_3, \quad [e_2, e_5] = e_5. \tag{5.1}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}(a\dot{z} + b\dot{w}), \quad \ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = \dot{z}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.2}$$

For the general case $A_{6,20}^{a\neq 0, b\neq 0}$, the symmetry Lie algebra is spanned by:

$$\begin{aligned} &e_1 = D_w, && e_2 = D_z, && e_3 = tD_t, && e_4 = D_t, && e_5 = tD_y, \\ &e_6 = D_p, && e_7 = D_y, && e_8 = D_q, && e_9 = D_x, && e_{10} = pD_p, \\ &e_{11} = wD_t, && e_{12} = zD_t, && e_{13} = wD_y, && e_{14} = zD_y, && e_{15} = qD_q, \\ &e_{16} = xD_x, && e_{17} = e^zD_q, && e_{18} = e^wD_x, && e_{19} = (wz - 2y)D_t, && e_{20} = (wz - 2y)D_y, \\ &e_{21} = e^{bw}e^{az}D_p. \end{aligned} \tag{5.3}$$

We make the following change of basis:

$$\begin{array}{lllll}\overline{e}_1 = e_4, & \overline{e}_2 = e_6, & \overline{e}_3 = e_7, & \overline{e}_4 = e_8, & \overline{e}_5 = e_9, \\ \overline{e}_6 = e_{11}, & \overline{e}_7 = e_{12}, & \overline{e}_8 = e_{13}, & \overline{e}_9 = e_{14}, & \overline{e}_{10} = e_{17}, \\ \overline{e}_{11} = e_{18}, & \overline{e}_{12} = e_{21}, & \overline{e}_{13} = e_1 + \frac{e_{14}}{2}, & \overline{e}_{14} = e_2 + \frac{e_{13}}{2}, & \overline{e}_{15} = e_3 - \frac{e_{20}}{2}, \\ \overline{e}_{16} = e_{10}, & \overline{e}_{17} = e_{15}, & \overline{e}_{18} = e_{16}, & \overline{e}_{19} = e_3 + \frac{e_{20}}{2}, & \overline{e}_{20} = e_5, \\ \overline{e}_{21} = e_{19}. & & & & \end{array} \tag{5.4}$$

The nonzero brackets of the symmetry algebra are given by

$$\begin{array}{llll}[e_1, e_{15}] = e_1, & [e_1, e_{19}] = e_1, & [e_1, e_{20}] = e_3, & [e_2, e_{16}] = e_2, \\ [e_3, e_{15}] = e_3, & [e_3, e_{19}] = -e_3, & [e_3, e_{21}] = -2e_1, & [e_4, e_{17}] = e_4, \\ [e_5, e_{18}] = e_5, & [e_6, e_{13}] = -e_1, & [e_6, e_{15}] = e_6, & [e_6, e_{19}] = e_6, \\ [e_6, e_{20}] = e_8, & [e_7, e_{14}] = -e_1, & [e_7, e_{15}] = e_7, & [e_7, e_{19}] = e_7, \\ [e_7, e_{20}] = e_9, & [e_8, e_{13}] = -e_3, & [e_8, e_{15}] = e_8, & [e_8, e_{19}] = -e_8, \\ [e_8, e_{21}] = -2e_6, & [e_9, e_{14}] = -e_3, & [e_9, e_{15}] = e_9, & [e_9, e_{19}] = -e_9, \\ [e_9, e_{21}] = -2e_7, & [e_{10}, e_{14}] = -e_{10}, & [e_{10}, e_{17}] = e_{10}, & [e_{11}, e_{13}] = -e_{11}, \\ [e_{11}, e_{18}] = e_{11}, & [e_{12}, e_{13}] = -be_{12}, & [e_{12}, e_{14}] = -ae_{12}, & [e_{12}, e_{16}] = e_{12}, \\ [e_{19}, e_{20}] = 2e_{20}, & [e_{19}, e_{21}] = -2e_{21}, & [e_{20}, e_{21}] = -2e_{19}. & \end{array} \tag{5.5}$$

We describe the symmetry algebra by the following proposition:

Proposition 2. *The symmetry Lie algebra is a twenty-one-dimensional Lie algebra. It is a semi-direct product of eighteen-dimensional solvable Lie algebra and* $sl(2,\mathbb{R})$. *The solvable part is* $(\mathbb{R}^{12} \rtimes \mathbb{R}^6)$ *a semi-direct product of* $\mathbb{R}^{12}$ *and* $\mathbb{R}^6$. *Therefore, the symmetry algebra can be identified as* $(\mathbb{R}^{12} \rtimes \mathbb{R}^6) \rtimes sl(2,\mathbb{R})$.

5.2. Algebra $A^a_{6,21}$

The nonzero brackets for the algebra $A^a_{6,21}$ are given by

$$[e_1, e_4] = e_4, \quad [e_1, e_5] = e_6, \quad [e_2, e_4] = ae_4, \quad [e_2, e_5] = e_5, \quad [e_2, e_6] = e_6, \quad [e_1, e_2] = e_3. \tag{5.6}$$

The geodesic equations are given by

$$\ddot{p} = \dot{p}(\dot{z} + a\dot{w}), \quad \ddot{q} = \dot{w}(\dot{q} - x\dot{z}) + \dot{z}\dot{x}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = \dot{z}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.7}$$

For the general case $A^{a\neq 0}_{6,21}$, the symmetry Lie algebra is spanned by

$$\begin{array}{llll}e_1 = D_t, & e_2 = tD_y, & e_3 = D_y, & e_4 = D_p, \\ e_5 = D_q, & e_6 = D_z, & e_7 = D_w, & e_8 = tD_t, \\ e_9 = pD_p, & e_{10} = wD_t, & e_{11} = zD_t, & e_{12} = wD_y, \\ e_{13} = zD_y, & e_{14} = xD_q, & e_{15} = zD_q + D_x, & e_{16} = qD_q + xD_x, \\ e_{17} = e^w D_q, & e_{18} = e^w D_x, & e_{19} = (wz - 2y)D_t, & e_{20} = (wz - 2y)D_y, \\ e_{21} = e^{aw} e^z D_p. & & & \end{array} \tag{5.8}$$

We make the following change of basis:

$$\begin{array}{llllll} \bar{e}_1 = e_1, & \bar{e}_2 = e_3, & \bar{e}_3 = e_4, & \bar{e}_4 = e_5, & \bar{e}_5 = e_{10}, & \bar{e}_6 = e_{11}, \\ \bar{e}_7 = e_{12}, & \bar{e}_8 = e_{13}, & \bar{e}_9 = e_{14}, & \bar{e}_{10} = e_{15}, & \bar{e}_{11} = e_{17}, & \bar{e}_{12} = e_{18}, \\ \bar{e}_{13} = e_{21}, & \bar{e}_{14} = e_6 + \frac{e_{21}}{2}, & \bar{e}_{15} = e_7 + \frac{e_{13}}{2}, & \bar{e}_{16} = e_8 - \frac{e_{20}}{2}, & \bar{e}_{17} = e_9, & \bar{e}_{18} = e_{16}, \\ \bar{e}_{19} = e_2, & \bar{e}_{20} = e_8 + \frac{e_{20}}{2}, & \bar{e}_{21} = e_{19}. & & & \end{array} \tag{5.9}$$

The nonzero brackets of the symmetry algebra are given by:

$$\begin{array}{llll} [e_1, e_{15}] = -e_1, & [e_1, e_{18}] = e_1, & [e_2, e_{18}] = e_2, & [e_3, e_5] = e_1, \\ [e_3, e_{15}] = -e_3, & [e_3, e_{18}] = e_3, & [e_4, e_5] = e_2, & [e_4, e_{14}] = -e_2, \\ [e_4, e_{18}] = e_4, & [e_6, e_{14}] = -e_9, & [e_6, e_{16}] = e_6, & [e_6, e_{19}] = e_8, \\ [e_6, e_{20}] = e_6, & [e_7, e_{15}] = -e_{10}, & [e_7, e_{16}] = e_7, & [e_7, e_{20}] = -e_7, \\ [e_7, e_{21}] = -2e_{12}, & [e_8, e_{14}] = -e_{10}, & [e_8, e_{16}] = e_8, & [e_8, e_{20}] = -e_8, \\ [e_8, e_{21}] = -2e_6, & [e_9, e_{16}] = e_9, & [e_9, e_{19}] = e_{10}, & [e_9, e_{20}] = e_9, \\ [e_{10}, e_{16}] = e_{10}, & [e_{10}, e_{20}] = -e_{10}, & [e_{10}, e_{21}] = -2e_9, & [e_{11}, e_{17}] = e_{11}, \\ [e_{12}, e_{15}] = -e_9, & [e_{12}, e_{16}] = e_{12}, & [e_{12}, e_{19}] = e_7, & [e_{12}, e_{20}] = e_{12}, \\ [e_{13}, e_{14}] = -e_{13}, & [e_{13}, e_{15}] = -ae_{13}, & [e_{13}, e_{17}] = e_{13}, & [e_{19}, e_{20}] = -2e_{19}, \\ [e_{19}, e_{21}] = -2e_{20}, & [e_{20}, e_{21}] = -2e_{21}. & & \end{array} \tag{5.10}$$

We describe the symmetry algebra by the following proposition:

Proposition 3. *The symmetry Lie algebra is a twenty-one-dimensional Lie algebra. It is a semi-direct product of eighteen-dimensional solvable Lie algebra and $sl(2,\mathbb{R})$. The nilradical is thirteen-dimensional decomposable Lie algebra. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz [5] and $\mathbb{R}^8$. The nilradical has a five-dimensional abelian complement. Therefore, the symmetry algebra can be identified as*

$$((A_{5,1} \oplus \mathbb{R}^8) \rtimes \mathbb{R}^5) \rtimes sl(2,\mathbb{R}),$$

where the nonzero brackets of $A_{5,1}$ are given by

$$[e_3, e_5] = e_1, \quad [e_4, e_5] = e_2. \tag{5.11}$$

5.3. Algebra $A^{a\epsilon}_{6,22}$ ($a\epsilon$: $a^2 + \epsilon^2 \neq 0, \epsilon = 0, 1$)

The nonzero brackets for the algebra $A^{a\epsilon}_{6,22}$ are given by

$$[e_1, e_3] = e_3, \quad [e_1, e_5] = e_6, \quad [e_2, e_4] = e_4, \quad [e_2, e_3] = ae_3, \quad [e_1, e_2] = \epsilon e_5. \tag{5.12}$$

5.3.1. $A^{\epsilon=0}_{6,22}$

The geodesic equations are given by

$$\ddot{p} = \dot{z}\dot{y}, \quad \ddot{q} = \dot{w}\dot{q}, \quad \ddot{x} = \dot{x}(\dot{z} + a\dot{w}), \quad \ddot{y} = 0, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.13}$$

For the general case $A_{6,22}^{a\neq0,\epsilon=0}$, the symmetry Lie algebra is spanned by

$$\begin{array}{llll}
e_1 = D_y, & e_2 = D_w, & e_3 = D_z, & e_4 = tD_t, \\
e_5 = D_x, & e_6 = D_t, & e_7 = tD_p, & e_8 = D_p, \\
e_9 = D_q, & e_{10} = xD_x, & e_{11} = wD_t, & e_{12} = yD_t, \\
e_{13} = zD_t, & e_{14} = wD_p, & e_{15} = yD_p, & e_{16} = zD_p, \\
e_{17} = qD_q, & e_{18} = pD_p + yD_y, & e_{19} = e^w D_q, & e_{20} = \frac{z^2}{2}D_p + zD_y, \\
e_{21} = \frac{zt}{2}D_p + tD_y, & e_{22} = (yz-2p)D_t, & e_{23} = (yz-2p)D_p, & e_{24} = \frac{wz}{2}D_p + wD_y, \\
e_{25} = e^{aw}e^{z}D_x, & e_{26} = (\frac{yz^2}{2} - pz)D_p + (yz-2p)D_y. & &
\end{array} \tag{5.14}$$

We consider the following change of basis:

$$\begin{array}{lllll}
\overline{e}_1 = e_1, & \overline{e}_2 = e_5, & \overline{e}_3 = e_6, & \overline{e}_4 = e_8, & \overline{e}_5 = e_9, \\
\overline{e}_6 = e_{11}, & \overline{e}_7 = e_{13}, & \overline{e}_8 = e_{14}, & \overline{e}_9 = e_{16}, & \overline{e}_{10} = e_{19}, \\
\overline{e}_{11} = e_{20}, & \overline{e}_{12} = e_{24}, & \overline{e}_{13} = e_{25}, & \overline{e}_{14} = e_2, & \overline{e}_{15} = e_3 + \frac{e_{15}}{2}, \\
\overline{e}_{16} = e_4 + e_{18}, & \overline{e}_{17} = e_{10}, & \overline{e}_{18} = e_{17}, & \overline{e}_{19} = e_4 + \frac{e_{23}}{2}, & \overline{e}_{20} = e_7, \\
\overline{e}_{21} = e_{12}, & \overline{e}_{22} = e_{15}, & \overline{e}_{23} = e_{18} + e_{23}, & \overline{e}_{24} = e_{21}, & \overline{e}_{25} = e_{22}, \\
\overline{e}_{26} = e_{26}. & & & &
\end{array} \tag{5.15}$$

The nonzero brackets of the symmetry algebra are given by

$$\begin{array}{lll}
[e_1, e_{15}] = \frac{e_4}{2}, & [e_1, e_{16}] = e_1, & [e_1, e_{19}] = \frac{e_9}{2}, \\
[e_1, e_{21}] = e_3, & [e_1, e_{22}] = e_4, & [e_1, e_{23}] = e_1 + e_9, \\
[e_1, e_{25}] = e_7, & [e_1, e_{26}] = e_{11}, & [e_2, e_{17}] = e_2, \\
[e_3, e_{16}] = e_3, & [e_3, e_{19}] = e_3, & [e_3, e_{20}] = e_4, \\
[e_3, e_{24}] = e_1 + \frac{e_9}{2}, & [e_4, e_{16}] = e_4, & [e_4, e_{19}] = -e_4, \\
[e_4, e_{23}] = -e_4, & [e_4, e_{25}] = -2e_3, & [e_4, e_{26}] = -2e_1 - e_9, \\
[e_5, e_{18}] = e_5, & [e_6, e_{14}] = -e_3, & [e_6, e_{16}] = e_6, \\
[e_6, e_{19}] = e_6, & [e_6, e_{20}] = e_8, & [e_6, e_{24}] = e_{12}, \\
[e_7, e_{15}] = -e_3, & [e_7, e_{16}] = e_7, & [e_7, e_{19}] = e_7, \\
[e_7, e_{20}] = e_9, & [e_7, e_{24}] = e_{11}, & [e_8, e_{14}] = -e_4, \\
[e_8, e_{16}] = e_8, & [e_8, e_{19}] = -e_8, & [e_8, e_{23}] = -e_8, \\
[e_8, e_{25}] = -2e_6, & [e_8, e_{26}] = -2e_{12}, & [e_9, e_{15}] = -e_4, \\
[e_9, e_{16}] = e_9, & [e_9, e_{19}] = -e_9, & [e_9, e_{23}] = -e_9, \\
[e_9, e_{25}] = -2e_7, & [e_9, e_{26}] = -2e_{11}, & [e_{10}, e_{14}] = -e_{10}, \\
[e_{10}, e_{18}] = e_{10}, & [e_{11}, e_{15}] = -e_1 - \frac{9}{2}, & [e_{11}, e_{16}] = e_{11}, \\
[e_{11}, e_{21}] = e_7, & [e_{11}, e_{22}] = e_9, & [e_{11}, e_{23}] = e_{11}, \\
[e_{12}, e_{14}] = -e_1 - \frac{9}{2}, & [e_{12}, e_{16}] = e_{12}, & [e_{12}, e_{21}] = e_6, \\
[e_{12}, e_{22}] = e_8, & [e_{12}, e_{23}] = e_{12}, & [e_{13}, e_{14}] = -ae_{13}, \\
[e_{13}, e_{15}] = -e_{13}, & [e_{13}, e_{17}] = e_{13}, & [e_{19}, e_{20}] = 2e_{20}, \\
[e_{19}, e_{21}] = -e_{21}, & [e_{19}, e_{22}] = e_{22}, & [e_{19}, e_{24}] = e_{24}, \\
[e_{19}, e_{25}] = -2e_{25}, & [e_{19}, e_{26}] = -e_{26}, & [e_{20}, e_{21}] = -e_{22}, \\
[e_{20}, e_{23}] = -e_{20}, & [e_{20}, e_{25}] = -2e_{19}, & [e_{20}, e_{26}] = -2e_{24}, \\
[e_{21}, e_{23}] = -e_{21}, & [e_{21}, e_{24}] = -e_{19} + e_{23}, & [e_{21}, e_{26}] = -e_{25}, \\
[e_{22}, e_{23}] = -2e_{22}, & [e_{22}, e_{24}] = -e_{20}, & [e_{22}, e_{25}] = -2e_{21}, \\
[e_{22}, e_{26}] = -2e_{23}, & [e_{23}, e_{24}] = -e_{24}, & [e_{23}, e_{25}] = -e_{25}, \\
[e_{23}, e_{26}] = -2e_{26}, & [e_{24}, e_{25}] = -e_{26}. &
\end{array} \tag{5.16}$$

We describe the symmetry algebra by the following proposition:

Proposition 4. *The symmetry Lie algebra is a twenty-six-dimensional Lie algebra. It is a semi-direct product of an eighteen-dimensional solvable Lie algebra and* $sl(3,\mathbb{R})$. *The solvable part is* $(\mathbb{R}^{13} \rtimes \mathbb{R}^5)$, *a semi-direct product of* $\mathbb{R}^{13}$ *and* $\mathbb{R}^5$. *Therefore, the symmetry algebra can be identified as* $(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3,\mathbb{R})$.

5.3.2. $A_{6,22}^{\epsilon=1}$

The geodesic equations are given by

$$\ddot{p} = \dot{p}(a\dot{z} + \dot{w}), \quad \ddot{q} = \dot{q}\dot{z}, \quad \ddot{x} = \dot{y}\dot{w}, \quad \ddot{y} = \dot{z}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.17}$$

For the general case $A_{6,22}^{a\neq 0,\epsilon=1}$, the symmetry Lie algebra is spanned by

$$\begin{array}{llll}
e_1 = D_t, & e_2 = tD_x, & e_3 = D_p, & e_4 = D_x, \\
e_5 = D_y, & e_6 = D_q, & e_7 = D_w, & e_8 = D_z, \\
e_9 = tD_t, & e_{10} = pD_p, & e_{11} = wD_t, & e_{12} = zD_t, \\
e_{13} = zD_x, & e_{14} = wD_x, & e_{15} = qD_q, & e_{16} = yD_x + zD_y, \\
e_{17} = e^z D_q, & e_{18} = twD_x + 2tD_y, & e_{19} = \frac{w^2}{2}D_x + wD_y, & e_{20} = wzD_x + 2zD_y, \\
e_{21} = (yz - 2y)D_t, & e_{22} = e^w e^{az} D_p, & e_{23} = (wy - \frac{w^2 z}{2})D_x + (-wz + 2y)D_y. &
\end{array} \tag{5.18}$$

We consider the following change of basis:

$$\begin{array}{lllll}
\overline{e}_1 = e_1, & \overline{e}_2 = e_2, & \overline{e}_3 = e_3, & \overline{e}_4 = e_4, & \overline{e}_5 = e_5, \\
\overline{e}_6 = e_6, & \overline{e}_7 = e_{11}, & \overline{e}_8 = e_{12}, & \overline{e}_9 = e_{13}, & \overline{e}_{10} = e_{14}, \\
\overline{e}_{11} = e_{16}, & \overline{e}_{12} = e_{17}, & \overline{e}_{13} = e_{19}, & \overline{e}_{14} = e_{20}, & \overline{e}_{15} = e_{22}, \\
\overline{e}_{16} = e_7, & \overline{e}_{17} = e_8, & \overline{e}_{18} = e_9 + \frac{e_{23}}{2}, & \overline{e}_{19} = e_{10}, & \overline{e}_{20} = e_{15}, \\
\overline{e}_{21} = e_9 - \frac{e_{23}}{2}, & \overline{e}_{22} = e_{18}, & \overline{e}_{23} = e_{21}. & &
\end{array} \tag{5.19}$$

The nonzero brackets of the symmetry algebra are given by

$$\begin{array}{lll}
[e_1, e_2] = e_4, & [e_1, e_{18}] = e_1, & [e_1, e_{21}] = e_1, \\
[e_1, e_{22}] = e_{10} + 2e_5, & [e_2, e_7] = -e_{10}, & [e_2, e_8] = -e_9, \\
[e_2, e_{18}] = -e_2, & [e_2, e_{21}] = -e_2, & [e_2, e_{23}] = 2e_{11} - e_{14}, \\
[e_3, e_{19}] = e_3, & [e_5, e_{11}] = e_4, & [e_5, e_{18}] = e_5 + \frac{e_{10}}{2}, \\
[e_5, e_{21}] = -e_5 - \frac{e_{10}}{2}, & [e_5, e_{23}] = -2e_1, & [e_6, e_{20}] = e_6, \\
[e_7, e_{16}] = -e_1, & [e_7, e_{18}] = e_7, & [e_7, e_{21}] = e_7, \\
[e_7, e_{22}] = 2e_{13}, & [e_8, e_{17}] = -e_1, & [e_8, e_{18}] = e_8, \\
[e_8, e_{21}] = e_8, & [e_8, e_{22}] = e_{14}, & [e_9, e_{17}] = -e_4, \\
[e_{10}, e_{16}] = -e_4, & [e_{11}, e_{13}] = -e_{10}, & [e_{11}, e_{14}] = -e_9, \\
[e_{11}, e_{17}] = -e_5, & [e_{11}, e_{18}] = -e_{11} + e_{14}, & [e_{11}, e_{21}] = e_{11} - e_{14}, \\
[e_{11}, e_{22}] = -2e_2, & [e_{11}, e_{23}] = -2e_8, & [e_{12}, e_{17}] = -e_{12}, \\
[e_{12}, e_{20}] = e_{12}, & [e_{13}, e_{16}] = -e_{10} - e_5, & [e_{13}, e_{18}] = e_{18}, \\
[e_{13}, e_{21}] = -e_{13}, & [e_{13}, e_{23}] = -2e_7, & [e_{14}, e_{16}] = -e_9, \\
[e_{14}, e_{17}] = -e_{10} - 2e_5, & [e_{14}, e_{18}] = e_{14}, & [e_{14}, e_{21}] = -e_{14}, \\
[e_{14}, e_{23}] = -4e_8, & [e_{15}, e_{16}] = -e_{15}, & [e_{15}, e_{17}] = -ae_{15}, \\
[e_{15}, e_{19}] = e_{15}, & [e_{16}, e_{18}] = \frac{e_{11}}{2} - \frac{e_{14}}{2}, & [e_{16}, e_{21}] = -\frac{e_{11}}{2} + \frac{e_{14}}{2}, \\
[e_{16}, e_{22}] = e_2, & [e_{16}, e_{23}] = e_8, & [e_{17}, e_{18}] = -\frac{e_{13}}{2}, \\
[e_{17}, e_{21}] = \frac{e_{13}}{2}, & [e_{17}, e_{23}] = e_7, & [e_{21}, e_{22}] = 2e_{22}, \\
[e_{21}, e_{23}] = -2e_{23}, & [e_{22}, e_{23}] = -4e_{21}. &
\end{array} \tag{5.20}$$

We describe the symmetry algebra by the following proposition:

Proposition 5. *The symmetry Lie algebra is a twenty-three-dimensional Lie algebra. It is a semi-direct product of twenty-dimensional solvable Lie algebra and $sl(2,\mathbb{R})$. The solvable part is $(\mathbb{R}^{15} \rtimes \mathbb{R}^5)$, a semi-direct product of $\mathbb{R}^{15}$ and $\mathbb{R}^5$. Therefore, the symmetry algebra can be identified as $(\mathbb{R}^{15} \rtimes \mathbb{R}^5) \rtimes sl(2,\mathbb{R})$.*

5.4. Algebra $A_{6,23}^{a\epsilon}$($a\epsilon$: $a \geq 0, \epsilon = 0, 1$)

The nonzero brackets for the algebra $A_{6,23}^{a\epsilon}$ are given by

$$\begin{array}{llll} [e_1, e_3] = e_3, & [e_1, e_4] = e_4, & [e_1, e_5] = e_6, & [e_2, e_3] = e_4, \\ [e_2, e_4] = -e_3, & [e_2, e_5] = ae_6, & [e_1, e_2] = \epsilon e_5. & \end{array} \tag{5.21}$$

5.4.1. Case 1: $A_{6,23}^{a,\epsilon=0}$

The geodesic equations when $\epsilon = 0$ are given by

$$\ddot{p} = \dot{p}\dot{z} - \dot{q}\dot{w}, \quad \ddot{q} = \dot{p}\dot{w} + \dot{q}\dot{z}, \quad \ddot{x} = \dot{y}(\dot{z} + a\dot{w}), \quad \ddot{y} = 0, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.22}$$

The symmetry Lie algebra is spanned by

$$\begin{array}{ll} e_1 = D_t, & e_2 = D_p, \\ e_3 = D_q, & e_4 = tD_x, \\ e_5 = D_x, & e_6 = D_y, \\ e_7 = D_w, & e_8 = D_z, \\ e_9 = tD_t, & e_{10} = wD_t, \\ e_{11} = yD_t, & e_{12} = zD_t, \\ e_{13} = yD_x, & e_{14} = zD_x, \\ e_{15} = wD_x, & e_{16} = pD_p + qD_q, \\ e_{17} = xD_x + yD_y, & e_{18} = qD_p - pD_q, \\ e_{19} = \frac{t(aw+z)}{2}D_x + tD_y, & e_{20} = ((aw + z)y - 2x)D_x, \\ e_{21} = \frac{w(aw+z)}{2}D_x + wD_y, & e_{22} = \frac{z(aw+z)}{2}D_x + zD_y, \\ e_{23} = \frac{((aw+z)y-2x)}{a}D_t, & e_{24} = \cos(w)e^z D_p + \sin(w)e^z D_q, \\ e_{25} = \sin(w)e^z D_p - \cos(w)e^z D_q, & e_{26} = \frac{(\frac{aw}{2}+\frac{z}{2})(awy+yz-2x)}{a}D_x + \frac{((aw+z)y-2x)}{a}D_y. \end{array} \tag{5.23}$$

We consider the following change of basis:

$$\begin{array}{llllll} \overline{e}_1 = e_1, & \overline{e}_2 = e_2, & \overline{e}_3 = e_3, & \overline{e}_4 = e_5, & \overline{e}_5 = e_6, & \overline{e}_6 = e_{10}, \\ \overline{e}_7 = e_{12}, & \overline{e}_8 = e_{14}, & \overline{e}_9 = e_{15}, & \overline{e}_{10} = e_{21}, & \overline{e}_{11} = e_{22}, & \overline{e}_{12} = e_{24}, \\ \overline{e}_{13} = e_{25}, & \overline{e}_{14} = e_7 + \frac{ae_{13}}{2}, & \overline{e}_{15} = e_8 + \frac{e_{13}}{2}, & \overline{e}_{16} = e_9 + e_{17}, & \overline{e}_{17} = e_{16}, & \overline{e}_{18} = e_{18}, \\ \overline{e}_{19} = e_4, & \overline{e}_{20} = e_9 + \frac{e_{20}}{2}, & \overline{e}_{21} = e_{11}, & \overline{e}_{22} = e_{13}, & \overline{e}_{23} = e_{17} + e_{20}, & \overline{e}_{24} = e_{19}, \\ \overline{e}_{25} = e_{23}, & \overline{e}_{26} = e_{26}. & & & & \end{array} \tag{5.24}$$

The nonzero brackets of the symmetry algebra are given by

$$\begin{array}{lll}
[e_1,e_{16}]=e_1, & [e_1,e_{19}]=e_4, & [e_1,e_{20}]=e_1,\\
[e_1,e_{24}]=\frac{ae_9}{2}+e_5+\frac{e_8}{2}, & [e_2,e_{17}]=e_2, & [e_2,e_{18}]=-e_3,\\
[e_3,e_{17}]=e_3, & [e_3,e_{18}]=e_2, & [e_4,e_{16}]=e_4,\\
[e_4,e_{20}]=-e_4, & [e_4,e_{23}]=-e_4, & [e_4,e_{25}]=-\frac{2e_1}{a},\\
[e_4,e_{26}]=-e_9-\frac{e_8}{a}-\frac{2e_5}{a}, & [e_5,e_{14}]=\frac{ae_4}{2}, & [e_5,e_{15}]=\frac{e_4}{2},\\
[e_5,e_{16}]=e_5, & [e_5,e_{20}]=\frac{ae_9}{2}+\frac{e_8}{2}, & [e_5,e_{21}]=e_1,\\
[e_5,e_{22}]=e_4, & [e_5,e_{23}]=ae_9+e_5+e_8, & [e_5,e_{25}]=\frac{e_7}{a}+e_6,\\
[e_5,e_{26}]=\frac{e_{11}}{a}+e_{10}, & [e_6,e_{14}]=-e_1, & [e_6,e_{16}]=e_6,\\
[e_6,e_{19}]=e_9, & [e_6,e_{20}]=e_6, & [e_6,e_{24}]=e_{10},\\
[e_7,e_{15}]=-e_1, & [e_7,e_{16}]=e_7, & [e_7,e_{19}]=e_8,\\
[e_7,e_{20}]=e_7, & [e_7,e_{24}]=e_{11}, & [e_8,e_{15}]=-e_4,\\
[e_8,e_{16}]=e_8, & [e_8,e_{20}]=-e_8, & [e_8,e_{23}]=-e_8,\\
[e_8,e_{25}]=-\frac{2e_7}{a}, & [e_8,e_{26}]=-\frac{2e_{11}}{a}, & [e_9,e_{14}]=-e_4,\\
[e_9,e_{16}]=e_9, & [e_9,e_{20}]=-e_9, & [e_9,e_{23}]=-e_9,\\
[e_9,e_{25}]=-\frac{2e_6}{a}, & [e_9,e_{26}]=-\frac{2e_{10}}{a}, & [e_{10},e_{14}]=-\frac{ae_9}{2}-e_5-\frac{e_8}{2},\\
[e_{10},e_{16}]=e_{10}, & [e_{10},e_{21}]=e_6, & [e_{10},e_{22}]=e_9,\\
[e_{10},e_{23}]=e_{10}, & [e_{11},e_{15}]=-\frac{ae_9}{2}-e_5-\frac{e_8}{2}, & [e_{11},e_{16}]=e_{11},\\
[e_{11},e_{21}]=e_7, & [e_{11},e_{22}]=e_8, & [e_{11},e_{23}]=e_{11}.
\end{array}$$

$$\begin{array}{lll}
[e_{12},e_{14}]=e_{13}, & [e_{12},e_{15}]=-e_{12}, & [e_{12},e_{17}]=e_{12},\\
[e_{12},e_{18}]=e_{13}, & [e_{13},e_{14}]=-e_{12}, & [e_{13},e_{15}]=-e_{13},\\
[e_{13},e_{17}]=e_{13}, & [e_{13},e_{18}]=-e_{12}, & [e_{19},e_{20}]=-2e_{19},\\
[e_{19},e_{21}]=-e_{22}, & [e_{19},e_{23}]=-e_{19}, & [e_{19},e_{25}]=-\frac{2e_{20}}{a},\\
[e_{19},e_{26}]=-\frac{2e_{24}}{a}, & [e_{20},e_{21}]=-e_{21}, & [e_{20},e_{22}]=e_{22},\\
[e_{20},e_{24}]=e_{24}, & [e_{20},e_{25}]=-2e_{25}, & [e_{20},e_{26}]=-e_{26},\\
[e_{21},e_{23}]=-e_{21}, & [e_{21},e_{24}]=-e_{20}+e_{23}, & [e_{21},e_{26}]=-e_{25},\\
[e_{22},e_{23}]=-2e_{22}, & [e_{22},e_{24}]=-e_{19}, & [e_{22},e_{25}]=-\frac{2e_{21}}{a},\\
[e_{22},e_{26}]=-\frac{2e_{23}}{a}, & [e_{23},e_{24}]=-e_{24}, & [e_{23},e_{25}]=-e_{25},\\
[e_{23},e_{26}]=-2e_{26}, & [e_{24},e_{25}]=-e_{26}. &
\end{array} \tag{5.25}$$

We describe the symmetry algebra by the following proposition:

Proposition 6. *The symmetry Lie algebra is a twenty-six- dimensional Lie algebra. It is a semi-direct product of eighteen-dimensional solvable Lie algebra and* $sl(3,\mathbb{R})$*. The solvable part is* $(\mathbb{R}^{13} \rtimes \mathbb{R}^5)$*, a semi-direct product of* $\mathbb{R}^{13}$ *and* $\mathbb{R}^5$*. Therefore, the symmetry algebra can be identified as* $(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3,\mathbb{R})$*.*

5.5. Case 2: $A_{6,23}^{a,\epsilon=1}$

5.5.1. $A_{6,23}^{a\neq 0,\epsilon=1}$

For $A_{6,23}^{a\neq 0,\epsilon=1}$, the geodesic equations are given by

$$\begin{array}{lll}
\ddot{p}=\dot{p}\dot{z}+\dot{w}\dot{q}, & \ddot{q}=-\dot{p}\dot{w}+\dot{q}\dot{z}, & \ddot{x}=\dot{y}(\dot{z}+a\dot{w}),\\
\ddot{y}=\dot{z}(\dot{z}+a\dot{w}), & \ddot{z}=0, & \ddot{w}=0.
\end{array} \tag{5.26}$$

The symmetry Lie algebra is spanned by

$$
\begin{array}{lll}
e_1 = D_w, & e_2 = D_q, & e_3 = D_p, \\
e_4 = D_y, & e_5 = tD_x, & e_6 = D_z, \\
e_7 = D_x, & e_8 = tD_t, & e_9 = D_t, \\
e_{10} = wD_x, & e_{11} = zD_x, & e_{12} = wD_t, \\
e_{13} = zD_t, & e_{14} = yD_x + zD_y, & e_{15} = pD_p + qD_q, \\
e_{16} = -qD_p + pD_q, & e_{17} = \frac{t(aw+z)D_x}{2} + tD_y, & e_{18} = \frac{(awz+z^2-2y)D_x}{a}, \\
e_{19} = \frac{(awz+z^2-2y)D_t}{a}, & e_{20} = \frac{(\frac{a^2w^2}{2}-\frac{z^2}{2}+y)D_x}{a} + wD_y, & e_{21} = \sin(w)e^zD_p + \cos(w)e^zD_q, \\
e_{22} = -\cos(w)e^zD_p + \sin(w)e^zD_q, & e_{23} = \frac{(\frac{aw}{2}+\frac{z}{2})(awz+z^2-2y)D_x}{a} + \frac{(awz+z^2-2y)D_y}{a}. &
\end{array}
\tag{5.27}
$$

We consider the following change of basis:

$$
\begin{array}{lllll}
\overline{e}_1 = e_4, & \overline{e}_2 = e_5, & \overline{e}_3 = e_7, & \overline{e}_4 = e_9, & \overline{e}_5 = e_{10}, \\
\overline{e}_6 = e_{11}, & \overline{e}_7 = e_{12}, & \overline{e}_8 = e_{13}, & \overline{e}_9 = e_{14}, & \overline{e}_{10} = e_{18}, \\
\overline{e}_{11} = e_{20}, & \overline{e}_{12} = e_2, & \overline{e}_{13} = e_3, & \overline{e}_{14} = e_{21}, & \overline{e}_{15} = e_{22}, \\
\overline{e}_{16} = e_1, & \overline{e}_{17} = e_6, & \overline{e}_{18} = e_8 - \frac{ae_{23}}{2}, & \overline{e}_{19} = e_{15}, & \overline{e}_{20} = e_{16}, \\
\overline{e}_{21} = e_8 + \frac{ae_{23}}{2}, & \overline{e}_{22} = e_{17}, & \overline{e}_{23} = e_{19}. & &
\end{array}
\tag{5.28}
$$

The nonzero brackets of the symmetry algebra are given by

$$
\begin{array}{ll}
[e_1, e_9] = e_3, & [e_1, e_{10}] = -\frac{2e_3}{a}, \\
[e_1, e_{11}] = \frac{e_3}{a}, & [e_1, e_{18}] = \frac{e_6}{2} + \frac{ae_5}{2} + e_1, \\
[e_1, e_{21}] = -\frac{e_6}{2} - \frac{ae_5}{2} - e_1, & [e_1, e_{23}] = -\frac{2e_4}{2}, \\
[e_2, e_4] = -e_3, & [e_2, e_7] = -e_5, \\
[e_2, e_8] = -e_6, & [e_2, e_{18}] = -e_2, \\
[e_2, e_{21}] = -e_2, & [e_2, e_{23}] = -e_{10}, \\
[e_4, e_{18}] = e_4, & [e_4, e_{21}] = e_4, \\
[e_4, e_{22}] = \frac{e_6}{2} + \frac{ae_5}{2} + e_1, & [e_5, e_{16}] = -e_3, \\
[e_6, e_{17}] = -e_3, & [e_7, e_{16}] = -e_4, \\
[e_7, e_{18}] = e_7, & [e_7, e_{21}] = e_7, \\
[e_7, e_{22}] = \frac{e_{10}}{2} + e_{11}, & [e_8, e_{17}] = -e_4, \\
[e_8, e_{18}] = e_8, & [e_8, e_{21}] = e_8, \\
[e_8, e_{22}] = \frac{ae_{10}}{2} + e_9, & [e_9, e_{10}] = -\frac{2e_6}{a}, \\
[e_9, e_{11}] = \frac{e_6}{a} - e_5, & [e_9, e_{17}] = -e_1, \\
[e_9, e_{18}] = ae_{10} + e_9, & [e_9, e_{21}] = -ae_{10} - e_9, \\
[e_9, e_{22}] = -e_2, & [e_9, e_{23}] = -\frac{2e_8}{a}, \\
[e_{10}, e_{11}] = \frac{2e_5}{a}, & [e_{10}, e_{16}] = -e_6, \\
[e_{10}, e_{17}] = -\frac{2e_6}{a} - e_5, & [e_{10}, e_{18}] = -e_{10}, \\
[e_{10}, e_{21}] = e_{10}, & [e_{10}, e_{22}] = \frac{2e_2}{a}, \\
[e_{11}, e_{16}] = -ae_5 - e_1, & [e_{11}, e_{17}] = \frac{e_6}{a}, \\
[e_{11}, e_{18}] = e_{10} + e_{11}, & [e_{11}, e_{21}] = -e_{10} - e_{11}, \\
[e_{11}, e_{22}] = -\frac{e_2}{a}, & [e_{11}, e_{23}] = -\frac{2e_7}{a}, \\
[e_{12}, e_{19}] = e_{12}, & [e_{12}, e_{20}] = -e_{13}, \\
[e_{13}, e_{19}] = e_{13}, & [e_{13}, e_{20}] = e_{12}, \\
[e_{14}, e_{16}] = e_{15}, & [e_{14}, e_{17}] = -e_{14}, \\
[e_{14}, e_{19}] = e_{14}, & [e_{14}, e_{20}] = e_{15}.
\end{array}
\tag{5.29}
$$

$$
\begin{array}{ll}
[e_{15}, e_{16}] = -e_{14}, & [e_{15}, e_{17}] = -e_{15}, \\
[e_{15}, e_{19}] = e_{15}, & [e_{15}, e_{20}] = -e_{14}, \\
[e_{16}, e_{18}] = -\frac{a^2 e_{10}}{2} - \frac{a e_9}{2}, & [e_{16}, e_{21}] = \frac{a^2 e_{10}}{2} + \frac{a e_9}{2}, \\
[e_{16}, e_{22}] = \frac{a e_2}{2}, & [e_{16}, e_{23}] = e_8 \\
[e_{17}, e_{18}] = -\frac{a e_{11}}{2} - a e_{10} - e_9, & [e_{17}, e_{21}] = \frac{a e_{11}}{2} + a e_{10} + e_9, \\
[e_{17}, e_{22}] = \frac{e_2}{2}, & [e_{17}, e_{23}] = \frac{2 e_8}{a} + e_7, \\
[e_{21}, e_{22}] = 2 e_{22}, & [e_{21}, e_{23}] = -2 e_{23}, \\
[e_{22}, e_{23}] = -\frac{2 e_{21}}{a}. &
\end{array}
\tag{5.30}
$$

We describe the symmetry algebra by the following proposition:

Proposition 7. *The symmetry Lie algebra is a twenty-three- dimensional semi-direct product of twenty-dimensional solvable Lie algebra $S_{1,20}$ and $sl(2,\mathbb{R})$. The nilradical a fifteen-dimensional nilpotant Lie algebra $N_{1,11} \oplus \mathbb{R}^4$, which is a direct sum of $N_{1,11}$, an eleven-dimensional nilpotent Lie algebra, and a four-dimensional abelian Lie algebra $\mathbb{R}^4$. The complement to the nilradical is a four-dimensional non-abelian. Therefore, the symmetry Lie algebra can be identified as $S_{1,20} \rtimes sl(2,\mathbb{R})$.*

5.6. Algebra $A_{6,24}$

The nonzero brackets for the algebra $A_{6,24}$ are given by

$$
[e_1, e_5] = e_5 + e_6, \quad [e_1, e_6] = e_6, \quad [e_2, e_4] = e_4, \quad [e_1, e_2] = e_3. \tag{5.31}
$$

The geodesic equations are given by

$$
\ddot{p} = \dot{p}\dot{z}, \quad \ddot{q} = \dot{w}(\dot{q} + \dot{x}), \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = \dot{z}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.32}
$$

The symmetry Lie algebra is spanned by

$$
\begin{array}{lll}
e_1 = D_t, & e_2 = tD_y, & e_3 = D_y, \\
e_4 = D_p, & e_5 = D_q, & e_6 = D_x, \\
e_7 = D_z, & e_8 = D_w, & e_9 = tD_t, \\
e_{10} = pD_p, & e_{11} = wD_t, & e_{12} = zD_t, \\
e_{13} = wD_y, & e_{14} = zD_y, & e_{15} = xD_q, \\
e_{16} = qD_q + xD_x, & e_{17} = e^w D_q, & e_{18} = e^z D_p, \\
e_{19} = (wz - 2y)D_t, & e_{20} = (wz - 2y)D_y, & e_{21} = (w-1)e^w D_q + e^w D_x.
\end{array}
\tag{5.33}
$$

We consider the following change of basis:

$$
\begin{array}{llll}
\overline{e}_1 = e_5, & \overline{e}_2 = e_{17}, & \overline{e}_3 = e_6, & \overline{e}_4 = e_{21}, \\
\overline{e}_5 = e_{15}, & \overline{e}_6 = e_1, & \overline{e}_7 = e_3, & \overline{e}_8 = e_4, \\
\overline{e}_9 = e_{11}, & \overline{e}_{10} = e_{12}, & \overline{e}_{11} = e_{13}, & \overline{e}_{12} = e_{14}, \\
\overline{e}_{13} = e_{18}, & \overline{e}_{14} = e_7 + \frac{e_{13}}{2}, & \overline{e}_{15} = e_8 + \frac{e_{14}}{2}, & \overline{e}_{16} = e_9 - \frac{e_{20}}{2}, \\
\overline{e}_{17} = e_{10}, & \overline{e}_{18} = e_{16}, & \overline{e}_{19} = e_2, & \overline{e}_{20} = e_9 + \frac{e_{20}}{2}, \\
\overline{e}_{21} = e_{19}, & & &
\end{array}
\tag{5.34}
$$

and the nonzero brackets of the symmetry algebra are given by

$$\begin{array}{llll}
[e_1,e_{18}]=e_1, & [e_2,e_{15}]=-e_2, & [e_2,e_{18}]=e_2, & [e_3,e_5]=e_1,\\
[e_3,e_{18}]=e_3, & [e_4,e_5]=e_2, & [e_4,e_{15}]=-e_2-e_4, & [e_4,e_{18}]=e_4,\\
[e_6,e_{16}]=e_6, & [e_6,e_{19}]=e_7, & [e_6,e_{20}]=e_6, & [e_7,e_{16}]=e_7,\\
[e_7,e_{20}]=-e_7, & [e_7,e_{21}]=-2e_6, & [e_8,e_{17}]=e_8, & [e_9,e_{15}]=-e_6,\\
[e_9,e_{16}]=e_9, & [e_9,e_{19}]=e_{11}, & [e_9,e_{20}]=e_9, & [e_{10},e_{14}]=-e_6,\\
[e_{10},e_{16}]=e_{10}, & [e_{10},e_{19}]=e_{12}, & [e_{10},e_{20}]=e_{10}, & [e_{11},e_{15}]=-e_7,\\
[e_{11},e_{16}]=e_{11}, & [e_{11},e_{20}]=-e_{11}, & [e_{11},e_{21}]=-2e_9, & [e_{12},e_{14}]=-e_7,\\
[e_{12},e_{16}]=e_{12}, & [e_{12},e_{20}]=-e_{12}, & [e_{12},e_{21}]=-2e_{10}, & [e_{13},e_{14}]=-e_{13},\\
[e_{13},e_{17}]=e_{13}, & [e_{19},e_{20}]=-2e_{19}, & [e_{19},e_{21}]=-2e_{20}, & [e_{20},e_{21}]=-2e_{21}.
\end{array} \tag{5.35}$$

We describe the symmetry algebra by the following proposition:

Proposition 8. *The symmetry Lie algebra is a twenty-one-dimensional Lie algebra. It is a semi-direct product of an eighteen-dimensional solvable Lie algebra and $sl(2,\mathbb{R})$. The nilradical is a thirteen-dimensional decomposable Lie algebra. In fact, the nilradical is a direct sum of $A_{5,1}$ in Winternitz [5] and $\mathbb{R}^8$. The nilradical has a five-dimensional abelian complement. Therefore, the symmetry algebra can be identified as $((A_{5,1}\oplus\mathbb{R}^8)\rtimes\mathbb{R}^5)\rtimes sl(2,\mathbb{R})$, where the nonzero brackets of $A_{5.1}$ are given by*

$$[e_3,e_5]=e_1,\quad [e_4,e_5]=e_2. \tag{5.36}$$

5.7. Algebra $A^{ab}_{6,25}$ (ab: $a^2+b^2\neq 0$)

The nonzero brackets for the algebra $A^{ab}_{6,25}$ are given by

$$\begin{array}{llll}
[e_1,e_4]=ae_4, & [e_1,e_5]=e_6, & [e_1,e_6]=-e_5, & [e_2,e_4]=be_4,\\
[e_2,e_5]=e_5, & [e_2,e_6]=e_6, & [e_1,e_2]=e_3. &
\end{array} \tag{5.37}$$

The geodesic equations are given by

$$\ddot{p}=\dot{p}(b\dot{w}+a\dot{z}),\quad \ddot{q}=-\dot{w}\dot{z},\quad \ddot{x}=\dot{x}\dot{z}-\dot{y}\dot{w},\quad \ddot{y}=\dot{x}\dot{w}-\dot{y}\dot{z},\quad \ddot{z}=0,\quad \ddot{w}=0. \tag{5.38}$$

For the general case $A^{a\neq 0,b\neq 0}_{6,25}$, the symmetry Lie algebra is spanned by

$$\begin{array}{lll}
e_1=D_t, & e_2=tD_q, & e_3=D_q,\\
e_4=D_y, & e_5=D_x, & e_6=D_p,\\
e_7=D_w, & e_8=D_z, & e_9=tD_t,\\
e_{10}=pD_p, & e_{11}=wD_q, & e_{12}=zD_q,\\
e_{13}=wD_t, & e_{14}=zD_t, & e_{15}=xD_x+yD_y,\\
e_{16}=(wz+2q)D_q, & e_{17}=(wz+2q)D_t, & e_{18}=e^{bw}e^{az}D_p.
\end{array} \tag{5.39}$$

We implement the following change of basis:

$$\begin{array}{llll}
\overline{e}_1=e_1, & \overline{e}_2=e_3, & \overline{e}_3=e_{13}, & \overline{e}_4=e_{11},\\
\overline{e}_5=-e_7+\frac{be_8}{a}, & \overline{e}_6=e_4, & \overline{e}_7=e_5, & \overline{e}_8=e_6,\\
\overline{e}_9=e_{11}+\frac{ae_{12}}{b}, & \overline{e}_{10}=e_{13}+\frac{ae_{14}}{b}, & \overline{e}_{11}=e_{18}, & \overline{e}_{12}=e_8-\frac{e_{11}}{2},\\
\overline{e}_{13}=e_9+\frac{e_{16}}{2}, & \overline{e}_{14}=e_{10}, & \overline{e}_{15}=e_{15}, & \overline{e}_{16}=e_2,\\
\overline{e}_{17}=e_9-\frac{e_{16}}{2}, & \overline{e}_{18}=e_{17}, & &
\end{array} \tag{5.40}$$

and the nonzero brackets of the symmetry algebra are given by

$$\begin{array}{lll}
[e_1,e_{13}]=e_1, & [e_1,e_{16}]=e_2, & [e_1,e_{17}]=e_1,\\
[e_2,e_{13}]=e_2, & [e_2,e_{17}]=-e_2, & [e_2,e_{18}]=2e_1,\\
[e_3,e_5]=e_1, & [e_3,e_{13}]=e_3, & [e_3,e_{16}]=e_4,\\
[e_3,e_{17}]=e_3, & [e_4,e_5]=e_2, & [e_4,e_{13}]=e_4,\\
[e_4,e_{17}]=-e_4, & [e_4,e_{18}]=2e_3, & [e_5,e_{12}]=\frac{e_2}{2},\\
[e_5,e_{13}]=-\frac{be_9}{2a}+\frac{be_4}{a}, & [e_5,e_{17}]=\frac{be_9}{2a}-\frac{be_4}{a}, & [e_5,e_{18}]=-\frac{be_{10}}{a}+\frac{2be_3}{a},\\
[e_6,e_{15}]=e_6, & [e_7,e_{15}]=e_7, & [e_8,e_{14}]=e_8,\\
[e_9,e_{12}]=-\frac{ae_2}{b}, & [e_9,e_{13}]=e_9, & [e_9,e_{17}]=-e_9,\\
[e_9,e_{18}]=2e_{10}, & [e_{10},e_{12}]=-\frac{ae_1}{b}, & [e_{10},e_{13}]=e_{10},\\
[e_{10},e_{16}]=e_9, & [e_{10},e_{17}]=e_{10}, & [e_{11},e_{12}]=-ae_{11},\\
[e_{11},e_{14}]=e_{11}, & [e_{16},e_{17}]=-2e_{16}, & [e_{16},e_{18}]=2e_{17},\\
[e_{17},e_{18}]=-2e_{18}. & &
\end{array} \tag{5.41}$$

We describe the symmetry algebra by the following proposition:.

Proposition 9. *The symmetry Lie algebra is an eighteen-dimensional Lie algebra. It is a semi-direct product of fifteen-dimensional solvable Lie algebra and* $sl(2,\mathbb{R})$*. The nilradical is an eleven-dimensional decomposable Lie algebra. In fact, the nilradical is a direct sum of* $A_{5,1}$ *in Winternitz [5] and* $\mathbb{R}^6$*. The nilradical has a four-dimensional abelian complement. Therefore, the symmetry algebra can be identified as* $((A_{5,1}\oplus\mathbb{R}^4)\rtimes\mathbb{R}^5)\rtimes sl(2,\mathbb{R})$*, where the nonzero brackets of* $A_{5.1}$ *are given by*

$$[e_3,e_5]=e_1,\quad [e_4,e_5]=e_2. \tag{5.42}$$

5.8. *Algebra* $A^a_{6,26}$

The nonzero brackets for the algebra $A^a_{6,26}$ are given by

$$[e_1,e_5]=ae_5+e_6,\quad [e_1,e_6]=ae_6-e_5,\quad [e_2,e_4]=e_4,\quad [e_1,e_2]=e_3. \tag{5.43}$$

The geodesic equations are given by

$$\ddot{p}=\dot{p}\dot{z},\quad \ddot{q}=\dot{w}\dot{z},\quad \ddot{x}=\dot{w}(a\dot{x}-\dot{y}),\quad \ddot{y}=\dot{w}(\dot{x}+a\dot{y}),\quad \ddot{z}=0,\quad \ddot{w}=0. \tag{5.44}$$

For the general case $A^{a\neq 0}_{6,26}$, the symmetry Lie algebra is spanned by

$$\begin{array}{llll}
e_1=D_t, & e_2=tD_q, & e_3=D_q, & e_4=D_x,\\
e_5=D_p, & e_6=D_y, & e_7=D_z, & e_8=D_w,\\
e_9=tD_t, & e_{10}=pD_p, & e_{11}=wD_q, & e_{12}=zD_q,\\
e_{13}=wD_t, & e_{14}=zD_t, & e_{15}=xD_x+yD_y, & e_{16}=e^zD_p,\\
e_{17}=yD_x-xD_y, & e_{18}=(wz+2q)D_q, & e_{19}=(wz+2q)D_t, & \\
e_{20}=e^{aw}\cos(w)D_x+e^{aw}\sin(w)D_y, & e_{21}=e^{aw}\sin(w)D_x-e^{aw}\cos(w)D_y. & &
\end{array} \tag{5.45}$$

We consider the following change of basis:

$$\begin{array}{lllll}
\overline{e}_1=e_1, & \overline{e}_2=e_3, & \overline{e}_3=e_4, & \overline{e}_4=e_5, & \overline{e}_5=e_6,\\
\overline{e}_6=e_{11}, & \overline{e}_7=e_{12}, & \overline{e}_8=e_{13}, & \overline{e}_9=e_{14}, & \overline{e}_{10}=e_{16},\\
\overline{e}_{11}=e_{20}, & \overline{e}_{12}=e_{21}, & \overline{e}_{13}=e_7+\frac{e_{11}}{2}, & \overline{e}_{14}=e_8+\frac{e_{12}}{2}, & \overline{e}_{15}=e_9-\frac{e_{18}}{2},\\
\overline{e}_{16}=e_{10}, & \overline{e}_{17}=e_{15}, & \overline{e}_{18}=e_{17}, & \overline{e}_{19}=e_2, & \overline{e}_{20}=e_9+\frac{e_{18}}{2},\\
\overline{e}_{21}=e_{19}. & & & &
\end{array} \tag{5.46}$$

The nonzero brackets of the symmetry algebra are given by

$$\begin{array}{lll}
[e_1, e_{15}] = e_1, & [e_1, e_{19}] = e_2, & [e_1, e_{20}] = e_1, \\
[e_2, e_{15}] = e_2, & [e_2, e_{20}] = -e_2, & [e_2, e_{21}] = -2e_1, \\
[e_3, e_{17}] = e_3, & [e_3, e_{18}] = -e_5, & [e_4, e_{16}] = e_4, \\
[e_5, e_{17}] = e_5, & [e_5, e_{18}] = e_3, & [e_6, e_{14}] = -e_2, \\
[e_6, e_{15}] = e_6, & [e_6, e_{20}] = -e_6, & [e_6, e_{21}] = -2e_8, \\
[e_7, e_{13}] = -e_2, & [e_7, e_{15}] = e_7, & [e_7, e_{20}] = -e_7, \\
[e_7, e_{21}] = -2e_9, & [e_8, e_{14}] = -e_1, & [e_8, e_{15}] = e_8, \\
[e_8, e_{19}] = e_6, & [e_8, e_{20}] = e_8, & [e_9, e_{13}] = -e_1, \\
[e_9, e_{15}] = e_9, & [e_9, e_{19}] = e_7, & [e_9, e_{20}] = e_9, \\
[e_{10}, e_{13}] = -e_{10}, & [e_{10}, e_{16}] = e_{10}, & [e_{11}, e_{14}] = -ae_{11} + e_{12}, \\
[e_{11}, e_{17}] = e_{11}, & [e_{11}, e_{18}] = e_{12}, & [e_{12}, e_{14}] = -ae_{12} - e_{11}, \\
[e_{12}, e_{17}] = e_{12}, & [e_{12}, e_{18}] = -e_{11}, & [e_{19}, e_{20}] = -2e_{19}, \\
[e_{19}, e_{21}] = -2e_{20}, & [e_{20}, e_{21}] = -2e_{21}. &
\end{array} \tag{5.47}$$

We describe the symmetry algebra by the following proposition:

Proposition 10. *The symmetry Lie algebra is a twenty-one-dimensional Lie algebra. It is a semi-direct product of an eighteen-dimensional solvable Lie algebra and $sl(2, \mathbb{R})$. The nilradical is twelve-dimensional abelian Lie algebra and has a six-dimensional abelian complement. Therefore, the symmetry algebra can be identified as:* $(\mathbb{R}^{12} \rtimes \mathbb{R}^6) \rtimes sl(2, \mathbb{R})$.

5.8.1. Case 1: $A_{6,26}^{a=0}$

The symmetry Lie algebra is spanned by

$$\begin{array}{lll}
e_1 = D_t, & e_2 = tD_q, & e_3 = D_q, \\
e_4 = D_p, & e_5 = D_x, & e_6 = D_y, \\
e_7 = D_z, & e_8 = D_w, & e_9 = tD_t, \\
e_{10} = pD_p, & e_{11} = wD_q, & e_{12} = zD_p, \\
e_{13} = wD_t, & e_{14} = zD_t, & e_{15} = xD_x + yD_y, \\
e_{16} = e^z D_p, & e_{17} = yD_x - xD_y, & e_{18} = \cos(w)D_x + \sin(w)D_y, \\
e_{19} = (wz - 2q)D_q, & e_{20} = (wz - 2q)D_t, & e_{21} = \sin(w)D_x - \cos(w)D_y, \\
\multicolumn{3}{l}{e_{22} = (-\cos(w)y + x\sin(w))D_x + (-\cos(w)x - y\sin(w))D_y,} \\
\multicolumn{3}{l}{e_{23} = (\cos(w)x + y\sin(w))D_x + (-\cos(w)y + x\sin(w))D_y.}
\end{array} \tag{5.48}$$

We implement the following change of basis

$$\begin{array}{llll}
\bar{e}_1 = e_1, & \bar{e}_2 = e_3, & \bar{e}_3 = e_4, & \bar{e}_4 = e_5, \\
\bar{e}_5 = e_6, & \bar{e}_6 = e_{11}, & \bar{e}_7 = e_{12}, & \bar{e}_8 = e_{13}, \\
\bar{e}_9 = e_{14}, & \bar{e}_{10} = e_{16}, & \bar{e}_{11} = e_{18}, & \bar{e}_{12} = e_{21}, \\
\bar{e}_{13} = e_7 + \frac{e_{11}}{2}, & \bar{e}_{14} = e_8 - \frac{e_{17}}{2} + \frac{e_{12}}{2}, & \bar{e}_{15} = e_9 - \frac{e_{19}}{2}, & \bar{e}_{16} = e_{10}, \\
\bar{e}_{17} = e_{15}, & \bar{e}_{18} = e_2, & \bar{e}_{19} = e_9 + \frac{e_{19}}{2}, & \bar{e}_{20} = e_{20}, \\
\bar{e}_{21} = e_{17}, & \bar{e}_{22} = e_{22}, & \bar{e}_{23} = e_{23}, &
\end{array} \tag{5.49}$$

and the nonzero brackets of the symmetry algebra are given by

$$
\begin{array}{llll}
[e_1,e_{15}]=e_1, & [e_1,e_{18}]=e_2, & [e_1,e_{19}]=-e_1, & [e_2,e_{15}]=e_2,\\
[e_2,e_{19}]=-e_2, & [e_2,e_{20}]=-2e_1, & [e_3,e_{16}]=e_3, & [e_4,e_{14}]=\frac{e_5}{2},\\
[e_4,e_{17}]=e_4, & [e_4,e_{21}]=-e_5, & [e_4,e_{22}]=e_{12}, & [e_4,e_{23}]=e_{11},\\
[e_5,e_{14}]=-\frac{e_4}{2}, & [e_5,e_{17}]=e_5, & [e_5,e_{21}]=e_4, & [e_5,e_{22}]=-e_{11},\\
[e_5,e_{23}]=e_{12}, & [e_6,e_{14}]=-e_2, & [e_6,e_{15}]=e_6, & [e_6,e_{19}]=-e_6,\\
[e_6,e_{20}]=-2e_8, & [e_7,e_{13}]=-e_2, & [e_7,e_{15}]=e_7, & [e_7,e_{19}]=-e_7,\\
[e_7,e_{20}]=-2e_9, & [e_8,e_{14}]=-e_1, & [e_8,e_{15}]=e_8, & [e_8,e_{18}]=e_6,\\
[e_8,e_{19}]=e_8, & [e_9,e_{13}]=-e_1, & [e_9,e_{15}]=e_9, & [e_9,e_{18}]=e_7,\\
[e_9,e_{19}]=e_9, & [e_{10},e_{13}]=-e_{10}, & [e_{10},e_{16}]=e_{10}, & [e_{11},e_{14}]=\frac{e_{12}}{2},\\
[e_{11},e_{17}]=e_{11}, & [e_{11},e_{21}]=e_{12}, & [e_{11},e_{22}]=-e_5, & [e_{11},e_{23}]=e_4,\\
[e_{12},e_{14}]=-\frac{e_{11}}{2}, & [e_{12},e_{17}]=e_{12}, & [e_{12},e_{21}]=-e_{11}, & [e_{12},e_{22}]=e_4,\\
[e_{12},e_{23}]=e_5, & [e_{18},e_{19}]=-e_{18}, & [e_{18},e_{20}]=-2e_{19}, & [e_{19},e_{20}]=-2e_{20},\\
[e_{21},e_{22}]=2e_{23}, & [e_{21},e_{23}]=-2e_{22}, & [e_{22},e_{23}]=-2e_{21}. &
\end{array}
\tag{5.50}
$$

We describe the symmetry algebra by the following proposition:

Proposition 11. *The symmetry Lie algebra is a twenty-three-dimensional Lie algebra. It is a semi-direct product of a seventeen-dimensional solvable Lie algebra and two copies of* $sl(2,\mathbb{R})$*. Furthermore, the symmetry Lie algebra has a twelve-dimensional abelian nilradical and five-dimensional abelian complement. Therefore, the symmetry algebra can be identified as*

$$(\mathbb{R}^{12} \rtimes \mathbb{R}^5) \rtimes (sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})).$$

5.9. *Algebra* $A^{\epsilon}_{6,27}$*:* $(\epsilon = 0, 1)$

The nonzero brackets for the algebra $A^{\epsilon}_{6,27}$ are given by

$$[e_1,e_3]=e_4,\quad [e_1,e_5]=e_6,\quad [e_1,e_6]=-e_5,\quad [e_2,e_5]=e_5,\quad [e_2,e_6]=e_6,\quad [e_1,e_2]=\epsilon e_3. \tag{5.51}$$

5.9.1. $A^{\epsilon=0}_{6,27}$

The geodesic equations where $\epsilon = 0$ are given by

$$\ddot{p}=\dot{p}\dot{w}-\dot{q}\dot{z},\quad \ddot{q}=\dot{p}\dot{z}+\dot{q}\dot{w},\quad \ddot{x}=0,\quad \ddot{y}=\dot{x}\dot{z},\quad \ddot{z}=0,\quad \ddot{w}=0. \tag{5.52}$$

The symmetry Lie algebra is spanned by

$$
\begin{array}{ll}
e_1 = D_t, & e_2 = tD_y,\\
e_3 = D_y, & e_4 = D_p,\\
e_5 = D_q, & e_6 = D_x,\\
e_7 = D_w, & e_8 = D_z,\\
e_9 = tD_t, & e_{10} = wD_t,\\
e_{11} = xD_t, & e_{12} = zD_t,\\
e_{13} = wD_y, & e_{14} = xD_y,\\
e_{15} = zD_y, & e_{16} = pD_p + qD_q,\\
e_{17} = xD_x + yD_y, & e_{18} = qD_p - pD_q,\\
e_{19} = tD_x + \frac{tz}{2}D_y, & e_{20} = zD_x + \frac{z^2}{2}D_y,\\
e_{21} = (xz-2y)D_t, & e_{22} = (xz-2y)D_y,\\
e_{23} = wD_x + \frac{wz}{2}D_y, & e_{24} = e^w\cos(z)D_p + e^w\sin(z)D_q,\\
e_{25} = e^w\sin(z)D_p - e^w\cos(z)D_q, & e_{26} = (xz-2y)D_x + (\frac{xz^2}{2} - yz)D_y.
\end{array}
\tag{5.53}
$$

We implement the following change of basis:

$$\begin{aligned}
&\bar{e}_1 = e_1, && \bar{e}_2 = e_3, && \bar{e}_3 = e_4, && \bar{e}_4 = e_5,\\
&\bar{e}_5 = e_6, && \bar{e}_6 = e_{10}, && \bar{e}_7 = e_{12}, && \bar{e}_8 = e_{13},\\
&\bar{e}_9 = e_{15}, && \bar{e}_{10} = e_{20}, && \bar{e}_{11} = e_{23}, && \bar{e}_{12} = e_{24},\\
&\bar{e}_{13} = e_{25}, && \bar{e}_{14} = e_7, && \bar{e}_{15} = e_8 + \tfrac{e_{14}}{2}, && \bar{e}_{16} = e_9 + e_{17},\\
&\bar{e}_{17} = e_{16}, && \bar{e}_{18} = e_{18}, && \bar{e}_{19} = e_2, && \bar{e}_{20} = e_9 + \tfrac{e_{22}}{2},\\
&\bar{e}_{21} = e_{11}, && \bar{e}_{22} = e_{14}, && \bar{e}_{23} = e_{17} + e_{22}, && \bar{e}_{24} = e_{19},\\
&\bar{e}_{25} = e_{21}, && \bar{e}_{26} = e_{26}, && &&
\end{aligned} \tag{5.54}$$

and the nonzero brackets of the symmetry algebra are given by

$$\begin{aligned}
&[e_1,e_{16}] = e_1, && [e_1,e_{19}] = e_2, && [e_1,e_{20}] = e_1,\\
&[e_1,e_{24}] = e_5 + \tfrac{e_9}{2}, && [e_2,e_{16}] = e_2, && [e_2,e_{20}] = -e_2,\\
&[e_2,e_{23}] = -e_2, && [e_2,e_{25}] = -2e_1, && [e_2,e_{26}] = -2e_5 - e_9,\\
&[e_3,e_{17}] = e_3, && [e_3,e_{18}] = -e_4, && [e_4,e_{17}] = e_4,\\
&[e_4,e_{18}] = e_3, && [e_5,e_{15}] = \tfrac{e_2}{2}, && [e_5,e_{16}] = e_5,\\
&[e_5,e_{20}] = \tfrac{e_9}{2}, && [e_5,e_{21}] = e_1, && [e_5,e_{22}] = e_2,\\
&[e_5,e_{23}] = e_5 + e_9, && [e_5,e_{25}] = e_7, && [e_5,e_{26}] = e_{10},\\
&[e_6,e_{14}] = -e_1, && [e_6,e_{16}] = e_6, && [e_6,e_{19}] = e_8,\\
&[e_6,e_{20}] = e_6, && [e_6,e_{24}] = e_{11}, && [e_7,e_{15}] = -e_1,\\
&[e_7,e_{16}] = e_7, && [e_7,e_{19}] = e_9, && [e_7,e_{20}] = e_7,\\
&[e_7,e_{24}] = e_{10}, && [e_8,e_{14}] = -e_2, && [e_8,e_{16}] = e_8,\\
&[e_8,e_{20}] = -e_8, && [e_8,e_{23}] = -e_8, && [e_8,e_{25}] = -2e_6,\\
&[e_8,e_{26}] = -2e_{11}, && [e_9,e_{15}] = -e_2, && [e_9,e_{16}] = e_9,\\
&[e_9,e_{20}] = -e_9, && [e_9,e_{23}] = -e_9, && [e_9,e_{25}] = -2e_7,\\
&[e_9,e_{26}] = -2e_{10}, && [e_{10},e_{15}] = -e_5 - \tfrac{e_9}{2}, && [e_{10},e_{16}] = e_{10},\\
&[e_{10},e_{21}] = e_7, && [e_{10},e_{22}] = e_9, && [e_{10},e_{23}] = e_{10},\\
&[e_{11},e_{14}] = -e_5 - \tfrac{e_9}{2}, && [e_{11},e_{16}] = e_{11}, && [e_{11},e_{21}] = e_6,\\
&[e_{11},e_{22}] = e_8, && [e_{11},e_{23}] = e_{11}, && [e_{12},e_{14}] = -e_{12},\\
&[e_{12},e_{15}] = e_{13}, && [e_{12},e_{17}] = e_{12}, && [e_{12},e_{18}] = e_{13},\\
&[e_{13},e_{14}] = -e_{13}, && [e_{13},e_{15}] = -e_{12}, && [e_{13},e_{17}] = e_{13},\\
&[e_{13},e_{18}] = -e_{12}, && [e_{19},e_{20}] = -2e_{19}, && [e_{19},e_{21}] = -e_{22},\\
&[e_{19},e_{23}] = -e_{19}, && [e_{19},e_{25}] = -2e_{20}, && [e_{19},e_{26}] = -2e_{24},\\
&[e_{20},e_{21}] = -e_{21}, && [e_{20},e_{22}] = e_{22}, && [e_{20},e_{24}] = e_{24},\\
&[e_{20},e_{25}] = -2e_{25}, && [e_{20},e_{26}] = -e_{26}, && [e_{21},e_{23}] = -e_{21},\\
&[e_{21},e_{24}] = -e_{20} + e_{23}, && [e_{21},e_{26}] = -e_{25}, && [e_{22},e_{23}] = -2e_{22},\\
&[e_{22},e_{24}] = -e_{19}, && [e_{22},e_{25}] = -2e_{21}, && [e_{22},e_{26}] = -2e_{23},\\
&[e_{23},e_{24}] = -e_{24}, && [e_{23},e_{25}] = -e_{25}, && [e_{23},e_{26}] = -2e_{26},\\
&[e_{24},e_{25}] = -e_{26}. && &&
\end{aligned} \tag{5.55}$$

We describe the symmetry algebra by the following proposition:

Proposition 12. *The symmetry Lie algebra is a twenty-six-dimensional semi-direct product of an eighteen solvable Lie algebra and eight-dimensional semi-simple* $sl(3,\mathbb{R})$. *Furthermore, the symmetry Lie algebra has a thirteen-dimensional abelian nilradical. Therefore, the symmetry algebra can be identified as:* $(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3,\mathbb{R})$.

5.9.2. $A_{6,27}^{\epsilon=1}$

The geodesic equations where $\epsilon = 1$ are given by

$$\ddot{p} = \dot{q}\dot{w}, \quad \ddot{q} = \dot{z}\dot{w}, \quad \ddot{x} = \dot{z}\dot{x} - \dot{w}\dot{y}, \quad \ddot{y} = \dot{z}\dot{y} + \dot{w}\dot{x}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{5.56}$$

The symmetry Lie algebra is spanned by

$$\begin{array}{llll} e_1 = D_z, & e_2 = D_p, & e_3 = D_x, & e_4 = D_w, \\ e_5 = D_y, & e_6 = D_q, & e_7 = tD_t, & e_8 = D_t, \\ e_9 = tD_p, & e_{10} = zD_p, & e_{11} = wD_p, & e_{12} = wD_t, \\ e_{13} = zD_t, & e_{14} = qD_p + zD_q, & e_{15} = xD_x + yD_y, & e_{16} = yD_x - xD_y, \\ e_{17} = twD_p + 2tD_q, & e_{18} = \frac{w^2}{2}D_p + wD_q, & e_{19} = wzD_p + 2zD_q, & e_{20} = (wz - 2q)D_t, \\ e_{21} = e^z\cos(w)D_x + e^z\sin(w)D_y, & e_{22} = e^z\sin(w)D_x - e^z\cos(w)D_y, & & \\ e_{23} = (qw - \frac{zw^2}{2})D_p + (-wz + 2q)D_q. & & & \end{array} \tag{5.57}$$

We implement the following change of basis:

$$\begin{array}{llll} \overline{e}_1 = e_2, & \overline{e}_2 = e_6, & \overline{e}_3 = e_8, & \overline{e}_4 = e_9, \\ \overline{e}_5 = e_{10}, & \overline{e}_6 = e_{11}, & \overline{e}_7 = e_{12}, & \overline{e}_8 = e_{13}, \\ \overline{e}_9 = e_{14}, & \overline{e}_{10} = e_{18}, & \overline{e}_{11} = e_{19}, & \overline{e}_{12} = e_3, \\ \overline{e}_{13} = e_5, & \overline{e}_{14} = e_{21}, & \overline{e}_{15} = e_{22}, & \overline{e}_{16} = e_1, \\ \overline{e}_{17} = e_4, & \overline{e}_{18} = e_7 + \frac{e_{23}}{2}, & \overline{e}_{19} = e_{15}, & \overline{e}_{20} = e_{16}, \\ \overline{e}_{21} = e_7 - \frac{e_{23}}{2}, & \overline{e}_{22} = e_{17}, & \overline{e}_{23} = e_{20}, & \end{array} \tag{5.58}$$

and the nonzero brackets of the symmetry algebra are given by

$$\begin{array}{lll} [e_2, e_9] = e_1, & [e_2, e_{18}] = e_2 + \frac{e_6}{2}, & [e_2, e_{21}] = -e_2 - \frac{e_6}{2}, \\ [e_2, e_{23}] = -2e_3, & [e_3, e_4] = e_1, & [e_3, e_{18}] = e_3, \\ [e_3, e_{21}] = e_3, & [e_3, e_{22}] = 2e_2 + e_6, & [e_4, e_7] = -e_6, \\ [e_4, e_8] = -e_5, & [e_4, e_{18}] = -e_4, & [e_4, e_{21}] = -e_4, \\ [e_4, e_{23}] = -e_{11} + 2e_9, & [e_5, e_{16}] = -e_1, & [e_6, e_{17}] = -e_1, \\ [e_7, e_{17}] = -e_3, & [e_7, e_{18}] = e_7, & [e_7, e_{21}] = e_7, \\ [e_7, e_{22}] = 2e_{10}, & [e_8, e_{16}] = -e_3, & [e_8, e_{18}] = e_8, \\ [e_8, e_{21}] = e_8, & [e_8, e_{22}] = e_{11}, & [e_9, e_{10}] = -e_6, \\ [e_9, e_{11}] = -2e_5, & [e_9, e_{16}] = -e_2, & [e_9, e_{18}] = e_{11} - e_9, \\ [e_9, e_{21}] = -e_{11} + e_9, & [e_9, e_{22}] = -2e_4, & [e_9, e_{23}] = -2e_8, \\ [e_{10}, e_{17}] = -e_2 - e_6, & [e_{10}, e_{18}] = e_{10}, & [e_{10}, e_{21}] = -e_{10}, \\ [e_{10}, e_{23}] = -2e_7, & [e_{11}, e_{16}] = -2e_2 - e_6, & [e_{11}, e_{17}] = -e_5, \\ [e_{11}, e_{18}] = e_{11}, & [e_{11}, e_{21}] = -e_{11}, & [e_{11}, e_{23}] = -4e_8, \\ [e_{12}, e_{19}] = e_{12}, & [e_{12}, e_{20}] = -e_{13}, & [e_{13}, e_{19}] = e_{13}, \\ [e_{13}, e_{20}] = e_{12}, & [e_{14}, e_{16}] = -e_{14}, & [e_{14}, e_{17}] = e_{15}, \\ [e_{14}, e_{19}] = e_{14}, & [e_{14}, e_{20}] = e_{15}, & [e_{15}, e_{16}] = -e_{15}, \\ [e_{15}, e_{17}] = -e_{14}, & [e_{15}, e_{19}] = e_{15}, & [e_{15}, e_{20}] = -e_{14}, \\ [e_{16}, e_{18}] = -\frac{e_{10}}{2}, & [e_{16}, e_{21}] = \frac{e_{10}}{2}, & [e_{16}, e_{23}] = e_7, \\ [e_{17}, e_{18}] = -\frac{e_{11}}{2} + \frac{e_9}{2}, & [e_{17}, e_{21}] = \frac{e_{11}}{2} - \frac{e_9}{2}, & [e_{17}, e_{22}] = e_4, \\ [e_{17}, e_{23}] = e_8, & [e_{21}, e_{22}] = 2e_{22}, & [e_{21}, e_{23}] = -2e_{23}, \\ [e_{22}, e_{23}] = -4e_{21}. & & \end{array} \tag{5.59}$$

We describe the symmetry algebra by the following proposition:

Proposition 13. *The symmetry Lie algebra is a twenty-three-dimensional semi-direct product of twenty-dimensional solvable Lie algebra* $S_{2,20}$*, and* $sl(2,\mathbb{R})$*. The nilradical is a fifteen-dimensional nilpotant Lie algebra* $N_{2,11} \oplus \mathbb{R}^4$*, which is a direct sum of* $N_{2,11}$*, an eleven-dimensional nilpotent Lie algebra, and a four-dimensional abelian Lie algebra* $\mathbb{R}^4$*. The complement of the nilradical is four-dimensional non-abelian. Therefore, the symmetry Lie algebra can be identified as* $S_{2,20} \rtimes sl(2,\mathbb{R})$*.*

6. Conclusions and future work

In this work, we have investigated the symmetry Lie algebra of the geodesic equations of the canonical connection on a Lie group corresponding to the eight classes of Lie algebra $A_{6,20}$–$A_{6,27}$ in [7]. In each case, we list the nonzero brackets of the given Lie algebra, the geodesic equations, and a basis for the symmetry Lie algebra in terms of vector fields. For every symmetry Lie algebra, we identify its nilradical, solvable complement, and semi-simple factor; a summary of our results is given in Table 1. In future work, we plan to study the symmetry Lie algebras for the rest of the six-dimensional Lie algebras $A_{6,28}$–$A_{6,40}$ in [7]. The results help to put symmetry Lie algebras into context since they are of very high dimension. It remains to use the symmetries to help integrate the geodesic equations. Another useful by-product is the construction of many large dimensional Levi decomposition Lie algebras, which is a topic of independent interest.

Table 1. Six-dimensional Lie algebras and identification of the symmetry algebra.

Six-dimensional Lie algebras	Dimension	Identification
$A^{ab}_{6,20}$ $(ab : a^2 + b^2 \neq 0)$	21	$(\mathbb{R}^{12} \rtimes \mathbb{R}^6) \rtimes sl(2,\mathbb{R})$
$A^{a}_{6,21}$	21	$((A_{5,1} \oplus \mathbb{R}^8) \rtimes \mathbb{R}^5) \rtimes sl(2,\mathbb{R})$
$A^{\epsilon=0}_{6,22}$	26	$(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3,\mathbb{R})$
$A^{\epsilon=1}_{6,22}$	23	$(R^{15} \rtimes \mathbb{R}^5) \rtimes sl(2,\mathbb{R})$
$A^{a,\epsilon=0}_{6,23}$	26	$(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3,\mathbb{R})$
$A^{a,\epsilon=1}_{6,23}$	23	$S_{1,20} \rtimes sl(2,\mathbb{R})$
$A_{6,24}$	21	$((A_{5.1} \oplus \mathbb{R}^8) \rtimes \mathbb{R}^5) \rtimes sl(2,\mathbb{R})$
$A^{ab}_{6,25}$ $(ab : a^2 + b^2 \neq 0)$	18	$((A_{5,1} \oplus \mathbb{R}^4) \rtimes \mathbb{R}^5) \rtimes sl(2,\mathbb{R})$
$A^{a}_{6,26}$	21	$(\mathbb{R}^{12} \rtimes \mathbb{R}^6) \rtimes sl(2,\mathbb{R})$
$A^{a=0}_{6,26}$	23	$(\mathbb{R}^{12} \rtimes \mathbb{R}^5) \rtimes (sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}))$
$A^{\epsilon=0}_{6,27}$	26	$(\mathbb{R}^{13} \rtimes \mathbb{R}^5) \rtimes sl(3,\mathbb{R})$
$A^{\epsilon=1}_{6,27}$	23	$S_{2,20} \rtimes sl(2,\mathbb{R})$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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