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Research article

A higher-order uniform accuracy scheme for nonlinear ψ-Volterra integral equations in two dimension with weakly singular kernel

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Abstract: In this paper, we proposed a higher-order uniform accuracy scheme for nonlinear ψ-Volterra integral equations in two dimension with weakly singular kernel by using the modified block-by-block method. First, we constructed a high order uniform accuracy scheme method in this paper by dividing the entire domain into some small sub-domains and approximating the integration function with biquadratic interpolation in each sub-domain. Second, we rigorously proved that the convergence order of the higher order uniform accuracy scheme was $O(h_s^{3+\sigma_1} + h_t^{3+\sigma_2})$ with $0 < \sigma_1, \sigma_2 < 1$ by using the discrete Gronwall inequality. Finally, two numerical examples were used to illustrate experimental results with different values of ψ to support the theoretical results.

Keywords: ψ-Volterra integral equations; high-order uniform accuracy scheme; block-by-block method; convergence analysis
Mathematics Subject Classification: 65R20, 65D30

1. Introduction

Volterra integral equations (VIEs) are important in many fields and they have been studied extensively. The purpose of this paper is to investigate the higher-order uniform accuracy numerical solution of nonlinear ψ-VIEs with weakly singular kernel in two dimension,

$$v(s,t) = g(s,t) + \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)} \int_a^s \int_c^t \frac{\omega(s,t,\tau,\eta,v(\tau,\eta))}{(\psi(s)-\psi(\tau))^{1-\sigma_1}(\psi(t)-\psi(\eta))^{1-\sigma_2}} d\psi(\eta) d\psi(\tau), \tag{1.1}$$

where $(s,t) \in H, 0 < \sigma_1, \sigma_2 < 1$, $v(s,t)$ is an unknown function defined in $H = [a,b] \times [c,d]$. g, ω are known functions and defined in H and $\Omega = H \times H \times R$, respectively. ψ is a monotonically increasing function and $\psi' > 0$.

Various numerical methods for solving the spectral form of the ψ-VIEs (1.1) have already been constructed by the direct quadrature methods [1, 2], spectral methods [3, 4], difference methods [5], operational matrix methods [6, 7], block-by-block method [8], and Taylor polynomials [9]. More numerical methods are available for readers to read [10–14]. For the numerical solution for the general ψ-VIEs (1.1), many researchers have studied this topic and achieved some research results. For example, in [16], the Chebyshev wavelets were employed to establish a computational procedure for ψ-VIEs. In [17], they established a Maxwell model with the ψ-Caputo fractional derivative. In [18], they presented a novel pharmacokinetic/pharmacodynamic model with ψ-Caputo fractional derivatives for the induction phase of anesthesia. In [19], they presented a generalization of the ψ-Hilfer fractional derivative. In [20], they investigated the existence, uniqueness, and Ulam-Hyers stabilities of mild solutions of ψ-fractional differential equations. They investigated the multiple ψ-type stability of fractional-order quaternion-valued neural networks in [21]. In [22], they studied a new class of impulsive boundary value problems with the generalized ψ-Caputo fractional derivatives. In [23], the stability of the autonomous linear $\alpha \in (0, 1)$ order ψ-Caputo fractional differential system was investigated. In [24], they presented a numerical method for solving the ψ-Caputo fractional differential equations. In [25], they solved a nonsingularity kernel VIEs by the fourth order Lagrange basis function.

Therefore, the main aim of this work is to propose an efficient high order time uniform numerical scheme for nonlinear ψ-VIEs with weakly singular kernel in two dimension (1.1) by using biquadratic interpolation in each sub-domain with the convergence order $O(h_s^{3+\sigma_1} + h_t^{3+\sigma_2})$ with $0 < \sigma_1, \sigma_2 < 1$. The modified block-by-block method is introduced to discrete nonlinear ψ-VIEs with weakly singular kernel in two dimension. The proposed scheme can avoid convergence accuracy degeneracy near the two boundary layers by coupled calculating the numerical solutions, which achieves the sharp convergence order without the fine mesh near the boundary layer. The proposed scheme is efficient and does not require coupled solutions in the other sub-domains. The numerical scheme will be established in this article to provide a paradigm for establishing a high-order scheme of the nonlinear ψ-VIEs with weakly singular kernel in two dimension and analyzing its convergence and stability of a high-order numerical scheme. It can also provide readers with a feasible method for constructing high order time numerical schemes and their theoretical analysis for similar nonlinear ψ-VIEs with weakly singular kernel in two dimension.

The article follows this structure: A higher-order uniform accuracy numerical scheme is introduced in Section 2. Section 3 estimates the truncation error of the higher-order uniform accuracy numerical scheme. Section 4 analyzes the convergence of the higher-order uniform accuracy numerical scheme. In Section 5, we demonstrate the efficiency of the higher-order uniform accuracy numerical scheme and supports our theoretical findings through two numerical experiments. Finally, some conclusions are drawn in Section 6.

2. Higher-order numerical approach for nonlinear ψ-fractional VIEs in two dimension

In this section, we will consider the following nonlinear ψ-fractional VIEs in two dimension (1.1) based on the idea of [26,27]. For $\omega(s, t, \tau, \eta, \nu(\tau, \eta))$, we assume that it satisfies the following Lipschitz continuity for the fifth variable

$$|\omega(s, t, \tau, \eta, \nu_1(\tau, \eta)) - \omega(s, t, \tau, \eta, \nu_2(\tau, \eta))| \le L|\nu_1(\tau, \eta) - \nu_2(\tau, \eta)|,\ L > 0. \tag{2.1}$$

Next, we will approximate the nonlinear ψ-fractional VIEs in two dimension of (1.1) by using biquadratic Lagrange interpolation. Denote positive integers L, K, and $h_s = \frac{b-a}{2L}$, $h_t = \frac{d-c}{2K}$. Let $s_i = a + ih_s(0 \le i \le 2L), t_j = c + jh_t(0 \le j \le 2K)$, $\nu_{i,j}$ be the numerical solution at (s_i, t_j), and $\omega^{i,j}(\tau, \eta, \nu(\tau, \eta)) = \omega(s_i, t_j, \tau, \eta, \nu(\tau, \eta))$, $g_{i,j} = g(s_i, t_j)$. By using simple calculations, one can obtain that $\nu_{i,0} = g(s_i, 0)$, $\nu_{0,j} = g(0, t_j)$.

Let $\hat{f}_{p,i}(s), p = 0, 1, 2, 0 \le i \le 2L - 2$ and $f_{q,j}(t), q = 0, 1, 2, 0 \le j \le 2K - 2$ be the quadratic Lagrange basis functions at points s_i, s_{i+1}, s_{i+2} and t_j, t_{j+1}, t_{j+2}, respectively,

$$\hat{f}_{0,i}(\tau) = \frac{(\psi(\tau) - \psi(s_{i+1}))(\psi(\tau) - \psi(s_{i+2}))}{(\psi(s_i) - \psi(s_{i+1}))(\psi(s_i) - \psi(s_{i+2}))}, \hat{f}_{1,i}(\tau) = \frac{(\psi(\tau) - \psi(s_i))(\psi(\tau) - \psi(s_{i+2}))}{(\psi(s_{i+1}) - \psi(s_i))(\psi(s_{i+1}) - \psi(s_{i+2}))},$$
$$\hat{f}_{2,i}(\tau) = \frac{(\psi(\tau) - \psi(s_i))(\psi(\tau) - \psi(s_{i+1}))}{(\psi(s_{i+2}) - \psi(s_i))(\psi(s_{i+2}) - \psi(s_{i+1}))}; f_{0,j}(\eta) = \frac{(\psi(\eta) - \psi(t_{j+1}))(\psi(\eta) - \psi(t_{j+2}))}{(\psi(t_j) - \psi(t_{j+1}))(\psi(t_j) - \psi(t_{j+2}))},$$
$$f_{1,j}(\eta) = \frac{(\psi(\eta) - \psi(t_j))(\psi(\eta) - \psi(t_{j+2}))}{(\psi(t_{j+1}) - \psi(t_j))(\psi(t_{j+1}) - \psi(t_{j+2}))}, f_{2,j}(\eta) = \frac{(\psi(\eta) - \psi(t_j))((\psi(\eta) - \psi(t_{j+1}))}{(\psi(t_{j+2}) - \psi(t_j))(\psi(t_{j+2}) - \psi(t_{j+1}))}.$$

Let $\nu_{i,j}$, $\nu_{i,2k+1}$, $\nu_{i,2k+2}$, $\nu_{2l+1,j}$, and $\nu_{2l+2,j}$ be known, where $0 \le i \le 2l, 1 \le l \le L-1, 0 \le j \le 2k, 1 \le k \le K-1, i, j, k, l$ are integers. We will construct an approximate scheme for $\nu_{2l+1,2k+1}$, $\nu_{2l+1,2k+2}$, $\nu_{2l+2,2k+1}$, and $\nu_{2l+2,2k+2}$.

For $\nu(s_{2l+1}, t_{2k+1})$, we have

$$\begin{aligned}
&\nu(s_{2l+1}, t_{2k+1}) = g_{2l+1,2k+1} \\
&+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1}) - \psi(\eta))^{1-\sigma_2}}\omega^{2l+1,2k+1}(\tau, \eta, \nu(\tau, \eta))d\psi(\eta)d\psi(\tau) \\
&+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{j=1}^{k}\int_a^{s_1}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1}) - \psi(\eta))^{1-\sigma_2}}\omega^{2l+1,2k+1}(\tau, \eta, \nu(\tau, \eta))d\psi(\eta)d\psi(\tau) \\
&+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{i=1}^{l}\int_{s_{2i-1}}^{s_{2i+1}}\int_c^{t_1}\frac{(\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1}) - \psi(\eta))^{1-\sigma_2}}\omega^{2l+1,2k+1}(\tau, \eta, \nu(\tau, \eta))d\psi(\eta)d\psi(\tau) \\
&+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{i=1}^{l}\sum_{j=1}^{k}\int_{s_{2i-1}}^{s_{2i+1}}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1}) - \psi(\eta))^{1-\sigma_2}} \\
&\times\omega^{2l+1,2k+1}(\tau, \eta, \nu(\tau, \eta))d\psi(\eta)d\psi(\tau) \doteq g_{2l+1,2k+1} + D_1 + D_2 + D_3 + D_4.
\end{aligned} \tag{2.2}$$

For D_1, one can obtain that

$$\begin{aligned}
D_1 &= \frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1}) - \psi(\eta))^{1-\sigma_2}}\omega^{2l+1,2k+1}(\tau, \eta, \nu(\tau, \eta))d\psi(\eta)d\psi(\tau) \\
&\approx \frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1}) - \psi(\eta))^{1-\sigma_2}}\sum_{p=0}^{2}\sum_{q=0}^{2}\hat{f}_{p,0}(\tau)f_{q,0}(\eta) \\
&\quad\times\omega^{2l+1,2k+1}(s_p, t_q, \nu_{p,q})d\psi(\eta)d\psi(\tau) \\
&= \sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+1}^{p,0}B_{2k+1}^{q,0}\omega^{2l+1,2k+1}(s_p, t_q, \nu_{p,q}),
\end{aligned} \tag{2.3}$$

where $A_{2l+1}^{p,0}$, $B_{2k+1}^{q,0}$ are defined by

$$A_{2l+1}^{p,0} = \frac{1}{\Gamma(\sigma_1)} \int_a^{s_1} (\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1 - 1} \hat{f}_{p,0}(\tau) d\psi(\tau), p = 0, 1, 2, \tag{2.4}$$

$$B_{2k+1}^{q,0} = \frac{1}{\Gamma(\sigma_2)} \int_c^{t_1} (\psi(t_{2k+1}) - \psi(\eta))^{\sigma_2 - 1} f_{q,0}(\eta) d\psi(\eta), q = 0, 1, 2. \tag{2.5}$$

Similarly, for D_2, D_3, and D_4, we have

$$D_2 \approx \sum_{j=1}^{k} \sum_{p=0}^{2} \sum_{q=0}^{2} A_{2l+1}^{p,0} B_{2k+1}^{q,j} \omega^{2l+1,2k+1}(s_p, t_{2j-1+q}, \nu_{p,2j-1+q}), \tag{2.6}$$

$$D_3 \approx \sum_{i=1}^{l} \sum_{p=0}^{2} \sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+1}^{q,0} \omega^{2l+1,2k+1}(s_{2i-1+p}, t_q, \nu_{2i-1+p,q}), \tag{2.7}$$

$$D_4 \approx \sum_{i=1}^{l} \sum_{j=1}^{k} \sum_{p=0}^{2} \sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+1}^{q,j} \omega^{2l+1,2k+1}(s_{2i-1+p}, t_{2j-1+q}, \nu_{2i-1+p,2j-1+q}), \tag{2.8}$$

where $A_{2l+1}^{p,i}$ and $B_{2k+1}^{q,j}$ are defined as follows:

$$A_{2l+1}^{p,i} = \frac{1}{\Gamma(\sigma_1)} \int_{s_{2i-1}}^{s_{2i+1}} (\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1 - 1} \hat{f}_{p,2i-1}(\tau) d\psi(\tau), p = 0, 1, 2; i = 1, 2, \ldots, l, \tag{2.9}$$

$$B_{2k+1}^{q,j} = \frac{1}{\Gamma(\sigma_2)} \int_{t_{2j-1}}^{t_{2j+1}} (\psi(t_{2k+1}) - \psi(\eta))^{\sigma_2 - 1} f_{q,2j-1}(\eta) d\psi(\eta), q = 0, 1, 2; j = 1, 2, \ldots, k, \tag{2.10}$$

and $A_{2l+1}^{p,0}$, $B_{2k+1}^{q,0}$ are defined by (2.4) and (2.5), respectively.

Bringing (2.3) and (2.6)–(2.8) into (2.2), we have

$$\begin{aligned} \nu_{2l+1,2k+1} = {} & g_{2l+1,2k+1} + \sum_{p=0}^{2} \sum_{q=0}^{2} A_{2l+1}^{p,0} B_{2k+1}^{q,0} \omega^{2l+1,2k+1}(s_p, t_q, \nu_{p,q}) \\ & + \sum_{j=1}^{k} \sum_{p=0}^{2} \sum_{q=0}^{2} A_{2l+1}^{p,0} B_{2k+1}^{q,j} \omega^{2l+1,2k+1}(s_p, t_{2j-1+q}, \nu_{p,2j-1+q}) \\ & + \sum_{i=1}^{l} \sum_{p=0}^{2} \sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+1}^{q,0} \omega^{2l+1,2k+1}(s_{2i-1+p}, t_q, \nu_{2i-1+p,q}) \\ & + \sum_{i=1}^{l} \sum_{j=1}^{k} \sum_{p=0}^{2} \sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+1}^{q,j} \omega^{2l+1,2k+1}(s_{2i-1+p}, t_{2j-1+q}, \nu_{2i-1+p,2j-1+q}). \end{aligned} \tag{2.11}$$

For $\nu(s_{2l+2}, t_{2k+1})$, $\nu(s_{2l+1}, t_{2k+2})$, and $\nu(s_{2l+2}, t_{2k+2})$, we have

$$\nu(s_{2l+2}, t_{2k+1}) \approx g_{2l+2,2k+1} + \sum_{i=0}^{l} \sum_{p=0}^{2} \sum_{q=0}^{2} A_{2l+2}^{p,i} B_{2k+1}^{q,0} \omega^{2l+2,2k+1}(s_{2i+p}, t_q, \nu_{2i+p,q})$$

$$+\sum_{i=0}^{l}\sum_{j=1}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+2}^{p,i}B_{2k+1}^{q,j}\omega^{2l+2,2k+1}(s_{2i+p},t_{2j-1+q},v_{2i+p,2j-1+q}), \quad (2.12)$$

$$\begin{aligned} v(s_{2l+1},t_{2k+2}) &\approx g_{2l+1,2k+2}+\sum_{j=0}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+1}^{p,0}B_{2k+2}^{q,j}\omega^{2l+1,2k+2}(s_p,t_{2j+q},v_{p,2j+q}) \\ &\quad +\sum_{i=1}^{l}\sum_{j=0}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+1}^{p,i}B_{2k+2}^{q,j}\omega^{2l+1,2k+2}(s_{2i-1+p},t_{2j+q},v_{2i-1+p,2j+q}), \end{aligned} \quad (2.13)$$

$$v(s_{2l+2},t_{2k+2}) \approx g_{2l+2,2k+2}+\sum_{i=0}^{l}\sum_{j=0}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+2}^{p,i}B_{2k+2}^{q,j}\omega^{2l+2,2k+2}(s_{2i+p},t_{2j+q},v_{2i+p,2j+q}), \quad (2.14)$$

where $A_{2l+2}^{p,i}$ and $B_{2k+2}^{q,j}$ are defined as follows:

$$A_{2l+2}^{p,i} = \frac{1}{\Gamma(\sigma_1)}\int_{s_{2i}}^{s_{2i+2}}(\psi(s_{2l+2})-\psi(\tau))^{\sigma_1-1}\hat{f}_{p,2i}(\tau)d\psi(\tau), p=0,1,2; i=0,1,\ldots,l, \quad (2.15)$$

$$B_{2k+2}^{q,j} = \frac{1}{\Gamma(\sigma_2)}\int_{t_{2j}}^{t_{2j+2}}(\psi(t_{2k+2})-\psi(\eta))^{\sigma_2-1}f_{q,2j}(\eta)d\psi(\eta), q=0,1,2; j=0,1,\ldots,k, \quad (2.16)$$

and $A_{2l+1}^{p,0}$, $A_{2l+1}^{p,i}$, $B_{2k+1}^{q,0}$, and $B_{2k+1}^{q,j}$ are defined in (2.4), (2.9), (2.5), and (2.10).

Similar to the same line of (2.11), we can obtain the other point's numerical scheme. A detailed derivation is given in Appendix A.

Combining with (A.1), (A.4)–(A.6), (A.9), (A.10)–(A.12), (A.13)–(A.16), (2.11)–(2.14), we obtain a high-order numerical scheme for (1.1) as $i_1, j_1 = 1,2$,

$$\begin{aligned} v_{i_1,j_1} &= g_{i_1,j_1}+\sum_{p=0}^{2}\sum_{q=0}^{2}A_{i_1}^{p,0}B_{j_1}^{q,0}\omega^{i_1,j_1}(s_p,t_q,v_{p,q}), \\ v_{2l+1,j_1} &= g_{2l+1,j_1}+\sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+1}^{p,0}B_{j_1}^{q,0}\omega^{2l+1,j_1}(s_p,t_q,v_{p,q}) \\ &\quad +\sum_{i=1}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+1}^{p,i}B_{j_1}^{q,0}\omega^{2l+1,j_1}(s_{2i-1+p},t_q,v_{2i-1+p,q}), \\ v_{2l+2,j_1} &= g_{2l+2,j_1}+\sum_{i=0}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+2}^{p,i}B_{j_1}^{q,0}\omega^{2l+2,j_1}(s_{2i+p},t_q,v_{2i+p,q}), \\ v_{i_1,2k+1} &= g_{i_1,2k+1}+\sum_{p=0}^{2}\sum_{q=0}^{2}A_{i_1}^{p,0}B_{2k+1}^{q,0}\omega^{i_1,2k+1}(s_p,t_q,v_{p,q}) \\ &\quad +\sum_{j=1}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{i_1}^{p,0}B_{2k+1}^{q,j}\omega^{i_1,2k+1}(s_p,t_{2j-1+q},v_{p,2j-1+q}), \\ v_{i_1,2k+2} &= g_{i_1,2k+2}+\sum_{j=0}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{i_1}^{p,0}B_{2k+2}^{q,j}\omega^{i_1,2k+2}(s_p,t_{2j+q},v_{p,2j+q}), \end{aligned}$$

$$
\begin{aligned}
\nu_{2l+1,2k+1} &= g_{2l+1,2k+1} + \sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,0} B_{2k+1}^{q,0} \omega^{2l+1,2k+1}(s_p, t_q, \nu_{p,q}) \\
&\quad + \sum_{j=1}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,0} B_{2k+1}^{q,j} \omega^{2l+1,2k+1}(s_p, t_{2j-1+q}, \nu_{p,2j-1+q}) \\
&\quad + \sum_{i=1}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+1}^{q,0} \omega^{2l+1,2k+1}(s_{2i-1+p}, t_q, \nu_{2i-1+p,q}) \\
&\quad + \sum_{i=1}^{l}\sum_{j=1}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+1}^{q,j} \omega^{2l+1,2k+1}(s_{2i-1+p}, t_{2j-1+q}, \nu_{2i-1+p,2j-1+q}), \\
\nu_{2l+2,2k+1} &= g_{2l+2,2k+1} + \sum_{i=0}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+2}^{p,i} B_{2k+1}^{q,0} \omega^{2l+2,2k+1}(s_{2i+p}, t_q, \nu_{2i+p,q}) \\
&\quad + \sum_{i=0}^{l}\sum_{j=1}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+2}^{p,i} B_{2k+1}^{q,j} \omega^{2l+2,2k+1}(s_{2i+p}, t_{2j-1+q}, \nu_{2i+p,2j-1+q}), \\
\nu_{2l+1,2k+2} &= g_{2l+1,2k+2} + \sum_{j=0}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,0} B_{2k+2}^{q,j} \omega^{2l+1,2k+2}(s_p, t_{2j+q}, \nu_{p,2j+q}) \\
&\quad + \sum_{i=1}^{l}\sum_{j=0}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+2}^{q,j} \omega^{2l+1,2k+2}(s_{2i-1+p}, t_{2j+q}, \nu_{2i-1+p,2j+q}), \\
\nu_{2l+2,2k+2} &= g_{2l+2,2k+2} + \sum_{i=0}^{l}\sum_{j=0}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+2}^{p,i} B_{2k+2}^{q,j} \omega^{2l+2,2k+2}(s_{2i+p}, t_{2j+q}, \nu_{2i+p,2j+q}).
\end{aligned}
\tag{2.17}
$$

3. Truncation error

Let $E_{i,j}$ be the truncation error of scheme (2.17) at point (s_i, t_j) as follows:

$$
E_{i,j} := \nu(s_i, t_j) - \bar{\nu}_{i,j}, \tag{3.1}
$$

where $\bar{\nu}_{i,j}$ is the numerical solution for $\nu(s_i, t_j)$. For instance, at the point (s_{2l+1}, t_{2k+1}), $\bar{\nu}_{i,j}$ is defined as follows:

$$
\begin{aligned}
\bar{\nu}_{2l+1,2k+1} = g_{2l+1,2k+1} &+ \sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,0} B_{2k+1}^{q,0} \omega^{2l+1,2k+1}(s_p, t_q, \nu(s_p, t_q)) \\
&+ \sum_{j=1}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,0} B_{2k+1}^{q,j} \omega^{2l+1,2k+1}(s_p, t_{2j-1+q}, \nu(s_p, t_{2j-1+q})) \\
&+ \sum_{i=1}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+1}^{q,0} \omega^{2l+1,2k+1}(s_{2i-1+p}, t_q, \nu(s_{2i-1+p}, t_q)) \\
&+ \sum_{i=1}^{l}\sum_{j=1}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+1}^{q,j} \omega^{2l+1,2k+1}(s_{2i-1+p}, t_{2j-1+q}, \nu(s_{2i-1+p}, t_{2j-1+q})).
\end{aligned}
\tag{3.2}
$$

In order to analyze error estimation for the scheme (2.17), we will introduce the following lemmas.

Lemma 3.1. *For all $\psi \in C^1(I)$, an increasing function such that $\psi'(s) > 0$, we have*

$$\int_a^{s_1} (\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1 - 1} d\psi(\tau) \leq 2^{1-\sigma_1} r^{\sigma_1} (2l+1)^{\sigma_1 - 1} h_s^{\sigma_1},$$
$$\int_c^{t_1} (\psi(t_{2k+1}) - \psi(\eta))^{\sigma_2 - 1} d\psi(\eta) \leq 2^{1-\sigma_2} \bar{r}^{\sigma_2} (2k+1)^{\sigma_2 - 1} h_t^{\sigma_2},$$

where

$$r = \max_{s\in[a,b]} \psi'(s), \bar{r} = \max_{t\in[c,d]} \psi'(t). \tag{3.3}$$

Proof. Using the integral mean value theorem and Lagrange mean value theorem, one can get

$$\begin{aligned}
&\int_a^{s_1} (\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1 - 1} d\psi(\tau) = (\psi(s_{2l+1}) - \varrho_1)^{\sigma_1 - 1} (\psi(s_1) - \psi(a)) \\
\leq\ & (\psi(s_{2l+1}) - \psi(s_1))^{\sigma_1 - 1} (\psi(s_1) - \psi(a)) = (\psi'(\theta_{2l+1}))^{\sigma_1 - 1} (s_{2l+1} - s_1)^{\sigma_1 - 1} \psi'(\theta_1)(s_1 - a) \\
=\ & (\psi'(\theta_{2l+1}))^{\sigma_1 - 1} (2lh_s)^{\sigma_1 - 1} \psi'(\theta_1)(s_1 - a) \\
=\ & (\psi'(\theta_{2l+1}))^{\sigma_1 - 1} (2l+1)^{\sigma_1 - 1} h_s^{\sigma_1 - 1} \psi'(\theta_1) h_s (\frac{2l}{2l+1})^{\sigma_1 - 1} \leq 2^{1-\sigma_1} r^{\sigma_1} (2l+1)^{\sigma_1 - 1} h_s^{\sigma_1},
\end{aligned}$$

where $\varrho_1 \in (\psi(a), \psi(s_1))$, $\theta_{2l+1} \in (s_1, s_{2l+1})$, $\theta_1 \in (a, s_1)$, and r is shown in (3.3).

Similarly, $\int_c^{t_1} (\psi(t_{2k+1}) - \psi(\eta))^{\sigma_2 - 1} d\psi(\eta) \leq 2^{1-\sigma_2} \bar{r}^{\sigma_2} (2k+1)^{\sigma_2 - 1} h_t^{\sigma_2}$, where $\bar{r}$ is shown in (3.3). Therefore, Lemma 3.1 has been proven. □

Lemma 3.2. *Let $E_{i,j}$ be defined by* (3.1), *$\omega(\cdot,\cdot,\cdot,\cdot,\nu(\cdot,\cdot)) \in C^4([a,b]\times[c,d])$, and $\nu(\cdot,\cdot) \in C^4([a,b]\times[c,d])$, then*

$$|E_{i,j}| \leq C(h_s^{3+\sigma_1} + h_t^{3+\sigma_2}),$$

where C is a constant independent of h_s, h_t.

Proof. For a detailed proof, see Appendix B. □

4. Convergence analysis

Lemma 4.1. *The coefficients are estimated as follows:*

$$\begin{aligned}
&|A_1^{p,0}| \leq Ch_s^{\sigma_1},\ |A_2^{p,0}| \leq Ch_s^{\sigma_1},\ |A_{2l+1}^{p,i}| \leq Ch_s^{\sigma_1}(2l-2i)^{\sigma_1-1}, |A_{2l+2}^{p,i}| \leq Ch_s^{\sigma_1}(2l+2-2i)^{\sigma_1-1},\\
&|A_{2l+1}^{p,l}| \leq C2^{\sigma_1} h_s^{\sigma_1},\ |A_{2l+2}^{p,l}| \leq C2^{\sigma_1} h_s^{\sigma_1}, |B_1^{q,0}| \leq Ch_t^{\sigma_2},\ |B_2^{q,0}| \leq Ch_t^{\sigma_2},\ |B_{2k+1}^{q,j}| \leq Ch_t^{\sigma_2}(2k-2j)^{\sigma_2-1},\\
&|B_{2k+2}^{q,j}| \leq Ch_t^{\sigma_2}(2k+2-2j)^{\sigma_2-1},\ |B_{2k+1}^{q,k}| \leq C2^{\sigma_2} h_t^{\sigma_2}, |B_{2k+2}^{q,k}| \leq C2^{\sigma_2} h_t^{\sigma_2},
\end{aligned}$$

where C is independent of h_s and h_t. In equation (A.2), (A.7), (2.4), (2.9), (2.15) *and* (A.3), (A.8), (2.5), (2.10), (2.16), *we have the definition of $A_1^{p,0}$, $A_2^{p,0}$, $A_{2l+1}^{p,0}$, $A_{2l+1}^{p,i}$, $A_{2l+2}^{p,i}$ and $B_1^{q,0}$, $B_2^{q,0}$, $B_{2k+1}^{q,0}$, $B_{2k+1}^{q,j}$, $B_{2k+2}^{q,j}$, where $p,q = 0,1,2$, $i = 0,1,\ldots,l-1$, $j = 0,1,\ldots,k-1$, $l = 1,2,\ldots,L-1$, and $k = 1,2,\ldots,K-1$.*

Proof. By employing the Lagrange mean value theorem, we have

$$\begin{aligned}|A_1^{0,0}| &= |\frac{1}{\Gamma(\sigma_1)}\int_a^{s_1}(\psi(s_1)-\psi(\tau))^{\sigma_1-1}\hat{f}_{0,0}(\tau)d\psi(\tau)|\\ &\le \frac{1}{\Gamma(\sigma_1)}\int_a^{s_1}(\psi(s_1)-\psi(\tau))^{\sigma_1-1}|\frac{(\psi(\tau)-\psi(s_1))(\psi(\tau)-\psi(s_2))}{(\psi(s_0)-\psi(s_1))(\psi(s_0)-\psi(s_2))}|d\psi(\tau)\\ &\le \frac{1}{\Gamma(\sigma_1)}\int_a^{s_1}(\psi(s_1)-\psi(\tau))^{\sigma_1-1}d\psi(\tau)=\frac{1}{\Gamma(\sigma_1+1)}(\psi(s_1)-\psi(a))^{\sigma_1}\\ &= \frac{1}{\Gamma(\sigma_1+1)}(\psi'(\xi_1)(s_1-a))^{\sigma_1}\le\frac{1}{\Gamma(\sigma_1+1)}r^{\sigma_1}h_s^{\sigma_1}\le Ch_s^{\sigma_1},\end{aligned}$$

where r is defined by (3.3), $\xi_1\in(a,s_1)$, and

$$\begin{aligned}|A_2^{0,0}| &= |\frac{1}{\Gamma(\sigma_1)}\int_a^{s_2}(\psi(s_2)-\psi(\tau))^{\sigma_1-1}\hat{f}_{0,0}(\tau)d\psi(\tau)|\\ &\le \frac{1}{\Gamma(\sigma_1)}\int_a^{s_2}(\psi(s_2)-\psi(\tau))^{\sigma_1-1}d\psi(\tau)=\frac{1}{\Gamma(\sigma_1+1)}(\psi(s_2)-\psi(a))^{\sigma_1}\\ &= \frac{1}{\Gamma(\sigma_1+1)}(\psi'(\xi_0')(s_2-a))^{\sigma_1}\le\frac{1}{\Gamma(\sigma_1+1)}(2r)^{\sigma_1}h_s^{\sigma_1}\le Ch_s^{\sigma_1},\end{aligned}$$

where $\xi_0'\in(a,s_2)$.

In the same way, we can get

$$|A_1^{1,0}|\le Ch_s^{\sigma_1},\ \ |A_1^{2,0}|\le Ch_s^{\sigma_1},\ \ |A_2^{1,0}|\le Ch_s^{\sigma_1},\ \ |A_2^{2,0}|\le Ch_s^{\sigma_1}.$$

For the estimation of $|A_{2l+1}^{0,0}|$, we can use the integral mean value theorem and Lagrange mean value theorem,

$$\begin{aligned}|A_{2l+1}^{0,0}| &= |\frac{1}{\Gamma(\sigma_1)}\int_a^{s_1}(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}\hat{f}_{0,0}(\tau)d\psi(\tau)|\\ &\le \frac{1}{\Gamma(\sigma_1)}\int_a^{s_1}(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}d\psi(\tau)\le\frac{h_s}{\Gamma(\sigma_1)}(\psi(s_{2l+1})-\psi(s_1))^{\sigma_1-1}\psi'(s_1)\\ &\le \frac{rh_s}{\Gamma(\sigma_1)}(\psi'(\xi_1)(s_{2l+1}-s_1))^{\sigma_1-1}\le\frac{r^{\sigma_1}}{\Gamma(\sigma_1)}(2l)^{\sigma_1-1}h_s^{\sigma_1}\le Ch_s^{\sigma_1}(2l)^{\sigma_1-1}.\end{aligned}$$

Similarly, we have

$$|A_{2l+1}^{1,0}|\le Ch_s^{\sigma_1}(2l)^{\sigma_1-1},\ \ |A_{2l+1}^{2,0}|\le Ch_s^{\sigma_1}(2l)^{\sigma_1-1}.$$

We can use the integral mean value theorem and Lagrange mean value theorem to get

$$\begin{aligned}|A_{2l+1}^{p,i}| &= |\frac{1}{\Gamma(\sigma_1)}\int_{s_{2i-1}}^{s_{2i+1}}(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}\hat{f}_{p,2i-1}(\tau)d\psi(\tau)|\\ &\le \frac{1}{\Gamma(\sigma_1)}\int_{s_{2i-1}}^{s_{2i+1}}(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}\psi'(\tau)d\tau\le\frac{2h_sr}{\Gamma(\sigma_1)}(\psi(s_{2l+1})-\psi(s_{2i+1}))^{\sigma_1-1}\\ &= \frac{2h_sr}{\Gamma(\sigma_1)}(\psi'(\xi_{2i-1}))^{\sigma_1-1}h_s^{\sigma_1-1}[2l+1-(2i+1)]^{\sigma_1-1}\end{aligned}$$

$$\leq \frac{2h_s}{\Gamma(\sigma_1)} r^{\sigma_1} h_s^{\sigma_1 - 1} (2l - 2i)^{\sigma_1 - 1} \leq C h_s^{\sigma_1} (2l - 2i)^{\sigma_1 - 1},$$

where $\xi_{2i-1} \in (s_{2i-1}, s_{2i+1})$.

In the same way, we have

$$|A_{2l+2}^{k,i}| \leq C h_s^{\sigma_1} (2l + 2 - 2i)^{\sigma_1 - 1}, i = 1, 2, \ldots, l - 1.$$

When $i = l$,

$$\begin{aligned} |A_{2l+1}^{p,l}| &= |\frac{1}{\Gamma(\sigma_1)} \int_{s_{2l-1}}^{s_{2l+1}} (\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1 - 1} \hat{f}_{p,2l-1}(\tau) d\tau| \\ &\leq \frac{1}{\Gamma(\sigma_1)} \int_{s_{2l-1}}^{s_{2l+1}} (\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1 - 1} |\hat{f}_{p,2l-1}(\tau)| d\psi(\tau) \\ &\leq \frac{1}{\Gamma(\sigma_1)} \int_{s_{2l-1}}^{s_{2l+1}} (\psi(s_{2l+1}) - \psi(\tau))^{\sigma_1 - 1} d\psi(\tau) = \frac{1}{\Gamma(\sigma_1 + 1)} (\psi(s_{2l+1}) - \psi(s_{2l-1}))^{\sigma_1} \\ &= \frac{1}{\Gamma(\sigma_1 + 1)} (\psi'(\xi_{2l+1})(s_{2l+1} - s_{2l-1}))^{\sigma_1} \leq \frac{h_s^{\sigma_1}}{\Gamma(\sigma_1 + 1)} (2r)^{\sigma_1} \leq C 2^{\sigma_1} h_s^{\sigma_1}. \end{aligned}$$

Similarly, we have $|A_{2l+2}^{p,l}| \leq C 2^{\sigma_1} h_s^{\sigma_1}$. Using the same approach, we can estimate $|B_1^{q,0}|, |B_2^{q,0}|, |B_{2k+1}^{q,0}|, |B_{2k+1}^{q,j}|, |B_{2k+2}^{q,j}|, |B_{2k+1}^{q,k}|, |B_{2k+2}^{q,k}|$ to be

$$\begin{aligned} &|B_1^{q,0}| \leq C h_t^{\sigma_2}, \ \ |B_2^{q,0}| \leq C h_t^{\sigma_2}, |B_{2k+1}^{q,j}| \leq C h_t^{\sigma_2} (2k - 2j)^{\sigma_2 - 1}, \\ &|B_{2k+2}^{q,j}| \leq C h_t^{\sigma_2} (2k + 2 - 2j)^{\sigma_2 - 1}, |B_{2k+1}^{q,k}| \leq C 2^{\sigma_2} h_t^{\sigma_2}, \ \ |B_{2k+2}^{q,k}| \leq C 2^{\sigma_2} h_t^{\sigma_2}. \end{aligned}$$

The proof of Lemma 4.1 is then completed. □

Theorem 4.1. *Let $v(s_i, t_j)$ be the exact solution of formula* (1.1) *and $v_{i,j}$ be the numerical solution. If w satisfies condition* (2.1)*, $\omega(\cdot, \cdot, \cdot, \cdot, v(\cdot, \cdot)) \in C^4([a, b] \times [c, d])$, and $v(\cdot, \cdot) \in C^4([a, b] \times [c, d])$, the step size satisfies the following formula:*

$$\begin{aligned} &2^{\sigma_1} C L_1 h_s^{\sigma_1} h_t^{\sigma_2} \leq 1, \ \ 2^{\sigma_2} C L_1 h_s^{\sigma_1} h_t^{\sigma_2} \leq 1, \\ &2^{\sigma_1} 2^{\sigma_2} C L_1 h_s^{\sigma_1} h_t^{\sigma_2} \leq 1, \ \ C L_1 \frac{(d - c)^{\sigma_2}}{\sigma_2} h_s^{\sigma_1} (2l)^{\sigma_1 - 1} \leq 1. \end{aligned} \tag{4.1}$$

We have

$$|v(s_i, t_j) - v_{i,j}| \leq C(h_s^{3+\sigma_1} + h_t^{3+\sigma_2}), i = 0, 1, \ldots, 2L; j = 0, 1, \ldots, 2K, \tag{4.2}$$

where C is independent of h_s and h_t.

Proof. $\forall i = 0, 1, \ldots, 2L; j = 0, 1, \ldots, 2K$; let $e_{i,j} = v(s_i, t_j) - v_{i,j} = v(s_i, t_j) - \bar{v}_{i,j} + \bar{v}_{i,j} - v_{i,j} = \bar{v}_{i,j} - v_{i,j} + E_{i,j}$. When $i, j = 1, 2$, we have $e_{i,j}$ as follows:

$$e_{i,j} = \sum_{p=0}^{2} \sum_{q=0}^{2} A_i^{p,0} B_j^{q,0} [\omega^{i,j}(s_p, t_q, v(s_p, t_q)) - \omega^{i,j}(s_p, t_q, v_{p,q})] + E_{i,j}.$$

According to (2.1) and Lemma 4.1, there is

$$|e_{i,j}| \le CL_1 h_s^{\sigma_1} h_t^{\sigma_2} \sum_{p=0}^{2}\sum_{q=0}^{2} |e_{p,q}| + |E_{i,j}|, i, j = 1, 2.$$

Combining the above four inequalities, we have

$$|e_{i,j}| \le CL_1(|E_{1,1}| + |E_{2,1}| + |E_{1,2}| + |E_{2,2}|), i, j = 1, 2.$$

That is,

$$|e_{i,j}| \le C(h_s^{3+\sigma_1} + h_t^{3+\sigma_2}), i, j = 1, 2. \tag{4.3}$$

When $i \ge 3,\ j = 1, 2$, we have

$$\begin{aligned}
e_{2l+1,1} &= \sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,0} B_1^{q,0}[\omega^{2l+1,1}(s_p, t_q, v(s_p, t_q)) - \omega^{2l+1,1}(s_p, t_q, v_{p,q})]\\
&+ \sum_{i=1}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,i} B_1^{q,0}[\omega^{2l+1,1}(s_{2i-1+p}, t_q, v(s_{2i-1+p}, t_q)) - \omega^{2l+1,1}(s_{2i-1+p}, t_q, v_{2i-1+p,q})] + E_{2l+1,1},\\
e_{2l+1,2} &= \sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,0} B_2^{q,0}[\omega^{2l+1,2}(s_p, t_q, v(s_p, t_q)) - \omega^{2l+1,2}(s_p, t_q, v_{p,q})]\\
&+ \sum_{i=1}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,i} B_2^{q,0}[\omega^{2l+1,2}(s_{2i-1+p}, t_q, v(s_{2i-1+p}, t_q)) - \omega^{2l+1,2}(s_{2i-1+p}, t_q, v_{2i-1+p,q})] + E_{2l+1,2},\\
e_{2l+2,1} &= \sum_{i=0}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+2}^{p,i} B_1^{q,0}[\omega^{2l+2,1}(s_{2i+p}, t_q, v(s_{2i+p}, t_q)) - \omega^{2l+2,1}(s_{2i+p}, t_q, v_{2i+p,q})] + E_{2l+2,1},\\
e_{2l+2,2} &= \sum_{i=0}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+2}^{p,i} B_2^{q,0}[\omega^{2l+2,2}(s_{2i+p}, t_q, v(s_{2i+p}, t_q)) - \omega^{2l+2,2}(s_{2i+p}, t_q, v_{2i+p,q})] + E_{2l+2,2},
\end{aligned}$$

where $A_{2l+1}^{p,0}, A_{2l+1}^{p,i}$ and $B_1^{q,0}, B_2^{q,0}$ are defined in (2.4), (2.9) and (A.3), (A.8), and it satisfies with Lemma 4.1 that we have

$$\begin{aligned}
|e_{2l+1,k_1}| \le\, & CL_1 \sum_{p=0}^{2}\sum_{q=0}^{2} h_s^{\sigma_1} h_t^{\sigma_2} (2l)^{\sigma_1-1} |e_{p,q}| + CL_1 \sum_{i=1}^{l-1}\sum_{p=0}^{2}\sum_{q=0}^{2} h_s^{\sigma_1}(2l-2i)^{\sigma_1-1} h_t^{\sigma_2} |e_{2i-1+p,q}|\\
& + CL_1 \sum_{p=0}^{2}\sum_{q=0}^{2} 2^{\sigma_1} h_s^{\sigma_1} h_t^{\sigma_2} |e_{2l-1+p,q}| + |E_{2l+1,k_1}|,\\
|e_{2l+2,k_1}| \le\, & CL_1 \sum_{p=0}^{2}\sum_{q=0}^{2} h_s^{\sigma_1} h_t^{\sigma_2} (2l+2)^{\sigma_1-1} |e_{p,q}| + CL_1 \sum_{i=1}^{l-1}\sum_{p=0}^{2}\sum_{q=0}^{2} h_s^{\sigma_1}(2l+2-2i)^{\sigma_1-1} h_t^{\sigma_2} |e_{2i+p,q}|\\
& + CL_1 \sum_{p=0}^{2}\sum_{q=0}^{2} 2^{\sigma_1} h_s^{\sigma_1} h_t^{\sigma_2} |e_{2l+p,q}| + |E_{2l+2,k_1}|,
\end{aligned}$$

where $k_1 = 1, 2$.

For the sake of convenience, we take

$$\|\bar{e}_i\| = max\{|e_{i,1}|, |e_{i,2}|, i = 0, 1, \ldots, 2L\}, \left\|\bar{E}_i\right\| = max\{|E_{i,1}|, |E_{i,2}|, i = 0, 1, \ldots, 2L\}.$$

Therefore, we get

$$\|\bar{e}_{2l+1}\| \leq CL_1 h_s^{\sigma_1} h_t^{\sigma_2} \sum_{i=0}^{2l} (2l+1-i)^{\sigma_1 - 1} \|\bar{e}_i\| + 2^{\sigma_1} CL_1 h_s^{\sigma_1} h_t^{\sigma_2} \|\bar{e}_{2l+1}\| + \|\bar{E}_{2l+1}\|, \tag{4.4}$$

$$\|\bar{e}_{2l+2}\| \leq CL_1 h_s^{\sigma_1} h_t^{\sigma_2} \sum_{i=0}^{2l+1} (2l+2-i)^{\sigma_1 - 1} \|\bar{e}_i\| + 2^{\sigma_1} CL_1 h_s^{\sigma_1} h_t^{\sigma_2} \|\bar{e}_{2l+2}\| + \|\bar{E}_{2l+2}\|. \tag{4.5}$$

For (4.4), we have

$$\|\bar{e}_{2l+1}\| \leq CL_1 h_s^{\sigma_1} h_t^{\sigma_2} \sum_{i=0}^{2l} (2l+1-i)^{\sigma_1 - 1} \|\bar{e}_i\| + C\|\bar{E}_{2l+1}\|. \tag{4.6}$$

For inequality (4.6), use Gronwall inequality [28]. We have

$$\begin{aligned} \|\bar{e}_{2l+1}\| &\leq CL_1 h_s^{\sigma_1} h_t^{\sigma_2} \sum_{i=0}^{2l} (2l+1-i)^{\sigma_1 - 1} \|\bar{e}_i\| + C\|\bar{E}_{2l+1}\| \\ &\leq C\|\bar{E}_{2l+1}\| E_{\sigma_1}(CL_1 h_t^{\sigma_2} \Gamma(\sigma_1)((2l+1)h_s)^{\sigma_1}) \leq C\|\bar{E}_{2l+1}\| E_{\sigma_1}(CL_1 (b-a)^{\sigma_1} h_t^{\sigma_2} \Gamma(\sigma_1)). \end{aligned}$$

Using the same method for (4.5), we can get

$$\|\bar{e}_{2l+2}\| \leq C\|\bar{E}_{2l+2}\| E_{\sigma_1}(CL_1 (b-a)^{\sigma_1} h_t^{\sigma_2} \Gamma(\sigma_1)).$$

When $i = 1, 2$, $j \geq 3$ computing $e_{i,j}$ is analogous to computing $e_{i,j}$ for $i \geq 3$, $j = 1, 2$; hence, we omitted it. According to the results above, combined with Lemma 3.2, we have

$$\begin{aligned} \|e_{i,j}\| &\leq C(h_s^{3+\sigma_1} + h_t^{3+\sigma_2}), i \geq 3, j = 1, 2, \\ \|e_{i,j}\| &\leq C(h_s^{3+\sigma_1} + h_t^{3+\sigma_2}), i = 1, 2, j \geq 3. \end{aligned} \tag{4.7}$$

Next, for $e_{i,j}$, $i, j \geq 3$, we obtain

$$\begin{aligned} e_{2l+1,2k+1} = &\sum_{p=0}^{2} \sum_{q=0}^{2} A_{2l+1}^{p,0} B_{2k+1}^{q,0} [\omega^{2l+1,2k+1}(s_p, t_q, v(s_p, t_q)) - \omega^{2l+1,2k+1}(s_p, t_q, v_{p,q})] \\ &+ \sum_{j=1}^{k} \sum_{p=0}^{2} \sum_{q=0}^{2} A_{2l+1}^{p,0} B_{2k+1}^{q,j} [\omega^{2l+1,2k+1}(s_p, t_{2j-1+q}, v(s_p, t_{2j-1+q})) \\ &- \omega^{2l+1,2k+1}(s_p, t_{2j-1+q}, v_{p,2j-1+q})] \\ &+ \sum_{i=1}^{l} \sum_{p=0}^{2} \sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+1}^{q,0} [\omega^{2l+1,2k+1}(s_{2i-1+p}, t_q, v(s_{2i-1+p}, t_q)) \end{aligned}$$

$$
\begin{aligned}
&-\omega^{2l+1,2k+1}(s_{2i-1+p}, t_q, \nu_{2i-1+p,q})] \\
&+\sum_{i=1}^{l}\sum_{j=1}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+1}^{q,j}[\omega^{2l+1,2k+1}(s_{2i-1+p}, t_{2j-1+q}, \nu(s_{2i-1+p}, t_{2j-1+q})) \\
&-\omega^{2l+1,2k+1}(s_{2i-1+p}, t_{2j-1+q}, \nu_{2i-1+p,2j-1+q})] + E_{2l+1,2k+1}, \\
e_{2l+2,2k+1} = &\sum_{i=0}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+2}^{p,i} B_{2k+1}^{q,0}[\omega^{2l+2,2k+1}(s_{2i+p}, t_q, \nu(s_{2i+p}, t_q)) - \omega^{2l+2,2k+1}(s_{2i+p}, t_q, \nu_{2i+p,q})] \\
&+\sum_{i=0}^{l}\sum_{j=1}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+2}^{p,i} B_{2k+1}^{q,j}[\omega^{2l+2,2k+1}(s_{2i+p}, t_{2j-1+q}, \nu(s_{2i+p}, t_{2j-1+q})) \\
&-\omega^{2l+2,2k+1}(s_{2i+p}, t_{2j-1+q}, \nu_{2i+p,2j-1+q})] + E_{2l+2,2k+1}, \\
e_{2l+1,2k+2} = &\sum_{j=0}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,0} B_{2k+2}^{q,j}[\omega^{2l+1,2k+2}(s_p, t_{2j+q}, \nu(s_p, t_{2j+q})) - \omega^{2l+1,2k+2}(s_p, t_{2j+q}, \nu_{p,2j+q})] \\
&+\sum_{i=1}^{l}\sum_{j=0}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+1}^{p,i} B_{2k+2}^{q,j}[\omega^{2l+1,2k+2}(s_{2i-1+p}, t_{2j+q}, \nu(s_{2i-1+p}, t_{2j+q})) \\
&-\omega^{2l+1,2k+2}(s_{2i-1+p}, t_{2j+q}, \nu_{2i-1+p,2j+q})] + E_{2l+1,2k+2}, \\
e_{2l+2,2k+2} = &\sum_{i=0}^{l}\sum_{j=0}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2} A_{2l+2}^{p,i} B_{2k+2}^{q,j}[\omega^{2l+2,2k+2}(s_{2i+p}, t_{2j+q}, \nu(s_{2i+p}, t_{2j+q})) \\
&-\omega^{2l+2,2k+2}(s_{2i+p}, t_{2j+q}, \nu_{2i+p,2j+q})] + E_{2l+2,2k+2},
\end{aligned}
$$

where $A_{2l+1}^{p,0}, A_{2l+1}^{p,i}, A_{2l+2}^{p,i}$ and $B_{2k+1}^{q,0}, B_{2k+1}^{q,j}, B_{2k+2}^{q,j}$ are defined in (2.4), (2.9), (2.15) and (2.5), (2.10), (2.16), and they are satisfied with Lemma 4.1. We have the following estimates for $e_{2l+1,2k+1}$,

$$
\begin{aligned}
|e_{2l+1,2k+1}| \leq\ & CL_1 \sum_{p=0}^{2}\sum_{q=0}^{2} h_s^{\sigma_1}(2l)^{\sigma_1-1} h_t^{\sigma_2}(2k)^{\sigma_2-1}|e_{p,q}| \\
&+CL_1 \sum_{j=1}^{k-1}\sum_{p=0}^{2}\sum_{q=0}^{2} h_s^{\sigma_1}(2l)^{\sigma_1-1} h_t^{\sigma_2}(2k-2j)^{\sigma_2-1}|e_{p,2j-1+q}| \\
&+CL_1 \sum_{p=0}^{2}\sum_{q=0}^{2} h_s^{\sigma_1}(2l)^{\sigma_1-1} h_t^{\sigma_2} 2^{\sigma_2}|e_{p,2k-1+q}| \\
&+CL_1 \sum_{i=1}^{l-1}\sum_{p=0}^{2}\sum_{q=0}^{2} h_s^{\sigma_1}(2l-2i)^{\sigma_1-1} h_t^{\sigma_2}(2k)^{\sigma_2-1}|e_{2i-1+p,q}| \\
&+CL_1 \sum_{p=0}^{2}\sum_{q=0}^{2} h_s^{\sigma_1} 2^{\sigma_1} h_t^{\sigma_2}(2k)^{\sigma_2-1}|e_{2l-1+p,q}| \\
&+CL_1 \sum_{i=1}^{l-1}\sum_{j=1}^{k-1}\sum_{p=0}^{2}\sum_{q=0}^{2} h_s^{\sigma_1}(2l-2i)^{\sigma_1-1} h_t^{\sigma_2}(2k-2j)^{\sigma_2-1}|e_{2i-1+p,2j-1+q}|
\end{aligned}
$$

$$+CL_1\sum_{p=0}^{2}\sum_{q=0}^{2}2^{\sigma_1}2^{\sigma_2}h_s^{\sigma_1}h_t^{\sigma_2}|e_{2l-1+p,2k-1+q}|+|E_{2l+1,2k+1}|.$$

Therefore, we can rearrange into the preferred form:

$$\begin{aligned}|e_{2l+1,2k+1}| \quad\leq\quad & CL_1\sum_{i=0}^{2l}\sum_{j=0}^{2k}h_s^{\sigma_1}(2l+1-i)^{\sigma_1-1}h_t^{\sigma_2}(2k+1-j)^{\sigma_2-1}|e_{i,j}|\\ &+CL_1h_s^{\sigma_1}(2l)^{\sigma_1-1}\sum_{j=0}^{2k}h_t^{\sigma_2}(2k+1-j)^{\sigma_2-1}|e_{2l+1,j}|\\ &+CL_1h_t^{\sigma_2}(2k)^{\sigma_2-1}\sum_{i=0}^{2l}h_s^{\sigma_1}(2l+1-i)^{\sigma_1-1}|e_{i,2k+1}|\\ &+CL_12^{\sigma_1+\sigma_2}h_s^{\sigma_1}h_t^{\sigma_2}|e_{2l+1,2k+1}|+|E_{2l+1,2k+1}|.\end{aligned}$$

If letting $\|e_i\|=\max\limits_{0\leq j\leq 2K}|e_{i,j}|, \|E_i\|=\max\limits_{0\leq j\leq 2K}|E_{i,j}|$, then we have

$$\begin{aligned}|e_{2l+1,2k+1}|\leq & CL_1\sum_{i=0}^{2l}\sum_{j=0}^{2k}h_s^{\sigma_1}(2l+1-i)^{\sigma_1-1}h_t^{\sigma_2}(2k+1-j)^{\sigma_2-1}\,\|e_i\|\\ &+CL_1h_s^{\sigma_1}(2l)^{\sigma_1-1}\sum_{j=0}^{2k}h_t^{\sigma_2}(2k+1-j)^{\sigma_2-1}\,\|e_{2l+1}\|\\ &+CL_1h_t^{\sigma_2}(2k)^{\sigma_2-1}\sum_{i=0}^{2l}h_s^{\sigma_1}(2l+1-i)^{\sigma_1-1}\,\|e_i\|+C\,\|E_{2l+1}\|\\ \leq & CL_1\sum_{i=0}^{2l}h_s^{\sigma_1}(2l+1-i)^{\sigma_1-1}\,\|e_i\|\int_{t_0}^{t_{2k+1}}(t_{2k+1}-\eta)^{\sigma_2-1}d\eta\\ &+CL_1h_s^{\sigma_1}(2l)^{\sigma_1-1}\,\|e_{2l+1}\|\int_{t_0}^{t_{2k+1}}(t_{2k+1}-\eta)^{\sigma_2-1}d\eta\\ &+CL_1h_t^{\sigma_2}(2k)^{\sigma_2-1}\sum_{i=0}^{2l}h_s^{\sigma_1}(2l+1-i)^{\sigma_1-1}\,\|e_i\|+C\,\|E_{2l+1}\|\\ = & CL_1\sum_{i=0}^{2l}h_s^{\sigma_1}(2l+1-i)^{\sigma_1-1}\,\|e_i\|\,\frac{t_{2k+1}^{\sigma_2}}{\sigma_2}+CL_1h_s^{\sigma_1}(2l)^{\sigma_1-1}\,\|e_{2l+1}\|\,\frac{t_{2k+1}^{\sigma_2}}{\sigma_2}\\ &+CL_1h_t^{\sigma_2}(2k)^{\sigma_2-1}\sum_{i=0}^{2l}h_s^{\sigma_1}(2l+1-i)^{\sigma_1-1}\,\|e_i\|+C\,\|E_{2l+1}\|\\ \leq & CL_1\frac{(d-c)^{\sigma_2}}{\sigma_2}\sum_{i=0}^{2l}h_s^{\sigma_1}(2l+1-i)^{\sigma_1-1}\,\|e_i\|+CL_1\frac{(d-c)^{\sigma_2}}{\sigma_2}h_s^{\sigma_1}(2l)^{\sigma_1-1}\,\|e_{2l+1}\|\\ &+CL_1h_t^{\sigma_2}(2k)^{\sigma_2-1}\sum_{i=0}^{2l}h_s^{\sigma_1}(2l+1-i)^{\sigma_1-1}\,\|e_i\|+C\,\|E_{2l+1}\|\,.\end{aligned}$$

Since the above formula for all $k = 1, 2, \ldots, K-1$, we can infer that it is universally applicable that

$$\begin{aligned}\|e_{2l+1}\| \;&\leq\; CL_1\frac{(d-c)^{\sigma_2}}{\sigma_2}h_s^{\sigma_1}\sum_{i=0}^{2l}(2l+1-i)^{\sigma_1-1}\|e_i\| + CL_1\frac{(d-c)^{\sigma_2}}{\sigma_2}h_s^{\sigma_1}(2l)^{\sigma_1-1}\|e_{2l+1}\| \\ &\quad + CL_1h_s^{\sigma_1}h_t^{\sigma_2}(2k)^{\sigma_2-1}\sum_{i=0}^{2l}(2l+1-i)^{\sigma_1-1}\|e_i\| + C\|E_{2l+1}\|.\end{aligned}$$

Hence,

$$\|e_{2l+1}\| \leq [CL_1\frac{(d-c)^{\sigma_2}}{\sigma_2}h_s^{\sigma_1} + CL_1h_s^{\sigma_1}h_t^{\sigma_2}(2k)^{\sigma_2-1}]\sum_{i=0}^{2l}(2l+1-i)^{\sigma_1-1}\|e_i\| + C\|E_{2l+1}\|. \tag{4.8}$$

Through the use of the discrete Gronwall inequality [28], the inequality (4.8) becomes

$$\begin{aligned}\|e_{2l+1}\| \;&\leq\; C\|E_{2l+1}\|E_{\sigma_1}\{[CL_1\frac{(d-c)^{\sigma_2}}{\sigma_2} + CL_1h_t^{\sigma_2}(2k)^{\sigma_2-1}]\Gamma(\sigma_1)((2l+1)h_s)^{\sigma_1}\} \\ &\leq\; C\|E_{2l+1}\|E_{\sigma_1}\{CL_1\Gamma(\sigma_1)(b-a)^{\sigma_1}[\frac{(d-c)^{\sigma_2}}{\sigma_2} + h_t^{\sigma_2}(2k)^{\sigma_2-1}]\}.\end{aligned}$$

Combining with Lemma 3.2, the above estimate becomes

$$|e_{2l+1,2k+1}| \leq C(h_s^{3+\sigma_1} + h_t^{3+\sigma_2}). \tag{4.9}$$

Using the same approach, we can derive the result

$$\begin{aligned}&|e_{2l+2,2k+1}| \leq C(h_s^{3+\sigma_1} + h_t^{3+\sigma_2}), \;\; |e_{2l+1,2k+2}| \leq C(h_s^{3+\sigma_1} + h_t^{3+\sigma_2}), \\ &|e_{2l+2,2k+2}| \leq C(h_s^{3+\sigma_1} + h_t^{3+\sigma_2}).\end{aligned} \tag{4.10}$$

Combining (4.3), (4.7), and (4.9)–(4.10), we complete the proof of Theorem 4.1. □

5. Numerical example

In this section, we will verify the correctness of the theory by using several numerical examples. We employ the proposed algorithm to solve the values of the VIEs with different values of $\psi(s)$ and the error and convergence order. All numerical examples will be implemented using Matlab software.

Example 5.1. *The two-dimensional linear ψ-type VIEs under consideration are as follows.*

$$v(s,t) = g(s,t) + \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_a^s\int_c^t \frac{(\psi(s)\psi(t) + \psi(\tau) + \psi(\eta))v(\tau,\eta)}{(\psi(s)-\psi(\tau))^{1-\sigma_1}(\psi(t)-\psi(\eta))^{1-\sigma_2}}d\psi(\eta)d\psi(\tau),$$

where $(s,t) \in [a,b]\times[c,d]$, and

$$\begin{aligned}g(s,t) = &(\psi(s))^4(\psi(t))^4 - \frac{576}{\Gamma(\sigma_1+5)\Gamma(\sigma_2+5)}(\psi(s))^{\sigma_1+5}(\psi(t))^{\sigma_2+5} \\ &- \frac{2880}{\Gamma(\sigma_1+5)\Gamma(\sigma_2+6)}(\psi(s))^{\sigma_1+4}(\psi(t))^{\sigma_2+5} - \frac{2880}{\Gamma(\sigma_1+6)\Gamma(\sigma_2+5)}(\psi(s))^{\sigma_1+5}(\psi(t))^{\sigma_2+4},\end{aligned}$$

with the exact solution $v(s,t) = (\psi(s))^4(\psi(t))^4$.

In this paper, we give two groups of data where σ_1 *and* σ_2 *take different values,which are* $\sigma_1 = 0.4$, $\sigma_2 = 0.6$ *and* $\sigma_1 = 0.3$, $\sigma_2 = 0.5$, *respectively. The steps in s-direction and t-direction are divided into* $h_s = \frac{b-a}{2L}, h_t = \frac{d-c}{2K}$. *Error is defined as follows:*

$$e_h = \max_{\substack{i=1,\ldots,2L \\ j=1,\ldots,2K}} |v(s_i,t_j) - v_{i,j}|.$$

when $\psi(s) = \log(s)$, *and take* $[a,b]\times[c,d] = [1,2]\times[1,2]$. *To evaluate the level of convergence achieved by the numerical schemes developed for different values of* σ_1 *and* σ_2, *we conducted tests as presented in Tables 1 and 2. The convergence order was determined using the formula* $log_2(\frac{e_h}{e_{h/2}})$. *According to the analysis conducted in this study, the theoretical convergence order of the numerical scheme developed is expressed as* $O(h_s^{3+\sigma_1} + h_t^{3+\sigma_2})$. *It can be observed that when* h_s *significantly exceeds* h_t, *the convergence order simplifies to* $O(h_s^{3+\sigma_1})$. *The tables presented in this paper (Table 1 and Table 2) consider different values for parameters L and K. In Table 1, we set* $K = 2L$ *with a corresponding order of* $3+\sigma_1$, *while in Table 2, we set* $L = 2K$ *with a corresponding order of* $3+\sigma_2$. *The values chosen for these tables are as follows:* $\sigma_1 = 0.4, \sigma_2 = 0.6$ *and* $\sigma_1 = 0.3, \sigma_2 = 0.5$. *From our observations in Table 1, it can be inferred that when* $\sigma_1 = 0.4, \sigma_2 = 0.6$, *the achieved order closely approximates 3.4; similarly, when* $\sigma_1 = 0.3, \sigma_2 = 0.5$, *the achieved order is close to 3.3, which aligns well with the theoretical expectation of* $3+\sigma_1$. *Similar conclusions can also be drawn from Table 2, with the convergence order close to the theoretical order* $3+\sigma_2$.

Table 1. Change of maximum error with step size and order of convergence with $K = 2L$.

L	$\sigma_1 = 0.4, \sigma_2 = 0.6$	Order	$\sigma_1 = 0.3, \sigma_2 = 0.5$	Order
8	$9.7735973460\times10^{-7}$	–	$1.9542229570\times10^{-6}$	–
16	$9.9153270337\times10^{-8}$	3.3011574072	$2.0826505593\times10^{-7}$	3.2301023735
32	$9.7507514562\times10^{-9}$	3.3460750454	$2.1637356351\times10^{-8}$	3.2668246481
64	$9.4336555234\times10^{-10}$	3.3696245807	$2.2198713909\times10^{-9}$	3.2849762407
128	$9.0416826803\times10^{-11}$	3.3831537288	$2.2624961626\times10^{-10}$	3.2944888455

Table 2. Change of maximum error with step size and order of convergence with $L = 2K$.

K	$\sigma_1 = 0.4, \sigma_2 = 0.6$	Order	$\sigma_1 = 0.3, \sigma_2 = 0.5$	Order
8	$7.3163516254\times10^{-7}$	–	$1.8606043702\times10^{-6}$	–
16	$7.1646985719\times10^{-8}$	3.3521464981	$1.8824121970\times10^{-7}$	3.3051168417
32	$6.5863218074\times10^{-9}$	3.4433611009	$1.8075405187\times10^{-8}$	3.3804826808
64	$5.8705672962\times10^{-10}$	3.4879011751	$1.6931334598\times10^{-9}$	3.4162603856
128	$5.1399086376\times10^{-11}$	3.5136853027	$1.5651222679\times10^{-10}$	3.4353484264

The Figure 1 below shows the error distribution of $\psi(s) = \log(s)$ *when* $K = 2L$ *and* $L = 128$. *The left diagram shows the discrepancy distribution of* $\sigma_1 = 0.4, \sigma_2 = 0.6$, *and the right diagram shows the discrepancy distribution of* $\sigma_1 = 0.3, \sigma_2 = 0.5$. *From the following two error graphs of Figure 1, we can see that when* $\sigma_1 = 0.4, \sigma_2 = 0.6$ *and* $\sigma_1 = 0.3, \sigma_2 = 0.5$, *although the errors are different, the error graphs are similar, and the errors can reach* 10^{-10}, *so it can be known from the error graphs that the numerical method used can better approximate* $v(s,t)$.

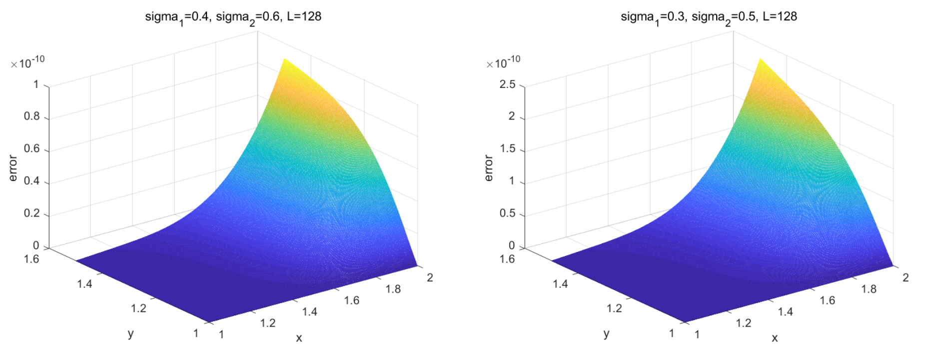

Figure 1. The error distribution for $L = 128$ is shown on the **left** with $\sigma_1 = 0.4, \sigma_2 = 0.6$, and on the **right** with $\sigma_1 = 0.3, \sigma_2 = 0.5$.

When $\psi(s) = s^\gamma$*, where* $\gamma = 1, 2$*, here we're going to take* $\gamma = 2$ *and take* $[a, b]\times[c, d] = [0, 1]\times[0, 1]$*. We list in the table the maximum errors and orders obtained for different values of* σ_1*,* σ_2*,* h_s*, and* h_t*. In this case, the values of* σ_1 *and* σ_2 *are the same as in the* $\psi(s) = \log(s)$*, and the values of K and L are going to be the same as* $\psi(s) = \log(s)$*. We can come to the same conclusion. When* $K = 2L$*, the orders of Table 3 are close to 3.4 and 3.3, respectively, which is close to the theoretical order* $3 + \sigma_1$*; when* $L = 2K$*, the orders of Table 4 are close to 3.6 and 3.5, respectively. That is, close to the theoretical order* $3 + \sigma_2$*.*

Table 3. Change of maximum error with step size and order of convergence with $K = 2L$.

L	$\sigma_1 = 0.4, \sigma_2 = 0.6$	Order	$\sigma_1 = 0.3, \sigma_2 = 0.5$	Order
8	$1.3928479154\times10^{-3}$	–	$2.8139154113\times10^{-3}$	–
16	$1.5068097519\times10^{-4}$	3.2084685586	$3.2493625228\times10^{-4}$	3.1143503452
32	$1.5407550712\times10^{-5}$	3.2897878307	$3.5457774736\times10^{-5}$	3.1959828070
64	$1.5258024479\times10^{-6}$	3.3359974520	$3.7451046675\times10^{-6}$	3.2430240537
128	$1.4831735151\times10^{-7}$	3.3628088901	$3.8830393190\times10^{-7}$	3.2697478185

Table 4. Change of maximum error with step size and order of convergence with $L = 2K$.

K	$\sigma_1 = 0.4, \sigma_2 = 0.6$	Order	$\sigma_1 = 0.3, \sigma_2 = 0.5$	Order
8	$1.3674595288\times10^{-3}$	–	$3.1438667735\times10^{-3}$	–
16	$1.3107931264\times10^{-4}$	3.3829862179	$3.2368563952\times10^{-4}$	3.2798748162
32	$1.1966003524\times10^{-5}$	3.4534267146	$3.1643283054\times10^{-5}$	3.3546221659
64	$1.0624704478\times10^{-6}$	3.0003202069	$5.1560512317\times10^{-6}$	3.3987109055
128	$9.2828275511\times10^{-8}$	3.5167145862	$2.7953305426\times10^{-7}$	3.4240256852

The Figure 2 below shows the error distribution of $\psi(s) = s^\gamma$*, where* $\gamma = 2$ *when* $K = 2L$ *and* $L = 128$*. The left diagram shows the discrepancy distribution of* $\sigma_1 = 0.4, \sigma_2 = 0.6$*, and the right diagram shows the discrepancy distribution of* $\sigma_1 = 0.3, \sigma_2 = 0.5$*. From the following two error graphs, we can see that when the values of* σ_1 *and* σ_2 *are different, although the errors are not the same, the error graphs are similar, the errors can reach* 10^{-7}*, and the effect is better, so it can be seen from the error graph that the numerical method used can better approximate* $v(s,t)$*.*

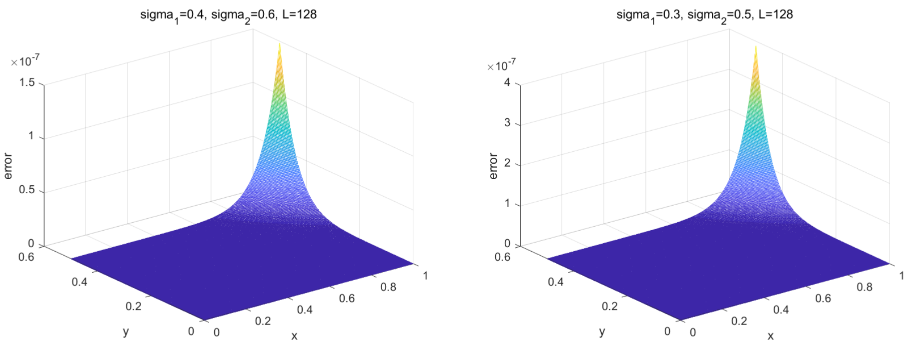

Figure 2. The error distribution for $L = 128$ is shown on the **left** with $\sigma_1 = 0.4, \sigma_2 = 0.6$, and on the **right** with $\sigma_1 = 0.3, \sigma_2 = 0.5$.

Example 5.2. *Consider the following nonlinear equation:*

$$v(s,t) = g(s,t) + \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)} \int_a^s \int_c^t \frac{(\psi(s)\psi(t) + \psi(\tau) + \psi(\eta))v^2(\tau,\eta)}{(\psi(s)-\psi(\tau))^{1-\sigma_1}(\psi(t)-\psi(\eta))^{1-\sigma_2}} d\psi(\eta) d\psi(\tau),$$

where $(s,t) \in [a,b]\times[c,d]$*, and*

$$\begin{aligned} g(s,t) = &(\psi(s))^2(\psi(t))^3 - \frac{17280}{\Gamma(\sigma_1+5)\Gamma(\sigma_2+7)}(\psi(s))^{\sigma_1+5}(\psi(t))^{\sigma_2+7} \\ &- \frac{86400}{\Gamma(\sigma_1+6)\Gamma(\sigma_2+7)}(\psi(s))^{\sigma_1+5}(\psi(t))^{\sigma_2+6} - \frac{120960}{\Gamma(\sigma_1+5)\Gamma(\sigma_2+8)}(\psi(s))^{\sigma_1+4}(\psi(t))^{\sigma_2+7}, \end{aligned}$$

with the exact solution is $v(s,t) = (\psi(s))^2(\psi(t))^3$*. We're still divided into two cases: first of all,* $\psi(s) = \log(s)$*, and second,* $\psi(s) = s^\gamma$*.*

Now, let's consider the case $\psi(s) = \log(s)$*, and take* $[a,b]\times[c,d] = [1,2]\times[1,2]$*. For this example, we repeat the calculation procedure of Example 5.1 using the appropriate scheme. The variation of its maximum error and order with* $K = 2L$ *or* $L = 2K$ *is shown in Table 5 or Table 6, and the values of L and K are the same as those in Example 5.1. These two tables again verify that the order is close to the theoretical order 3.4, 3.3, i.e.,* $3+\sigma_1$*, and 3.6, 3.5, i.e.,* $3+\sigma_2$*.*

Table 5. Change of maximum error with step size and order of convergence with $K = 2L$.

L	$\sigma_1 = 0.4, \sigma_2 = 0.6$	Order	$\sigma_1 = 0.3, \sigma_2 = 0.5$	Order
8	$3.2646193718\times10^{-7}$	–	$5.8807323880\times10^{-7}$	–
16	$3.1941197659\times10^{-8}$	3.3534244812	$6.0426050469\times10^{-8}$	3.2827532862
32	$3.0647534788\times10^{-9}$	3.3815754718	$6.1160141951\times10^{-9}$	3.3045069883
64	$2.9079247343\times10^{-10}$	3.3977091981	$6.1417454456\times10^{-10}$	3.3158711313
128	$2.7419067017\times10^{-11}$	3.4067385418	$6.1444627164\times10^{-11}$	3.3212899496

Table 6. Change of maximum error with step size and order of convergence with $L = 2K$.

K	$\sigma_1 = 0.4, \sigma_2 = 0.6$	Order	$\sigma_1 = 0.3, \sigma_2 = 0.5$	Order
8	$8.1677686153\times10^{-7}$	–	$1.6145922298\times10^{-6}$	–
16	$7.5881499251\times10^{-8}$	3.4281219081	$1.5779327084\times10^{-7}$	3.3550622664
32	$6.7981849594\times10^{-9}$	3.4805266643	$1.4912196727\times10^{-8}$	3.4034709794
64	$5.9532231922\times10^{-10}$	3.5134067255	$1.3807975596\times10^{-9}$	3.4329190728
128	$5.1339349438\times10^{-11}$	3.5355340638	$1.2619894019\times10^{-10}$	3.4517301201

Now, let's think about $\psi(s) = s^{\gamma}$, $\gamma = 2$ *and take* $[a, b] \times [c, d] = [0, 1] \times [0, 1]$. *From the following two tables, we know that the errors gradually become smaller and the order also tends to stabilize, which is close to 3.4,3.3 and 3.6,3.5.*

Table 7. Change of maximum error with step size and order of convergence with $K = 2L$.

L	$\sigma_1 = 0.4, \sigma_2 = 0.6$	Order	$\sigma_1 = 0.3, \sigma_2 = 0.5$	Order
8	$1.6250872088\times10^{-3}$	–	$4.3711285405\times10^{-3}$	–
16	$1.6880955354\times10^{-4}$	3.2670486824	$4.6795802277\times10^{-4}$	3.2235547770
32	$1.6830042009\times10^{-5}$	3.3262858708	$4.9400832297\times10^{-5}$	3.2437718682
64	$1.6319260485\times10^{-6}$	3.3663911902	$5.0905853166\times10^{-6}$	3.2786318959
128	$1.5567500733\times10^{-7}$	3.3899664301	$5.1643556231\times10^{-7}$	3.3011712925

Table 8. Change of maximum error with step size and order of convergence with $L = 2K$.

K	$\sigma_1 = 0.4, \sigma_2 = 0.6$	Order	$\sigma_1 = 0.3, \sigma_2 = 0.5$	Order
8	$4.7092542214\times10^{-3}$	–	$1.4079854053\times10^{-2}$	–
16	$4.6737446488\times10^{-4}$	3.3328477869	$1.3967936080\times10^{-3}$	3.3334416124
32	$4.3871206886\times10^{-5}$	3.4132326141	$1.3909889695\times10^{-4}$	3.3279359776
64	$3.9489534212\times10^{-6}$	3.4737321390	$1.3364857961\times10^{-5}$	3.3795945693
128	$3.4618023848\times10^{-7}$	3.5118750743	$1.2497697712\times10^{-6}$	3.4187102494

6. Conclusions

In this paper, the modified block-by-block method is used to solve the numerical solution of fractional ψ-Volterra integral equations. The convergence of order of the high order numerical scheme is analyzed rigorously by using the discrete Gronwall inequality. Through experimental verification, we find that this method has high accuracy and stability and can effectively deal with high dimensionality fractional ψ-Volterra integral equations.

We furthermore discussed the sophisticated numerical scheme of high order for high dimensionality fractional ψ-Volterra integral equations with singular solution by using graded mesh. We also noticed that when the grid division is very fine, the computational complexity of the format is very high. Therefore, we will plan to introduce the fast high-order algorithms for solving the high dimensionality fractional ψ-Volterra integral equations by using the fast Fourier transform.

Thus, we can apply it to a wider range of practical problems. There are still some limitations and shortcomings in this study, that is, the calculation amount is too big. In the future, we can continue to study that the modified block-by-block method can be further improved to enhance its computational efficiency and stability [29]. Additionally, the modified block-by-block method can be applied to practical problems and its application value in engineering and science can be explored.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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Appendix A: Construction of numerical scheme at points (s_i, t_j) for $i = 1, 2$ or $j = 1, 2$.

For $v(s_1, t_1)$, one has

$$\begin{aligned} v(s_1,t_1) &= g_{1,1} + \frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_1)-\psi(\tau))^{\sigma_1-1}}{(\psi(t_1)-\psi(\eta))^{1-\sigma_2}}\omega^{1,1}(\tau,\eta,v(\tau,\eta))d\psi(\eta)d\psi(\tau) \\ &\approx g_{1,1} + \sum_{p=0}^{2}\sum_{q=0}^{2}\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_1)-\psi(\tau))^{\sigma_1-1}}{(\psi(t_1)-\psi(\eta))^{1-\sigma_2}} \\ &\quad \times\hat{f}_{p,0}(\tau)f_{q,0}(\eta)\omega^{1,1}(s_p,t_q,v_{p,q})d\psi(\eta)d\psi(\tau) \\ &= g_{1,1} + \sum_{p=0}^{2}\sum_{q=0}^{2}A_1^{p,0}B_1^{q,0}\omega^{1,1}(s_p,t_q,v_{p,q}), \end{aligned} \tag{A.1}$$

where

$$A_1^{p,0} = \frac{1}{\Gamma(\sigma_1)}\int_a^{s_1}(\psi(s_1)-\psi(\tau))^{\sigma_1-1}\hat{f}_{p,0}(\tau)d\psi(\tau), p = 0, 1, 2, \tag{A.2}$$

$$B_1^{q,0} = \frac{1}{\Gamma(\sigma_2)}\int_c^{t_1}(\psi(t_1)-\psi(\eta))^{\sigma_2-1}f_{q,0}(\eta)d\psi(\eta), q = 0, 1, 2. \tag{A.3}$$

Similar to (A.1), one can obtain $v(s_2, t_1)$, $v(s_1, t_2)$, and $v(s_2, t_2)$ as follows:

$$v(s_2,t_1) \approx g_{2,1} + \sum_{p=0}^{2}\sum_{q=0}^{2}A_2^{p,0}B_1^{q,0}\omega^{2,1}(s_p,t_q,v_{p,q}), \tag{A.4}$$

$$v(s_1,t_2) \approx g_{1,2} + \sum_{p=0}^{2}\sum_{q=0}^{2}A_1^{p,0}B_2^{q,0}\omega^{1,2}(s_p,t_q,v_{p,q}), \tag{A.5}$$

$$v(s_2,t_2) \approx g_{2,2} + \sum_{p=0}^{2}\sum_{q=0}^{2}A_2^{p,0}B_2^{q,0}\omega^{2,2}(s_p,t_q,v_{p,q}), \tag{A.6}$$

where

$$A_2^{p,0} = \frac{1}{\Gamma(\sigma_1)}\int_a^{s_2}(\psi(s_2)-\psi(\tau))^{\sigma_1-1}\hat{f}_{p,0}(\tau)d\psi(\tau), p = 0, 1, 2, \tag{A.7}$$

$$B_2^{q,0} = \frac{1}{\Gamma(\sigma_2)}\int_c^{t_2}(\psi(t_2)-\psi(\eta))^{\sigma_2-1}f_{q,0}(\eta)d\psi(\eta), q = 0, 1, 2. \tag{A.8}$$

Therefore, $v_{1,1}$, $v_{2,1}$, $v_{1,2}$, and $v_{2,2}$ can be simultaneously solved from (A.1), (A.4), (A.5), and (A.6).

For $v(s_{2l+r_1}, t_{r_2})$, $l = 1, 2, \ldots, L-1$ and $v(s_{r_1}, t_{2k+r_2})$, $r_1, r_2 = 1, 2$, $k = 1, 2, \ldots, K-1$, under assumption that v_{i,r_2}, $i = 0, 1, \ldots, 2l$ and $v_{r_1,j}$, $j = 0, 1, \ldots, 2k$ already known, we have

$$\begin{aligned} v(s_{2l+1},t_1) =& g_{2l+1,1} + \frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_1)-\psi(\eta))^{1-\sigma_2}}\omega^{2l+1,1}(\tau,\eta,v(\tau,\eta))d\psi(\eta)d\psi(\tau) \\ &+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{i=1}^{l}\int_{s_{2i-1}}^{s_{2i+1}}\int_c^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_1)-\psi(\eta))^{1-\sigma_2}}\omega^{2l+1,1}(\tau,\eta,v(\tau,\eta))d\psi(\eta)d\psi(\tau) \end{aligned}$$

$$\approx g_{2l+1,1} + \sum_{p=0}^{2}\sum_{q=0}^{2}\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_1)-\psi(\eta))^{1-\sigma_2}}$$
$$\times \hat{f}_{p,0}(\tau)f_{q,0}(\eta)\omega^{2l+1,1}(s_p,t_q,\nu_{p,q})d\psi(\eta)d\psi(\tau)$$
$$+\sum_{i=1}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2}\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\int_{s_{2i-1}}^{s_{2i+1}}\int_c^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_1)-\psi(\tau))^{1-\sigma_2}}$$
$$\times \hat{f}_{p,2i-1}(\tau)f_{q,0}(\eta)\omega^{2l+1,1}(s_{2i-1+p},t_q,\nu_{2i-1+p,q})d\psi(\eta)d\psi(\tau)$$
$$= g_{2l+1,1} + \sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+1}^{p,0}B_1^{q,0}\omega^{2l+1,1}(s_p,t_q,\nu_{p,q})$$
$$+\sum_{i=1}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+1}^{p,i}B_1^{q,0}\omega^{2l+1,1}(s_{2i-1+p},t_q,\nu_{2i-1+p,q}), \tag{A.9}$$

where $B_1^{q,0}$ is given by (A.3) and $A_{2l+1}^{p,0}, A_{2l+1}^{p,i}$ are defined by (2.4) and (2.9).

For $\nu(s_{2l+2},t_1)$, $\nu(s_{2l+1},t_2)$, and $\nu(s_{2l+2},t_2)$, we have

$$\nu(s_{2l+2},t_1) = g_{2l+2,1} + \frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{i=0}^{l}\int_{s_{2i}}^{s_{2i+2}}\int_c^{t_1}\frac{(\psi(s_{2l+2})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_1)-\psi(\eta))^{1-\sigma_2}}\omega^{2l+2,1}(\tau,\eta,\nu(\tau,\eta))d\psi(\eta)d\psi(\tau)$$
$$\approx g_{2l+2,1} + \sum_{i=0}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+2}^{p,i}B_1^{q,0}\omega^{2l+2,1}(s_{2i+p},t_q,\nu_{2i+p,q}), \tag{A.10}$$

$$\nu(s_{2l+1},t_2) = g_{2l+1,2} + \frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\int_a^{s_1}\int_c^{t_2}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_2)-\psi(\eta))^{1-\sigma_2}}\omega^{2l+1,2}(\tau,\eta,\nu(\tau,\eta))d\psi(\eta)d\psi(\tau)$$
$$+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{i=1}^{l}\int_{s_{2i-1}}^{s_{2i+1}}\int_c^{t_2}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_2)-\psi(\eta))^{1-\sigma_2}}\omega^{2l+1,2}(\tau,\eta,\nu(\tau,\eta))d\psi(\eta)d\psi(\tau)$$
$$\approx g_{2l+1,2} + \sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+1}^{p,0}B_2^{q,0}\omega^{2l+1,2}(s_p,t_q,\nu_{p,q})$$
$$+\sum_{i=1}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+1}^{p,i}B_2^{q,0}\omega^{2l+1,2}(s_{2i-1+p},t_q,\nu_{2i-1+p,q}), \tag{A.11}$$

$$\nu(s_{2l+2},t_2) = g_{2l+2,2} + \frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{i=0}^{l}\int_{s_{2i}}^{s_{2i+2}}\int_c^{t_2}\frac{(\psi(s_{2l+2})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_2)-\psi(\eta))^{1-\sigma_2}}\omega^{2l+2,2}(\tau,\eta,\nu(\tau,\eta))d\psi(\eta)d\psi(\tau)$$
$$\approx g_{2l+2,2} + \sum_{i=0}^{l}\sum_{p=0}^{2}\sum_{q=0}^{2}A_{2l+2}^{p,i}B_2^{q,0}\omega^{2l+2,2}(s_{2i+p},t_q,\nu_{2i+p,q}), \tag{A.12}$$

where $B_1^{q,0}$, $B_2^{q,0}$, $A_{2l+1}^{p,0}$, $A_{2l+1}^{p,i}$, and $A_{2l+2}^{p,i}$ are given by (A.3), (A.8), (2.4), (2.9), and (2.15), respectively.

For $\nu(s_1,t_{2k+1})$, similar to (A.10), one can obtain

$$\nu(s_1,t_{2k+1}) \approx g_{1,2k+1} + \sum_{p=0}^{2}\sum_{q=0}^{2}A_1^{p,0}B_{2k+1}^{q,0}\omega^{1,2k+1}(s_p,t_q,\nu_{p,q})$$

$$+\sum_{j=1}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2}A_1^{p,0}B_{2k+1}^{q,j}\omega^{1,2k+1}(s_p,t_{2j-1+q},\nu_{p,2j-1+q}), \tag{A.13}$$

$$\nu(s_2,t_{2k+1})\approx g_{2,2k+1}+\sum_{p=0}^{2}\sum_{q=0}^{2}A_2^{p,0}B_{2k+1}^{q,0}\omega^{2,2k+1}(s_p,t_q,\nu_{p,q})$$

$$+\sum_{j=1}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2}A_2^{p,0}B_{2k+1}^{q,j}\omega^{2,2k+1}(s_p,t_{2j-1+q},\nu_{p,2j-1+q}), \tag{A.14}$$

$$\nu(s_1,t_{2k+2})\approx g_{1,2k+2}+\sum_{j=0}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2}A_1^{p,0}B_{2k+2}^{q,j}\omega^{1,2k+2}(s_p,t_{2j+q},\nu_{p,2j+q}), \tag{A.15}$$

$$\nu(s_2,t_{2k+2})\approx g_{2,2k+2}+\sum_{j=0}^{k}\sum_{p=0}^{2}\sum_{q=0}^{2}A_2^{p,0}B_{2k+2}^{q,j}\omega^{2,2k+2}(s_p,t_{2j+q},\nu_{p,2j+q}), \tag{A.16}$$

where $A_1^{p,0}$, $A_2^{p,0}$ are given by (A.2), (A.7). $B_{2k+1}^{q,0}$, $B_{2k+1}^{q,j}$, and $B_{2k+2}^{q,j}$ are defined by (2.5), (2.10), and (2.16).

Appendix B: The proof of Lemma 3.2.

Proof. Let Q_1, Q_2 be defined as follows:

$$Q_1=\max_{\substack{s,\tau\in[a,b]\\ t,\eta\in[c,d]}}(|\partial_\tau^3\omega(s,t,\tau,\eta,\nu(\tau,\eta))|,|\partial_\eta^3\omega(s,t,\tau,\eta,\nu(\tau,\eta))|), \tag{B.1}$$

$$Q_2=\max_{\substack{s,\tau\in[a,b]\\ t,\eta\in[c,d]}}(|\partial_\tau^4\omega(s,t,\tau,\eta,\nu(\tau,\eta))|,|\partial_\eta^4\omega(s,t,\tau,\eta,\nu(\tau,\eta))|). \tag{B.2}$$

For $E_{2l+1,2k+1}$, we have

$$E_{2l+1,2k+1}=\nu(s_{2l+1},t_{2k+1})-\bar{\nu}_{2l+1,2k+1}$$

$$=\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}[\omega^{2l+1,2k+1}(\tau,\eta,\nu(\tau,\eta))$$

$$-\sum_{p=0}^{2}\sum_{q=0}^{2}\hat{f}_{p,0}(\tau)f_{q,0}(\eta)\omega^{2l+1,2k+1}(s_p,t_q,\nu(s_p,t_q))]d\psi(\eta)d\psi(\tau)$$

$$+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{j=1}^{k}\int_a^{s_1}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}[\omega^{2l+1,2k+1}(\tau,\eta,\nu(\tau,\eta))$$

$$-\sum_{p=0}^{2}\sum_{q=0}^{2}\hat{f}_{p,0}(\tau)f_{q,2j-1}(\eta)\omega^{2l+1,2k+1}(s_p,t_{2j-1+q},\nu(s_p,t_{2j-1+q}))]d\psi(\eta)d\psi(\tau)$$

$$+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{i=1}^{l}\int_{s_{2i-1}}^{s_{2i+1}}\int_c^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}[\omega^{2l+1,2k+1}(\tau,\eta,\nu(\tau,\eta))$$

$$-\sum_{p=0}^{2}\sum_{q=0}^{2}\hat{f}_{p,2i-1}(\tau)f_{q,0}(\eta)\omega^{2l+1,2k+1}(s_{2i-1+p},t_q,\nu(s_{2i-1+p},t_q))]d\psi(\eta)d\psi(\tau)$$

$$+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{i=1}^{l}\sum_{j=1}^{k}\int_{s_{2i-1}}^{s_{2i+1}}\int_{t_{2j-1}}^{s_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}[\omega^{2l+1,2k+1}(\tau,\eta,\nu(\tau,\eta))$$

$$-\sum_{p=0}^{2}\sum_{q=0}^{2}\hat{f}_{p,2i-1}(\tau)f_{q,2j-1}(\eta)\omega^{2l+1,2k+1}(s_{2i-1+p},t_{2j-1+q},\nu(s_{2i-1+p},t_{2j-1+q}))]d\psi(\eta)d\psi(\tau)$$

$$= \frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\int_{a}^{s_1}\int_{c}^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}T^{(1)}_{2l+1,2k+1}d\psi(\eta)d\psi(\tau)$$

$$+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{j=1}^{k}\int_{a}^{s_1}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}T^{(2)}_{2l+1,2k+1}d\psi(\eta)d\psi(\tau)$$

$$+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{i=1}^{l}\int_{s_{2i-1}}^{s_{2i+1}}\int_{c}^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}T^{(3)}_{2l+1,2k+1}d\psi(\eta)d\psi(\tau)$$

$$+\frac{1}{\Gamma(\sigma_1)}\frac{1}{\Gamma(\sigma_2)}\sum_{i=1}^{l}\sum_{j=1}^{k}\int_{s_{2i-1}}^{s_{2i+1}}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}T^{(4)}_{2l+1,2k+1}d\psi(\eta)d\psi(\tau)$$

$$\doteq E^1_{2l+1,2k+1}+E^2_{2l+1,2k+1}+E^3_{2l+1,2k+1}+E^4_{2l+1,2k+1}. \tag{B.3}$$

According to the Taylor theorem, there is

$$\begin{aligned}
T^{(1)}_{2l+1,2k+1} &= \omega^{2l+1,2k+1}(\tau,\eta,\nu(\tau,\eta))-\sum_{p=0}^{2}\sum_{q=0}^{2}\hat{f}_{p,0}(\tau)f_{q,0}(\eta)\omega^{2l+1,2k+1}(s_p,t_q,\nu(s_p,t_q))\\
&= \omega^{2l+1,2k+1}(\tau,\eta,\nu(\tau,\eta))-\sum_{p=0}^{2}\hat{f}_{p,0}(\tau)\omega^{2l+1,2k+1}(s_p,\eta,\nu(s_p,\eta))\\
&\quad+\sum_{p=0}^{2}\hat{f}_{p,0}(\tau)\omega^{2l+1,2k+1}(s_p,\eta,\nu(s_p,\eta))-\sum_{p=0}^{2}\sum_{q=0}^{2}\hat{f}_{p,0}(\tau)f_{q,0}(\eta)\omega^{2l+1,2k+1}(s_p,t_q,\nu(s_p,t_q))\\
&= \frac{1}{3!}\partial_\tau^3\omega^{2l+1,2k+1}(\varepsilon_1(\tau),\eta,\nu(\varepsilon_1(\tau),\eta))\prod_{p=0}^{2}(\psi(\tau)-\psi(s_p))\\
&\quad+\sum_{p=0}^{2}\frac{\hat{f}_{p,0}(\tau)}{3!}\partial_\eta^3\omega^{2l+1,2k+1}(s_p,\xi_1(\eta),\nu(s_p,\xi_1(\eta)))\prod_{q=0}^{2}(\psi(\eta)-\psi(t_q)),
\end{aligned} \tag{B.4}$$

where $(\varepsilon_1(\tau),\xi_1(\eta))\in[a,s_1]\times[c,t_1]$.

For $(\tau,\eta)\in[a,s_1]\times[t_{2j-1},t_{2j+1}]$, there is $(\varepsilon_2(\tau),\xi_{j1}(\eta))\in[a,s_1]\times[t_{2j-1},t_{2j+1}]$,

$$\begin{aligned}
T^{(2)}_{2l+1,2k+1}=&\frac{1}{3!}\partial_\tau^3\omega^{2l+1,2k+1}(\varepsilon_2(\tau),\eta,\nu(\varepsilon_2(\tau),\eta))\prod_{p=0}^{2}(\psi(\tau)-\psi(s_p))\\
&+\sum_{p=0}^{2}\frac{\hat{f}_{p,0}(\tau)}{3!}\partial_\eta^3\omega^{2l+1,2k+1}(s_p,\xi_{j1}(\eta),\nu(s_p,\xi_{j1}(\eta)))\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q})).
\end{aligned} \tag{B.5}$$

In the same way, we have $(\tau,\eta)\in[s_{2i-1},s_{2i+1}]\times[c,t_1]$, and $(\varepsilon_{i1}(\tau),\xi_2(\eta))\in[s_{2i-1},s_{2i+1}]\times[c,t_1]$,

$$\begin{aligned}T^{(3)}_{2l+1,2k+1} &= \frac{1}{3!}\partial^3_\tau\omega^{2l+1,2k+1}(\varepsilon_{i1}(\tau),\eta,\nu(\varepsilon_{i1}(\tau),\eta))\prod_{p=0}^{2}(\psi(\tau)-\psi(s_{2i-1+p}))\\ &+\sum_{p=0}^{2}\frac{\hat f_{p,2i-1}(\tau)}{3!}\partial^3_\eta\omega^{2l+1,2k+1}(s_{2i-1+p},\xi_2(\eta),\nu(s_{2i-1+p},\xi_2(\eta)))\prod_{q=0}^{2}(\psi(\eta)-\psi(t_q)).\end{aligned}\tag{B.6}$$

For $\forall(\tau,\eta)\in[s_{2i-1},s_{2i+1}]\times[t_{2j-1},t_{2j+1}]$, there is $(\varepsilon_{i2}(\tau),\xi_{j2}(\eta))\in[s_{2i-1},s_{2i+1}]\times[t_{2j-1},t_{2j+1}]$,

$$\begin{aligned}&T^{(4)}_{2l+1,2k+1}\\ =&\frac{1}{3!}\partial^3_\tau\omega^{2l+1,2k+1}(\varepsilon_{i2}(\tau),\eta,\nu(\varepsilon_{i2}(\tau),\eta))\prod_{p=0}^{2}(\psi(\tau)-\psi(s_{2i-1+p}))\\ &+\sum_{p=0}^{2}\frac{\hat f_{p,2i-1}(\tau)}{3!}\partial^3_\eta\omega^{2l+1,2k+1}(s_{2i-1+p},\xi_{j2}(\eta),\nu(s_{2i-1+p},\xi_{j2}(\eta)))\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q})).\end{aligned}\tag{B.7}$$

By putting (B.4) into the first term to the right of formula (B.3) gives the following result:

$$\begin{aligned}&\left|E^1_{2l+1,2k+1}\right|\\ \le\ &\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\mid\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\frac{1}{3!}\partial^3_\tau\omega^{2l+1,2k+1}(\varepsilon_1(\tau),\eta,\nu(\varepsilon_1(\tau),\eta))\\ &\times\prod_{p=0}^{2}(\psi(\tau)-\psi(s_p))d\psi(\eta)d\psi(\tau)\mid+\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\mid\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\\ &\times\sum_{p=0}^{2}\frac{\hat f_{p,0}(\tau)}{3!}\partial^3_\eta\omega^{2l+1,2k+1}(s_p,\xi_1(\eta),\nu(s_p,\xi_1(\eta)))\prod_{q=0}^{2}(\psi(\eta)-\psi(t_q))d\psi(\eta)d\psi(\tau)\mid\\ &\doteq E^1_1+E^2_1.\end{aligned}\tag{B.8}$$

Using direct and simple calculations, we have

$$\mid\prod_{p=0}^{2}(\psi(\tau)-\psi(s_p))|\le r^3h_s^3.\tag{B.9}$$

Similarily,

$$\mid\prod_{q=0}^{2}(\psi(\eta)-\psi(t_q))|\le \bar r^3h_t^3.\tag{B.10}$$

By Lemma 3.1 and (B.9), we have

$$E^1_1 = \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\mid\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\frac{1}{3!}$$

$$
\begin{aligned}
&\times\partial_\tau^3\omega^{2l+1,2k+1}(\varepsilon_1(\tau),\eta,\nu(\varepsilon_1(\tau),\eta))\prod_{p=0}^{2}(\psi(\tau)-\psi(s_p))d\psi(\eta)d\psi(\tau)\,|\\
\leq\ & \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1 2^{1-\sigma_1}r^{\sigma_1}(2l+1)^{\sigma_1-1}h_s^{\sigma_1}r^3h_s^3 2^{1-\sigma_2}\bar{r}^{\sigma_2}(2k+1)^{\sigma_2-1}h_t^{\sigma_2}\\
=\ & \frac{2^{2-\sigma_1-\sigma_2}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1 r^{3+\sigma_1}\bar{r}^{\sigma_2}(2l+1)^{\sigma_1-1}(2k+1)^{\sigma_2-1}h_s^{3+\sigma_1}h_t^{\sigma_2},
\end{aligned}\tag{B.11}
$$

where Q_1 is defined by (B.1).

By Lemma 3.1 and (B.10), we have

$$
\begin{aligned}
E_1^2 =\ & \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\,|\int_a^{s_1}\int_c^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\sum_{p=0}^{2}\frac{\hat{f}_{p,0}(\tau)}{3!}\\
&\times\partial_\eta^3\omega^{2l+1,2k+1}(s_p,\xi_1(\eta),\nu(s_p,\xi_1(\eta)))\prod_{q=0}^{2}(\psi(\eta)-\psi(t_q))d\psi(\eta)d\psi(\tau)\,|\\
\leq\ & \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1 2^{1-\sigma_1}r^{\sigma_1}(2l+1)^{\sigma_1-1}h_s^{\sigma_1}2^{1-\sigma_2}\bar{r}^{\sigma_2}(2k+1)^{\sigma_2-1}h_t^{\sigma_2}\bar{r}^3h_t^3\\
=\ & \frac{2^{2-\sigma_1-\sigma_2}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1 r^{\sigma_1}\bar{r}^{3+\sigma_2}(2l+1)^{\sigma_1-1}(2k+1)^{\sigma_2-1}h_s^{\sigma_1}h_t^{3+\sigma_2}.
\end{aligned}\tag{B.12}
$$

According to (B.11) and (B.12), we get $E^1_{2l+1,2k+1}$ as follows:

$$
\left|E^1_{2l+1,2k+1}\right|\leq\frac{2^{2-\sigma_1-\sigma_2}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1(2l+1)^{\sigma_1-1}(2k+1)^{\sigma_2-1}(r^{3+\sigma_1}\bar{r}^{\sigma_2}h_s^{3+\sigma_1}h_t^{\sigma_2}+r^{\sigma_1}\bar{r}^{3+\sigma_2}h_s^{\sigma_1}h_t^{3+\sigma_2}).\tag{B.13}
$$

Bring equation (B.5) to the second term to the right of equation (B.3), and we have the following:

$$
\begin{aligned}
\left|E^2_{2l+1,2k+1}\right|\leq\ & \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\sum_{j=1}^{k}|\int_a^{s_1}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\frac{1}{3!}\\
&\times\partial_\tau^3\omega^{2l+1,2k+1}(\varepsilon_2(\tau),\eta,\nu(\varepsilon_2(\tau),\eta))\prod_{p=0}^{2}(\psi(\tau)-\psi(s_p))d\psi(\eta)d\psi(\tau)|\\
&+\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\sum_{j=1}^{k}|\int_a^{s_1}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\sum_{p=0}^{2}\frac{\hat{f}_{p,0}(\tau)}{3!}\\
&\times\partial_\eta^3\omega^{2l+1,2k+1}(s_p,\xi_{j1}(\eta),\nu(s_p,\xi_{j1}(\eta)))\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)d\psi(\tau)|\\
&\doteq E_2^1+E_2^2.
\end{aligned}\tag{B.14}
$$

Using Lemma 3.1 and equation (B.9), we have

$$
E_2^1 \leq \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\sum_{j=1}^{k}\int_a^{s_1}\int_{t_{2j-1}}^{t_{2j+1}}|\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\frac{1}{3!}
$$

$$
\begin{aligned}
&\times\partial_\tau^3\omega^{2l+1,2k+1}(\varepsilon_2(\tau),\eta,\nu(\varepsilon_2(\tau),\eta))\prod_{p=0}^{2}(\psi(\tau)-\psi(s_p))|d\psi(\eta)d\psi(\tau)\\
\leq\ &\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1r^3h_s^3\sum_{j=1}^{k}\int_a^{s_1}\int_{t_{2j-1}}^{t_{2j+1}}|\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}|d\psi(\eta)d\psi(\tau)\\
\leq\ &\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1r^3h_s^32^{1-\sigma_1}r^{\sigma_1}(2l+1)^{\sigma_1-1}h_s^{\sigma_1}\sum_{j=1}^{k}\int_{t_{2j-1}}^{t_{2j+1}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}d\psi(\eta)\\
\leq\ &\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1r^3h_s^32^{1-\sigma_1}r^{\sigma_1}(2l+1)^{\sigma_1-1}h_s^{\sigma_1}\int_{t_1}^{t_{2k+1}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}d\psi(\eta)\\
=\ &\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2+1)}Q_1r^{3+\sigma_1}h_s^{3+\sigma_1}2^{1-\sigma_1}(2l+1)^{\sigma_1-1}(\psi(t_{2k+1})-\psi(t_1))^{\sigma_2}\\
\leq\ &\frac{2^{1-\sigma_1}}{\Gamma(\sigma_1)\Gamma(\sigma_2+1)}Q_1(2l+1)^{\sigma_1-1}r^{3+\sigma_1}\overline{r}^{\sigma_2}(d-c)^{\sigma_2}h_s^{\sigma_1+3}.
\end{aligned}\tag{B.15}
$$

For E_2^2, we estimate this by adding one term and subtracting one term,

$$
\begin{aligned}
E_2^2\ =\ &\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\sum_{j=1}^{k}|\int_a^{s_1}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\sum_{p=0}^{2}\frac{\hat{f}_{p,0}(\tau)}{3!}\\
&\times\partial_\eta^3\omega^{2l+1,2k+1}(s_p,\xi_{j1}(\eta),\nu(s_p,\xi_{j1}(\eta)))\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)d\psi(\tau)|\\
\leq\ &\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\sum_{j=1}^{k}|\int_a^{s_1}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\sum_{p=0}^{2}\frac{\hat{f}_{p,0}(\tau)}{3!}\\
&\times\partial_\eta^3\omega^{2l+1,2k+1}(s_p,t_{2j},\nu(s_p,t_{2j}))\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)d\psi(\tau)|\\
&+\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\sum_{j=1}^{k}|\int_a^{s_1}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\sum_{p=0}^{2}\frac{\hat{f}_{p,0}(\tau)}{3!}\\
&\times[\partial_\eta^3\omega^{2l+1,2k+1}(s_p,\xi_{j1}(\eta),\nu(s_p,\xi_{j1}(\eta)))-\partial_\eta^3\omega^{2l+1,2k+1}(s_p,t_{2j},\nu(s_p,t_{2j}))]\\
&\times\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)d\psi(\tau)|\doteq L1+L2.
\end{aligned}\tag{B.16}
$$

By definition of $\hat{f}_{p,i}(\tau)$ and $\tau\in(s_i,s_{i+2})$, $p=0,1,2;i=0,1,\ldots,2l$, we know $|\hat{f}_{p,i}(\tau)|\leq 1$; hence, $|\sum_{p=0}^{2}\hat{f}_{p,i}(\tau)|\leq 3$. We get

$$
\begin{aligned}
L1\ \leq\ &\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1\sum_{j=1}^{k}|\int_a^{s_1}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\sum_{p=0}^{2}\frac{\hat{f}_{p,0}(\tau)}{3!}\\
&\times\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)d\psi(\tau)|
\end{aligned}
$$

$$\leq \frac{2^{1-\sigma_1}(2l+1)^{\sigma_1-1}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1 r^{\sigma_1}h_s^{\sigma_1}\sum_{j=1}^{k}|\int_{t_{2j-1}}^{t_{2j}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)$$
$$+\int_{t_{2j}}^{t_{2j+1}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)|$$
$$\leq \frac{2^{1-\sigma_1}(2l+1)^{\sigma_1-1}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1 r^{\sigma_1}h_s^{\sigma_1}\sum_{j=1}^{k-2}|\int_{t_{2j-1}}^{t_{2j}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)$$
$$+\int_{t_{2j}}^{t_{2j+1}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)|$$
$$+\frac{2^{1-\sigma_1}(2l+1)^{\sigma_1-1}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1 r^{\sigma_1}h_s^{\sigma_1}(|\int_{t_{2k-3}}^{t_{2k-1}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2k-3+q}))d\psi(\eta)|$$
$$+|\int_{t_{2k-1}}^{t_{2k+1}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2k-1+q}))d\psi(\eta)|)$$
$$\doteq \frac{2^{1-\sigma_1}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1 r^{\sigma_1}(2l+1)^{\sigma_1-1}h_s^{\sigma_1}[L_{11}+L_{12}]. \tag{B.17}$$

Next, we will reckon L_{11}. To begin, we reckoned $\int_{t_{2j-1}}^{t_{2j}}\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)$. By the Lagrange mean value theorem, the integral mean value theorem, and continuity of the function on a closed interval, we have

$$\int_{t_{2j-1}}^{t_{2j}}\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)$$
$$= \int_{t_{2j-1}}^{t_{2j}}\psi'(\zeta_0^j)\psi'(\zeta_1^j)\psi'(\zeta_2^j)\psi'(\eta)(\eta-t_{2j-1})(\eta-t_{2j})(\eta-t_{2j+1})d\eta$$
$$= \psi'(\zeta_0^j(\gamma_j))\psi'(\zeta_1^j(\gamma_j))\psi'(\zeta_2^j(\gamma_j))\psi'(\gamma_j)\int_{t_{2j-1}}^{t_{2j}}(\eta-t_{2j-1})(\eta-t_{2j})(\eta-t_{2j+1})d\eta$$
$$= [\psi'(\widehat{\zeta_j})]^4\int_{t_{2j-1}}^{t_{2j}}(\eta-t_{2j-1})(\eta-t_{2j})(\eta-t_{2j+1})d\eta = \frac{1}{4}[\psi'(\widehat{\zeta_j})]^4 h_t^4, \tag{B.18}$$

where $\zeta_i^j \in (t_{2j-1}, t_{2j}), i = 0, 1, 2; \gamma_j \in (t_{2j-1}, t_{2j}), \widehat{\zeta_j} \in (t_{2j-1}, t_{2j})$.

Using the same method of (B.18), we estimate $\int_{t_{2j}}^{t_{2j+1}}\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta)$ and obtain

$$\int_{t_{2j}}^{t_{2j+1}}\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))d\psi(\eta) = \frac{1}{4}[\psi'(\widehat{\eta_j})]^4 h_t^4, \quad \widehat{\eta_j} \in (t_{2j}, t_{2j+1}). \tag{B.19}$$

Combining (B.18) and (B.19), we obtain

$$L_{11} = \frac{1}{4}h_t^4\sum_{j=1}^{k-2}|(\psi(t_{2k+1})-\psi(\bar{\eta}_j))^{\sigma_2-1}[\psi'(\widehat{\zeta_j})]^4 - (\psi(t_{2k+1})-\psi(\bar{\bar{\eta}}_j))^{\sigma_2-1}[\psi'(\widehat{\eta_j})]^4|$$

$$
\begin{aligned}
&= \frac{1}{4}h_t^4\sum_{j=1}^{k-2}|(\psi(t_{2k+1})-\psi(\bar{\eta}_j))^{\sigma_2-1}[\psi'(\widehat{\zeta_j})]^4-(\psi(t_{2k+1})-\psi(\bar{\bar{\eta}}_j))^{\sigma_2-1}[\psi'(\widehat{\zeta_j})]^4\\
&\quad +(\psi(t_{2k+1})-\psi(\bar{\bar{\eta}}_j))^{\sigma_2-1}[\psi'(\widehat{\zeta_j})]^4-(\psi(t_{2k+1})-\psi(\bar{\bar{\eta}}_j))^{\sigma_2-1}[\psi'(\widehat{\eta_j})]^4|\\
&= \frac{1}{4}h_t^4\sum_{j=1}^{k-2}|[\psi'(\widehat{\zeta_j})]^4[(\psi(t_{2k+1})-\psi(\bar{\eta}_j))^{\sigma_2-1}-(\psi(t_{2k+1})-\psi(\bar{\bar{\eta}}_j))^{\sigma_2-1}]\\
&\quad +(\psi(t_{2k+1})-\psi(\bar{\bar{\eta}}_j))^{\sigma_2-1}[(\psi'(\widehat{\zeta_j}))^2+(\psi'(\widehat{\eta_j}))^2][\psi'(\widehat{\zeta_j})+\psi'(\widehat{\eta_j})][\psi'(\widehat{\zeta_j})-\psi'(\widehat{\eta_j})]|\\
&\le \frac{1}{4}h_t^4(\bar{r}^4\sum_{j=1}^{k-2}|(\psi(t_{2k+1})-\psi(\bar{\eta}_j))^{\sigma_2-1}-(\psi(t_{2k+1})-\psi(\bar{\bar{\eta}}_j))^{\sigma_2-1}|\\
&\quad +4\bar{r}^3\sum_{j=1}^{k-2}|(\psi(t_{2k+1})-\psi(\bar{\bar{\eta}}_j))^{\sigma_2-1}[\psi'(\widehat{\zeta_j})-\psi'(\widehat{\eta_j})]|)\doteq\frac{1}{4}h_t^4(\bar{r}^4y1+4\bar{r}^3y2).
\end{aligned} \tag{B.20}
$$

First, we will estimate $y1$,

$$
\begin{aligned}
y1 &= \sum_{j=1}^{k-2}|(\psi(t_{2k+1})-\psi(\bar{\eta}_j))^{\sigma_2-1}-(\psi(t_{2k+1})-\psi(\bar{\bar{\eta}}_j))^{\sigma_2-1}|\\
&= \sum_{j=1}^{k-2}|(\sigma_2-1)(\psi(t_{2k+1})-\psi(\eta_j))^{\sigma_2-2}(\psi(\bar{\bar{\eta}}_j)-\psi(\bar{\eta}_j))|\\
&\le (1-\sigma_2)\bar{r}^{\sigma_2-1}\sum_{j=1}^{k-2}(t_{2k+1}-\eta_j)^{\sigma_2-2}(\bar{\bar{\eta}}_j-\bar{\eta}_j)\\
&\le (1-\sigma_2)\bar{r}^{\sigma_2-1}\sum_{j=1}^{k-2}(t_{2k+1}-t_{2j+1})^{\sigma_2-2}2h_t\\
&\le (1-\sigma_2)\bar{r}^{\sigma_2-1}\int_{t_0}^{t_{2k-1}}(t_{2k+1}-\eta)^{\sigma_2-2}d\eta\\
&= (1-\sigma_2)\bar{r}^{\sigma_2-1}\frac{-1}{\sigma_2-1}[(t_{2k+1}-t_{2k-1})^{\sigma_2-1}-(t_{2k+1}-t_0)^{\sigma_2-1}]\\
&\le \bar{r}^{\sigma_2-1}(t_{2k+1}-t_{2k-1})^{\sigma_2-1}=2^{\sigma_2-1}\bar{r}^{\sigma_2-1}h_t^{\sigma_2-1},
\end{aligned} \tag{B.21}
$$

where $\bar{\eta}_j\in(t_{2j-1},t_{2j})$, $\bar{\bar{\eta}}_j\in(t_{2j},t_{2j+1})$, and $\eta_j\in(\bar{\eta}_j,\bar{\bar{\eta}}_j)$.

Second, we will estimate $y2$. If $\psi'(\eta)$ satisfies the Lipschitz condition, i.e.,

$$|\psi'(\eta_1)-\psi'(\eta_2)|\le\widehat{L}|\eta_1-\eta_2|,\widehat{L}>0,$$

then, we have

$$
\begin{aligned}
y2 &= \sum_{j=1}^{k-2}|(\psi(t_{2k+1})-\psi(\bar{\bar{\eta}}_j))^{\sigma_2-1}|\cdot|\psi'(\widehat{\zeta_j})-\psi'(\widehat{\eta_j})|\\
&\le \widehat{L}\sum_{j=1}^{k-2}(\psi(t_{2k+1})-\psi(\bar{\bar{\eta}}_j))^{\sigma_2-1}\cdot 2h_t\le\widehat{L}\sum_{j=1}^{k-2}(\psi(t_{2k+1})-\psi(t_{2j+1}))^{\sigma_2-1}\cdot 2h_t
\end{aligned}
$$

$$
\begin{aligned}
&\leq \widehat{L}\bar{r}^{\sigma_2-1}\sum_{j=1}^{k-2}(t_{2k+1}-t_{2j+1})^{\sigma_2-1}\cdot 2h_t \leq \widehat{L}\bar{r}^{\sigma_2-1}\int_{t_0}^{t_{2k+1}}(t_{2k+1}-\eta)^{\sigma_2-1}d\eta \\
&= \widehat{L}\bar{r}^{\sigma_2-1}\frac{1}{\sigma_2}(t_{2k+1}-t_0)^{\sigma_2} \leq \frac{1}{\sigma_2}\widehat{L}\bar{r}^{\sigma_2-1}(d-c)^{\sigma_2}h_t^{\sigma_2}. \qquad \text{(B.22)}
\end{aligned}
$$

Bring (B.21) and (B.22) into (B.20), and we have the estimation of L_{11} as follows:

$$
\begin{aligned}
L_{11} &\leq \frac{1}{4}h_t^4[\bar{r}^4 2^{\sigma_2-1}\bar{r}^{\sigma_2-1}h_t^{\sigma_2-1}+\frac{4}{\sigma_2}\bar{r}^3\widehat{L}\bar{r}^{\sigma_2-1}(d-c)^{\sigma_2}h_t^{\sigma_2}] \\
&= \frac{1}{4}\bar{r}^{3+\sigma_2}2^{\sigma_2-1}h_t^{3+\sigma_2}+\frac{1}{\sigma_2}\bar{r}^{2+\sigma_2}\widehat{L}(d-c)^{\sigma_2}h_t^{4+\sigma_2}. \qquad \text{(B.23)}
\end{aligned}
$$

For L_{12}, using (3.3) and (B.10), we have

$$
\begin{aligned}
L_{12} &\leq \bar{r}^3h_t^3[\int_{t_{2k-3}}^{t_{2k-1}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}d\psi(\eta)+\int_{t_{2k-1}}^{t_{2k+1}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}d\psi(\eta)] \\
&= \bar{r}^3h_t^3\int_{t_{2k-3}}^{t_{2k+1}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}d\psi(\eta) \leq \frac{4^{\sigma_2}}{\sigma_2}\bar{r}^{3+\sigma_2}h_t^{3+\sigma_2}. \qquad \text{(B.24)}
\end{aligned}
$$

Combine (B.23) and (B.24) to (B.17). Therefore, $L1$ is as follows:

$$
\begin{aligned}
L1 &\leq \frac{2^{1-\sigma_1}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1r^{\sigma_1}(2l+1)^{\sigma_1-1}h_s^{\sigma_1}(\frac{1}{4}\bar{r}^{3+\sigma_2}2^{\sigma_2-1}h_t^{3+\sigma_2} \\
&\quad +\frac{1}{\sigma_2}\bar{r}^{2+\sigma_2}\widehat{L}(d-c)^{\sigma_2}h_t^{4+\sigma_2}+\frac{4^{\sigma_2}}{\sigma_2}\bar{r}^{3+\sigma_2}h_t^{3+\sigma_2}) \\
&= \frac{2^{1-\sigma_1}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1r^{\sigma_1}(2l+1)^{\sigma_1-1}\bar{r}^{3+\sigma_2}[\frac{2^{\sigma_2-1}}{4}+\frac{1}{\bar{r}\sigma_2}\widehat{L}(d-c)^{\sigma_2}h_t+\frac{4^{\sigma_2}}{\sigma_2}]h_s^{\sigma_1}h_t^{3+\sigma_2}. \qquad \text{(B.25)}
\end{aligned}
$$

Using the mean value theorem and Lemma 3.1, we can estimate $L2$ as follows:

$$
\begin{aligned}
L2 &\leq \frac{Q_2h_t}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_a^{s_1}\frac{|\sum_{p=0}^{2}\hat{f}_{p,0}(\tau)|}{3(\psi(s_{2l+1})-\psi(\tau))^{1-\sigma_1}}d\psi(\tau) \\
&\quad \times\sum_{j=1}^{k}\int_{t_{2j-1}}^{t_{2j+1}}\frac{|\prod_{q=0}^{2}(\psi(\eta)-\psi(t_{2j-1+q}))|}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}d\psi(\eta) \\
&\leq \frac{2^{1-\sigma_1}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_2r^{\sigma_1}(2l+1)^{\sigma_1-1}h_s^{\sigma_1}\bar{r}^3h_t^4\sum_{j=1}^{k}\int_{t_{2j-1}}^{t_{2j+1}}(\psi(t_{2k+1})-\psi(\eta))^{\sigma_2-1}d\psi(\eta) \\
&\leq \frac{2^{1-\sigma_1}}{\Gamma(\sigma_1)\Gamma(\sigma_2+1)}Q_2r^{\sigma_1}(2l+1)^{\sigma_1-1}\bar{r}^{3+\sigma_2}(d-c)^{\sigma_2}h_s^{\sigma_1}h_t^4, \qquad \text{(B.26)}
\end{aligned}
$$

where Q_2 is defined in (B.2).

Combine (B.25) and (B.26) to obtain E_2^2,

$$
E_2^2 \leq \frac{2^{1-\sigma_1}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1r^{\sigma_1}(2l+1)^{\sigma_1-1}\bar{r}^{3+\sigma_2}[\frac{2^{\sigma_2-1}}{4}+\frac{1}{\bar{r}\sigma_2}\widehat{L}(d-c)^{\sigma_2}h_t+\frac{4^{\sigma_2}}{\sigma_2}]h_s^{\sigma_1}h_t^{3+\sigma_2}
$$

$$+\frac{2^{1-\sigma_1}}{\Gamma(\sigma_1)\Gamma(\sigma_2+1)}Q_2 r^{\sigma_1}(2l+1)^{\sigma_1-1}\bar{r}^{3+\sigma_2}(d-c)^{\sigma_2}h_s^{\sigma_1}h_t^4. \quad (B.27)$$

Take (B.15), (B.27) and put it into (B.14), and we have

$$\begin{aligned}\left|E^2_{2l+1,2k+1}\right| \leq\ & \frac{2^{1-\sigma_1}(2l+1)^{\sigma_1-1}}{\Gamma(\sigma_1)\Gamma(\sigma_2+1)}Q_1 r^{3+\sigma_1}\bar{r}^{\sigma_2}(d-c)^{\sigma_2}h_s^{\sigma_1+3} \\ & +\frac{2^{1-\sigma_1}(2l+1)^{\sigma_1-1}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1 r^{\sigma_1}\bar{r}^{3+\sigma_2}[\frac{2^{\sigma_2-1}}{4}+\frac{1}{\bar{r}\sigma_2}\widehat{L}(d-c)^{\sigma_2}h_t+\frac{4^{\sigma_2}}{\sigma_2}]h_s^{\sigma_1}h_t^{3+\sigma_2} \\ & +\frac{2^{1-\sigma_1}(2l+1)^{\sigma_1-1}}{\Gamma(\sigma_1)\Gamma(\sigma_2+1)}Q_2 r^{\sigma_1}\bar{r}^{3+\sigma_2}(d-c)^{\sigma_2}h_s^{\sigma_1}h_t^4. \quad (B.28)\end{aligned}$$

Next, we will estimate $E^3_{2l+1,2k+1}$ and $E^4_{2l+1,2k+1}$. Similar to the estimate for $E^1_{2l+1,2k+1}$, $E^2_{2l+1,2k+1}$, by putting (B.6) into the third term on the right side of formula (B.3), we have

$$\begin{aligned}\left|E^3_{2l+1,2k+1}\right| \leq\ & \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\sum_{i=1}^{l}\Big|\int_{s_{2i-1}}^{s_{2i+1}}\int_{c}^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\frac{1}{3!} \\ & \times\partial_\tau^3\omega^{2l+1,2k+1}(\varepsilon_{i1}(\tau),\eta,\nu(\varepsilon_{i1}(\tau),\eta))\prod_{p=0}^{2}(\psi(\tau)-\psi(s_{2i-1+p}))d\psi(\eta)d\psi(\tau)\Big| \\ & +\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\sum_{i=1}^{l}\Big|\int_{s_{2i-1}}^{s_{2i+1}}\int_{c}^{t_1}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\sum_{p=0}^{2}\frac{\hat{f}_{p,2i-1}(\tau)}{3!} \\ & \times\partial_\eta^3\omega^{2l+1,2k+1}(s_{2i-1+p},\xi_2(\eta),\nu(s_{2i-1+p},\xi_2(\eta)))\prod_{q=0}^{2}(\psi(\eta)-\psi(t_q))d\psi(\eta)d\psi(\tau)\Big| \\ \leq\ & \frac{2^{1-\sigma_2}}{\Gamma(\sigma_1)\Gamma(\sigma_2)}Q_1\bar{r}^{\sigma_2}(2k+1)^{\sigma_2-1}r^{3+\sigma_1}[\frac{2^{\sigma_1-1}}{4}+\frac{1}{r\sigma_1}\widehat{L}(b-a)^{\sigma_1}h_s \\ & +\frac{4^{\sigma_1}}{\sigma_1}]h_s^{3+\sigma_1}h_t^{\sigma_2}+\frac{2^{1-\sigma_2}}{\Gamma(\sigma_1+1)\Gamma(\sigma_2)}Q_2 r^{3+\sigma_1}\bar{r}^{\sigma_2}(2k+1)^{\sigma_2-1}(b-a)^{\sigma_1}h_s^4h_t^{\sigma_2} \\ & +\frac{1}{\Gamma(\sigma_1+1)\Gamma(\sigma_2)}Q_1 2^{1-\sigma_2}(2k+1)^{\sigma_2-1}r^{\sigma_1}\bar{r}^{3+\sigma_2}(b-a)^{\sigma_1}h_t^{3+\sigma_2}. \quad (B.29)\end{aligned}$$

Bring (B.7) to the fourth item on the right side of (B.3), and we have

$$\begin{aligned}\left|E^4_{2l+1,2k+1}\right| \leq\ & \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\sum_{i=1}^{l}\sum_{j=1}^{k}\Big|\int_{s_{2i-1}}^{s_{2i+1}}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\frac{1}{3!} \\ & \times\partial_\tau^3\omega^{2l+1,2k+1}(\varepsilon_{i2}(\tau),\eta,\nu(\varepsilon_{i2}(\tau),\eta))\prod_{p=0}^{2}(\psi(\tau)-\psi(s_{2i-1+p}))d\psi(\eta)d\psi(\tau)\Big| \\ & +\frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\sum_{i=1}^{l}\sum_{j=1}^{k}\Big|\int_{s_{2i-1}}^{s_{2i+1}}\int_{t_{2j-1}}^{t_{2j+1}}\frac{(\psi(s_{2l+1})-\psi(\tau))^{\sigma_1-1}}{(\psi(t_{2k+1})-\psi(\eta))^{1-\sigma_2}}\sum_{p=0}^{2}\frac{\hat{f}_{p,2i-1}(\tau)}{3!} \\ & \times\partial_\eta^3\omega^{2l+1,2k+1}(s_{2i-1+p},\xi_{j2}(\eta),\nu(s_{2i-1+p},\xi_{j2}(\eta)))\end{aligned}$$

$$\times \prod_{q=0}^{2} (\psi(\eta) - \psi(t_{2j-1+q}))d\psi(\eta)d\psi(\tau)|$$
$$\leq \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)} Q_1 \bar{r}^{\sigma_2} \frac{(d-c)^{\sigma_2}}{\sigma_2} r^{3+\sigma_1} [\frac{2^{\sigma_1-1}}{4} + \frac{1}{r\sigma_1} \widehat{L}(b-a)^{\sigma_1} h_s + \frac{4^{\sigma_1}}{\sigma_1}] h_s^{3+\sigma_1}$$
$$+ \frac{1}{\Gamma(\sigma_1+1)\Gamma(\sigma_2+1)} Q_2 r^{3+\sigma_1} \bar{r}^{\sigma_2} (b-a)^{\sigma_1} (d-c)^{\sigma_2} h_s^4$$
$$+ \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)} Q_1 r^{\sigma_1} \frac{(b-a)^{\sigma_1}}{\sigma_1} \bar{r}^{3+\sigma_2} [\frac{2^{\sigma_2-1}}{4} + \frac{1}{\bar{r}\sigma_2} \widehat{L}(d-c)^{\sigma_2} h_t + \frac{4^{\sigma_2}}{\sigma_2}] h_t^{3+\sigma_2}$$
$$+ \frac{1}{\Gamma(\sigma_1+1)\Gamma(\sigma_2+1)} Q_2 r^{\sigma_1} \bar{r}^{3+\sigma_2} (b-a)^{\sigma_1} (d-c)^{\sigma_2} h_t^4. \quad \text{(B.30)}$$

Plug (B.13), (B.28), (B.29), and (B.30) into (B.3). We get

$$|E_{2l+1,2k+1}| \leq C(h_s^{3+\sigma_1} + h_t^{3+\sigma_2}),$$

where C is independent of h_s and h_t.

The estimation of the other cases for $E_{i,j}$ are similar to $E_{2l+1,2k+1}$. Here, we will omit them. Therefore, we have already proven Lemma 3.2.