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*Research article*

## Complex dynamics of a nonlinear impulsive control predator-prey model with Leslie-Gower and B-D functional response

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**Abstract:** A good pulse control strategy should depend on the numbers of pests and natural enemies as determined via an integrated pest control strategy. Taking this into consideration, here, a nonlinear impulsive predator-prey model with improved Leslie-Gower and Beddington-DeAngelis functional response terms is qualitatively analyzed. The existence of a periodic solution for pest eradication has been obtained and the critical condition of global asymptotic stability has been established by using the impulsive differential equation Floquet theory. Furthermore, the conditions for the lasting survival of the system has been proved by applying a comparison theorem for differential equations. Additionally, a stable positive periodic solution has been obtained by applying bifurcation theory. To understand how nonlinear pulses affect the dynamic behavior of a system, MATLAB was used to conduct numerical simulations to show that the model has very complex dynamical behavior.

**Keywords:** nonlinear impulsive; global asymptotic stability; bifurcation; numerical simulation

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### 1. Introduction

In recent years, predator-prey systems with integrated pest control have been extensively studied to realize the effective control of pests [1–4]. Generally, IPM (Integrated Pest Management) mainly uses chemical control, biological control and artificial control to establish an efficient comprehensive system to control pests. Among them, biological control is mainly the use of natural enemies, which involves their harvesting and capture pests, while chemical control is the use of pesticides and other chemicals to quickly kill pests in large quantities. Regarding implementing pest control, the combination of two strategies can yield an optimal control strategy [5, 6].

The functional response is defined as the number of prey killed by a predator per unit of time, and it describes the amount of biological transfer between different trophic levels, i.e., the effects of the predator population on the prey. Many scholars have considered predator-prey models with different

functional responses from many aspects. For example, the nonlinear functional response first appeared in the Lotka-Volterra [7] predation system. Also, Wang et al. [8] and Cheng et al. [9] considered Holling-I functional responses in a predator-prey model; Luo et al. [10] and Liu et al. [11] focused on Holling type II functional responses. Other scholars [12, 13] studied the Holling-III functional response; through their research, it was found that functional response terms not only rely on prey behavior, but they are also related to predator activity. Wang [14] researched the predator-prey model with ratio-dependent functional response terms, and by applying a ratio-dependent functional response, it was found that the predator density could not be zero, i.e., the predator and prey could not be extinct at the same time. Thus, the Beddington-DeAngelis functional response term [15, 16] has garnered much attention.

To facilitate research, many scholars have set the periodic release of natural enemies in the pest-natural enemy system as an invariant constant [17]; in fact, pest outbreaks are generally transient, uncertain and unpredictable, so pest control strategies incorporate pulse disturbances to simulate the effects of the sudden change of actual factors on the state of the system. Moreover, many anthropogenic phenomena in nature can be described by impulsive differential equations, and these phenomena include fishing and stocking in fish farming, spraying pesticides in agriculture the regular release of natural enemies, insulin injection in diabetes, etc. In [18, 19], the authors considered a predator-prey model with linear pulses, employing modified versions of the Leslie-Gower and Holling-II functional responses, but these impulse effects tend to be nonlinear. However, the excessive use of pesticides or biological predators can potentially yield significant damage to the environment and economy. Nonetheless, optimal impulse control management requires that the IPM rely on the densities of both pests and natural enemies in the field; therefore, predator-prey models with nonlinear impulse control are more practical and reasonable. To investigate how nonlinear impulse control affects the dynamic behavior of a system, in [20], Li et al. studied a predator-prey model with nonlinear pulses and a Beddington-DeAngelis functional response, finding that the models with nonlinear impulse control exhibit richer dynamic behavior than those with linear pulses. The Leslie-Gower functional response is an important functional response, which means that the predator population does not unlimitedly grow because of the limitations of natural resources, and many predators do not live on just one kind of food. The authors of [21, 22] studied the dynamic behavior of predatory models with Leslie-Gower and Holling-II terms. However, these studies applied the assumption that the impulse control is linear or constant. Because the outbreak of pests and diseases occurs rapidly, it is difficult for many countries or regions, which have limited capacity for agricultural resources such as pesticides, labor, and biological resources, to effectively control an outbreak of pests. In order to take the resource limitation into account, the instant killing rate should be a monotonically increasing saturation function, and the natural enemies that are released should be based on the current pest and prey densities in the field. With the goal of realizing optimal pulse management, we studied a nonlinear impulsive control predator-prey model with a Leslie-Gower term and Beddington-DeAngelis term, which more comprehensively consider the effects of human control and environmental factors on population density:

$$\begin{cases} \frac{dx}{dt} = (r_1 - bx(t) - \frac{\alpha y(t)}{a_1 + a_2 x(t) + a_3 y(t)})x(t), & t \neq nT, \\ \frac{dy}{dt} = (r_2 - \frac{\beta y(t)}{x(t) + k})y(t), & t \neq nT, \\ x(t^+) = (1 - \frac{\delta x(t)}{x(t) + h})x(t), & t = nT, \\ y(t^+) = qy(t) + \frac{w_1 x(t)}{1 + \theta x(t)} + w_2, & t = nT, \end{cases} \quad (1.1)$$

where  $x(t)$  and  $y(t)$  are the populations of pests and natural enemies at time  $t$ , respectively  $\alpha, \beta, a_1, a_2, a_3, k, h, q, \theta, \delta, w_1, w_2$  are model parameters that assume only positive values,  $r_1, r_2$  represent the birth rates of pests and natural enemies,  $b$  describes the intraspecific competition for pests,  $\delta$  measures the maximum rate at which pests are killed by spraying insecticides ( $0 < \delta \leq 1$ ),  $h$  is the semi-saturation constant,  $w_1, w_2$  denote the maximum releases of predators based on prey density and predator population density, and  $\frac{\alpha x(t)y(t)}{a_1 + a_2 x(t) + a_3 y(t)}$  is the B-D functional response term; IPM strategies are adopted at each discrete time point.

## 2. Global stability of periodic solutions

First, we give some notations, definitions and lemmas which will be useful for our main results.

Assume that

$$R_+ = [0, +\infty), R_+^2 = \{X = (x(t), y(t)) | x(t), y(t) \geq 0\},$$

$$V_0 = \{V : R_+ \times R_+^2 \rightarrow R_+\}$$

are continuous on  $(nT, (n+1)T]$  and  $f = (f_1, f_2)^T$  is the right-hand mapping of the first two equations of the system (1.1); obviously, the smoothness of  $f$  guarantees the existence of a unique solution. The following lemma [23, 24] is given.

**Lemma 2.1.** *Let  $X(t) = (x(t), y(t))$  be a solution of system (1.1), and if the initial value  $X(0^+) \geq 0$ , then  $X(t) \geq 0$  for all  $t \geq 0$ ; furthermore, if  $X(0^+) > 0$ , then  $X(t) > 0$  for all  $t \geq 0$ .*

*Proof.* Obviously for  $y(t)$ , through scaling we have the following:

$$\begin{cases} \frac{dy(t)}{dt} = (r_2 - \frac{\beta y(t)}{x(t) + k})y(t), & t \neq nT, \\ y(t^+) \geq qy(t) + w_2, & t = nT. \end{cases} \quad (2.1)$$

It is easy to see that

$$y(t) \geq y(0^+) \exp \int_0^t (r_2 - \frac{\beta y(s)}{x(s) + k}) ds$$

for  $t \in (0, T]$ .

At time  $t = T$ ,  $y(T^+) = qy(T) + w_2 > qy(T) > 0$ ; thus, when  $t \in (T, 2T]$ , the initial value becomes  $y(T^+) \geq 0$ . For  $x(t)$ , through scaling we get the following:

$$\begin{cases} \frac{dx(t)}{dt} = (r_1 - bx(t) - \frac{\alpha y(t)}{a_1 + a_2 x(t) + a_3 y(t)})x(t), & t \neq nT, \\ x(t^+) > (1 - \delta)x(t), & t = nT. \end{cases} \quad (2.2)$$

Thus, we have

$$x(t) \geq x(0^+) \exp \int_0^t (r_1 - bx(s) - \frac{\alpha y(s)}{a_1 + a_2 x(s) + a_3 y(s)}) ds,$$

for  $t \in (0, T]$ .

At time  $t = T$ ,  $x(T^+) > (1 - \delta)x(T) > 0$ ; thus, when  $t \in (T, 2T]$ , the initial value becomes  $x(T^+) \geq 0$ .

Similar to the previous proof we can determine when  $X(0^+) \geq 0$  and  $X(t) \geq 0$  for all  $t \geq 0$ . Similarly, when  $X(0^+) > 0$  we can get that  $X(t) > 0$  for all  $t > 0$ .  $\square$

**Lemma 2.2.** Define  $g : R_+ \times R_+ \rightarrow R$  as continuous on  $(nT, (n+1)T]$  when  $n \in Z_+, u \in R_+$ , and

$$\lim_{(t,y) \rightarrow (nT^+, u)} g(t, y) = g(nT^+, u),$$

$\psi_n : R_+ \rightarrow R_+$  is a non-decreasing mapping; suppose that the following inequalities hold:

$$\begin{cases} D^+ V(t, x) \leq g(t, V(t, x)), & t \neq nT, \\ V(t, x) \leq \psi_n(V(t, x)), & t = nT, \end{cases} \quad (2.3)$$

and that the impulsive differential equations given by

$$\begin{cases} \frac{du(t)}{dt} = g(t, u(t)), & t \neq nT, \\ u(t^+) = \psi_n(u(t)), & t = nT, \\ u(0^+) = u_0 > 0, \end{cases} \quad (2.4)$$

have a maximum solution  $r(t) \in [0, +\infty)$ ; when  $t \geq 0$  and  $V(0^+, x_0) \leq u_0$ , we have that  $V(t, x) \leq r(t)$ .

The elimination of pest populations is an important goal of IPM strategies; therefore, studying the existence and global stability of periodic solutions plays a crucial role in the investigation of their dynamic behavior. Thus, we consider the subsystem of (1.1):

$$\begin{cases} \frac{dy(t)}{dt} = (r_2 - \frac{\beta}{k}y(t))y(t), & t \neq nT, \\ y(t^+) = qy(t) + w_2, & t = nT. \end{cases} \quad (2.5)$$

Now, we shall give the basic properties of the subsystem:

**Theorem 2.3.** If  $1 - q \exp(r_2 T) < 0$ , system (2.5) has a globally asymptotically stable periodic solution

$$y^*(t) = \frac{r_2 y^* e^{r_2(t-nT)}}{r_2 - \frac{\beta}{k} y^* (1 - e^{r_2(t-nT)})}, \quad t \in (nT, (n+1)T], \quad (2.6)$$

where  $y^* = A + \sqrt{A^2 + B}$  with  $A = \frac{w_2}{2} + \frac{r_2(1-qe^{r_2 T})}{2\frac{\beta}{k}(1-e^{r_2 T})}$  and  $B = \frac{r_2 w_2}{-\frac{\beta}{k}(1-e^{r_2 T})}$  as two positive constants.

*Proof.* By applying the idea of variable separation and difference equations to solve the system (2.5), we have

$$y((n+1)T) = \frac{r_2 y(nT^+) e^{r_2 T}}{r_2 - \frac{\beta}{k} y(nT^+) (1 - e^{r_2 T})}, \quad (2.7)$$

and  $y((n+1)T^+) = qy((n+1)T) + w_2$ ; then, the following equation is obtained

$$y((n+1)T^+) = qy((n+1)T) + w_2 = q \frac{r_2 y(nT^+) e^{r_2 T}}{r_2 - \frac{\beta}{k} y(nT^+) (1 - e^{r_2 T})} + w_2. \quad (2.8)$$

Let  $y(nT^+) = y((n+1)T^+) = y^*$ . Substituting  $y^*$  into the above equation yields the following quadratic equation:

$$y^{*2} + (-w_2 + \frac{r_2(1 - qe^{r_2 T})}{-\frac{\beta}{k}(1 - e^{r_2 T})})y^* + \frac{r_2 w_2}{\frac{\beta}{k}(1 - e^{r_2 T})} = 0. \quad (2.9)$$

We have that  $y^* = A + \sqrt{A^2 + B}$ , where

$$A = \frac{w_2}{2} + \frac{r_2(1 - qe^{r_2 T})}{2\frac{\beta}{k}(1 - e^{r_2 T})}, \quad B = \frac{r_2 w_2}{-\frac{\beta}{k}(1 - e^{r_2 T})}.$$

Apparently when  $1 - qe^{r_2 T} < 0$ , system (2.5) has a periodic solution  $y^*(t)$  for the case with pests.

According to the above analysis, we can easily get the expression for the unique pest-free periodic solution of system (1.1), i.e.,  $(0, y^*(t))$ . In what follows, we will give the sufficient condition for the global stability of the pest-free periodic solution  $(0, y^*(t))$  of system (1.1).  $\square$

**Theorem 2.4.** *The periodic solution  $(0, y^*(t))$  of the system (1.1) is locally asymptotically stable for the case with pests if*

$$T = T_{\max} < \frac{\alpha}{r_1(a_1\frac{\beta}{k} + a_3r_2)} \ln \frac{a_1r_2 - \frac{\beta}{k}a_1y^* + (a_1\frac{\beta}{k} + a_3r_2)e^{r_2T}y^*}{a_1r_2 + a_3r_2y^*}. \quad (2.10)$$

*Proof.* Here we first demonstrate the local stability of the periodic solution for pests.

The local stability of the periodic solution for pests may be determined by considering the behavior of small-amplitude perturbations  $(u(t), v(t))$  of the solution. Define  $u(t) = x(t)$ ,  $v(t) = y(t) - y^*(t)$ , where  $u(t), v(t)$  are small perturbations. After neglecting the higher-order terms, it can be written as

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \varphi(t) \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, \quad 0 \leq t < T, \quad (2.11)$$

where  $\varphi(t)$  satisfies

$$\frac{d\varphi(t)}{dt} = \begin{pmatrix} r_1 - \frac{\alpha y^*(t)}{a_1 + a_3 y^*(t)} & 0 \\ * & r_2 - \frac{2\beta}{k} y^*(t) \end{pmatrix} \cdot \varphi(t), \quad (2.12)$$

and  $\varphi(0) = I$  is the identity matrix. Hence, the fundamental solution matrix is given by

$$\varphi(t) = \begin{pmatrix} e^{\int_0^t r_1 - \frac{\alpha}{a_1 + a_3 y^*(t)} y^*(t) dt} & 0 \\ * & e^{\int_0^t r_2 - \frac{2\beta}{k} y^*(t) dt} \end{pmatrix}. \quad (2.13)$$

Then, linearizing the third and fourth equation of system (1.1), we have

$$\begin{pmatrix} u(nT^+) \\ u(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda_1 & q \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix} = A(T) \begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}. \quad (2.14)$$

Let

$$M = A(T)\varphi(T) = \begin{pmatrix} e^{\int_0^T r_1 - \frac{\alpha}{a_1 + a_3 y^*(t)} \cdot y^*(t) dt} & 0 \\ * & q e^{\int_0^T r_2 - \frac{2\beta}{k} y^*(t) dt} \end{pmatrix}. \quad (2.15)$$

If each eigenvalue of  $M$  has an absolute value that is less than one, the solution is locally stable; note that (\*) is not important, so it is not considered. Below we define

$$\lambda_1 = \exp\left(\int_0^T r_1 - \frac{\alpha}{a_1 + a_3 y^*(t)} \cdot y^*(t) dt\right), \quad \lambda_2 = q \exp\left(\int_0^T r_2 - \frac{2\beta}{k} y^*(t) dt\right), \quad (2.16)$$

as two eigenvalues of the matrix  $M$  because

$$\begin{aligned} \frac{kr_2}{\beta} - y^*(1 + e^{r_2(t-nT)}) &< \frac{kr_2}{\beta} - y^* \\ &= \frac{kr_2}{\beta} - \left(\frac{w_2}{2} + \frac{kr_2(1 - qe^{r_2T})}{2\beta(1 - e^{r_2T})}\right) + \sqrt{\left(\frac{w_2}{2} + \frac{kr_2(1 - qe^{r_2T})}{2\beta(1 - e^{r_2T})}\right)^2 + \frac{kr_2 w_2}{\beta(e^{r_2T} - 1)}} \\ &< \frac{kr_2}{\beta} - \frac{w_2}{2} - \frac{kr_2}{2\beta} - \frac{kr_2}{2\beta} \\ &= -\frac{w_2}{2} < 0, \\ r_2 - \frac{2\beta}{k} y^*(t) &= r_2 \frac{\frac{kr_2}{\beta} - y^*(1 + e^{r_2(t-nT)})}{\frac{kr_2}{\beta} - y^*(1 - e^{r_2(t-nT)})} < 0. \end{aligned} \quad (2.17)$$

It is easy to see that  $\lambda_2 < 1$ , according to the Floquet theory for impulsive differential equations,  $(0, y^*(t))$  is locally stable if

$$T < \frac{\alpha}{a_3 r_2 + \frac{\beta}{k} a_1} \ln \frac{a_1 r_2 - \frac{\beta}{k} a_1 + (a_2 r_2 + \frac{\beta}{k} a_1) e^{r_2 T} y^*}{a_1 r_2 + a_3 r_2 y^*}. \quad (2.18)$$

Next, we will prove the global attraction of the periodic solution for pests. Just prove that the arbitrary solution  $(x(t), y(t))$  tends to  $(0, y^*(t))$  when  $t \rightarrow \infty$ .

(1) Prove that  $x(t) \rightarrow 0$ .

Since  $x = \frac{r_1}{b}$ , it follows that  $\frac{dx}{dt} < 0$ ; then  $0 < x < \frac{r_1}{b}$ . Take a small enough  $\varepsilon > 0$  so that

$$\eta \triangleq \exp\left(\int_0^T r_1 - \frac{\alpha(y^*(t) - \varepsilon)}{a_1 + a_2(k + \varepsilon) + a_3(y^*(t) - \varepsilon)} dt\right) < 1. \quad (2.19)$$

It follows from the system (1.1) that

$$\begin{aligned} \frac{dx(t)}{dt} &= \left(r_1 - bx(t) - \frac{\alpha y(t)}{a_1 + a_2 x(t) + a_3 y(t)}\right)x(t) \\ &\leq \left(r_1 - \frac{\alpha y(t)}{a_1 + a_2 x(t) + a_3 y(t)}\right)x(t) \\ &\leq \left(r_1 - \frac{\alpha(y^*(t) - \varepsilon)}{a_1 + a_2(k + \varepsilon) + a_3(y^*(t) - \varepsilon)}\right)x(t), \end{aligned} \quad (2.20)$$

further, we consider the following impulsive differential equation:

$$\begin{cases} \frac{dx(t)}{dt} \leq \left( r_1 - \frac{\alpha(y^*(t) - \varepsilon)}{a_1 + a_2(k + \varepsilon) + a_3(y^*(t) - \varepsilon)} \right) x(t), & t \neq nT, \\ x(t^+) = \left( 1 - \frac{\delta x(t)}{x(t) + h} \right) x(t), & t = nT. \end{cases} \quad (2.21)$$

By applying the impulsive differential equation theorem and the comparison theorem, within the interval  $((nT), (n+1)T)$ , we get

$$\begin{aligned} x((n+1)T) &\leq x(nT) \exp\left( \int_0^T r_1 - \frac{\alpha(y^*(t) - \varepsilon)}{a_1 + a_2(k + \varepsilon) + a_3(y^*(t) - \varepsilon)} dt \right) \\ &\leq x(nT)\eta. \end{aligned} \quad (2.22)$$

Then,  $x(nT) \leq x(0^+)\eta^n$  and  $x(nT) \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

(2) Prove that  $y(t) \rightarrow y^*(t)$ .

From above  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  for any  $\varepsilon_1 > 0$ . Additionally, there must exist a  $T_1 > 0$ , such that  $0 < x(t) < \varepsilon_1$  and  $t > T_1$ . Without loss of generality, we may assume that  $0 < x(t) < \varepsilon_1$  for all  $t > 0$ ; from system (2.5) we have

$$y(t)\left(r_2 - \frac{\beta y(t)}{k}\right) < \frac{dy(t)}{dt} = \left(r_2 - \frac{\beta y(t)}{x(t) + k}\right)y(t) < y(t)\left(r_2 - \frac{\beta y(t)}{\varepsilon_1 + k}\right). \quad (2.23)$$

Then we consider the following equations

$$\begin{cases} \frac{dz_1(t)}{dt} = \left( r_2 - \frac{\beta}{k} z_1(t) \right) z_1(t), & t \neq nT, \\ z_1(t^+) = qz_1(t)w_2, & t = nT, \end{cases} \quad (2.24)$$

and

$$\begin{cases} \frac{dz_2(t)}{dt} = \left( r_2 - \frac{\beta}{\varepsilon_1 + k} z_2(t) \right) z_2(t), & t \neq nT, \\ z_2(t^+) = qz_2(t)w_2, & t = nT, \end{cases} \quad (2.25)$$

where

$$\begin{aligned} z_1^*(t) = y^*(t) &= \frac{r_2 z_1^*(1 - e^{r_2(t-nT)})}{r_2 - \frac{\beta}{k} z_1^*(1 - e^{r_2(t-nT)})}, & z_1^* &= A_1 + \sqrt{A_1^2 + B_1}, \\ A_1 &= \frac{w_2}{2} + \frac{kr_2(1 - qe^{r_2T})}{2\beta(1 - e^{r_2T})}, & B_1 &= \frac{-kr_2w_2}{\beta(1 - e^{r_2T})}, \end{aligned}$$

and

$$\begin{aligned} z_2^*(t) &= \frac{r_2 z_2^* e^{r_2(t-nT)}}{r_2 - \frac{\beta}{\varepsilon_1 + k} z_2^*(1 - e^{r_2(t-nT)})}, & z_2^* &= A_2 + \sqrt{A_2^2 + B_2}, \\ A_2 &= \frac{w_2}{2} + \frac{(k + \varepsilon_1)r_2(1 - qe^{r_2T})}{2\beta(1 - e^{r_2T})}, & B_2 &= \frac{-(k + \varepsilon_1)r_2w_2}{\beta(1 - e^{r_2T})}. \end{aligned}$$

For arbitrarily small  $\varepsilon_2 > 0$ , according to the comparison theorem we have

$$z_1^*(t) - \varepsilon_2 < y(t) < z_2^*(t) + \varepsilon_2.$$

Let  $\varepsilon_1 \rightarrow 0$  as  $t \rightarrow \infty$ ; we have that  $y^*(t) - \varepsilon_2 < y(t) < y^*(t) + \varepsilon_2$ ; hence,  $y(t) \rightarrow y^*(t)$  for  $t$  large enough. Thus the proof is completed.  $\square$

### 3. Permanence

In order to understand the long-term survival of the prey and predator populations, it is important to investigate the permanence of system (1.1). First, we give the definition of permanence as follows:

**Definition 3.1.** *The system (1.1) is permanent if there exists  $M \geq m > 0$  for any solution  $(x(t), y(t))$  of (1.1) with  $x_0 > 0$  such that*

$$m \leq \liminf_{t \rightarrow \infty} x(t) \leq \limsup_{t \rightarrow \infty} x(t) \leq M, m \leq \liminf_{t \rightarrow \infty} y(t) \leq \limsup_{t \rightarrow \infty} y(t) \leq M. \tag{3.1}$$

**Theorem 3.2.** *System (1.1) is permanent if*

$$T > \frac{\alpha}{r_1(a_1 \frac{\beta}{k} + a_3 r_2)} \ln \frac{a_1 r_2 - \frac{\beta}{k} a_1 y^* + (a_1 \frac{\beta}{k} + a_3 r_2) e^{r_2 T} y^*}{a_1 r_2 + a_3 r_2 y^*}. \tag{3.2}$$

*Proof.* Suppose that  $(x(t), y(t))$  is any solution of system (1.1). Apparently, given the above analysis, we have that  $x < M_1 < \frac{r_1}{b}$  for  $t$  large enough. For convenience, let  $x(t) \leq M$  for any  $t > 0$ . Then, from Theorem 2.4, we can easily obtain that  $y^*(t) - \varepsilon_2 < y(t) < y^*(t) + \varepsilon_2$  for  $t$  large enough. Let  $m_2 = y^*(t) - \varepsilon_2$  and  $M_2 = y^*(t) + \varepsilon_2$ ; then we have that  $m_2 \leq y(t) \leq M_2$ .

Assume that  $y(t) \geq m_2$  for any  $t > 0$ . We need to prove that there exists a positive constant  $m_1$  such that  $x(t) \geq m_1$  when  $t$  is large enough. We will prove it in two steps:

1) Take a positive number  $m_3 > 0$  and  $\varepsilon_3 > 0$  arbitrarily small enough. Let

$$\eta_2 = \exp((r - bm_3 - \frac{\alpha \varepsilon_3}{a_1})T) - \frac{\alpha}{a_1} \frac{k + a_3}{\beta} \ln \frac{r_2 - \frac{\beta}{k+m_3} z_3^* (1 - \exp(r_2 T))}{r_2} > 1, \tag{3.3}$$

where

$$z_3^* = \sqrt{A_3^2 + B_3}$$

and

$$A = \frac{w_2}{2} + \frac{r_2(1 - qe^{r_2 T})}{2 \frac{\beta}{k+m_3} (1 - e^{r_2 T})}, \quad B = \frac{r_2 w_2}{-\frac{\beta}{k+m_3} (1 - e^{r_2 T})}.$$

Next we will prove that there must exist a  $t_3 > 0$ . Set  $x(t_3) \geq m_3$  for  $t > t_3$ . Otherwise, from Theorem 2.4, we obtain that  $y(t) \leq z_3(t)$  and  $z_3(t) \rightarrow z_3^*(t)$  as  $t \rightarrow \infty$ , where  $z_3(t)$  is the solution of system (3.4).

$$\begin{cases} \frac{dz_3(t)}{dt} = z_3(t)(r_2 - \frac{\beta}{k+m_3} z_3(t)), & t \neq nT, \\ z_3(t^+) \leq qz_3(t) + \frac{w_1 m_3}{1 + \theta m_3} + w_2, & t = nT, \end{cases} \tag{3.4}$$

and

$$z_3^*(t) = \frac{r_2 z_3^* \exp(r_2(t - nT))}{r_2 - \frac{\beta}{k+m_3} z_3^* (1 - \exp(r_2(t - nT)))}, \quad t \in (nT, (n+1)T), n \in Z_+. \tag{3.5}$$

Thus there is a  $T_1$  such that  $y(t) \leq z_3(t) \leq z_3^*(t) + \varepsilon_3$  as  $t > T_1$ . Therefore we establish (3.6) as below:

$$\begin{cases} \frac{dx(t)}{dt} \geq x(t)(r_1 - bm_3 - \frac{\alpha(z_3^*(t) + \varepsilon_3)}{a_1}), & t \neq nT, \\ x(t^+) = (1 - \frac{\delta x(t)}{x(t) + h})x(t), & t = nT. \end{cases} \tag{3.6}$$



Let  $N \in \mathbb{Z}_+$ ,  $NT > T$ ; integrating (3.6) on  $(nT, (n+1)T)$ ,  $n \geq N$ , we have

$$\begin{aligned} \int_{nT}^{(n+1)T} \frac{dx(t)}{x(t)} dt &\geq \int_{nT}^{(n+1)T} \left( r_1 - bm_3 - \frac{\alpha(z_3^*(t) + \varepsilon_3)}{a_1} \right) dt, \\ \ln x((n+1)T) - \ln x(nT) &\geq \int_{nT}^{(n+1)T} \left( r_1 - bm_3 - \frac{\alpha(z_3^*(t) + \varepsilon_3)}{a_1} \right) dt, \\ \ln \frac{x((n+1)T)}{x(nT)} &\geq \int_{nT}^{(n+1)T} \left( r_1 - bm_3 - \frac{\alpha(z_3^*(t) + \varepsilon_3)}{a_1} \right) dt, \\ x((n+1)T) &\geq x(nT) \exp \left( \int_{nT}^{(n+1)T} \left( r_1 - bm_3 - \frac{\alpha(z_3^*(t) + \varepsilon_3)}{a_1} \right) dt \right) \\ &\geq x(nT) \exp \left( (r - bm_3 - \frac{\alpha\varepsilon_3}{a_1})T - \frac{\alpha}{a_1} \frac{k+a_3}{\beta} \ln \frac{r_2 - \frac{\beta}{k+m_3} z_3^*(1 - \exp(r_2 T))}{r_2} \right). \end{aligned} \quad (3.7)$$

Then, given (3.3), substituting the right-hand side of (3.7) with  $\eta_2$ , we get that  $x((n+1)T) \geq x(nT)\eta_2$ . Then  $x((N+n)T) > x(nT)\eta_2^n \rightarrow \infty$  as  $n \rightarrow \infty$ , which is a contradiction; hence, there must exist a  $t_3 > 0$  such that  $x(t_3) \geq m_3$ .

2) Assume that  $x(t) \geq m_3$  for all  $t > t_3$  and prove that the conclusion holds; otherwise, let  $t^* = \inf t > t_3, x(t) < m_3$ , and we have that  $x(t) \geq m_3$  as  $t \in [t_3, t^*]$ . It is divided into the following two cases: points in time of impulse and no impulse.

Case (1):  $t^* = nT, n_1 \in \mathbb{Z}_+$ . We have that  $q_1 m_1 \leq q_1 x(t^*) = x(t^{*+}) \leq m_3$ . Select  $n_2, n_3 \in \mathbb{Z}_+$  such that

$$\left(1 - \frac{\delta}{h}\right)^{n_2} \exp(\rho n_2 T) \eta^{n_3} > 1,$$

where  $\rho = r_1 - bm_3 - \frac{\alpha}{a_1} M_2 < 0$ . Let  $T' = n_2 T + n_3 T$ ; there must exist a  $t_4 \in (t^*, t^* + T')$  such that  $x(t_4) \geq m_3$ . Otherwise, assume that  $x(t) < m_3$  for any  $t \in (t^*, t^* + T')$ ; since (3.4) has an initial solution  $z_3(t^{*+}) = y(t^{*+})$ , we can get

$$y(t) < z_3(t) \leq z_3^*(t) + \varepsilon_3, t \in [t^* + n_2 T, t^* + T'].$$

According to the above, we know that

$$x(t^* + T') \geq x(t^* + n_2 T) \eta^{n_3}.$$

Since  $y(t) \leq M_2$ , when  $t \in [t^*, t^* + n_2 T]$ , we have

$$\begin{cases} \frac{dx(t)}{dt} \geq x(t) \left( r - bm_3 - \frac{\alpha}{a_1} M_2 \right), & t \neq nT, \\ x(t^+) = \left( 1 - \frac{\delta x(t)}{x(t) + h} \right) x(t), & t = nT. \end{cases} \quad (3.8)$$

Integrating (3.8) on the interval  $[t^*, t^* + n_2 T]$ , we obtain

$$\begin{aligned} x(t^* + n_2 T) &\geq \left(1 - \frac{\delta}{h}\right)^{n_2-1} \exp(\rho n_2 T) x(t^{*+}) \\ &= \left(1 - \frac{\delta}{h}\right)^{n_2} \exp(\rho n_2 T) x(t^*) \\ &\geq m_3 \left(1 - \frac{\delta}{h}\right)^{n_2} \exp(\rho n_2 T). \end{aligned} \quad (3.9)$$

So,

$$x(t^* + T') \geq x(t^* + n_2 T) \eta^{n_3} = m_3 \left(1 - \frac{\delta}{h}\right)^{n_2} \exp(\rho n_2 T) \eta^{n_3} > m_3, \quad (3.10)$$

which contradicts the above assumption and means that there must exist a  $t_4 \in (t^*, t^* + T')$  such that  $x(t_4) > m_3$ .

Let  $\bar{t} = \inf\{t > t^*, x(t) \geq m_3\}$ ; we have that  $x(t) < m_3$  when  $t \in (t^*, \bar{t}]$  and  $x(\bar{t}) = m_3$ . Assume that  $\bar{t} = \inf\{t > t^*, x(t) \geq m_3\}$ ; from (3.8), it follows that

$$\begin{aligned} x(t) &\geq x(t^{*+}) \left(1 - \frac{\delta}{h}\right)^{k-1} \exp(\rho(t - t^*)) \\ &\geq m_3 \left(1 - \frac{\delta}{h}\right)^{n_2+n_3} \exp(\rho(n_2 + n_3)T) = m'_1, \quad t \in (t^*, \bar{t}). \end{aligned} \quad (3.11)$$

Therefore we get that  $x(t) \geq m'_1$  for  $t \in (t^*, \bar{t}]$  when  $t > \bar{t}$ . We can still prove that the conclusion  $x(\bar{t}) \geq m_3$  holds by employing the same method as above.

Case (2):  $t^* \neq nT, n \in \mathbb{Z}^+$ . Then  $x(t) \geq m_3$  for  $t \in [t_3, t^*)$  and  $x(t^*) = m_3$ . Assume that

$$t^* \in (n'_1 T, (n'_1 + 1)T], \quad n'_1 \in \mathbb{Z}^+ \text{ for } t \in ((t^*, (n'_1 + 1)T)).$$

There are two possible cases for  $x(t)$ .

(H1): If  $x(t) < m_3$  for all  $t \in (t^*, (n'_1 + 1)T]$ , it is similar to Case (1). We can prove that there exists a  $t_5 \in [(n'_1 + 1)T, (n'_1 + 1)T + T']$  such that  $x(t_5) \geq m_3$ . Otherwise assume that  $x(t) < m_3$  for all  $t \in [(n'_1 + 1)T, (n'_1 + 1)T + T']$ . Consider the initial solution  $z_3((n'_1 + 1)T^*) = y((n'_1 + 1)T^*)$  of (3.4) with the initial values on  $[t^*, (n'_1 + 1 + n_2 + n_3)T + T']$ ; we get that

$$x((n'_1 + 1 + n_2 + n_3)T) \geq x((n'_1 + 1 + n_2)T) \eta_2^{n_3}. \quad (3.12)$$

Integrating (3.8), we have

$$x((n'_1 + 1 + n_2)T) \eta_2^{n_3} \geq m_3 \left(1 - \frac{\delta}{h}\right)^{n_2} \exp(\rho(1 + n_2)T). \quad (3.13)$$

Therefore,

$$x((n'_1 + 1 + n_2 + n_3)T) \geq m_3 \left(1 - \frac{\delta}{h}\right)^{n_2} \exp(\rho(1 + n_2)T) > m_3. \quad (3.14)$$

This contradicts the assumption.

Let  $\hat{t} = \inf\{t > t^*, x(t) > m_3\}$ . We have that  $x(t) < m_3$  when  $t \in (t^*, \hat{t})$  and  $x(\hat{t}) = m_3$ . For  $t \in (t^*, \hat{t})$ , suppose that

$$t \in (t^* + (l - 1)T, t^* + lT), \quad l \leq n_2 + n_3, \quad l \in \mathbb{Z}^+.$$

We have

$$\begin{aligned} x(t) &\geq m_3 \left(1 - \frac{\delta}{h}\right)^{l-1} \exp(\rho l T) \\ &\geq m_3 \left(1 - \frac{\delta}{h}\right)^{n_2+n_3} \exp(\rho(1 + n_2 + n_3)T) = m_1 < m'_1. \end{aligned} \quad (3.15)$$

Thus,  $x(\hat{t}) \geq m_3$  for  $t > \hat{t}$ . By repeating the above steps we can obtain that  $x(t) \geq m_1$  as  $t \geq t_3$ .

(H2): If there exists a  $t \in (t^*, (n'_1 + 1)T)$ , we have that  $x(t) \geq m_3$ . Let  $t' = \inf \{t > t^*, x(t) > m_3\}$  such that  $x(t) < m_3, t \in [t^*, t')$ , but  $x(t') = m_3$  when  $t \in [t^*, t')$ ; thus, we get

$$x(t) \geq x(t^*) \exp((t - t^*)\rho) \geq m_3 \exp(T\rho) > m_1,$$

since  $x(\hat{t}) \geq m_3$ . By repeating the steps above we can obtain that  $x(t) \geq m_1$  as  $t > t_3$ . In summary, when  $t \geq t_3$ , we can prove that  $x(t) \geq m_3$ .  $\square$

#### 4. Bifurcation

Next, we will research the existence of a nontrivial solution near the pest-eradication solution by applying the bifurcation theorem of Rabinowitz.

**Theorem 4.1.** *System (1.1) has a positive nontrivial periodic solution when  $T = T^*$ , and it is supercritical if  $\alpha a_2 \leq 4ba_1a_3$ .*

*Proof.* It is computationally convenient to exchange  $x$  and  $y$  and change the period  $T$  to  $\tau$ .

$$\begin{cases} \frac{dx}{dt} = (r_2 - \frac{\beta x(t)}{y(t) + k})x(t), & t \neq nT, \\ \frac{dy(t)}{dt} = (r_1 - by(t) - \frac{\alpha x(t)}{a_1 + a_2y(t) + a_3x(t)})y(t), & t \neq nT, \\ x(t^+) = qx(t) + \frac{w_1y(t)}{1 + \theta y(t)} + w_2, & t = nT, \\ y(t^+) = (1 - \frac{\delta y(t)}{y(t) + h})y(t), & t = nT, \end{cases} \quad (4.1)$$

let  $\Phi$  be the solution of the non-impulsive system associated with system (4.1); also, we get that  $X(t) = \Phi(t, X_0)$  with  $X_0 = X(0)$ . Define the mappings  $\Theta_1, \Theta_2 : R^2 \rightarrow R^2$  by

$$\begin{aligned} \Theta_1(x, y) &= qx + \frac{w_1y}{1 + \theta y} + w_2, \\ \Theta_2(x, y) &= (1 - \frac{\delta y}{y + h})y, \\ \zeta(t) &= (y^*(t), 0), \end{aligned} \quad (4.2)$$

and the mappings  $F_1, F_2 : R^2 \rightarrow R^2$  by

$$\begin{aligned} F_1(x, y) &= (r_2 - \frac{\beta x(t)}{y(t) + k})x(t), \\ F_2(x, y) &= (r_1 - by(t) - \frac{\alpha x(t)}{a_1 + a_2y(t) + a_3x(t)})y(t). \end{aligned} \quad (4.3)$$

Thus, by computation, we get

$$d'_0 = 1 - (\frac{\partial \Theta_2}{\partial y} \cdot \frac{\partial \Phi_2}{\partial y})_{(\tau_0, X_0)} = 1 - (1 - \frac{\delta y(y + 2h)}{(y + h)^2}) \exp(\int_0^{\tau_0} r_1 - \frac{\alpha y^*(t)}{a_1 + a_3 y^*(t)} dt), \quad (4.4)$$

where  $\tau_0$  is the root of  $d_0' = 0$ . In fact, it is easy to see that  $d_0'$  is equivalent to  $\tau_0 = T^*$ .

$$\begin{aligned}
 a_0' &= 1 - \left( \frac{\partial \Theta_1}{\partial x} \cdot \frac{\partial \Phi_1}{\partial x} \right)_{(\tau_0, X_0)} \\
 &= 1 - q \exp\left( \int_0^{\tau_0} r_2 - \frac{2\beta y^*(t)}{k} dt \right) > 0, \\
 b_0' &= - \left( \frac{\partial \Theta_1}{\partial x} \cdot \frac{\partial \Phi_1}{\partial y} + \frac{\partial \Theta_2}{\partial y} \cdot \frac{\partial \Phi_2}{\partial y} \right)_{(\tau_0, X_0)} \\
 &= - \left( q \frac{\partial \Phi_1}{\partial y} + \frac{1}{(1 + \theta y)^2} \frac{\partial \Phi_2}{\partial y} \right), \\
 \frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial x \partial y} &= \int_0^{\tau_0} \exp\left( \int_s^{\tau_0} \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) \frac{\partial^2 F_2(\zeta(t))}{\partial x \partial y} \exp\left( \int_0^s \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) ds \\
 &= - \int_0^{\tau_0} \exp\left( \int_s^{\tau_0} \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) \frac{\alpha a_1}{(a_1 + a_3 y^*(t))^2} \exp\left( \int_0^s \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) ds \\
 &< 0.
 \end{aligned} \tag{4.5}$$

By the same method, we have

$$\begin{aligned}
 \frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial y^2} &= \int_0^{\tau_0} \exp\left( \int_s^{\tau_0} \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) \frac{\partial^2 F_2(\zeta(t))}{\partial y^2} \exp\left( \int_0^s \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) ds \\
 &+ \int_0^{\tau_0} \left( \exp\left( \int_s^{\tau_0} \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) \frac{\partial^2 F_2(\zeta(s))}{\partial y \partial x} \right. \\
 &\times \int_0^s \exp\left( \int_v^s \frac{\partial F_1(\zeta(t))}{\partial x} dt \right) \frac{\partial F_1(\zeta(s))}{\partial y} \exp\left( \int_0^v \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) dv \Big) ds \\
 &= \int_0^{\tau_0} \exp\left( \int_s^{\tau_0} \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) \left( -2b + \frac{2\alpha a_2 y^*(t)}{(a_1 + a_3 y^*(t))^2} \right) \exp\left( \int_0^s \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) ds \\
 &- \int_0^{\tau_0} \exp\left( \int_s^{\tau_0} \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) \frac{\alpha a_1}{(a_1 + a_3 y^*(t))^2} \\
 &\times \int_0^s \exp\left( \int_v^s \frac{\partial F_1(\zeta(t))}{\partial x} dt \right) \frac{\beta y^*(t)^2}{k^2} \exp\left( \int_0^v \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) dv, \\
 \frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial \tau \partial y} &= \frac{\partial F_2(\zeta(\tau_0))}{\partial y} \exp\left( \int_0^{\tau_0} \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) \\
 &= \left( r_1 - \frac{\alpha y^*(t)}{a_1 + a_3 y^*(t)} \right) \exp\left( \int_0^{\tau_0} \left( r_1 - \frac{\alpha y^*(t)}{a_1 + a_3 y^*(t)} \right) dt \right),
 \end{aligned} \tag{4.6}$$

so we can get

$$\begin{aligned}
 B &= \frac{\partial^2 \Theta_2}{\partial x \partial y} \left( \frac{\partial \Phi_1(\tau_0, X_0)}{\partial \tau} + \frac{\partial \Phi_1(\tau_0, X_0)}{\partial x} \frac{1}{a_0'} \frac{\partial \Theta_1(\tau_0, X_0)}{\partial \tau} \right) \frac{\partial \Phi_2(\tau_0, X_0)}{\partial y} \\
 &- \frac{\partial \Theta_2}{\partial y} \left( \frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial x \partial y} \frac{1}{a_0'} \frac{\partial \Theta_1}{\partial x} \frac{\partial \Phi_1(\tau_0, X_0)}{\partial \tau} + \frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial \tau \partial y} \right) \\
 &= - \left( 1 - \frac{y\delta(y+2h)}{(y+h)^2} \right) \left( \frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial x \partial y} \frac{1}{a_0'} q y^{*'}(\tau_0) \right)
 \end{aligned}$$

$$\begin{aligned}
& + \left( r_1 - \frac{\alpha y^*(t)}{a_1 + a_3 y^*(t)} \right) \exp\left( \int_0^{\tau_0} \left( r_1 - \frac{\alpha y^*(t)}{a_1 + a_3 y^*(t)} \right) dt \right), \\
C = & -2 \frac{\partial^2 \Theta_2}{\partial x \partial y} \left( \frac{a_0'}{b_0'} \frac{\partial \Phi_1(\tau_0, X_0)}{\partial x} + \frac{\partial \Phi_1(\tau_0, X_0)}{\partial y} \right) \frac{\partial \Phi_2(\tau_0, X_0)}{\partial y} - \frac{\partial^2 \Theta_2}{\partial y^2} \left( \frac{\partial \Phi_2(\tau_0, X_0)}{\partial y} \right)^2 \\
& + 2 \frac{a_0'}{b_0'} \frac{\partial \Theta_2}{\partial y} \frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial x \partial y} - \frac{\partial \Theta_2}{\partial y} \frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial y^2} \\
= & -\frac{2\delta}{h} \left( \exp\left( \int_0^{\tau_0} \frac{\partial F_2(\zeta(t))}{\partial y} dt \right) \right)^2 + 2 \frac{a_0'}{b_0'} \left( 1 - \frac{\delta y(y+2h)}{(y+h)^2} \right) \frac{\partial^2 \Phi_2}{\partial x \partial y} - \left( 1 - \frac{\delta y(y+2h)}{(y+h)^2} \right) \frac{\partial^2 \Phi_2}{\partial y^2}. \quad (4.7)
\end{aligned}$$

To determine the sign of  $B$  and  $C$ , first assume that

$$\rho(t) = r_1 - \frac{\alpha y^*(t)}{a_1 + a_3 y^*(t)}$$

which implies that

$$\rho'(t) = \frac{\alpha a_1 y'^*(t)}{(a_1 + a_3 y^*(t))^2} > 0.$$

Then,  $\rho(t)$  is a strictly increasing function. Given that  $d_0' = 0$ , since

$$\int_0^{\tau_0} \rho(t) dt = \ln\left( 1 / \left( 1 - \frac{\delta y(y+2h)}{(y+h)^2} \right) \right) > 0,$$

it is obvious that  $\rho(\tau_0) > 0$ ; so,  $\frac{\partial^2 \Phi_2(\tau_0, X_0)}{\partial y \partial \tau_0} > 0$  and  $B < 0$ . Through analysis, we define the following:

$$\varpi(t) = -2b + \frac{2\alpha a_2 t}{(a_1 + a_3 t)^2}.$$

Apparently  $-2b + \frac{2\alpha a_2 t}{(a_1 + a_3 t)^2} \leq 0$  only such that  $\alpha a_2 \leq 4ba_1 a_3$ . This means that

$$\varpi(y^*(t)) = -2b + \frac{2\alpha a_2 y^*(t)}{(a_1 + a_3 y^*(t))^2} \leq 0$$

for all  $y^*(t) > 0$ . So we get

$$\frac{\partial^2 \Phi(\tau_0, X_0)}{\partial y^2} < 0.$$

From (4.7) we have that  $C > 0$ .

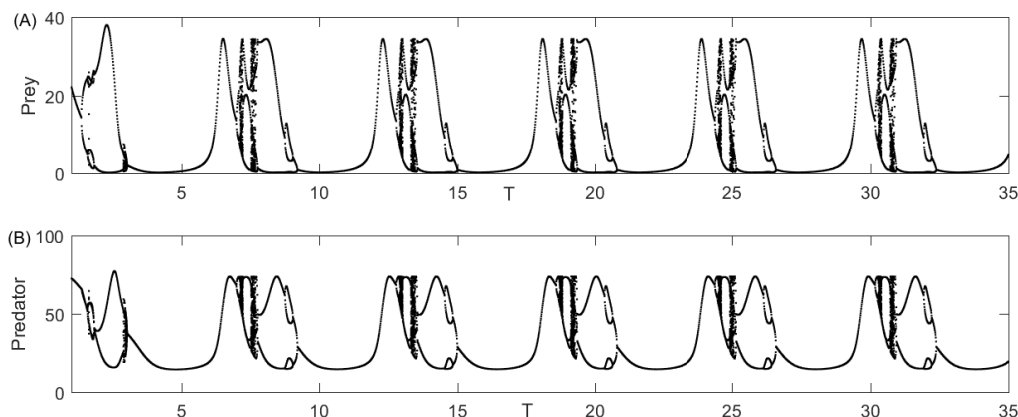
In summary,  $BC < 0$ , and there is a supercritical bifurcation when  $T = T^*$ .  $\square$

## 5. Numerical simulation

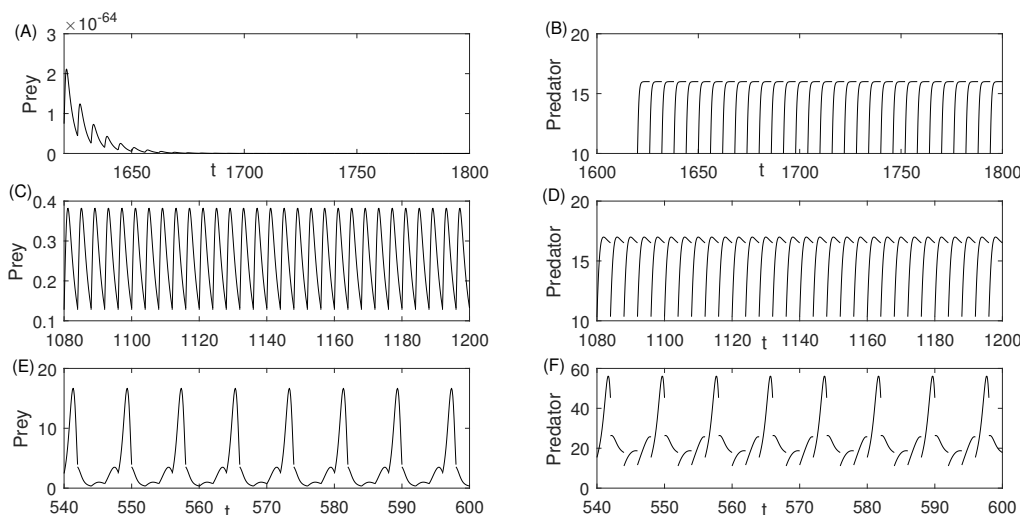
In order to confirm our theoretical results and facilitate interpretation, we will focus on the complex dynamics. The properties of dynamic systems can be obtained by conducting bifurcation analysis.

First, the influence of impulse period  $T$  on the dynamic characteristics of the system was studied. From Figure 1, we can ascertain that dynamic behaviors of system (1.1) become increasingly complicated as parameter  $T$  increases, and these include behaviors such as period-doubling bifurcation,

period-halving bifurcation, chaotic bands, meanwhile, multiple attractors may coexist with the same  $T$ . Figure 2 shows that the pest free periodic solution has global stability when the impulse period stays below some threshold value in (A,B), and that different attractors can coexist with the same  $T = 2.5$  when we select different initial values i.e.,  $(x_0, y_0) = (1.3, 2.1), (x_0, y_0) = (3.5, 6.8)$  in (C,D,E,F). The different solutions coexist for a wide range of parameters, which suggests that the final steady state of pest and predator populations depends on their initial densities, and these results confirm that changes in the impulse period  $T$  can greatly alter the dynamics of the system (1.1).



**Figure 1.** Bifurcation diagrams for system (1.1) with respect to  $T$ . The parameter values are as follows:  $r_1 = 7.65; r_2 = 2; b = 0.12; k = 5; a_1 = 5; a_2 = 1; a_3 = 0.00001; \alpha = 2.5; \beta = 0.75; h = 0.27; q = 0.45; \theta = 12.5; \text{deta} = 0.1; w_1 = 1.25; w_2 = 1.2$ .

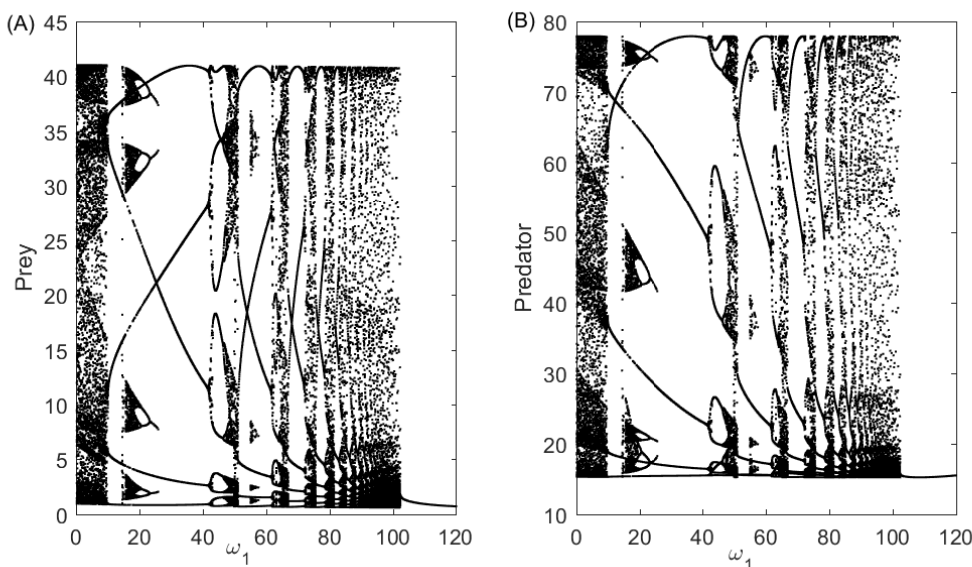


**Figure 2.** The different periodic solutions of system (1.1). The parameter values are identical to those in Figure 1.

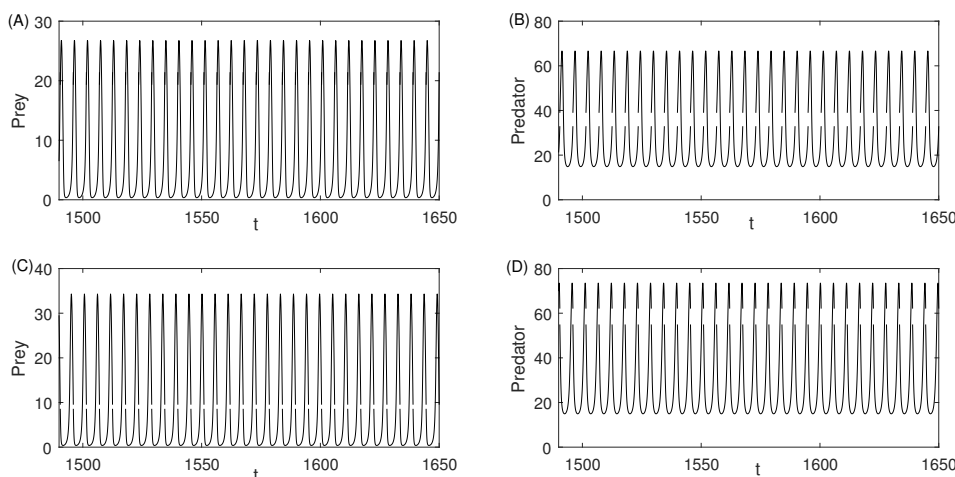
It can be ascertained from Figure 3 that the nonlinear impulse parameter  $w_1$  affect the dynamics of the system (1.1), and that with the increase of parameter  $w_1$ , the system (1.1) undergoes period-

doubling bifurcation, chaotic bifurcation, period-halving bifurcation and multi-stability. Figure 4 shows that attractors with different amplitudes appeared at  $w_1 = 10.2$ . Therefore, the initial values of both pest and predator populations are critical.

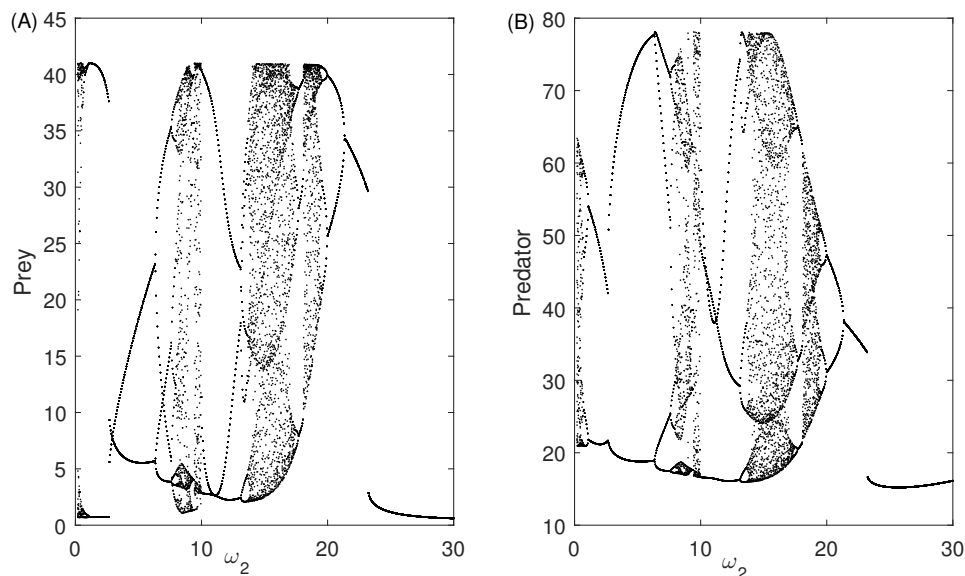
The results shown in Figure 5 reveal how the maximal release of predators, denoted by  $w_2$  affects the dynamics of system (1.1). The properties of the positive periodic solution are maintained until  $w_2 = 2.9$ , at which point a period-doubling bifurcation occurs. Additionally, as  $w_2$  increases, a series of period bifurcations cause the system (1.1) to move from periodicity to chaos. However, when  $w_2 \geq 20.2$ , the chaos disappears.



**Figure 3.** Bifurcation diagrams for system (1.1) with respect to  $w_1$ . The parameter values are as follows:  $r_1 = 7.65; r_2 = 2; b = 0.12; k = 5; a_1 = 5; a_2 = 1; a_3 = 0.00001; \alpha = 2.5; \beta = 0.75; h = 0.27; q = 0.45; \theta = 12.5; \delta = 0.1; w_2 = 1.2$ .



**Figure 4.** Two coexisting attractors for system (1.1) with  $w_1 = 10.2$ ; the other parameter values are identical to those in Figure 3.



**Figure 5.** Bifurcation diagrams for system (1.1) with respect to  $w_2$ , where  $w_1 = 7.85$ ,  $T = 5.5$ ; the other parameters are identical to those in Figure 1.

## 6. Conclusions

Regarding the predator-prey systems with an integrated pest management scheme, the impulse control strategies in many previous studies were linear, while ignoring the influence of nonlinear impulse on pest control such as the effects of limited resources. The number of natural enemies to be released must be based on the densities of the pest and predator populations in the field, so we have provided a nonlinear impulsive control predator-prey model with Leslie-Gower and B-D terms. The global stability of the pest extinction periodic solution of system (1.1) has been studied by applying the Floquet theory for differential equations and the comparison theorem, and the conditions for the permanence of system (1.1) has been obtained in detail. We have also proven that a nontrivial periodic solution can be obtained via a supercritical bifurcation once a threshold condition is reached within the context of bifurcation theory. In order to reveal how nonlinear impulses affect successful pest control strategies, we performed numerical simulations to show that system (1.1) has very complex dynamic behaviors, such as period-halving bifurcations, period-doubling bifurcation, chaos, and non-unique attractors. Comparing the bifurcation diagrams for parameters  $w_1$  and  $w_2$ , it was found that small perturbations can cause periodic oscillations of different amplitudes and periods in the populations of pests and natural enemies, that is, the influence of nonlinear impulses on the dynamic behavior of the model is very sensitive, which shows that the predator-prey models with nonlinear impulses have richer dynamic properties. These bifurcation diagrams show that impulse period, insecticide dose, and initial population densities are critical to successful control strategies.

In view of these results, we believe that systems with nonlinear impulsive control are more reasonable and have more complex dynamic behavior than systems with linear impulsive control. Furthermore, nonlinear impulsive control should be taken into account when investigating predator-prey systems with IPM. However, a more realistic case is that the density dependent releasing function



should depend on the densities of the prey and predator populations, i.e., the higher the density of pests or the lower the density of predators in the field, the higher the number of predators that should be released and vice versa. We will work on this in the near future.

### Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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### Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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