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Research article

Solitonic effect on relativistic string cloud spacetime attached with strange quark matter

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Abstract: In this research paper, we discussed some geometric axioms of a relativistic string cloud spacetime attached with strange quark matter. We determined the conformal η-Ricci soliton on a relativistic string cloud spacetime attached with strange quark matter with a $\varphi(\mathcal{R}ic)$-vector field. In addition, we illustrated some physical significance of conformal pressure P in terms of conformal η-Ricci soliton with the same vector field. Besides this, we deduced a generalized Liouville equation from the conformal η-Ricci soliton. Furthermore, we examine the harmonic relevance of conformal η-Ricci soliton on string cloud spacetime attached with strange quark matter with a potential function ψ. Finally, we turned up a necessary and sufficient condition for the 1-form η, which is the g-dual of the vector field γ on a string cloud spacetime attached with strange quark matter to be a solution for the Schrödinger-Ricci equation.

Keywords: conformal η-Ricci soliton; string cloud spacetime; strange quark matter; $\varphi(\mathcal{R}ic)$-vector field; Schrödinger-Ricci equation
Mathematics Subject Classification: 53B30, 53C44, 53C50, 53C80

1. Introduction

The general theory of relativity (GTR) states that space curves because of matter. In an attempt to provide an explanation for events that Newtonian physics might not be able to adequately explain, Einstein put forth radical revisions to the way that people think about time, space, and gravity. Einstein attempted to find a unified field theory, in which the properties of all matter and energy could be expressed in a single Eq (1.1).

Einstein formulated the GTR in 1915, which explains how the force of gravity arises from the curvature of spacetime caused by mass. The famous Einstein field equation (EFEs) represents the gravitational field equation that governs the scale of spacetime. Within Einstein's equation, the cosmological constant is commonly used to describe the present state of the universe. Moreover, a version of Einstein's gravitational field equation without the cosmological constant can be expressed as stated in [1].

$$S - \frac{R}{2} g = \kappa \mathcal{T}, \tag{1.1}$$

where g denotes the Riemannian metric and $\mathcal{T}$ is the energy-momentum tensor.

In order to achieve Einstein's objective of a static universe, the Ricci tensor S and scalar curvature R of spacetime play crucial roles. The gravitational constant, denoted as κ, is chosen to be $8\pi G$, where G represents the universal gravitational constant.

Moreover, the importance of GTR is rich with possibilities for further exploration. Mathematical relativists seek to understand the nature of singularities and the fundamental properties of Einstein's equations, while numerical relativists run increasingly powerful computer simulations such as those describing merging black holes [2].

There are several results that open important applications in connection with another concept for example eigenproblems [3]. The eigenproblem was first formulated as a result of research on rigid body motion, which is directly related to planet motion. The general goal of an eigenproblem is to minimize, within certain bounds, the greatest eigenvalue of a matrix that relies affinely on a variable. The topic was studied by Euler [4], Lagrange [5], Laplace [6], Fourier [7], and Cauchy [8].

In addition, to gain the necessary S and R, as well as incorporate the cosmological constant, one must employ Einstein's equation. This equation provides the mathematical framework for understanding the relationship between the distribution of matter and energy in the universe and the curvature of spacetime.

Alternatively, the universe can be described using a one-dimensional entity known as a string. These cosmic strings, which can exist throughout the universe, are believed to have a significant connection to the ongoing expansion of the universe [9]. String theory, which predicts quantum gravity, interprets particles and the fundamental forces as vibrations of tiny, super-symmetric strings. Consequently, investigations into the gravitational effects of string-like matter have emerged.

To explore the relationship between black hole entropy and string state counting, Letelier initially examined general solutions of string clouds that exhibit spherical symmetry [10]. These solutions were then extended to Einstein-Gauss-Bonnet theory in the Letelier spacetime [12] and third-order Lovelock gravity [13]. Additionally, numerous other extended solutions have been investigated in this context [14–16].

The spacetime framework in the GTR and cosmology both involve the representation of a time-constrained, four-dimensional connected Lorentzian manifold. This particular classification of pseudo-Riemannian manifolds with a Lorentzian metric of signature $(-, +, +, +)$ is essential in GTR [1, 17]. In order to analyze the behavior of vectors within this manifold, the geometry of Lorentzian manifolds is introduced, making them a powerful tool for studying GTR.

If the Ricci tensor takes a specific form, Lorentzian manifolds are referred to as quasi-Einstein manifolds, and in the context of perfect fluid spacetime, they are known as perfect fluid spacetime [18, 19].

$$S = A_1 g + A_2 \eta \otimes \eta. \tag{1.2}$$

In this context, the presence of scalars A_1 and A_2 is notable, along with the existence of a 1-form η that shares metric properties with a unit time-like vector field. Furthermore, the spacetime manifold is characterized as a Lorentzian manifold, permitting the existence of a vector field that resembles time.

Chaki laid out the concept of a generalized quasi-Einstein manifold (GQE) [20].

Definition 1.1. *[20] A non-flat Riemannian manifold* (M^n, g) $(n > 2)$ *is said to be a generalized quasi-Einstein Lorentzian manifold* (GQE) *if its Ricci tensor* S *of type* $(0, 2)$ *is nonzero and satisfies the expression*

$$S = A_1 g + A_2 \eta \otimes \eta + A_3 \theta \otimes \theta, \tag{1.3}$$

wherein A_1, A_2 and A_3 are scalars of which $A_2 \neq 0$, $A_3 \neq 0$ and η, θ are 1-from such that

$$g(p, \gamma) = \eta(p), \quad g(p, \zeta) = \theta(p)$$

for any vector field $p \in \chi(M^n, g)$.

The unit vectors γ and ζ, associated with the 1-forms η and θ respectively, are mutually orthogonal. Furthermore, γ and ζ serve as the generators of the manifold. When $A_3 = 0$, the manifold (M^n, g) simplifies to a perfect fluid spacetime.

Formally, the GTR can be reformulated using the effective momentum tensor [21]. This momentum tensor describes the energy density, isotropic pressure, anisotropic pressure, and energy flow in the presence of a suitable time-like vector field [17]. As suggested by [22] and [23], the structure of the momentum tensor resembles that of an imperfect fluid, specifically a viscous fluid spacetime's energy-momentum tensor. This concept of imperfect fluid spacetime extends beyond the conventional model and provides additional features that can be utilized in models such as perfect fluid spacetime.

In contrast, physical matter symmetry in the GTR is specifically related to spacetime geometry. More specifically, the classification of solutions to Einstein's field equations is typically made simpler by the metric of symmetry. Soliton, which is related to the geometrical flow of spacetime geometry, is a significant symmetry [24].

Hamilton [25] first time introduced the concept of Ricci flow in 1988. The limit of the solutions to the Ricci flow is revealed to be the Ricci soliton. Moreover, the classification of solutions that are self-similar to geometric flows has received a lot of attention recently.

In [26], Fischer introduced a novel geometric flow called conformal Ricci flow, which is a modification of the classical Ricci flow. This variant replaces the unit volume constraint in the original equation with a scalar curvature constraint. A specific set of solutions, where the metric evolves through dilation and diffeomorphisms, plays a crucial role in studying the singularities of the flow as they serve as acceptable models for singularities. These solutions are also referred to as solitons.

The equation for the conformal Ricci flow, as demonstrated in [26], is given by:

$$\frac{\partial}{\partial t}g(t) = -2Sg(t) - \left(P + \frac{2}{n}\right)g(t). \tag{1.4}$$

Here, $R(g) = -1$ represents the scalar curvature of the manifold (M^n, g), and P is a non-dynamical scalar field, which is dependent on time t, and the dimension of the manifold is denoted by n.

The Navier-Stokes equations of fluid mechanics are equivalent to the conformal Ricci flow equations, and as a result, the time-dependent scalar field P is referred to as a conformal pressure, which, like the actual physical pressure in fluid mechanics, maintains the fluid's incompressibility. The conformal Ricci flow equations' equilibrium points are Einstein metrics with the Einstein constant. $\frac{-1}{n}$.

In 2015, Basu and Bhattacharyya [27] introduced the concept of a conformal Ricci soliton, and the corresponding equation is given as follows.

$$\frac{1}{2}\mathfrak{L}_F g + S + \left[\Lambda - \left(\frac{P}{2} + \frac{1}{n}\right)\right]g = 0. \tag{1.5}$$

If the data $(g, F, \Lambda - \left(P + \frac{2}{n}\right))$ satisfies the Eq (1.5), it is referred to as a conformal Ricci soliton on M [29]. In this context, Λ represents a real constant, and $\mathfrak{L}_F$ denotes the Lie derivative operator along the vector field F. The conformal Ricci soliton can be classified as shrinking, steady, or expanding if

1) $\Lambda < 0$,
2) $\Lambda = 0$ and
3) $\Lambda > 0$, respectively.

A more general concept known as conformal η-Ricci soliton (conformal η-RS) was introduced by Siddiqi [28] and is denoted by the following expression.

$$\mathfrak{L}_F g + 2S + \left[2\Lambda - \left(P + \frac{2}{n}\right)\right]g + 2\Omega\eta \otimes \eta = 0, \tag{1.6}$$

wherein $\mathfrak{L}_F$ is the Lie derivative along the direction of the soliton vector field F, S is the Ricci tensor, n is the dimension of the manifold, Ω is a real constant. The conformal η-Ricci soliton in particular simplifies to the conformal Ricci soliton if $\Omega = 0$ [32].

Authors in [29] investigated spacetime in terms of the Ricci soliton. Blaga used η-Ricci and η-Einstein solitons to show the features of the ideal fluid spacetime in [30]. Venkatesha and Aruna also addressed about Ricci solitons on ideal fluid spacetime in [31]. Siddiqi [28] presented the concept of conformal η-Ricci solitons in 2018. Some properties of perfect fluid spacetime with almost Ricci-Bourguignon soliton and conformal η-Ricci solitons were investigated by Siddiqi and his coauthors in [32, 33]. Furthermore, using Ricci-Yamabe solitions and Ricci soliton, respectively, Siddiqi et al. investigated thermodynamical fluid spacetime [34] and magneto-fluid spacetime [35]. In [36], Alkhaldi et al. also worked on conformal η-Ricci solitons. Alkhaldi and collaborators [37] have recently explored Ricci-Yamabe solitons on imperfect fluid generalized Robertson Walker spacetime.

As a result, motivated by previous research, we investigate relativistic string cloud spacetime attached to strange quark matter in terms of a conformal η-Ricci soliton in this paper.

2. String cloud spacetime

This section discusses the essential characteristics of the spacetime filled with the energy-momentum tensor of the string cloud type, referred to as "string cloud spacetime" [11]. It is widely recognized that space matter can be understood as a fluid capable of encompassing various substances within spacetime, such as density and string tension [12]. The energy-momentum tensor associated with the string cloud plays a significant role in standard cosmological models, where the material composition of the universe is modeled as behaving akin to a string cloud spacetime [38]. Furthermore, when considering a perfect fluid spacetime, the presence of heat conduction and viscosity is absent. However, for the purposes of this study, we propose the assumption that the growth of spacetime is influenced by the energy-momentum tensor of the string cloud [9].

As seen in the following equation, the string cloud energy-momentum tensor [9, 38]

$$\mathcal{T}(p,q) = \rho\eta(p)\eta(q) - \lambda\theta(p)\theta(q), \tag{2.1}$$

where $p, q \in \chi(M^4, g)$, ρ is the energy density for the string cloud fluid with particles attached to them, the string tension is λ, and they are connected by

$$\rho = \rho_0 + \lambda, \tag{2.2}$$

wherein ρ_0 is the rest energy density of particles.

The concept of a compact star or a quark star, which is sustained by the degenerate pressure of quark matter, has been raised for a star that is smaller than neutron stars. Numerous authors have studied such a quark star. According to Alcock et al. [39] and Haensel et al. [40], some neutron stars may actually be weird stars made completely of strange materials. Researcher Cheng et al. [41] explored the features of strange quark stars. Yavuz et al. [42] examined the strange quark matter that is connected to the string cloud in a spacetime that is spherically symmetric and allows conformal motion.

The equation of state (EoS) employed to model quark matter is often based on the phenomenological bag model. In this model, quarks are conceptualized as a degenerate Fermi gas confined to a region of space with vacuum energy density V_{ed}. Within the framework of this model, the quark matter consists of electrons, massless quarks u, massive quarks s, and quarks d. According to the bag model [43], when quarks are massless and noninteracting, there will be a quark pressure present.

$$p_Q = \frac{\rho_Q}{3}, \tag{2.3}$$

where ρ_Q is the quark energy density.

In addition, the total energy density and the total pressure are given as

$$\rho_t = \rho_Q + V_{ed}, \tag{2.4}$$

$$p_t = p_Q - V_{ed}. \tag{2.5}$$

The strange quark matter EoS is [44, 47]

$$p_t = \frac{1}{3}(\rho_t - 4V_{ed}). \tag{2.6}$$

Various vibrational models [45] are seen as different masses or spins because the string is free to vibrate. We shall therefore focus on quarks rather than the particles in the cloud of string. In this instance, we obtain from (2.2)

$$\rho = \rho_Q + \lambda + V_{ed}. \tag{2.7}$$

We can deduce the energy-momentum tensor for the *strange quark matter attached to the string cloud* from (2.1) and (2.7) by writing [42].

$$\mathcal{T}(p,q) = (\rho_Q + \lambda + V_{ed})\eta(p)\eta(q) - \lambda\theta(p)\theta(q), \tag{2.8}$$

where $\eta(p)$ is the string's four-velocity and $\theta(p)$ is the direction vector of the anisotropy. Moreover, string cloud spacetime attached to strange quark matter, admitting the unit space-like vector field θ perpendicular to the unit time-like vector field η, such that

$$\eta(p) = g(p,\gamma) \text{ and } \theta(q) = g(q,\zeta)$$

are two nonzero 1-form. Also, γ and ζ are orthogonal vector fields that hold

$$g(\gamma,\gamma) = -1 = \eta(\gamma) \text{ and } g(\zeta,\zeta) = 1 = \theta(\zeta),\ g(\zeta,\gamma) = 0.$$

Considering the Eqs (1.1) and (2.8), we can derive the EFEs for a relativistic string cloud spacetime coupled with strange quark matter as follows:

$$S(p,q) = \frac{R}{2}g(p,q) + \kappa(\rho_Q + \lambda + V_{ed})\eta(p)\eta(q) - \kappa\lambda\theta(p)\theta(q). \tag{2.9}$$

Equation (2.9) follows that the relativistic string cloud spacetime attached with strange quark matter under consideration is a Lorentzian $(GQE)_4$ manifold with $\frac{R}{2}$, $\kappa(\rho_Q + \lambda + V_{ed})$, and $\kappa\lambda$ as associated scalars, η, and θ as associated 1-forms.

The relativistic string cloud spacetime connected to strange quark matter is characterized by the presence of the vacuum energy density V_{ed}, the quark energy density of the fluid ρ, and the string tension λ, all of which satisfy the EFEs. Therefore, we can express the following.

Theorem 2.1. *A string cloud spacetime attached with strange quark matter obeys the EFEs with vacuum energy density V_{ed}, the quark energy density of the fluid ρ and the string tension λ is a $(GQE)_4$-spacetime.*

Following the contraction of Eq (2.9), we find the following:

Theorem 2.2. *If a string cloud spacetime attached with strange quark matter obeying the EFEs with vacuum energy density V_{ed} and quark energy density ρ_Q, then the scalar curvature is*

$$R = -\kappa(\rho_Q + V_{ed}). \tag{2.10}$$

The conclusions of Theorems 2.1 and 2.2 can be summarized as follows:

Corollary 2.3. *A string cloud spacetime attached with strange quark matter obeys the EFEs with constant scalar curvature R is a bulk viscous fluid spacetime [46].*

Since γ and ζ are orthogonal unit vector fields and $g(\gamma, \zeta) = 0$, we obtain

$$S(p, \gamma) = (A_1 + A_2)\eta(p), \tag{2.11}$$

$$S(p, \zeta) = (A_1 + A_3)\theta(p), \tag{2.12}$$

where $A_1 = \frac{R}{2}$, $A_2 = \kappa(\rho_Q + \lambda + V_{ed})$, and $A_3 = -\kappa\lambda$.

Now, in view of (2.2) and (2.10) one can state the following.

Theorem 2.4. *If a string cloud spacetime attached with strange quark matter obeys EFEs with constant scalar curvature R and satisfies relation (2.2), then under this situation energy density ρ of the cloud fluid is $\frac{1}{2}\left(3\rho_0 + \frac{\mathcal{R}}{\kappa}\right)$ and string tension λ is $\frac{1}{2}\left(\rho_0 + \frac{\mathcal{R}}{\kappa}\right)$.*

Next, in light of (2.3)–(2.6), we turn up the following values:

$$\rho_Q = -\left(\frac{R}{\kappa} + V_{ed}\right),\ p_Q = -\frac{1}{3}\left(\frac{R}{\kappa} + V_{ed}\right), \tag{2.13}$$

and

$$\rho_t = -\frac{R}{\kappa},\ p_t = -\frac{1}{3}\left(\frac{R}{\kappa} - 4V_{ed}\right). \tag{2.14}$$

Thus, we can articulate the following results.

Theorem 2.5. *In the bag model, if a relativistic string cloud spacetime attached with strange quark matter obeys EFEs, then the quark energy density ρ_Q and quark pressure p_Q are governed by (2.13).*

Corollary 2.6. *If a relativistic string cloud spacetime attached with strange quark matter obeys EoS for strange quark matter, then the total energy density ρ_t and total pressure p_t are governed by (2.14).*

3. Conformal η-Ricci soliton on string cloud spacetime attached with strange quark matter

This section deals with conformal η-RS on string cloud spacetime attached with strange quark matter with a $\varphi(\mathcal{R}ic)$-vector field. Therefore, we suggest the subsequent definition.

Definition 3.1. *A vector field φ on a Riemannian manifold M is said to be a $\varphi(\mathcal{R}ic)$-vector field if it satisfies [48]*

$$\nabla_u \varphi = \sigma \mathcal{R}ic\ u, \tag{3.1}$$

where ∇ is the Levi-Civita connection, σ is a constant, and $\mathcal{R}ic$ is the Ricci operator defined by

$$S(p, q) = g(\mathcal{R}ic\ p, q).$$

If $\sigma \neq 0$ and $\sigma = 0$ in (3.1), then the vector field φ is said to be a proper $\varphi(\mathcal{R}ic)$-vector field and covariantly constant, respectively.

It follows from the definition of the Lie derivative and from (3.1) that one has

$$(\mathfrak{L}_\varphi g)(p, q) = 2\sigma S(p, q). \tag{3.2}$$

Adopting (3.2) and (2.9) in (1.6), we turn up

$$S(p,q) = \frac{\alpha}{\sigma} g(p,q) + \frac{\beta}{\sigma}\eta(p)\eta(q) + \frac{\delta}{\sigma}\theta(p)\theta(q), \tag{3.3}$$

where $\alpha = -\left(\Lambda - \left(\frac{P}{2} + \frac{1}{4}\right) + \frac{A}{2}\right)$, $\beta = -\left(\Omega + \frac{B}{2}\right)$, and $\delta = -\frac{C}{2}$ are scalars. Thus, in light of (1.3), the following outcome can be stated.

Theorem 3.2. *If a string cloud spacetime (M^4, g) attached with strange quark matter obeying the $EFEs$ admitting a conformal η-RS $(M^4, g, \varphi, \Lambda, \Omega)$, such that the vector field is a proper $\varphi(\mathcal{R}ic)$-vector field, then $(M^4, g, \varphi, \Lambda, \Omega)$ is a $(GQE)_4$ spacetime.*

Now, putting $u = v = \gamma$ and using (2.9) and (3.3) together, we find

$$\Lambda = \frac{R}{2}\left(\sigma - \frac{1}{2}\right) + \kappa(\rho_Q + \lambda + V_{ed})\left(\sigma + \frac{1}{2}\right) + \Omega - \left(\frac{P}{2} + \frac{1}{4}\right). \tag{3.4}$$

Thus, one can articulate the following results.

Theorem 3.3. *If a string cloud spacetime (M^4, g) attached with strange quark matter with a unit time-like proper $\varphi(\mathcal{R}ic)$-vector field γ admits a conformal η-RS $(M^4, g, \gamma = \varphi, \Lambda, \Omega)$, then the conformal η-Ricci soliton is shrinking, steady or expanding as*

1) $\frac{R}{2}\left(\sigma - \frac{1}{2}\right) + \kappa(\rho_Q + \lambda + V_{ed})\left(\sigma + \frac{1}{2}\right) + \Omega < \left(\frac{P}{2} + \frac{1}{4}\right)$,

2) $\frac{R}{2}\left(\sigma - \frac{1}{2}\right) + \kappa(\rho_Q + \lambda + V_{ed})\left(\sigma + \frac{1}{2}\right) + \Omega > \left(\frac{P}{2} + \frac{1}{4}\right)$, and

3) $\frac{R}{2}\left(\sigma - \frac{1}{2}\right) + \kappa(\rho_Q + \lambda + V_{ed})\left(\sigma + \frac{1}{2}\right) + \Omega = \left(\frac{P}{2} + \frac{1}{4}\right)$, respectively.

Corollary 3.4. *If a relativistic string cloud spacetime (M^4, g) attached with strange quark matter with a unit time-like covariantly constant $\varphi(\mathcal{R}ic)$-vector field γ admits a conformal η-RS $(M^4, g, \gamma = \varphi, \Lambda, \Omega)$, then the conformal η-Ricci soliton is shrinking, steady or expanding as*

1) $\left(\frac{\kappa(\rho_Q+\lambda+V_{ed})}{2} + \Omega\right) < \left(\frac{P}{2} + \frac{1}{4} + \frac{A}{2}\right)$,

2) $\left(\frac{\kappa(\rho_Q+\lambda+V_{ed})}{2} + \Omega\right) > \left(\frac{P}{2} + \frac{1}{4} + \frac{A}{2}\right)$, and

3) $\left(\frac{\kappa(\rho_Q+\lambda+V_{ed})}{2} + \Omega\right) = \left(\frac{P}{2} + \frac{1}{4} + \frac{A}{2}\right)$, respectively.

4. Physical significance of conformal pressure

Given that time-dependent scalar field, P is known as the conformal pressure and that the true physical pressure in fluid mechanics is what keeps the fluids in compressibility. The conformal pressure $P \geq 0$ is negative outside of an equilibrium point and zero inside. Moreover, the metric g is an equilibrium point or Einstein, providing a nonlinear restoring force (for more details, see [26]).

Now, Theorem 3.2 and Eq (3.4) entail the following:

Theorem 4.1. *If a string cloud spacetime* (M^4, g) *attached with strange quark matter obeys the EFEs admitting a conformal* η*-RS* $(M^4, g, \varphi, \Lambda, \Omega)$*, such that the unit time-like vector field is a proper* $\varphi(\mathcal{R}ic)$*-vector field, then, the conformal pressure is*

$$P = \frac{R}{2}\left(\sigma - \frac{1}{2}\right) + \kappa(\rho_Q + \lambda + V_{ed})\left(\sigma + \frac{1}{2}\right) + 2\Omega - 2(\Lambda + 1). \tag{4.1}$$

Theorem 4.2. *If a string cloud spacetime* (M^4, g) *attached with strange quark matter obeying the EFEs admitting a conformal* η*-RS* $(M^4, g, \varphi, \Lambda, \Omega)$*, such that the unit time-like vector field is a proper* $\varphi(\mathcal{R}ic)$*-vector field, then, the metric g is an equilibrium point or Einstein, if and only if,*

$$\frac{R}{2}\left(\sigma - \frac{1}{2}\right) + \kappa(\rho_Q + \lambda + V_{ed})\left(\sigma + \frac{1}{2}\right) + 2\Omega = 2(\Lambda + 1). \tag{4.2}$$

Corollary 4.3. *If a relativistic string cloud spacetime* (M^4, g) *attached with strange quark matter obeys the Einstein's field equation admitting an expanding conformal* η*-RS* $(M^4, g, \varphi, \Lambda < 0, \Omega)$*, with a unit time-like proper* $\varphi(\mathcal{R}ic)$*-vector field, then the conformal pressure is positive.*

In addition, we turn an interesting corollary for a dynamical system.

Corollary 4.4. *If a string cloud spacetime* (M^4, g) *attached with strange quark matter obeys the EFEs admitting a conformal* η*-RS* $(M^4, g, \varphi, \Lambda, \Omega)$ *with a time-like proper* $\varphi(\mathcal{R}ic)$*-vector field, then the metric g is an equilibrium point and acts as a nonlinear restoring force.*

5. Generalized Liouville equation of a relativistic string cloud spacetime attached with strange quark matter

Let (M^4, g) be a relativistic string cloud spacetime attached with strange quark matter and $(g, \gamma, \Lambda, \Omega)$ be a conformal η-RS in (M^4, g). From (1.6) and (2.9), we get

$$\left[-\frac{\kappa(\rho_Q + V_{ed})}{2} + \Lambda - \frac{1}{2}\left(P + \frac{1}{2}\right)\right] g(u, v) + [\kappa(\rho_Q + \lambda + V_{ed}) + \Omega]\eta(u)\eta(v) \tag{5.1}$$

$$-\kappa\lambda\theta(u)\theta(v) + \frac{1}{2}g(\nabla_u\gamma, v) + g(u, \nabla_v\gamma) = 0.$$

for any $u, v \in \chi(M)$.

After contracting (5.1), we obtain:

$$\mathcal{D}iv(\gamma) = 4\Lambda + \Omega - \kappa[(3\rho_Q - 3V_{ed}) - \lambda] - (2P + 1). \tag{5.2}$$

Next, we have the following remark.

Remark. *With a smooth function* $\psi \in C^\infty(M^4, g)$ *and a vector field* v*, a straightforward calculation gives*

$$\mathcal{D}iv(\psi v) = \pi(d\psi) + \psi\mathcal{D}iv v. \tag{5.3}$$

The function $\psi \in C^{\infty}(M^4, g)$ is said to be a last multiplier of vector field ν with respect to g if $div(\psi\nu) = 0$. The corresponding equation

$$\pi(dlog\psi) = -\mathcal{D}iv(\nu) \tag{5.4}$$

is said to be the **generalized Liouville equation** of the vector field ν with respect to g (for more details see [49, 50]).

Now, extrapolate from the statement and equation above (5.3), and we gain the following result.

Theorem 5.1. *Let (M^4, g) be a relativistic string cloud spacetime attached with strange quark matter admitting a conformal η-RS $(g, \gamma, \Lambda, \Omega)$ with a unit time-like vector field γ and ψ is the last multiplier of γ, and let η be the g-dual 1-form of the vector field γ, then the generalized Liouville equation of a string cloud spacetime attached with strange quark matter satisfied by ψ and γ is*

$$\gamma(dln\psi) = \kappa[(3\rho_Q - 3V_{ed}) - \lambda] + (2P + 1) - (4\Lambda + \Omega). \tag{5.5}$$

Corollary 5.2. *Let (M^4, g) be a relativistic string cloud spacetime attached with strange quark matter admitting a conformal η-RS $(g, \gamma, \Lambda, \Omega)$ with a unit time-like vector field γ and ψ is the last multiplier of γ, and let η be the g-dual 1-form of the vector field γ. If the vector field γ is incompressible or Killing, then the conformal η-RS is expanding, steady, and shrinking as*

(i) $\frac{\kappa}{4}[\Big(3\rho_Q - 3V_{ed}) - \lambda] + \frac{(2P+1)}{4}\Big) < \frac{\Omega}{4}$,
(ii) $\frac{\kappa}{4}[\Big(3\rho_Q - 3V_{ed}) - \lambda] + \frac{(2P+1)}{4}\Big) > \frac{\Omega}{4}$, and
(iii) $\frac{\kappa}{4}[\Big(3\rho_Q - 3V_{ed}) - \lambda] + \frac{(2P+1)}{4}\Big) = \frac{\Omega}{4}$, respectively.

6. Harmonic characteristics of conformal η-Ricci soliton in string cloud spacetime attached with strange quark matter

In this last section, we characterized conformal η-Ricci soliton on string cloud spacetime attached with strange quark matter in some specific conditions when the g-dual of γ, the 1-form η, is a harmonic or *Schrödinger-Ricci* harmonic. Additionally, we provide a condition that is both necessary and sufficient for η to be a solution of the *Schrödinger-Ricci* equation.

Let η be the g-dual 1-form of the given unit time-like vector field γ, with $g(p, \gamma) = \eta(p)$ and $g(\gamma, \gamma) = -1$, then γ is said to be the solution of the *Schrödinger-Ricci* equation if it holds

$$\mathcal{D}iv(\mathfrak{L}_{\gamma}) = 0, \tag{6.1}$$

where $\mathfrak{L}_{\gamma}g$ is the Lie derivative in the direction of vector field γ. In [51], Chow et al. studied the divergence of the Lie derivative such that

$$\mathcal{D}iv(\mathfrak{L}_{\gamma}g) = (\Delta + S)(\gamma) + d(\mathcal{D}iv(\gamma)), \tag{6.2}$$

where Δ indicates the Laplace-Hodge operator with respect to the metric g and S is the Ricci curvature tensor field. Now, by the definition of conformal η-RS, we have

$$\mathfrak{L}_{\gamma}g + 2S + (2\Lambda - (P + \frac{2}{n}))g + 2\Omega\eta \otimes \eta = 0. \tag{6.3}$$

After calculating the trace of the Eq (6.3), we get

$$\mathcal{D}iv(\gamma) + R + 4\Lambda - (4P + 2) + \Omega\,|\gamma|^2 = 0, \tag{6.4}$$

wherein R is the scalar curvature. By using a direct calculation, we find

$$\mathcal{D}iv(\eta \otimes \eta) = \mathcal{D}iv(\gamma)\eta + \nabla_\gamma \eta. \tag{6.5}$$

Using (6.2) and estimating the divergence of (6.3), we find

$$\mathcal{D}iv(\mathfrak{L}_\gamma g) + d(R) + 2\Omega[\mathcal{D}iv(\gamma)\eta + \nabla_\gamma \eta] = 0. \tag{6.6}$$

Schrödinger-Ricci solution: We assert that a 1-form π is a solution of the *Schrödinger-Ricci* equation, if

$$(\Delta + S)(\pi) + d(\mathcal{D}iv(\pi)) = 0. \tag{6.7}$$

As a result, we have the following.

Theorem 6.1. *If $(g, \gamma, \Lambda, \Omega)$ is a conformal η-RS in a string cloud spacetime (M^4, g) attached with strange quark matter with η being the g-dual of the time-like vector field γ, then η is a solution of the Schrödinger-Ricci equation if, and only if,*

$$d(\rho_Q + V_{ed}) = \frac{2\Omega}{-\kappa}\left\{[4\Lambda - (4P + 2) - \kappa(\rho_Q + V_{ed})]\eta - \nabla_\gamma \eta\right\}. \tag{6.8}$$

Proof. Applying (6.3)–(6.5), and (2.10), and in light of the formula

$$2\mathcal{D}iv(S) = d(R),$$

it continues that η is a solution of the Schrödinger-Ricci equation if, and only if (6.6) satisfies.

Schrödinger-Ricci harmonic forms: We assert that a 1-form π is a *Schrödinger-Ricci harmonic form* if [52]

$$(\Delta + S)(\pi) = 0. \tag{6.9}$$

In addition, if $\Omega = 0$, which yields the conformal Ricci soliton or

$$\nabla_\gamma \eta = [4\Lambda - (4P + 2) - \kappa(\rho_Q + V_{ed})]\eta, \tag{6.10}$$

implies that $\Omega = 4\Lambda - (4P + 2) - \kappa(\rho_Q + V_{ed})$. As a result, we get the following outcome.

Theorem 6.2. *If $(g, \gamma, \Lambda, \Omega)$ is a conformal η-RS in a string cloud spacetime (M^4, g) attached with strange quark matter with η being the g-dual of the time-like vector field γ, then, η is the Schrödinger-Ricci harmonic form if and only if $\Omega = 0$, which produces conformal RS or*

$$\nabla_\gamma \eta = [4\Lambda - (4P + 2) - \kappa(\rho_Q + V_{ed})]\eta, \tag{6.11}$$

which implies that $\Omega = 4\Lambda - (4P + 2) - \kappa(\rho_Q + V_{ed})$.

7. Example of a spacetime admitting conformal η-Ricci soliton

Example 7.1. *Let* $M = \left\{(x, y, z, t) \in \mathbb{R}^4 : t \neq 0\right\}$, *where* (x, y, z, t) *are the standard coordinates of* $\mathbb{R}^4$.

Let (e_1, e_2, e_3, e_4), be the set of linearly independent vector fields of M defined as

$$e_1 = t\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right),\ e_2 = t\frac{\partial}{\partial y},\ e_3 = t\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right),\ e_4 = (t)^3\frac{\partial}{\partial t}.$$

Let g be the Riemannian metric M defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,\ g(e_4, e_4) = -1,\ g(e_i, e_j) = 0,\ \text{for } i \neq j, i,\ j = 1, 2, 3, 4.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_4)$ for any $Z \in \chi(M)$. Also, let φ be the $(1, 1)$ tensor field, defined by

$$\varphi(e_1) = e_1,\ \varphi(e_2) = e_2,\ \varphi(e_3) = e_3,\ \varphi(e_4) = 0,\ \xi = (t)^3\frac{\partial}{\partial t}.$$

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Thus, using the linearity of φ and g, we have

$$[e_1, e_2] = -(t)e_2,\ [e_1, e_4] = -(t)^2e_1,\ [e_2, e_4] = -(t)^2e_2,\ [e_3, e_4] = -(t)^2e_3.$$

Then, for $e_4 = \xi$ and using Koszul's formula for the Lorentzian metric g, we have

$$\nabla_{e_1}e_1 = -(t)^2e_4,\ \nabla_{e_2}e_1 = te_2,\ \nabla_{e_1}e_4 = -(t)^2e_1,\ \nabla_{e_2}e_4 = -(t)^2e_2,$$

$$\nabla_{e_3}e_4 = -(t)^2e_3,\ \nabla_{e_3}e_3 = -(t)^2e_4, \nabla_{e_2}e_2 = -(t)^2e_4 - te_1. \tag{7.1}$$

From (7.1), we find that the structure (φ, ξ, η, g) is a Lorentzian structure on M. Consequently, $M^4(\varphi, \xi, \eta, g)$ is a Lorentzian manifold (four dimensional spacetime model).

The nonvanishing components of Riemannian curvature and the Ricci tensors are given by

$$R(e_1, e_4)e_1 = (t)^4e_4,\ R(e_2, e_4)e_2 = (t)^4e_4,\ R(e_3, e_4)e_3 = (t)^4e_4,$$

$$R(e_1, e_3)e_3 = (t)^4e_1,\ R(e_1, e_3)e_1 = -(t)^4e_3,\ R(e_2, e_3)e_2 = -(t)^4e_3,$$

$$R(e_1, e_4)e_4 = (t)^4e_1,\ R(e_2, e_4)e_4 = (t)^4e_2,\ R(e_1, e_2)e_2 = [(t)^4 - (t)^2]e_1,$$

$$R(e_2, e_3)e_3 = (t)^4e_2,\ R(e_3, e_4)e_4 = (t)^4e_3,\ R(e_1, e_2)e_1 = -[(t)^4 - (t)^2]e_2.$$

From the above expression of the curvature tensor, we can easily calculate the non-vanishing components of the Ricci tensor S

$$S(e_1, e_1) = 3(t)^4 - (t)^2,\ S(e_2, e_2) = 3(t)^4 - (t)^2.$$

Similarly, we have

$$S(e_3, e_3) = 3(t)^4,\ S(e_4, e_4) = 3(t)^4. \tag{7.2}$$

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) + S(e_4, e_4) = 2[6(t)^4 - (t)^2].$$

Now, from Eq (1.6), we obtain

$$2[g(e_i,e_i)+\eta(e_i)\eta(e_i)]+2S(e_i,e_i)+\left(2\Lambda-\left(P+\frac{2}{n}\right)\right)g(e_i,e_i)+2\Omega\eta(e_i)\eta(e_i)=0,$$

for all $i\in\{1,2,3,4\}$, and we have

$$2[(1+\delta_{i4}]+2\left\{3(t)^4-(t)^2\right\}+[2\mu-(P+\frac{2}{n}))]+2\omega\delta_{i4}=0,$$

for all $i\in\{1,2,3,4\}$. We get: $\Lambda=-[3(t^4-(t)^2]+(P-\frac{1}{2})]$, $P=\Lambda+[3(t^4-(t)^2]-\frac{1}{2})$, and $\Omega=-\frac{3}{2}$.

Thus, the data (g,ξ,Λ,Ω) is a conformal η-Ricci soliton on (M^4,ϕ,ξ,η,g), which is expanding if $-[3(t^4-(t)^2]<(P-\frac{1}{2})$, shrinking if $-[3(t^4-(t)^2]>(P-\frac{1}{2})$, or steady if $[3(t^4-(t)^2]+(P-\frac{1}{2})]=0$.

8. Conclusions

This research paper focused on the investigation of various geometric aspects within the framework of a relativistic string cloud spacetime attached with strange quark matter. Several key results were obtained and discussed in the context of this study.

To begin, we determined the existence of a conformal η-Ricci soliton on the relativistic string cloud spacetime when combined with strange quark matter and a $\varphi(\mathcal{R}ic)$-vector field. This finding highlights the presence of a specific geometric structure that exhibits soliton-like behavior in the conformal setting.

Moreover, we explored the physical significance of the conformal pressure P in relation to the conformal η-Ricci soliton with the same vector field. By establishing this connection, we gained insights into the role of conformal pressure in the behavior and properties of the soliton on the string cloud spacetime.

Furthermore, we deduced a generalized Liouville equation derived from the conformal η-Ricci soliton. This equation provides a deeper understanding of the underlying dynamics and relationships associated with the soliton and its geometric properties.

Additionally, we investigated the harmonic relevance of the conformal η-Ricci soliton on the string cloud spacetime attached with strange quark matter by introducing a harmonic potential function ψ. This analysis shed light on the harmonic aspects and potential energy considerations associated with the soliton within the studied spacetime.

Finally, we established necessary and sufficient conditions for the 1-form η, which represents the g-dual of the vector field γ on the string cloud spacetime attached with strange quark matter, to be a solution for the Schrödinger-Ricci equation. This condition provides insights into the relationship between the geometric properties of spacetime and the underlying mathematical equations.

In summary, this research paper contributes to the understanding of the geometric axioms and properties within the framework of a relativistic string cloud spacetime attached with strange quark matter. The obtained results deepen our knowledge of the soliton behavior, conformal pressure, harmonic relevance, and Schrödinger-Ricci equation in this particular context. These findings pave the way for further investigations and potential applications in the field of theoretical physics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors assert that they do not have any known competing financial interests or personal relationships that could have influenced the work reported in this paper.

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