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Research article

Probabilistic type 2 Bernoulli and Euler polynomials

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Abstract: Assume that the moment-generating function of the random variable Y exists in a neighborhood of the origin. The aim of this paper is to investigate the probabilistic type 2 Bernoulli polynomials associated with Y and the probabilistic type 2 Euler polynomials associated with Y, along with the probabilistic type 2 cosine-Bernoulli polynomials associated with Y, the probabilistic type 2 sine-Bernoulli polynomials associated with Y, the probabilistic type 2 cosine-Euler polynomials associated with Y, and the probabilistic type 2 sine-Euler polynomials associated with Y. We deal with their properties, related identities and explicit expressions.

Keywords: probabilistic type 2 Bernoulli polynomials; probabilistic type 2 Euler polynomials; probabilistic type 2 cosine-Bernoulli polynomials; probabilistic type 2 sine-Bernoulli polynomials; probabilistic type 2 cosine-Euler polynomials; probabilistic type 2 sine-Euler polynomials
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1. Introduction

We discuss the probabilistic extensions of type 2 Bernoulli polynomials and type 2 Euler polynomials, namely the probabilistic type 2 Bernoulli polynomials associated with Y and the probabilistic type 2 Euler polynomials associated with Y. Here, Y is a random variable whose moment-generating function exists in a neighborhood of the origin. Along with them, we also consider the probabilistic type 2 cosine-Bernoulli polynomials associated with Y, the probabilistic type 2 sine-Bernoulli polynomials associated with Y, the probabilistic type 2 cosine-Euler polynomials associated with Y, and the probabilistic type 2 sine-Euler polynomials associated with Y. We study their properties, related identities and explicit expressions.

The details of this paper are as follows. In Section 1, we recall the type 2 Bernoulli polynomials and the type 2 Euler polynomials. We remind the reader of the type 2 cosine-Bernoulli polynomials, the type 2 sine-Bernoulli polynomials, the type 2 cosine-Euler polynomials and the type 2 sine-Euler polynomials. We recall the Stirling numbers of the second kind and the central factorial numbers of the second kind. Assume that Y is a random variable whose moment-generating function exists in some neighborhood of the origin. Let $(Y_j)_{j\geq 1}$ be a sequence of mutually independent copies of the random variable Y, and let $S_k = Y_1 + Y_2 + \cdots + Y_k$, $(k \geq 1)$, with $S_0 = 0$. We remind the reader of the probabilistic Stirling numbers of the second kind associated Y and the probabilistic central factorial numbers of the second kind associated with Y, $T_Y(n,k)$. Section 2 contains the main results of this paper. We define the probabilistic type 2 Bernoulli polynomials associated with Y, $B_{n,Y}(x)$, the probabilistic type 2 Euler polynomials associated with Y, $E_{n,Y}(x)$, the probabilistic type 2 cosine-Bernoulli polynomials associated with Y, $B^c_{n,Y}(x,y)$ and the probabilistic type 2 sine-Bernoulli polynomials associated with Y, $B^s_{n,Y}(x,y)$. In Theorem 2.1, we derive some expressions of $B^c_{n,Y}(x,y)$ and $B^s_{n,Y}(x,y)$ in terms of $B_{n,Y}(x)$. We deduce one identity involving $B^c_{n,Y}(x,y)$ and another involving $B^s_{n,Y}(x,y)$ in Theorem 2.3. We define the probabilistic type 2 Bernoulli polynomials of order α associated with Y, $B^{(\alpha)}_{n,Y}(x)$, the probabilistic cosine-Bernoulli polynomials of order α associated with Y, $B^{(c,\alpha)}_{n,Y}(x,y)$, and the probabilistic sine-Bernoulli polynomials of order α associated with Y, $B^{(s,\alpha)}_{n,Y}(x,y)$. For any positive integer k, we show that $B^{(-k)}_{n,Y} = B^{(-k)}_{n,Y}(0) = \frac{1}{\binom{n+k}{k}}T_Y(n+k,k)$ in Theorem 2.4. In Theorem 2.5, we find certain expressions for $B^{(c,\alpha)}_{n,Y}(x,y)$ and $B^{(s,\alpha)}_{n,Y}(x,y)$ in terms of $B^{(\alpha)}_{n,Y} = B^{(\alpha)}_{n,Y}(0)$. For any positive integer k, we derive explicit expressions for $B^{(c,-k)}_{n,Y}(x,y)$ and $B^{(s,-k)}_{n,Y}(x,y)$ involving $T_Y(n,k)$ in Theorem 2.6. We define the probabilistic type 2 cosine-Euler polynomials associated with Y, $E^c_{n,Y}(x,y)$, and the probabilistic type 2 sine-Euler polynomials associated with Y, $E^s_{n,Y}(x,y)$. In Theorem 2.7, we get explicit expressions for $E^c_{n,Y}(x,y)$ and $E^c_{n,Y}(x,y)$ in terms of $E_{n,Y} = E_{n,Y}(0)$.

Now, we give a brief account of the literature in the References. In [2, 6, 7, 16–18], one can find general facts on probability, combinatorics, polynomials and functions. The r-Stirling numbers of the first kind count restricted permutations, while those of the second kind count restricted partitions. When $r = 0$ or $r = 1$, they reduce to the usual Stirling numbers of the first and second kinds. The reader may refer to [3, 4, 12, 13] for the r-Stirling numbers, the degenerate r-Stirling numbers, and probabilistic extensions of Stirling and degenerate Stirling numbers of the second kind. We refer the reader to [1, 5, 9, 19, 20] for the Bell numbers and polynomials, and for probabilistic extensions of the Bell, central Bell and degenerate Bell polynomials. We investigate probabilistic extensions of the Bernoulli and Euler polynomials in [10]. In [8, 11, 14], we study the type 2 degenerate Bernoulli and Euler polynomials (see [15]). The rest of this section is devoted to recalling the facts that are needed throughout this paper.

It is known that the type 2 Bernoulli polynomials $B_n(x)$ and the type 2 Euler polynomials $E_n(x)$ are respectively defined by

$$e^{xt}\frac{t}{2}\operatorname{csch}\frac{t}{2} = \frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.1}$$

and

$$e^{xt}\operatorname{sech}\frac{t}{2} = \frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad \text{(see [11, 14])}. \tag{1.2}$$

When $x = 0$, $B_n = B_n(0)$ and $E_n = E_n(0)$ are respectively called the type 2 Bernoulli numbers and the type 2 Euler numbers. From (1.1) and (1.2), we note that

$$\sum_{k=0}^{n-1}\left(k+\frac{1}{2}\right)^m=\frac{1}{m+1}(B_{m+1}(n)-B_{m+1}),\quad (n\geq 1), \tag{1.3}$$

and

$$\sum_{k=0}^{n-1}(-1)^k\left(k+\frac{1}{2}\right)^m=\frac{E_m(n)+E_m}{2},\ \text{for } n\equiv 1 \ (\text{mod } 2), \tag{1.4}$$

(see [11, 14]). By (1.1), the type 2 Bernoulli polynomials of the complex variable $x+iy$ are given by

$$\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}e^{(x+iy)t}=\sum_{n=0}^{\infty}B_n(x+iy)\frac{t^n}{n!},\quad (i=\sqrt{-1}). \tag{1.5}$$

The type 2 cosine-Bernoulli polynomials $B_n^c(x,y)$ and the type 2 sine-Bernoulli polynomials $B_n^s(x,y)$ are respectively defined by

$$\sum_{n=0}^{\infty}\left(\frac{B_n(x+iy)+B_n(x-iy)}{2}\right)\frac{t^n}{n!}=\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}e^{xt}\cos yt=\sum_{n=0}^{\infty}B_n^c(x,y)\frac{t^n}{n!}, \tag{1.6}$$

and

$$\sum_{n=0}^{\infty}\left(\frac{B_n(x+iy)-B_n(x-iy)}{2i}\right)\frac{t^n}{n!}=\frac{t}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}e^{xt}\sin yt=\sum_{n=0}^{\infty}B_n^s(x,y)\frac{t^n}{n!}. \tag{1.7}$$

The type 2 cosine-Euler polynomials $E_n^c(x,y)$ and the type 2 sine-Euler polynomials $E_n^s(x,y)$ are respectively given by

$$\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}e^{xt}\cos yt=\sum_{n=0}^{\infty}E_n^c(x,y)\frac{t^n}{n!}, \tag{1.8}$$

$$\frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}e^{xt}\sin yt=\sum_{n=0}^{\infty}E_n^s(x,y)\frac{t^n}{n!},\quad \text{(see [11, 14])}. \tag{1.9}$$

For $n\geq 0$, the Stirling numbers of the second kind are defined by

$$x^n=\sum_{k=0}^{n}\left\{ n \atop k \right\}(x)_k,\quad \text{(see [3, 4, 12, 13])}, \tag{1.10}$$

where $(x)_0=1,\ (x)_k=x(x-1)\cdots(x-k+1),\ (k\geq 1)$.

Let n be a nonnegative integer. Then the central factorial numbers of the second kind are given by

$$\frac{1}{k!}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})^k=\sum_{n=k}^{\infty}T(n,k)\frac{t^n}{n!},\quad (n\geq k\geq 0),\quad \text{(see [11, 20])}. \tag{1.11}$$

From (1.11), we have

$$x^n=\sum_{k=0}^{n}T(n,k)x^{[k]},\quad (n\geq 0),\quad \text{(see [9, 11])}, \tag{1.12}$$

where

$$x^{[0]} = 1, \quad x^{[n]} = x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right)\cdots\left(x - \frac{n}{2} + 1\right), \quad (n \geq 1).$$

Assume that Y is a random variable such that the moment generating function of Y,

$$E[e^{Yt}] = \sum_{n=0}^{\infty} E[Y^n]\frac{t^n}{n!}, \quad (|t| < r), \quad (\text{see [3, 9, 13, 18]}),$$

exists for some $r > 0$.

Let $(Y_j)_{j\geq 1}$ be a sequence of mutually independent copies of random variable Y, and let

$$S_k = S_1 + S_2 + \cdots + S_k, \ (n \geq 1), \quad \text{with } S_0 = 0.$$

For $n \geq k > 0$, the probabilistic Stirling numbers of the second kind associated with Y are defined by

$$\left\{ n \atop k \right\}_Y = \frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}E[S_m^n], \quad (\text{see [3, 9, 13]}). \tag{1.13}$$

When $Y = 1$, we have $\left\{ n \atop k \right\}_Y = \left\{ n \atop k \right\}$.

The probabilistic central factorial numbers of the second kind associated with Y are given by

$$\frac{1}{k!}\left(E\left[e^{\frac{Y}{2}t}\right] - E\left[e^{-\frac{Y}{2}t}\right]\right)^k = \sum_{n=k}^{\infty} T_Y(n,k)\frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see [20]}). \tag{1.14}$$

The probabilistic Bernoulli polynomials associated with Y are defined by

$$\beta_n^Y(x) = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\beta_{n-m}^Y\left\{ m \atop k \right\}_Y (x)_k \quad (n \geq 0), \quad (\text{see [9]}). \tag{1.15}$$

When $Y = 1$, we have $\beta_n^Y(x) = \beta_n(x)$, where $\beta_n(x)$ are the ordinary Bernoulli polynomials given by

$$\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty}\beta_n(x)\frac{t^n}{n!}, \quad (\text{see [1, 2, 6, 17]}). \tag{1.16}$$

The probabilistic Euler polynomials associated with Y are defined by

$$\mathcal{E}_n^Y(x) = \sum_{m=0}^{n}\binom{n}{m}\mathcal{E}_{n-m}^Y\sum_{k=0}^{m}\left\{ m \atop k \right\}_Y (x)_k, \quad (n \geq 0), \quad (\text{see [9]}). \tag{1.17}$$

When $Y = 1$, we have $\mathcal{E}_n^Y(x) = \mathcal{E}_n(x)$, where $\mathcal{E}_n(x)$ are the ordinary Euler polynomials given by

$$\frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty}\mathcal{E}_n(x)\frac{t^n}{n!}, \quad (\text{see [1, 2, 6, 7, 17]}).$$

2. Probabilistic type 2 Bernoulli and Euler polynomials

Let $(Y_j)_{j\geq 1}$ be a sequence of mutually independent copies of the random variable Y, and let

$$S_0 = 0, \quad S_k = S_1 + S_2 + \cdots + S_k, \quad (k \in \mathbb{N}).$$

Now, we define the probabilistic type 2 Bernoulli polynomials associated with Y, $B_{n,Y}(x)$, and the probabilistic type 2 Euler polynomials associated with Y, $E_{n,Y}(x)$, respectively, by

$$\frac{t}{E[e^{\frac{Y}{2}t}] - E[e^{-\frac{Y}{2}t}]} e^{xt} = \sum_{n=0}^{\infty} B_{n,Y}(x) \frac{t^n}{n!}, \tag{2.1}$$

and

$$\frac{2}{E[e^{\frac{Y}{2}t}] + E[e^{-\frac{Y}{2}t}]} e^{xt} = \sum_{n=0}^{\infty} E_{n,Y}(x) \frac{t^n}{n!}. \tag{2.2}$$

When $Y = 1$, we have $B_{n,Y}(x) = B_n(x)$ and $E_{n,Y}(x) = E_n(x)$. For $x = 0$, $B_{n,Y} = B_{n,Y}(0)$ and $E_{n,Y} = E_{n,Y}(0)$ are respectively called the probabilistic type 2 Bernoulli numbers associated with Y and the probabilistic type 2 Euler numbers associated with Y.

From (2.1), we note that

$$\frac{t}{E[e^{\frac{Y}{2}t}] - E[e^{-\frac{Y}{2}t}]} e^{(x+iy)t} = \sum_{n=0}^{\infty} B_{n,Y}(x+iy) \frac{t^n}{n!}. \tag{2.3}$$

Thus, by (2.3), we get

$$\sum_{n=0}^{\infty} \left(\frac{B_{n,Y}(x+iy) + B_{n,y}(x-iy)}{2} \right) \frac{t^n}{n!} = \frac{t}{E[e^{\frac{Y}{2}t}] - E[e^{-\frac{Y}{2}t}]} e^{xt} \cos yt, \tag{2.4}$$

and

$$\sum_{n=0}^{\infty} \left(\frac{B_{n,Y}(x+iy) - B_{n,y}(x-iy)}{2i} \right) \frac{t^n}{n!} = \frac{t}{E[e^{\frac{Y}{2}t}] - E[e^{-\frac{Y}{2}t}]} e^{xt} \sin yt. \tag{2.5}$$

We define the probabilistic type 2 cosine-Bernoulli polynomials associated with Y and the probabilistic type 2 sine-Bernoulli polynomials associated with Y as follows:

$$\frac{t}{E[e^{\frac{Y}{2}t}] - E[e^{-\frac{Y}{2}t}]} e^{xt} \cos yt = \sum_{n=0}^{\infty} B_{n,Y}^{c}(x,y) \frac{t^n}{n!}, \tag{2.6}$$

and

$$\frac{t}{E[e^{\frac{Y}{2}t}] - E[e^{-\frac{Y}{2}t}]} e^{xt} \sin yt = \sum_{n=0}^{\infty} B_{n,Y}^{s}(x,y) \frac{t^n}{n!}. \tag{2.7}$$

From (2.4)–(2.7), we obtain

$$\frac{B_{n,Y}(x+iy) + B_{n,Y}(x-iy)}{2} = B_{n,Y}^{c}(x,y),$$

$$\frac{B_{n,Y}(x+iy)-B_{n,Y}(x-iy)}{2i}=B^{s}_{n,Y}(x,y),$$

where n is a nonnegative integer.

From (2.6), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}B^{c}_{n}(x,y)\frac{t^n}{n!}&=\frac{t}{E[e^{\frac{Y}{2}t}]-E[e^{-\frac{Y}{2}t}]}e^{xt}\cos yt\\&=\sum_{l=0}^{\infty}B_{l,Y}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{y^{2m}(-1)^m}{(2m)!}t^{2m}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}(-1)^m B_{n-2m,Y}(x)y^{2m}\frac{t^n}{n!}.\end{aligned}\tag{2.8}$$

From (2.7), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}B^{s}_{n,Y}(x,y)\frac{t^n}{n!}&=\frac{t}{E[e^{\frac{Y}{2}t}]-E[e^{-\frac{Y}{2}t}]}e^{xt}\sin yt\\&=\sum_{l=0}^{\infty}B_{l,Y}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{(-1)^m y^{2m+1}}{(2m+1)!}t^{2m+1}\\&=\sum_{n=1}^{\infty}\sum_{m=0}^{[\frac{n-1}{2}]}\binom{n}{2m+1}(-1)^m B_{n-2m-1,Y}(x)y^{2m+1}\frac{t^n}{n!}.\end{aligned}\tag{2.9}$$

Therefore, by (2.8) and (2.9), we obtain the following theorem.

Theorem 2.1. *We have the following expressions:*

$$B^{c}_{n,Y}(x,y)=\sum_{m=0}^{[\frac{n}{2}]}(-1)^m\binom{n}{2m}B_{n-2m,Y}(x)y^{2m},\quad (n\geq 0),$$

and

$$B^{s}_{n,Y}(x,y)=\sum_{m=0}^{[\frac{n-1}{2}]}(-1)^m\binom{n}{2m+1}B_{n-2m-1,Y}(x)y^{2m+1},\quad (n\geq 1).$$

By (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,Y}(x)\frac{t^n}{n!}&=\frac{t}{E[e^{\frac{Y}{2}t}]-E[e^{-\frac{Y}{2}t}]}e^{xt}\\&=\sum_{l=0}^{\infty}B_{l,Y}\frac{t^l}{l!}\sum_{m=0}^{\infty}x^m\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}B_{l,Y}x^{n-l}\frac{t^n}{n!}.\end{aligned}\tag{2.10}$$

The next theorem follows immediately from (2.10).

Theorem 2.2. *For $n \geq 0$, we have*

$$B_{n,Y}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{l,Y} x^{n-l}.$$

On the one hand, from (2.6) and (2.7), we have

$$\begin{aligned} e^{tx}\cos yt &= \frac{1}{t}(E[e^{\frac{Y}{2}t}] - E[e^{-\frac{Y}{2}t}]) \sum_{l=0}^{\infty} B^{c}_{l,Y}(x,y)\frac{t^l}{l!} \\ &= \frac{1}{t}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k E[Y^k](1-(-1)^k)\frac{t^k}{k!}\sum_{l=0}^{\infty} B^{c}_{l,Y}(x,y)\frac{t^l}{l!} \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+1}\frac{1}{k+1}E[Y^{k+1}](1+(-1)^k)\frac{t^k}{k!}\sum_{l=0}^{\infty} B^{c}_{l,Y}(x,y)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{k+1}\frac{E[Y^{k+1}]}{k+1}\binom{n}{k}(1+(-1)^k)B^{c}_{n-k,Y}(x,y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{k+1}E[Y^{k+1}]\binom{n+1}{k+1}(1+(-1)^k)B^{c}_{n-k,Y}(x,y)\frac{t^n}{n!}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} e^{tx}\sin yt &= \frac{1}{t}(E[e^{\frac{Y}{2}t}] - E[e^{-\frac{Y}{2}t}]) \sum_{l=0}^{\infty} B^{s}_{l,Y}(x,y)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{k+1}E[Y^{k+1}]\binom{n+1}{k+1}(1+(-1)^k)B^{s}_{n-k,Y}(x,y)\frac{t^n}{n!}. \end{aligned} \tag{2.12}$$

On the other hand, we also have

$$\begin{aligned} e^{tx}\cos yt &= \sum_{l=0}^{\infty}\frac{x^l}{l!}t^l\sum_{m=0}^{\infty}\frac{(-1)^m y^{2m}}{(2m)!}t^{2m} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}(-1)^m\binom{n}{2m}y^{2m}x^{n-2m}\frac{t^n}{n!}, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} e^{tx}\sin yt &= \sum_{l=0}^{\infty}\frac{x^l}{l!}t^l\sum_{m=0}^{\infty}\frac{(-1)^m y^{2m+1}}{(2m+1)!}t^{2m+1} \\ &= \sum_{n=1}^{\infty}\sum_{m=0}^{[\frac{n-1}{2}]}(-1)^m\binom{n}{2m+1}y^{2m+1}x^{n-2m-1}\frac{t^n}{n!}. \end{aligned} \tag{2.14}$$

Taking (2.11)–(2.14) altogether, we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have the following identities:*

$$\frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{k+1}(1+(-1)^k)\binom{n+1}{k+1}E[Y^{k+1}]B^{c}_{n-k,Y}(x,y)=\sum_{m=0}^{[\frac{n}{2}]}(-1)^m\binom{n}{2m}y^{2m}x^{n-2m},$$

and

$$\frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{k+1}(1+(-1)^k)\binom{n+1}{k+1}E[Y^{k+1}]B^{s}_{n-k,Y}(x,y)=\sum_{m=0}^{[\frac{n}{2}]}(-1)^m\binom{n}{2m+1}y^{2m+1}x^{n-2m}.$$

For $\alpha \in \mathbb{R}$, the *probabilistic type 2 Bernoulli polynomials of order α associated with Y* are defined by

$$\left(\frac{t}{E[e^{\frac{Y}{2}t}]-E[e^{-\frac{Y}{2}t}]}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}B^{(\alpha)}_{n,Y}(x)\frac{t^n}{n!}. \tag{2.15}$$

For $x = 0$, $B^{(\alpha)}_{n,Y} = B^{(\alpha)}_{n,Y}(0)$ are called the probabilistic type 2 Bernoulli numbers of order α associated with Y. For $k \in \mathbb{N}$ and $x = 0$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}B^{(-k)}_{n,Y}\frac{t^n}{n!}&=\frac{k!}{t^k}\frac{1}{k!}(E[e^{\frac{Y}{2}t}]-E[e^{-\frac{Y}{2}t}])^k \\ &=\frac{k!}{t^k}\sum_{n=k}^{\infty}T_Y(n,k)\frac{t^n}{n!} \\ &=\sum_{n=0}^{\infty}\frac{1}{\binom{n+k}{k}}T_Y(n+k,k)\frac{t^n}{n!}.\end{aligned} \tag{2.16}$$

The next theorem follows readily from (2.16).

Theorem 2.4. *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$B^{(-k)}_{n,Y}=\frac{1}{\binom{n+k}{k}}T_Y(n+k,k).$$

For any $\alpha \in \mathbb{R}$, we define the *probabilistic cosine-Bernoulli polynomials of order α associated with Y* and the *probabilistic sine-Bernoulli polynomials of order α associated with Y*, respectively, by

$$\left(\frac{t}{E[e^{\frac{Y}{2}t}]-E[e^{-\frac{Y}{2}t}]}\right)^{\alpha}e^{xt}\cos yt=\sum_{n=0}^{\infty}B^{(c,\alpha)}_{n,Y}(x,y)\frac{t^n}{n!}, \tag{2.17}$$

and

$$\left(\frac{t}{E[e^{\frac{Y}{2}t}]-E[e^{-\frac{Y}{2}t}]}\right)^{\alpha}e^{xt}\sin yt=\sum_{n=0}^{\infty}B^{(s,\alpha)}_{n,Y}(x,y)\frac{t^n}{n!}. \tag{2.18}$$

Then we obtain

$$B^{(c,\alpha)}_{n,Y}(x,y)=\frac{B^{(\alpha)}_{n,Y}(x+iy)+B^{(\alpha)}_{n,Y}(x-iy)}{2}, \tag{2.19}$$

$$B^{(s,\alpha)}_{n,Y} = \frac{B^{(\alpha)}_{n,Y}(x+iy) - B^{(\alpha)}_{n,Y}(x-iy)}{2i},$$

where n is a nonnegative integer.

From (2.15), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\Big(\frac{B^{(\alpha)}_{n,Y}(x+iy) + B^{(\alpha)}_{n,Y}(x-iy)}{2}\Big)\frac{t^n}{n!} &= \Big(\frac{t}{E[e^{\frac{Y}{2}t}] - E[e^{-\frac{Y}{2}t}]}\Big)^{\alpha} e^{xt}\cos yt \\ &= \sum_{k=0}^{\infty} B^{(\alpha)}_{k,Y}\frac{t^k}{k!}\sum_{l=0}^{\infty}\sum_{m=0}^{[\frac{l}{2}]}\binom{l}{2m}(-1)^m y^{2m}x^{l-2m}\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{m=0}^{[\frac{l}{2}]}\binom{n}{l}\binom{l}{2m}(-1)^m \times y^{2m}x^{l-2m}B^{(\alpha)}_{n-l,Y}\frac{t^n}{n!},\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\Big(\frac{B^{(\alpha)}_{n,Y}(x+iy) - B^{(\alpha)}_{n,Y}(x-iy)}{2i}\Big)\frac{t^n}{n!} &= \Big(\frac{t}{E[e^{\frac{Y}{2}t}] - E[e^{-\frac{Y}{2}t}]}\Big)^{\alpha} e^{xt}\sin yt \\ &= \sum_{l=0}^{\infty} B^{(\alpha)}_{l,Y}\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}y^{2m+1}x^{2m+1}\sum_{j=0}^{\infty}\frac{x^j}{j!}t^j \\ &= \sum_{l=0}^{\infty} B^{(\alpha)}_{l,Y}\frac{t^l}{l!}\sum_{k=1}^{\infty}\sum_{m=0}^{[\frac{k-1}{2}]}\binom{k}{2m+1}(-1)^m \times y^{2m+1}x^{k-2m-1}\frac{t^k}{k!} \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{m=0}^{[\frac{k-1}{2}]}\binom{n}{k}\binom{k}{2m+1}(-1)^m \times y^{2m+1}x^{k-2m-1}B^{(\alpha)}_{n-k,Y}\frac{t^n}{n!}.\end{aligned} \tag{2.21}$$

Therefore, by (2.19)–(2.21), we have the following theorems.

Theorem 2.5. *We have the following expressions:*

$$B^{(c,\alpha)}_{n,Y}(x,y) = \sum_{l=0}^{n}\sum_{m=0}^{[\frac{l}{2}]}(-1)^m\binom{n}{l}\binom{l}{2m}B^{(\alpha)}_{n-l,Y}y^{2m}x^{l-2m}, \quad (n \geq 0),$$

and

$$B^{(s,\alpha)}_{n,Y}(x,y) = \sum_{k=1}^{n}\sum_{m=0}^{[\frac{k-1}{2}]}(-1)^m\binom{n}{k}\binom{k}{2m+1}B^{(\alpha)}_{n-k,Y}y^{2m+1}x^{k-2m-1}, \quad (n \geq 1).$$

For $k \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B^{(c,-k)}_{n,Y}(x,y)\frac{t^n}{n!} &= \frac{k!}{t^k}\frac{1}{k!}(E[e^{\frac{Y}{2}t}] - E[e^{-\frac{Y}{2}t}])^k e^{xt}\cos yt \\ &= \sum_{l=0}^{\infty}\frac{T_Y(l+k,k)}{\binom{l+k}{k}}\frac{t^l}{l!}\sum_{j=0}^{\infty}\sum_{m=0}^{[\frac{j}{2}]}\binom{j}{2m}y^{2m}x^{j-2m}(-1)^m\frac{t^j}{j!}\end{aligned} \tag{2.22}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{m=0}^{[\frac{j}{2}]}\frac{\binom{j}{2m}T_Y(n-j+k,k)}{\binom{n-j+k}{k}}\binom{n}{j}(-1)^m y^{2m}x^{j-2m}\frac{t^n}{n!},$$

and

$$\sum_{n=0}^{\infty}B_{n,Y}^{(s,-k)}(x,y)\frac{t^n}{n!} = \frac{k!}{t^k}\frac{1}{k!}(E[e^{\frac{Y}{2}t}]-E[e^{-\frac{Y}{2}t}])^k e^{xt}\sin yt \tag{2.23}$$

$$= \sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{m=0}^{[\frac{j-1}{2}]}\frac{\binom{j}{2m+1}T_Y(n-j+k,k)}{\binom{n-j+k}{k}}\binom{n}{j}(-1)^m y^{2m+1}x^{l-2m-1}\frac{t^n}{n!}.$$

Therefore, we deduce the next theorem from (2.22) and (2.23).

Theorem 2.6. *For $k \in \mathbb{N}$, we have the following expressions:*

$$B_{n,Y}^{(c,-k)}(x,y) = \sum_{j=0}^{n}\sum_{m=0}^{[\frac{j}{2}]}\frac{\binom{j}{2m}T_Y(n-j+k,k)}{\binom{n-j+k}{k}}(-1)^m\binom{n}{j}y^{2m}x^{l-2m}, \quad (n \geq 0),$$

and

$$B_{n,Y}^{(s,-k)}(x,y) = \sum_{j=1}^{n}\sum_{m=0}^{[\frac{j-1}{2}]}\frac{\binom{j}{2m+1}T_Y(n-j+k,k)}{\binom{n-j+k}{k}}(-1)^m\binom{n}{j}y^{2m+1}x^{l-2m-1}, \quad (n \geq 1).$$

By using (2.2), we get

$$\sum_{n=0}^{\infty}\left(\frac{E_{n,Y}(x+iy)+E_{n,Y}(x-iy)}{2}\right)\frac{t^n}{n!} = \frac{2}{E[e^{\frac{Y}{2}t}]+E[e^{-\frac{Y}{2}t}]}e^{xt}\cos yt, \tag{2.24}$$

and

$$\sum_{n=0}^{\infty}\left(\frac{E_{n,Y}(x+iy)-E_{n,Y}(x-iy)}{2i}\right)\frac{t^n}{n!} = \frac{2}{E[e^{\frac{Y}{2}t}]+E[e^{-\frac{Y}{2}t}]}e^{xt}\sin yt. \tag{2.25}$$

Now, we define the *probabilistic type 2 cosine-Euler polynomials associated with Y* and the *probabilistic type 2 sine-Euler polynomials associated with Y*, respectively, by

$$\frac{2}{E[e^{\frac{Y}{2}t}]+E[e^{-\frac{Y}{2}t}]}e^{xt}\cos yt = \sum_{n=0}^{\infty}E_{n,y}^{c}(x,y)\frac{t^n}{n!}, \tag{2.26}$$

and

$$\frac{2}{E[e^{\frac{Y}{2}t}]+E[e^{-\frac{Y}{2}t}]}e^{xt}\sin yt = \sum_{n=0}^{\infty}E_{n,Y}^{s}(x,y)\frac{t^n}{n!}. \tag{2.27}$$

By (2.26) and (2.27), we get

$$\sum_{n=0}^{\infty}E_{n,Y}^{c}(x,y)\frac{t^n}{n!} = \frac{2}{E[e^{\frac{Y}{2}t}]+E[e^{-\frac{Y}{2}t}]}e^{xt}\cos yt \tag{2.28}$$

$$= \sum_{l=0}^{\infty} E_{l,Y} \frac{t^l}{l!} \sum_{j=0}^{\infty} \sum_{m=0}^{[\frac{j}{2}]} \binom{j}{2m} y^{2m} (-1)^m x^{j-2m} \frac{t^j}{j!}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{m=0}^{[\frac{j}{2}]} \binom{n}{j} \binom{j}{2m} (-1)^m E_{n-j,Y} y^{2m} x^{j-2m} \frac{t^n}{n!},$$

and

$$\sum_{n=0}^{\infty} E_{n,Y}^{s}(x,y) \frac{t^n}{n!} = \frac{2}{E[e^{\frac{Y}{2}t}] + E[e^{-\frac{Y}{2}t}]} e^{xt} \sin yt \tag{2.29}$$

$$= \sum_{l=0}^{\infty} E_{l,Y} \frac{t^l}{l!} \sum_{j=1}^{\infty} \sum_{m=0}^{[\frac{j-1}{2}]} \binom{j}{2m+1} x^{j-2m-1} y^{2m+1} (-1)^m \frac{t^j}{j!}$$

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{n} \sum_{m=0}^{[\frac{j-1}{2}]} \binom{n}{j} \binom{j}{2m+1} (-1)^m E_{n-j,Y} y^{2m+1} x^{j-2m-1} \frac{t^n}{n!}.$$

Therefore, by (2.28) and (2.29), we get the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$E_{n,Y}^{c}(x,y) = \sum_{j=0}^{n} \sum_{m=0}^{[\frac{j}{2}]} (-1)^m \binom{n}{j} \binom{j}{2m} E_{n-j,Y} y^{2m} x^{j-2m},$$

and

$$E_{n,Y}^{s}(x,y) = \sum_{j=1}^{n} \sum_{m=0}^{[\frac{j-1}{2}]} (-1)^m \binom{n}{j} \binom{j}{2m+1} E_{n-j,Y} y^{2m+1} x^{j-2m-1}.$$

3. Conclusions

Let Y be a random variable such that the moment-generating function of Y exists in a neighborhood of the origin. We investigated probabilistic extensions of the type 2 Bernoulli polynomials and the type 2 Euler polynomials, namely the probabilistic type 2 Bernoulli polynomials associated with Y and the probabilistic type 2 Euler polynomials associated with Y. Along with these, we also considered the probabilistic type 2 cosine-Bernoulli polynomials associated with Y, the probabilisitic type 2 sine-Bernoulli polynomials associated with Y, the probabilistic type 2 cosine-Euler polynomials associated with Y, and the probabilistic type 2 sine-Euler polynomials associated with Y. We discussed their properties, related identities and explicit expressions.

As one of our future projects, we would like to continue to study probabilistic extensions of many special polynomials and numbers and to find their applications to physics, science and engineering as well as to mathematics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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