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Research article

New technique for solving the numerical computation of neutral fractional functional integro-differential equation based on the Legendre wavelet method

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Abstract: The aim of this work is to solve a numerical computation of the neutral fractional functional integro-differential equation based on a new approach to the Legendre wavelet method. The concept of fractional derivatives was examined in the sense of Caputo. The properties of the Legendre wavelet and function approximation were employed to determine the approximate solution of a given dynamical system. Moreover, the error estimations and convergence analysis of the truncated Legendre wavelet expansion for the proposed problem were discussed. The validity and applicability of this proposed technique to numerical computation were shown by illustrative examples. Eventually, the results of this technique demonstrate its great effectiveness and reliability.

Keywords: fractional derivatives; Legendre wavelet; numerical computation; error analysis
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1. Introduction

Fractional calculus is an extension of classical calculus that deals with arbitrary non-integer order integrations and differentiations. In 1695, Guillaume de L'Hôpital sent a letter to Gottfried Wilhelm Leibniz. In his message, an important question about the order of the derivative emerged: What might be a derivative of order $1/2$? Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn". After this extraordinary conversation, numerous notable mathematicians were interested in fractional calculus, and many of them either directly or indirectly contributed to its advancement. A wide variety of applications can be found in many fields of science and engineering using fractional calculus. In particular, aerodynamics, biology, control theory, rheology, diffusion equations, signal processing, and image processing fall under these categories. The readers can see the following monographs on fractional calculus and its application [1–8]. Working with these models and comprehending analytical solutions is quite challenging. Due to this, a number of numerical methods have been proposed by numerous researchers: operational matrices of piecewise constant orthogonal function [9], Chebyshev polynomials [10], Bernoulli wavelet method [11], Adomain decomposition [12], homotopy analysis method [13], differential transform method [14], finite difference method [15], variational iteration method [16], and one-leg-θ method [17].

Many complicated phenomena in nature and technology can be described by functional differential equations, which have become increasingly prevalent (see [18–21]). Functional differential equations can be used to analyze previous and current behavior being possible to use the construct models in a variety of fields, including medicine, mechanics, biology, and economics. Differential equations that involve both advanced and delayed arguments are referred to as neutral functional differential equations. Because of their ability to balance past and future action in a neutral manner, they are known as neutral. Unlike ordinary differential equations, where the derivative at a point depends only on the values of the function at that point and earlier, in neutral functional differential equations, the derivative depends not only on the function and its derivatives at the present time but also at some past and future times. In real-world phenomena, neutral functional differential equations help to generalize the formulation of any difficult problem.

The idea behind wavelets is to decompose a signal into different frequency components, allowing for localized analysis both in time and frequency domains. The concept was pioneered by Jean Morlet and Alex Grossmann in the year of 1981. Legendre wavelets are wavelets constructed using Legendre polynomials as their basis functions. The Legendre polynomials are orthogonal on the interval $[-1, 1]$, which makes them suitable for wavelet analysis on bounded domains. By scaling and translating the Legendre polynomials, one can generate a family of wavelet functions. Wavelet methods have attracted considerable attention in recent years for solving differential equations as well as integral equations of integer and non-integer order. It is highly beneficial to use the Legendre wavelets method (LWM) in various fields due to its versatility and effectiveness in solving differential equations and analyzing dynamic systems. Legendre wavelets often involve the development of algorithms for efficient computation and implementation. Researchers explore techniques for fast wavelet transforms using Legendre wavelets. Their ability to represent signals and images with localized features makes them suitable for applications in areas such as medical imaging, remote sensing, and digital communications. In earlier studies, many authors were involved in the field of Legendre wavelets. Antoine et al. [22] studied wavelet transforms and their applications. Recently, Etemad et al. [23] proposed a new fractal-fractional version of giving-up-smoking model. Most recently, Kanwal et al. [24] discussed the dynamics of a model of polluted lakes via fractal-fractional operators with two different numerical algorithms. Li et al. discussed the normalized ground states for Sobolev critical nonlinear Schrödinger equation in the L^2-supercritical case and mass critical growth in [25, 26]. Meng et al. [27] addressed the LWM for solving fractional integro-differential equations. Mohammadi et al. [28] studied a new Legendre wavelet operational matrix of derivatives and its applications in solving the singular ordinary differential equations. Rahimkhani et al. [29] discussed a numerical solution of fractional delay differential equations by using generalized fractional-order Bernoulli wavelet. Rehman et al. [30] investigated the Legendre wavelet method for solving fractional differential equations. Yi et al. [31] studied the LWM for the numerical solution of fractional integro-differential equations with weakly singular kernel. We discuss the new technique for solving the numerical computation of neutral fractional functional integro-differential equations based on LWM. As a fact, prior studies did not examine this idea, which prompted us to conduct extensive research and write the present manuscript. Hence, we considered the following form of neutral fractional functional integro-differential equations:

$$\begin{aligned} D^{\theta}(\varrho(t) - \kappa(t, \varrho_t)) &= \chi(t, \varrho_t, \int_0^t \xi(t, s, \varrho_s) ds), \ t \in \mathcal{I} := [0, 1]. \\ \varrho(t) &= \phi(t), \ t \in \mathcal{B} := (-r, 0]. \end{aligned} \tag{1.1}$$

Let $\mathcal{X}$ be a Hilbert space. D^{θ} denotes Caputo fractional derivative of order $0 < \theta < 1$. Let $\varrho(t)$ be a state variable in $\mathcal{X}$. The neutral term $\kappa : \mathcal{I} \times \mathcal{B} \to \mathcal{X}$ is given continuous function. The delay term $\varrho_t \in \mathcal{B}$, is defined by $\varrho_t(s) = \varrho(t + s)$, $-r < s \leq 0$ and 'r' is a positive constant. Let the functions $\chi : \mathcal{I} \times \mathcal{B} \times \mathcal{X} \to \mathcal{X}$ and $\xi : \mathcal{I} \times \mathcal{X} \times \mathcal{B} \to \mathcal{X}$ be given continuous, where χ represents the function of three variables and ξ represents an integral involving a function over the interval $[0, 1]$. Further, the initial condition $\varrho_0 = \phi = \{\phi(s) : -r < s \leq 0\}$ is considered to be a continuous function.

As a result of studying our proposed method, we have found the following characteristics to be most significant:

(1) Function approximation, unit step function, Laplace transform method, and Legendre polynomial are used to obtain the approximate solution of our given dynamical system. The main benefit of the suggested work is to reduce the error estimation as well as the complexity of solving the numerical computations.
(2) This numerical method provides a means to approximate solutions to complex models, even in cases where analytical solutions are infeasible, and it offers flexibility in modeling different types of systems. It helps to represent the solution to the dynamical system as a series expansion using Legendre wavelets and it helps to identify the error estimation and convergence analysis to assess the accuracy and reliability of numerical solutions. It allows for the validation and verification of working models by comparing numerical results with experimental data.
(3) The Legendre wavelets are orthogonal functions defined on a bounded interval, making them suitable for approximating functions defined on a finite domain and substituting the series expansion of the solution into the given dynamical equations. This step transforms the differential equations into a system of algebraic equations involving the coefficients of the Legendre wavelet expansion.
(4) The novelty of this work is solving the numerical computation of neutral fractional functional integro-differential equations based on our proposed technique. This technique evolves from another type of Legendre polynomial and is helpful to reduce the error estimation of our problem when compared with previous research studies [32–39]. We also discuss how several numerical examples are used to enhance the effectiveness of our proposed method. This could include the absolute errors, exact solution, and approximate solution of our given problem in the time domain. We present the graphical representation of the outputs.

We present this manuscript in the following manner: In Section 2, we provide the system description, basic definition of the Legendre polynomial, fractional derivatives and their properties.

In Section 3, we discuss the primary definitions of wavelets, Legendre wavelets, and function approximation. In Section 4, we estimate the numerical computation of the neutral fractional functional integro-differential equation based on our new technique. In Section 5, we examine the error estimation and convergence analysis of the truncated Legendre wavelet expansions based on the Caputo fractional derivative. In Section 6, we discuss the numerical examples and graphical illustrations for our proposed method. In Section 7, we discuss the results and discussion of our findings. Additionally, in Section 8, we provide the conclusion of this manuscript.

2. Preliminaries and notations

Definition 2.1 ([40]). *The Riemann-Liouville fractional integral of order $\theta > 0$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^n$ is defined by*

$$I^{\theta}_{0^+}\varphi(\mathtt{t}) = \frac{1}{\Gamma(\theta)}\int_0^t \frac{\varphi(s)}{(t-s)^{1-\theta}}ds,\ t > 0.$$

Definition 2.2 ([40]). *The Caputo fractional derivative of order $\theta > 0$ is described as:*

$$D^{\theta}_{0^+}\varphi(t) = \frac{1}{\Gamma(\rho-\theta)}\int_0^t \frac{\varphi^{(\rho)}(s)}{(t-s)^{\theta-\rho+1}}ds,\ t > 0,\ \rho - 1 < \theta < \rho,\ \rho \in \mathbb{N}.$$

Definition 2.3 ([41]). *(Bessel's inequality) Let X be a Hilbert space and suppose that $\{e_1, e_2, ...\}$ is an orthonormal sequence in X. Then, for any x in X, one has*

$$\sum_{k=1}^{\infty} |< x, e_k >| \quad \leq \quad ||x||^2,$$

where, $< . >$ denotes inner product in the Hilbert space X.

Definition 2.4 ([42]). *(Legendre polynomial) The Legendre polynomial of degree δ is defined in the following form:*

$$\mathfrak{P}_{\delta}(t) = \frac{1}{2^{\delta}}\sum_{\zeta=0}^{\delta}\binom{\delta}{\zeta}^2 (t-1)^{\delta-\zeta}(t+1)^{\zeta},\ \textit{where}\ \delta \in \mathbb{Z}^+,\ \binom{\delta}{\zeta} = \frac{\delta!}{\zeta!(\delta-\zeta)!}. \tag{2.1}$$

Proposition 1 ([43]). *The Riemann-Liouville fractional integral and Caputo fractional derivative are connected by the following relation:*

(A1) *The operators I^{θ} and D^{θ} are linear.*
(A2) $D^{\theta} * I^{\theta}\varphi(t) = \varphi(t)$.
(A3) $I^{\theta} * D^{\theta}\varphi(t) = \varphi(t) - \sum_{i=0}^{\rho-1}\varphi^{(i)}(0)\frac{t^i}{i!}$.

3. Legendre wavelet

Wavelets are a family of functions acquired from a single function through scaling (represented by index σ) and translation (represented by index β). The description of the continuous wavelet Ψ, defined and given by [43],

$$\Psi_{\sigma,\beta}(t) = |\sigma|^{-\frac{1}{2}}\Psi\Big(\frac{t-\beta}{\sigma}\Big),\ \sigma,\beta \in \mathbb{R}, \sigma \neq 0.$$

If we restrict the indexes σ and β to discrete values as $\sigma = \sigma_0^{-\omega}$, $\beta = \eta\beta_0\sigma_0^{-\omega}$, $\sigma_0, \theta_0 > 0$ and η, ω are positive integer, we have the following family of discrete wavelets Ψ, defined and given by [43],

$$\Psi_{\eta,\omega}(t) = |\sigma_0|^{\frac{\eta}{2}}\Psi(\sigma_0^{\eta}t - \omega\beta_0). \tag{3.1}$$

Then, Eq (3.1) forms a wavelet basis for $L^2(\mathbb{R})$. The Legendre wavelets $\Psi_{\beta,\delta}(t) = \Psi(\eta, \beta, \delta, t)$ have four arguments, $\beta = 1, 2, 3, ..., 2^{\eta-1}$, η can assume any positive integer, δ is the degree of Legendre polynomial, $\delta = 0, 1, 2, 3, ..., \mathcal{K} - 1$, β is the translation index, and t is the normalized time. They are defined on the interval [0, 1] and given by [44]

$$\Psi_{\beta,\delta}(t) = \begin{cases} 2^{\frac{\eta}{2}}\sqrt{\delta + \frac{1}{2}}\mathfrak{P}_{\delta}(2^{\eta}t - 2\beta + 1), & \frac{2\beta-2}{2^{\eta}} \leq t < \frac{2\beta}{2^{\eta}}. \\ 0, & otherwise. \end{cases} \tag{3.2}$$

Where the coefficient $\sqrt{\delta + \frac{1}{2}}$ is orthonormality, and $\mathfrak{P}_{\delta}$ is the Legendre polynomial of degree δ. A function $\varphi(t) \in L^2[0, 1]$ may be expanded by the Legendre wavelet series in the form of

$$\varphi(t) = \sum_{\beta=1}^{\infty}\sum_{\delta=0}^{\infty} c_{\beta,\delta}\Psi_{\beta,\delta}. \tag{3.3}$$

If the infinite series Eq (3.3) is truncated, then it can be written as

$$\varphi_{2^{\eta-1},\mathcal{K}}(t) \simeq \sum_{\beta=1}^{2^{\omega-1}}\sum_{\delta=0}^{\mathcal{K}-1} c_{\beta,\delta}\Psi_{\beta,\delta} = C^T\Psi(t), \tag{3.4}$$

where, C and Ψ are $2^{\eta-1}\mathcal{K} \times 1$ matrices given by

$$C = [c_{10}, c_{11}, ..c_{1\mathcal{K}-1}, c_{20}, c_{21}, ...c_{2\mathcal{K}-1}, ...c_{2^{\eta-1}0}, c_{2^{\eta-1}1}, ..., c_{2^{\eta-1}\mathcal{K}-1}]^T.$$

$$\Psi = [\Psi_{10}, \Psi_{11}, ..\Psi_{1\mathcal{K}-1}, \Psi_{20}, \Psi_{21}, ...\Psi_{2\mathcal{K}-1}, ...\Psi_{2^{\eta-1}0}, \Psi_{2^{\eta-1}1}, ..., \Psi_{2^{\eta-1}\mathcal{K}-1}]^T.$$

4. A new approach to solve numerical computation

In this clause, the Riemann-Liouvile fractional integral is computed making use of the Legendre wavelet. Let Λ_μ represent the unit step function demonstrated as follows:

$$\Lambda_\mu(t) = \begin{cases} 1 \; if \; t \geq \mu. \\ 0 \; if \; t < \mu. \end{cases}$$

Legendre wavelet is represented in terms of the unit step function and can be expressed as follows:

$$\Psi_{\beta,\delta}(t) = \left(\Lambda_{\frac{2\beta-2}{2^{\eta}}}(t) - \Lambda_{\frac{2\beta}{2^{\eta}}}(t)\right)2^{\frac{\eta}{2}}\sqrt{\delta + \frac{1}{2}}\mathfrak{P}_{\delta}(2^{\eta}t - 2\beta + 1).$$

Considering the Laplace transform following property,

$$\mathcal{L}\{\Lambda_\mu(t)\varphi(t)\} \;\; = \;\; e^{-\mu s}\mathcal{L}\{\varphi(t + \mu)\}.$$

We obtain,

$$\begin{aligned}\mathcal{L}\{\Psi_{\beta,\delta}(t)\} &= 2^{\frac{\eta}{2}}\sqrt{\delta+\frac{1}{2}}\Big(e^{-\frac{2\beta-2}{2^\eta}s}\mathcal{L}\Big\{\mathfrak{P}_\delta\Big(2^\eta\Big(t+\frac{2\beta-2}{2^\eta}\Big)-2\beta+1\Big)\Big\}\\ &\quad -e^{-\frac{2\beta}{2^\eta}s}\mathcal{L}\Big\{\mathfrak{P}_\delta\Big(2^\eta\Big(t+\frac{2\beta}{2^\eta}\Big)-2\beta+1\Big)\Big\}\Big)\\ &= 2^{\frac{\eta}{2}}\sqrt{\delta+\frac{1}{2}}\Big(e^{-\frac{2\beta-2}{2^\eta}s}\mathcal{L}\Big\{\mathfrak{P}_\delta\Big(2^\eta t-1\Big)\Big\}\\ &\quad -e^{-\frac{2\beta}{2^\eta}s}\mathcal{L}\Big\{\mathfrak{P}_\delta\Big(2^\eta t+1\Big)\Big\}\Big). \qquad (4.1)\end{aligned}$$

Now, we consider

$$\begin{aligned}\mathfrak{P}_\delta(2^\eta t-1) &= \frac{1}{2^\delta}\sum_{\zeta=0}^{\delta}\binom{\delta}{\zeta}^2(2^\eta t-2)^{\delta-\zeta}(2^\eta t)^\zeta\\ &= \frac{1}{2^\delta}\sum_{\zeta=0}^{\delta}\binom{\delta}{\zeta}^2\times\sum_{\varpi=0}^{\delta-\zeta}\binom{\delta-\zeta}{\varpi}(2^\eta t)^{\delta-\zeta-\varpi}(-2)^\varpi(2^\eta t)^\zeta\\ &= \frac{1}{2^\delta}\sum_{\zeta=0}^{\delta}\sum_{\varpi=0}^{\delta-\zeta}\binom{\delta}{\zeta}^2\binom{\delta-\zeta}{\varpi}2^{\eta(\delta-\varpi)}(-2)^\varpi t^{\delta-\varpi}.\end{aligned}$$

By using the Laplace transform, we obtain

$$\mathcal{L}\{\mathfrak{P}_\delta(2^\eta t-1)\} = \frac{1}{2^\delta}\sum_{\zeta=0}^{\delta}\sum_{\varpi=0}^{\delta-\zeta}\binom{\delta}{\zeta}^2\binom{\delta-\zeta}{\varpi}2^{\eta(\delta-\varpi)}\frac{(-2)^\varpi(\delta-\varpi)!}{s^{\delta-\varpi+1}}. \qquad (4.2)$$

Similarly, we get

$$\begin{aligned}\mathfrak{P}_\delta(2^\eta t+1) &= \frac{1}{2^\delta}\sum_{\zeta=0}^{\delta}\binom{\delta}{\zeta}^2(2^\eta t)^{\delta-\zeta}(2^\eta t+2)^\zeta\\ &= \frac{1}{2^\delta}\sum_{\zeta=0}^{\delta}\sum_{\varpi=0}^{\zeta}\binom{\delta}{\zeta}^2\binom{\zeta}{\varpi}2^{\eta(\delta-\varpi)}2^\varpi t^{\delta-\varpi}.\\ \mathcal{L}\{\mathfrak{P}_\delta(2^\eta t+1)\} &= \frac{1}{2^\delta}\sum_{\zeta=0}^{\delta}\sum_{\varpi=0}^{\zeta}\binom{\delta}{\zeta}^2\binom{\zeta}{\varpi}2^{\eta(\delta-\varpi)}\frac{(2)^\varpi(\delta-\varpi)!}{s^{\delta-\varpi+1}}. \qquad (4.3)\end{aligned}$$

Substituting the Eqs (4.2) and (4.3) in (4.1), we have

$$\begin{aligned}\mathcal{L}\{\Psi_{\beta,\delta}(t)\} &= 2^{\frac{\eta}{2}-\delta}\sqrt{\delta+\frac{1}{2}}\sum_{\zeta=0}^{\delta}\binom{\delta}{\zeta}^2\Big[e^{-\frac{2\beta-2}{2^\eta}s}\sum_{\varpi=0}^{\delta-\zeta}\binom{\delta-\zeta}{\varpi}2^{\eta(\delta-\varpi)}\times\frac{(-2)^\varpi(\delta-\varpi)!}{s^{\delta-\varpi+1}}\\ &\quad -e^{-\frac{2\beta}{2^\eta}s}\sum_{\varpi=0}^{\zeta}\binom{\zeta}{\varpi}2^{\eta(\delta-\varpi)}\times\frac{(2)^\varpi(\delta-\varpi)!}{s^{\delta-\varpi+1}}\Big]. \qquad (4.4)\end{aligned}$$

Now, we compute the Riemann-Liouville fractional integral in terms of the Legendre wavelet,

$$\begin{aligned}\mathcal{L}\{I^{\theta}\Psi_{\beta,\delta}(t)\} &= \frac{1}{\Gamma(\theta)}\mathcal{L}\{t^{\theta-1}\}\times\mathcal{L}\{\Psi_{\beta,\delta}(t)\}\\ &= \frac{1}{(\theta-1)!}\frac{(\theta-1)!}{s^{\theta}}\mathcal{L}\{\Psi_{\beta,\delta}(t)\}\\ &= \frac{1}{s^{\theta}}\mathcal{L}\{\Psi_{\beta,\delta}(t)\}.\end{aligned} \tag{4.5}$$

Applying inverse Laplace transform to Eq (4.5), then we get

$$\begin{aligned}I^{\theta}\Psi_{\beta,\delta}(t) &= \Big(2^{\frac{\eta}{2}-\delta}\sqrt{\delta+\frac{1}{2}}\sum_{\zeta=0}^{\delta}\binom{\delta}{\zeta}^{2}\Big[\Lambda_{\frac{2\beta-2}{2^{\eta}}}\sum_{\varpi=0}^{\delta-\zeta}\binom{\delta-\zeta}{\varpi}2^{\eta(\delta-\varpi)}(\delta-\varpi)!\\ &\quad\times\frac{(-2)^{\varpi}\Big(t-\frac{2\beta-2}{2^{\eta}}\Big)^{\delta-\varpi+\theta}}{\Gamma(\delta-\varpi+\theta+1)}-\Lambda_{\frac{2\beta}{2^{\eta}}}\sum_{\varpi=0}^{\zeta}\binom{\zeta}{\varpi}2^{\eta(\delta-\varpi)}(\delta-\varpi)!\\ &\quad\times\frac{(2)^{\varpi}\Big(t-\frac{2\beta}{2^{\eta}}\Big)^{\delta-\varpi+\theta}}{\Gamma(\delta-\varpi+\theta+1)}\Big]\Big).\end{aligned} \tag{4.6}$$

Using the Eq (4.6), we compute the approximate solution of Eq (1.1).

$$D^{\theta}(\varrho(t)-\kappa(t,\varrho_{t})\approx C^{T}\Psi(t).$$

$$where\ \Psi(t)=\Big\{\Psi_{\beta,\delta}(t),1\le\beta\le 2^{\eta-1},0\le\delta\le\mathcal{K}-1\Big\}^{T}.$$

$$Let,\ \tilde{\Psi}(t)=\Big\{I^{\theta}\Psi_{\beta,\delta}(t),1\le\beta\le 2^{\eta-1},0\le\delta\le\mathcal{K}-1\Big\}^{T}.$$

From Proposition 1, we have

$$\varrho(t)\approx\begin{cases}C^{T}\tilde{\Psi}(t)+\kappa(t,\varrho_{t})+\sum_{\sigma=0}^{n-1}\varrho^{(\sigma)}(0)\frac{t^{\sigma}}{\sigma!}, & t\in[0,1],\\ \phi(t), & t\in\mathcal{B}.\end{cases}$$

5. Error analysis

In this section, we compute the error estimation and convergence analysis of the given dynamical system (1.1) together with the truncated Legendre wavelet Eq (3.4).

Lemma 5.1 ([42])**.** *The orthogonal properties of the Legendre polynomial are defined as*

$$\int_{-1}^{1}\mathfrak{P}_{\delta}(z)\mathfrak{P}_{\theta}(z)dz=\begin{cases}0, & if\ \delta\ne\theta,\\ \frac{2}{2\delta+1}, & if\ \delta=\theta,\end{cases}$$

where δ and θ are degrees of the Legendre polynomial.

Remark 5.2 ([42])**.** $(2\delta+1)\mathfrak{P}_{\delta}=d(\mathfrak{P}_{\delta+1}-\mathfrak{P}_{\delta-1})$.

Theorem 5.3. *The function $\varphi \in L^2[0,1]$ is continuous such that $|D^\theta\varphi(t)|$ is bounded with second derivative for all $t \in [0,1]$, then we have the following estimation:*

$$\|D^\theta\varphi(t) - D^\theta\varphi_{2^{\eta-1},\mathcal{K}}(t)\|_2^2 \leq \frac{16\mathcal{L}}{2^{5\eta+1}}\left[\frac{4\mathcal{K}^2+4\mathcal{K}+9}{(2\mathcal{K}+1)^2(2\mathcal{K}+5)(2\mathcal{K}-3)(2\mathcal{K}+3)(2\mathcal{K}-1)}\right].$$

Proof. Let $D^\theta\varphi(t)$ be a continuous function in the interval [0, 1], and since the Legendre wavelets are orthonormal, then we have

$$\begin{aligned}
\|D^\theta\varphi(t) - D^\theta\varphi_{2^{\eta-1},\mathcal{K}}(t)\|_2^2 &= \|\sum_{\beta=1}^{2^{\eta-1}}\sum_{\delta=\mathcal{K}}^{\infty} c_{\beta,\delta}\Psi_{\beta,\delta}(t)\|_2^2 \\
&= \left\langle \sum_{\beta=1}^{2^{\eta-1}}\sum_{\delta=\mathcal{K}}^{\infty} c_{\beta,\delta}\Psi_{\beta,\delta}(t), \sum_{\beta=1}^{2^{\eta-1}}\sum_{\delta=\mathcal{K}}^{\infty} c_{\beta,\delta}\Psi_{\beta,\delta}(t)\right\rangle \\
&= \sum_{\beta=1}^{2^{\eta-1}}\sum_{\delta=\mathcal{K}}^{\infty} \left|c_{\beta,\delta}\right|^2 \left\|\Psi_{\beta,\delta}(t)\right\|^2.
\end{aligned}$$

By utilizing the orthogonal properties of the Legendre wavelet, then we attain the following form

$$\|D^\theta\varphi(t) - D^\theta\varphi_{2^{\eta-1},\mathcal{K}}(t)\|_2^2 = \sum_{\beta=1}^{2^{\eta-1}}\sum_{\delta=\mathcal{K}}^{\infty} \left|c_{\beta,\delta}\right|^2 \tag{5.1}$$

where

$$c_{\beta,\delta} = < D^\theta\varphi(t), \Psi_{\beta,\delta}(t) > = \int_0^1 D^\theta\varphi(t)\Psi_{\beta,\delta}(t)dt$$

and $< . >$ denotes inner product space of $L^2[0,1]$, then we can obtain

$$c_{\beta,\delta} = \int_0^1 D^\theta\varphi(t)\Psi_{\beta,\delta}(t)dt = \int_{\frac{2\beta-2}{2^\eta}}^{\frac{2\beta}{2^\eta}} D^\theta\varphi(t)2^{\frac{\eta}{2}}\sqrt{\delta+\frac{1}{2}}\mathfrak{P}_\delta(2^\eta t - 2\beta + 1)dt.$$

Using a change of variable put $s = (2^\eta t - 2\beta + 1)$, we have

$$c_{\beta,\delta} = 2^{-\frac{\eta}{2}}\sqrt{\delta+\frac{1}{2}}\int_{-1}^1 D^\theta\varphi\Big(\frac{s+2\beta-1}{2^\eta}\Big)\mathfrak{P}_\delta(s)ds.$$

By using Lemma 5.1, Remark 5.2, and integration by parts, then we get

$$\begin{aligned}
c_{\beta,\delta} &= 2^{-\frac{\eta}{2}}\sqrt{\delta+\frac{1}{2}}\int_{-1}^1 D^\theta\varphi\Big(\frac{s+2\beta-1}{2^\eta}\Big)\frac{d\,(\mathfrak{P}_{\delta+1}(s) - \mathfrak{P}_{\delta-1}(s))}{2\delta+1}ds \\
&= -\left(\frac{1}{2^{3\eta+1}(2\delta+1)}\right)^{\frac{1}{2}}\int_{-1}^1 D^{\theta+1}\varphi\Big(\frac{s+2\beta-1}{2^\eta}\Big)(\mathfrak{P}_{\delta+1}(s) - \mathfrak{P}_{\delta-1}(s))ds.
\end{aligned}$$

Again by using Lemma 5.1, Remark 5.2, and integration by parts, then we obtain

$$
\begin{aligned}
c_{\beta,\delta} &= \left(\frac{1}{2^{5\eta+1}(2\delta+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} D^{\theta+2}\varphi\left(\frac{s+2\beta-1}{2^{\eta}}\right) \\
&\quad \times\left(\frac{\mathfrak{P}_{\delta+2}(s)-\mathfrak{P}_{\delta}(s)}{2\delta+3}-\frac{\mathfrak{P}_{\delta}(s)-\mathfrak{P}_{\delta-2}(s)}{2\delta-1}\right)ds, \\
|c_{\beta,\delta}| &= \left|\left(\frac{1}{2^{5\eta+1}(2\delta+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} \mathcal{L}ds\right. \\
&\quad \left.\int_{-1}^{1}\left(\frac{\mathfrak{P}_{\delta+2}(s)-\mathfrak{P}_{\delta}(s)}{2\delta+3}-\frac{\mathfrak{P}_{\delta}(s)-\mathfrak{P}_{\delta-2}(s)}{2\delta-1}\right)ds\right| \\
&\leq \left|\left(\frac{1}{2^{5\eta+1}(2\delta+1)}\right)^{\frac{1}{2}}\right|\left|\int_{-1}^{1} \mathcal{L}ds\right. \\
&\quad \int_{-1}^{1}\left|\left(\frac{\mathfrak{P}_{\delta+2}(s)-\mathfrak{P}_{\delta}(s)}{2\delta+3}-\frac{\mathfrak{P}_{\delta}(s)-\mathfrak{P}_{\delta-2}(s)}{2\delta-1}\right)\right|ds, \\
|c_{\beta,\delta}|^2 &\leq \left(\frac{2\mathcal{L}}{2^{5\eta+1}(2\delta+1)}\right) \\
&\quad \int_{-1}^{1}\left|\left(\frac{\mathfrak{P}_{\delta+2}(s)-\mathfrak{P}_{\delta}(s)}{2\delta+3}-\frac{\mathfrak{P}_{\delta}(s)-\mathfrak{P}_{\delta-2}(s)}{2\delta-1}\right)\right|^2 ds \\
&= \frac{16\mathcal{L}}{2^{5\eta+1}(2\delta+1)^2}\left[\frac{4\delta^2+4\delta+9}{(2\delta+3)(2\delta+5)(2\delta-1)(2\delta-3)}\right], \\
\|D^{\theta}\varphi(t)-D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}}(t)\|_2^2 &= \sum_{\beta=1}^{2^{\eta-1}}\sum_{\delta=\mathcal{K}}^{\infty}|c_{\beta,\delta}|^2 \leq \frac{16\mathcal{L}}{2^{5\eta+1}(2\delta+1)^2}\left[\frac{4\delta^2+4\delta+9}{(2\delta+3)(2\delta+5)(2\delta-1)(2\delta-3)}\right], \\
\|D^{\theta}\varphi(t)-D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}}(t)\|_2^2 &\leq \frac{16\mathcal{L}}{2^{5\eta+1}}\left[\frac{4\mathcal{K}^2+4\mathcal{K}+9}{(2\mathcal{K}+1)^2(2\mathcal{K}+5)(2\mathcal{K}-3)(2\mathcal{K}+3)(2\mathcal{K}-1)}\right], \qquad (5.2)
\end{aligned}
$$

where $\mathcal{L} = \left|D^{\theta+2}\varphi\left(\frac{s+2\beta-1}{2^{\eta}}\right)\right|$. From the above inequality (5.2), we can conclude that the function $D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}}(t) \to D^{\theta}\varphi(t)$ in $L^2[0,1]$ as η or $\mathcal{K} \to \infty$. □

Note: From Theorem 5.3, we observed that the error analysis of the solution of the neutral fractional functional integro-differential equation can be reduced to the error analysis of the function approximation. In the following theorem, we prove the convergence of the function approximation.

Theorem 5.4. *For each $\eta \in \mathbb{Z}^+$ and the function $D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}}$ converges to $D^{\theta}\varphi(t)$ as $\mathcal{K} \to \infty$ in $L^2[0,1]$.*

Proof. Initially, let us assume $\mathcal{K}^* > \mathcal{K}$ and, using Theorem 5.1, then we obtain,

$$
\begin{aligned}
\|D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}^*}(t)-D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}}(t)\|_{L^2[0,1]}^2 &= \|D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}^*}(t)-D^{\theta}\varphi(t)+D^{\theta}\varphi(t)-D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}}(t)\|_{L^2[0,1]}^2 \\
&\leq \|D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}^*}(t)-D^{\theta}\varphi(t)\|_{L^2[0,1]}^2 \\
&\quad +\|D^{\theta}\varphi(t)-D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}}(t)\|_{L^2[0,1]}^2 \\
&= \sum_{\beta=1}^{2^{\eta-1}}\sum_{\delta=\mathcal{K}^*}^{\infty}\left|c_{\beta,\delta}\right|^2+\sum_{\beta=1}^{2^{\eta-1}}\sum_{\delta=\mathcal{K}}^{\infty}\left|c_{\beta,\delta}\right|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\beta=1}^{2^{\eta-1}} \sum_{\delta=\mathcal{K}^*}^{\infty} \left|\left\langle D^{\theta}\varphi(t), \Psi_{\beta,\delta}\right\rangle\right|^2 + \sum_{\beta=1}^{2^{\eta-1}} \sum_{\delta=\mathcal{K}}^{\infty} \left|\left\langle D^{\theta}\varphi(t), \Psi_{\beta,\delta}\right\rangle\right|^2 \\
&\leq 2\|\varphi\|^2 < \infty, \ (from\ Bessel's\ inequality).
\end{aligned}
$$

Therefore, $D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}}(t) \to D^{\theta}\varphi^*(t)$ in $L^2[0,1]$. Since it is a Cauchy sequence, now we demonstrate that $D^{\theta}\varphi^*(t) = D^{\theta}\varphi(t)$ for any indices m, n we possess,

$$
\begin{aligned}
\left\langle D^{\theta}\varphi^*(t) - D^{\theta}\varphi(t), \Psi_{m,n}(t)\right\rangle_{L^2[0,1]} &= \left\langle D^{\theta}\varphi^*(t), \Psi_{m,n}(t)\right\rangle_{L^2[0,1]} - \left\langle D^{\theta}\varphi(t), \Psi_{m,n}(t)\right\rangle_{L^2[0,1]} \\
&= \left\langle \sum_{\beta=1}^{2^{\eta-1}} \sum_{\delta=0}^{\mathcal{K}-1} c_{\beta,\delta}\Psi_{\beta,\delta} \sum_{\delta=\mathcal{K}}^{\infty} c_{\beta,\delta}\Psi_{\beta,\delta}, \Psi_{m,n}\right\rangle_{L^2[0,1]} - c_{m,n} \\
&= \left\langle \lim_{\mathcal{K}\to\infty} \sum_{\beta=1}^{2^{\eta-1}} \sum_{\delta=0}^{\mathcal{K}} c_{\beta,\delta}\Psi_{\beta,\delta}, \Psi_{m,n}\right\rangle_{L^2[0,1]} - c_{m,n} \\
&= \lim_{\mathcal{K}\to\infty} c_{m,n} - c_{m,n} \\
&= 0.
\end{aligned}
$$

Therefore $D^{\theta}\varphi^*(t) - D^{\theta}\varphi(t) = 0$. Hence the function $D^{\theta}\varphi_{2^{\eta-1},\mathcal{K}}(t) \to D^{\theta}\varphi(t)$ as $\mathcal{K} \to \infty$. □

Remark 5.5. *Theorem 5.3 stipulates that the error bound of the solution can be obtained if one is able to bound the second derivative of the solution over the interval [0, 1]. Theorem 5.4 indicates that the error always tends to zero in the case that a bound for the second derivative of the solution cannot be found.*

6. Numerical and graphical illustration

Example 6.1. *Consider the fractional differential equation of order $\theta \in (2,3]$ is defined as:*

$$
\begin{aligned}
D^{\theta}\varrho(t) + \varrho(t-0.3) &= -\varrho(t) + e^{-t+0.3} + \int_0^1 sin(2\pi t)dt, t \in [0,1]. \\
\varrho(t) &= e^{-t}, t < 0. \\
\varrho(0) &= -\varrho^{'}(0) = \varrho^{''}(0) = 1.
\end{aligned}
\tag{6.1}
$$

The exact solution for Eq (6.1) is $\varrho(t) = e^{-t}$, when $\theta = 3$. In Table 1, compare the values of the present method and previous methods in [29, 43]. Table 2 explores the numerical results for Eq (6.1). Figures 1 and 2 represent the exact solution, approximate solution, and absolute error of our given problem with the parameters $\eta = 2$ and $\mathcal{K} = 7$.

Table 1. Comparison between the exact solution of our present method and methods in [29, 43] with $\eta = 2$ and $\mathcal{K} = 7$.

t	Present method	Bernoulli [29]	LWM [43]
0.2	0.81873	0.8187	0.818731
0.4	0.67032	0.4067	0.406703
0.6	0.54881	0.6054	0.605488
0.8	0.44933	0.4493	0.449329
1	0.36788	0.3688	0.368794

Table 2. Numerical results for Example 6.1 with $\eta = 2$ and $\mathcal{K} = 7$.

t	Exact	Approximate solution for $\eta = 2$ and $\mathcal{K} = 7$	Absolute error
0	1	1	0
0.1	0.904840000000000	0.904794000000000	4.599999999999049e-05
0.2	0.818730000000000	0.818793000000000	6.300000000003525e-05
0.3	0.740820000000000	0.740792000000000	2.800000000002800e-05
0.4	0.670320000000000	0.670329000000000	8.999999999925734e-06
0.5	0.606530000000000	0.606529000000000	1.000000000028756e-06
0.6	0.548810000000000	0.548809000000000	1.000000000028756e-06
0.7	0.496590000000000	0.496589000000000	9.999000000000000e-07
0.8	0.449330000000000	0.449329000000000	1.000000000028756e-07
0.9	0.406570000000000	0.406569000000000	9.999999999732445e-07
1	0.367880000000000	0.367879000000000	9.999900000000000e-07

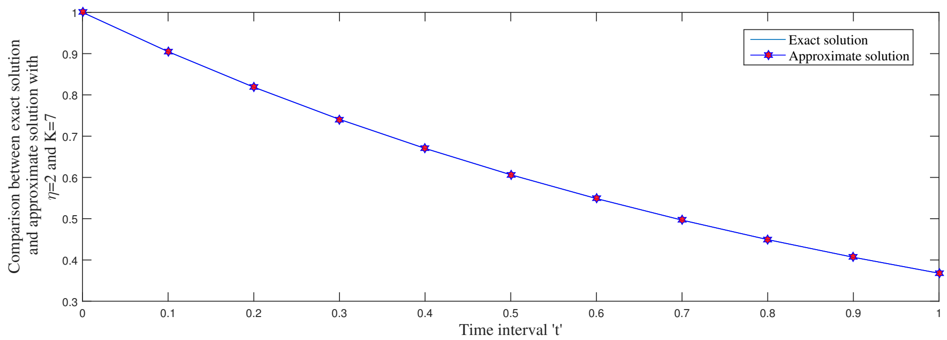

Figure 1. The illustration compares the solutions of Example 6.1.

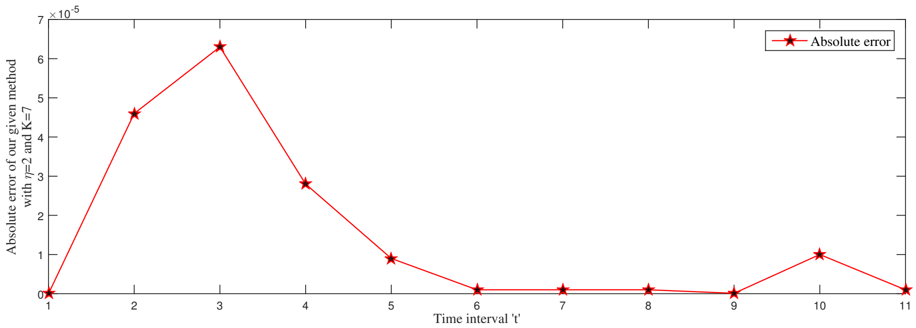

Figure 2. The graph represents the absolute error of Example 6.1.

Example 6.2. *Consider the initial value problem of the Caputo fractional derivative with higher order* (1, 2]*:*

$$\begin{aligned} D^{\theta}\varrho(t) - \frac{7}{2}\varrho(t) &= -\varrho\left(\frac{t}{4}\right) + 2,\ t \in [0, 1]. \\ \varrho(0) &= 0. \\ \varrho'(0) &= 0. \end{aligned} \tag{6.2}$$

The exact solution for Eq (6.2) is $\varrho(t) = 2t^2$, when $\theta = 2$. Table 3 explores the numerical results of Eq (6.2) using the parameter values $\eta = 1$ and $\mathcal{K} = 7$. Figures 3 and 4 represent the solution and the absolute error of our given problem with the parameters $\eta = 1$ and $\mathcal{K} = 7$.

Table 3. Numerical results for Example 6.2 with $\eta = 1$ and $\mathcal{K} = 7$.

t	Exact	Approximate solution for $\eta = 1$ and $\mathcal{K} = 7$	Absolute error
0	0	4.971002329224e-12	4.971002329224510e-12
0.1	0.020000000000000	0.019999999999950	5.000166947155549e-14
0.2	0.080000000000000	0.079999999999957	4.300726441641700e-14
0.3	0.180000000000000	0.179999999999957	4.299338662860919e-14
0.4	0.320000000000000	0.319999999999950	5.001554725936330e-14
0.5	0.500000000000000	0.499999999995029	4.971023592759138e-12
0.6	0.720000000000000	0.719999999999999	9.992007221626409e-16
0.7	0.980000000000000	0.979999999999999	9.992007221626409e-16
0.8	1.280000000000000	1.279999999999999	1.110223024625157e-15
0.9	1.620000000000000	1.619999999999999	1.110223024625157e-15
1	2.000000000000000	1.999999999999999	1.110223024625157e-15

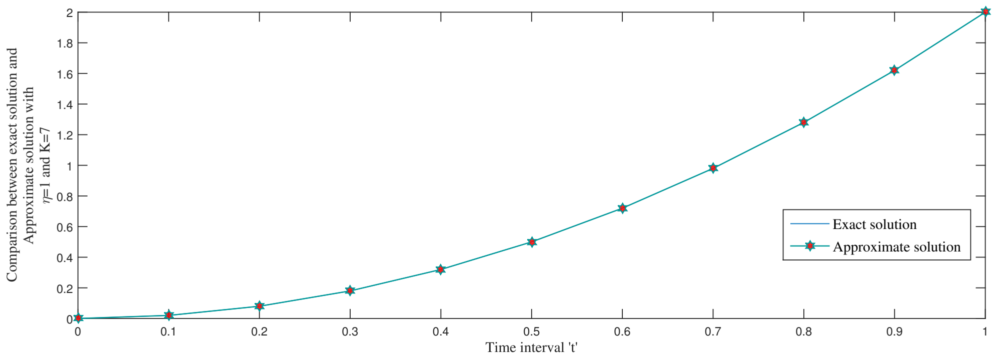

Figure 3. The illustration compares the solutions of Example 6.2.

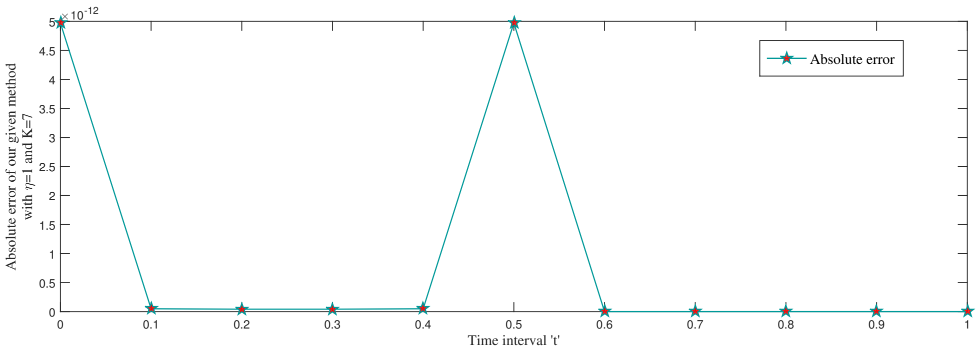

Figure 4. The graph represents the absolute error of Example 6.2.

Example 6.3 ([14]). *Consider the initial value problem of nonlinear fractional differential equation* *with higher order* $(1, 2]$:

$$\begin{aligned} D^{\theta}\varrho(t) &= 1 - 2\varrho^2\left(\frac{t}{2}\right)\ t \in [0, 1]. \\ \varrho(0) &= 1. \\ \varrho'(0) &= 0. \end{aligned} \tag{6.3}$$

The exact solution for Eq (6.3) is $\varrho(t) = cos(t)$, when $\theta = 2$. Table 4 explores numerical results for Eq (6.3) using the parameter values $\eta = 2$ and $\mathcal{K} = 8$. Figures 5 and 6 represent the solution and the absolute error of our given problem with the parameters $\eta = 2$ and $\mathcal{K} = 8$.

Table 4. Numerical results for Example 6.3 with $\eta = 2$ and $\mathcal{K} = 8$.

t	Exact	Approximate solution for $\eta = 2$ and $\mathcal{K} = 8$	Absolute error
0	1	0.999999999999750	2.50022225145585e-13
0.1	0.995000000000000	0.995009999999975	9.99999997497447e-06
0.2	0.980070000000000	0.980119999999975	4.99999999750145e-05
0.3	0.955340000000000	0.955333999999998	6.00000000194889e-06
0.4	0.921060000000000	0.921119999999975	5.99999999749690e-05
0.5	0.877580000000000	0.877579999991422	8.57802717746381e-12
0.6	0.825340000000000	0.825339999999975	2.49800180540660e-14
0.7	0.764840000000000	0.764839999999976	2.39808173319034e-14
0.8	0.696710000000000	0.696709999999976	2.40918396343659e-14
0.9	0.621610000000000	0.621609999999975	2.49800180540660e-14
1	0.540300000000000	0.540299999991422	8.57802717746381e-12

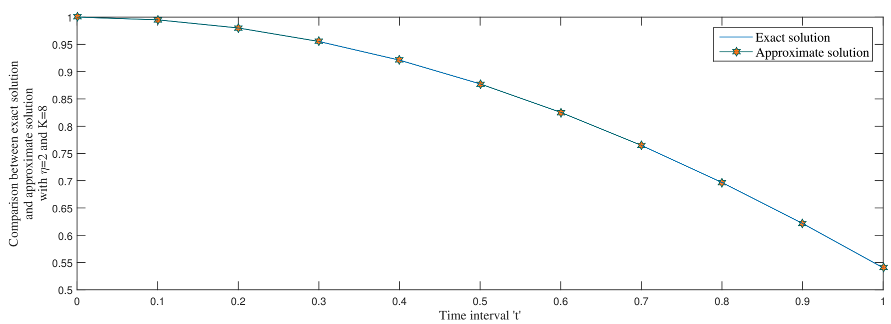

Figure 5. The illustration compares the solutions of Example 6.3.

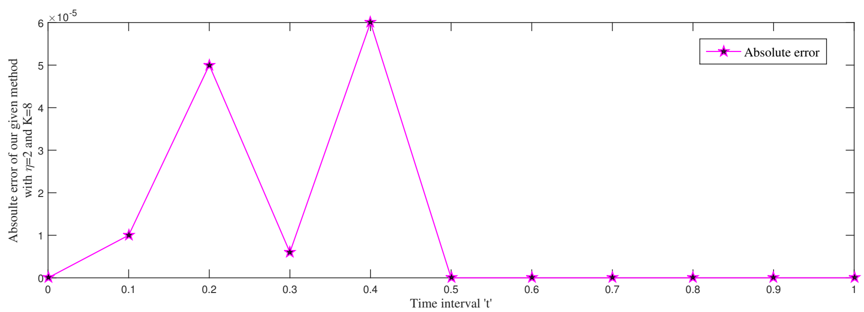

Figure 6. The graph represents the absolute error of Example 6.3.

Example 6.4 ([43]). *Consider the initial value problem of fractional differential equation with proportional delay.*

$$\begin{aligned} D^{\theta}\varrho(t) &= -\varrho(t) + \frac{1}{10}\varrho\left(\frac{4}{5}t\right) + \frac{1}{2}D^{\theta}\varrho\left(\frac{4}{5}t\right) + \left(\frac{8}{25}t - \frac{1}{2}\right)e^{-\frac{4}{5}t} + e^{-t}\ t \in [0,1]. \\ \varrho(0) &= 0. \end{aligned} \tag{6.4}$$

The exact solution for Eq (6.4) is $\varrho(t) = te^{-t}$, when $\theta = 1$. In Table 5, we provide a comparison of error estimation between our present method and methods in [17, 19, 45] with step size $h = 0.01$. Table 6 explores the numerical results for Eq (6.4) with $\eta = 2$ and $\mathcal{K} = 6$. Figures 7 and 8 represent the solution and absolute error of our given problem with the parameters $\eta = 2$ and $\mathcal{K} = 6$.

Table 5. Comparison of error estimation between our present method and other methods with step size $h = 0.01$.

t	One-leg θ-method [17]	Variational iteration method [19]	Rung-Kutta method [45]	Present method
0.1	4.65e-03	1.30e-03	8.68e-04	1.99e-05
0.3	2.57e-02	2.63e-03	1.90e-03	4.99e-05
0.5	4.43e-02	2.83e-03	2.28e-03	2.76e-12
0.7	5.31e-02	2.39e-03	2.27e-03	3.99e-15
0.9	5.35e-02	1.64e-03	2.03e-03	4.40e-14

Table 6. Numerical results for Example 6.4 with $\eta = 2$ and $\mathcal{K} = 6$.

t	Exact	Approximate solution for $\eta = 2$ and $\mathcal{K} = 6$	Absolute error
0	0	0	0
0.1	0.090480000000000	0.090499999999970	1.99999999699885e-05
0.2	0.163750000000000	0.163799999999960	4.99999999599987e-05
0.3	0.222250000000000	0.222299999999990	4.99999999900025e-05
0.4	0.268130000000000	0.268129999999999	9.99200722162641e-16
0.5	0.303270000000000	0.303269999997240	2.75995892806691e-12
0.6	0.329290000000000	0.329289999999956	4.40203429263875e-14
0.7	0.347610000000000	0.347609999999996	3.99680288865056e-15
0.8	0.359460000000000	0.359459999999996	3.99680288865056e-15
0.9	0.365910000000000	0.365909999999956	4.40203429263875e-14
1	0.367880000000000	0.367879999997240	2.75995892806691e-12

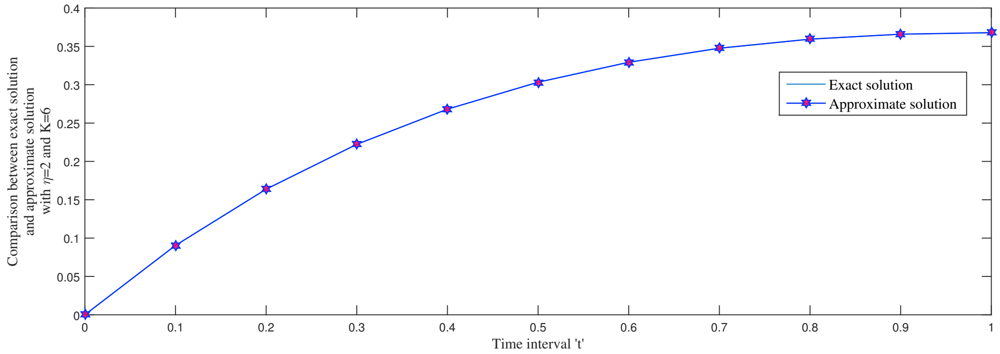

Figure 7. The illustration compares the solutions of Example 6.4.

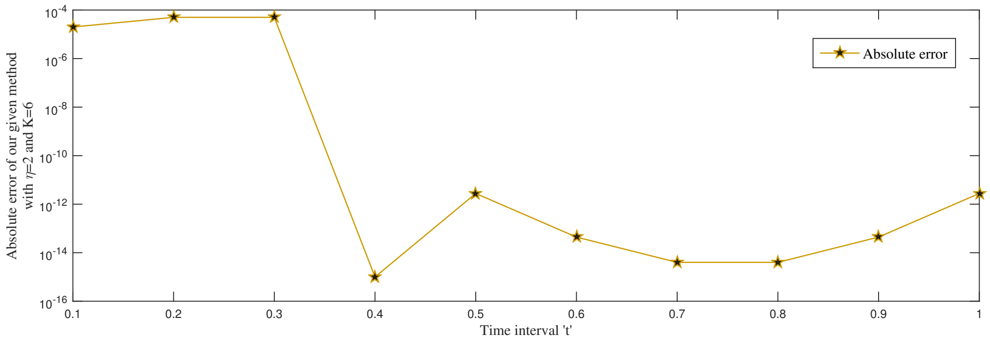

Figure 8. The graph represents the absolute error of Example 6.4.

7. Results and discussion

In this section, we present the results of applying the Legendre wavelet method to solve the neutral fractional functional integro-differential Eq (1.1). We discuss the accuracy, efficiency, and applicability of the proposed technique, along with comparisons with existing methods.

(1) We investigated the performance of the proposed method in several dynamical systems under different initial conditions and analyzed how these errors change with given parameters (see Examples 6.1–6.4). In the suggested method, the parameter scaling 'η' and translation '$\mathcal{K}$' were kept constant primarily for simplicity and computational efficiency. Legendre wavelets are constructed from Legendre polynomials, which have orthogonal properties. By keeping the translation and scaling parameters constant, the resulting wavelet basis functions remain orthogonal to each other.

(2) Additionally, the computational efficiency of the proposed technique was compared with other numerical methods such as the one-leg-θ method [17], the variational iteration method [19], the Bernoulli method [29], the Legendre wavelet method [43], and the Rung-Kutta method [45].

(3) We evaluated the efficiency of the Legendre wavelet method in terms of error estimations. We also examined the advantages and limitations of the proposed approach relative to other methods, considering factors such as exact solutions, approximate solutions, and absolute errors.

(4) Table 1 compares our proposed method's exact solution with other numerical techniques. The

output of the present method is closely related to other numerical techniques such as the Bernoulli method [29] and the Legendre wavelet method [43]. Tables 2–4 and 6 showed the absolute error of our proposed technique with different parameters using Legendre polynomials, Theorem 5.3, and MATLAB software R2014. The corresponding MATLAB code for our numerical computation is given in Appendices Al and A2. This approach, based on the numerical and graphical findings, has the following characteristics: The error estimation is very low compared with other techniques (see Table 5). Our analytical and numerical solutions are relatively identical when using the present method, which also reduces the complexity of solving the given system. According to the above observation, the suggested work has demonstrated great effectiveness and reliability.

(5) This comprehensive analysis of the proposed technique addresses its accuracy, efficiency, applicability, and comparative performance relative to existing methods. By using this method, readers will be able to comprehend the contributions and implications of the research findings on a more profound level.

8. Conclusions

In this manuscript, we proposed a new technique to solve the neutral fractional functional integro-differential equation in the sense of the Caputo fractional derivative. The numerical computation is obtained from our proposed technique based on the Legendre polynomial and the Legendre wavelet. The convergence analysis is obtained from Bessel's inequality. Our findings highlight several key insights into the performance and applicability of the proposed technique. First, we have shown that the Legendre wavelet method offers a promising framework for accurately approximating solutions to neutral fractional functional integro-differential equations. Furthermore, our computational experiments have provided valuable insights into the behavior and performance of the proposed technique. Based on our observations, the method is effective in handling large-scale and computationally demanding problems since it has desirable convergence features and computational efficiency. Our study contributes to the existing literature by introducing a novel technique for solving numerical computations of a given dynamical system based on the Legendre wavelet method. By addressing important theoretical and practical challenges, our findings enhance our understanding of fractional calculus and computational mathematics, paving the way for further research and applications in this exciting and rapidly evolving field. Interpretation and validation of numerical results obtained using this proposed work pose challenges. It would be difficult to verify the accuracy of the solutions or to assess their physical relevance, especially in the absence of analytical solutions or experimental data for comparison. The applicability of this proposed method is limited to certain types of problems or equations. It may not be suitable for all classes of neutral fractional functional integro-differential equations, particularly those with highly irregular or discontinuous solutions. Future research efforts are needed to address the identified limitations and improve the robustness and applicability of the technique.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. A. Akgül, S. H. A. Khoshnaw, Application of fractional derivative on non-linear biochemical reaction models, *Int. J. Intell. Networks*, **1** (2020), 52–58. https://doi.org/10.1016/j.ijin.2020.05.001
2. İ. Avcı, A. Hussain, T. Kanwal, Investigating the impact of memory effects on computer virus population dynamics: a fractal-fractional approach with numerical analysis, *Chaos Soliton. Fract.*, **174** (2023), 113845. https://doi.org/10.1016/j.chaos.2023.113845
3. R. L. Bagley, P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, *J. Rheol.*, **27** (1983), 201–210. https://doi.org/10.1122/1.549724
4. R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, 2000. https://doi.org/10.1142/3779
5. C. Ionescu, A. Lopes, D. Copot, J. A. T. Machado, J. H. T. Bates, The role of fractional calculus in modeling biological phenomena: a review, *Commun. Nonlinear Sci. Numer. Simul.*, **51** (2017), 141–159. https://doi.org/10.1016/j.cnsns.2017.04.001
6. R. L. Magin, Fractional calculus in bioengineering, *Crit. Rev. Biomed. Eng.*, **32** (2006), 1–104. https://doi.org/10.1615/critrevbiomedeng.v32.i1.10
7. N. S. Papageorgiou, J. Zhang, W. Zhang, Solutions with sign information for noncoercive double phase equations, *J. Geom. Anal.*, **34** (2024), 14. https://doi.org/10.1007/s12220-023-01463-y
8. I. Podlubny, *Fractional differential equation: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering, Vol. 340, 1999.
9. E. Babolian, A. S. Shamloo, Numerical solution of Volterra integral and integro-differential equations of convolution type by using operational matrices of piecewise constant orthogonal functions, *J. Comput. Appl. Math.*, **214** (2008), 495–508. https://doi.org/10.1016/j.cam.2007.03.007
10. A. H. Bhrawy, M. M. Tharwat, A. Yildirim, A new formula for fractional integrals of Chebyshev polynomials: application for solving multi-term fractional differential equations, *Appl. Math. Model.*, **37** (2013), 4245–4252. https://doi.org/10.1016/j.apm.2012.08.022
11. D. Chouchan, V. Mishra, H. M. Srivastava, Bernoulli wavelet method for numerical solution of anomalous infiltration and diffusion modeling by nonlinear fractional differential equations of variable order, *Results Appl. Math.*, **10** (2021), 100146. https://doi.org/10.1016/j.rinam.2021.100146

12. V. Daftardar-Gejji, H. Jafari, Solving a multi-order fractional differential equation using Adomian decomposition, *Appl. Math. Comput.*, **189** (2007), 541–548. https://doi.org/10.1016/j.amc.2006.11.129
13. M. Dehghan, J. Manafian, A. Saadatmandi, Solving nonlinear fractional partial differential equations using the homotopy analysis method, *Numer. Methods Partial Differ. Equations*, **26** (2010), 448–479. https://doi.org/10.1002/num.20460
14. V. S. Ert*ü*rk, S. Momani, Solving systems of fractional differential equations using differential transform method, *J. Comput. Appl. Math.*, **215** (2008), 142–151. https://doi.org/10.1016/j.cam.2007.03.029
15. M. Meerschaert, C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations, *Appl. Numer. Math.*, **56** (2006), 80–90. https://doi.org/10.1016/j.apnum.2005.02.008
16. Z. M. Odibat, S. Momani, Applications of variational iteration method to nonlinear differential equations of fractional order, *Int. J. Nonlinear Sci. Numer. Simul.*, **7** (2006), 27–34. https://doi.org/10.1515/IJNSNS.2006.7.1.27
17. W. S. Wang, S. F. Li, On the one-leg θ-methods for solving nonlinear neutral functional differential equations, *Appl. Math. Comput.*, **193** (2007), 285–301. https://doi.org/10.1016/j.amc.2007.03.064
18. M. Bohner, O. Tunç, C. Tunç, Qualitative analysis of Caputo fractional integro-differential equations with constant delays, *Comp. Appl. Math.*, **40** (2021), 214. https://doi.org/10.1007/s40314-021-01595-3
19. X. Chen, L. Wang, The variational iteration method for solving a neutral functional-differential equations with proportional delays, *Comput. Appl. Math.*, **59** (2008), 2696–2702. https://doi.org/10.1016/j.camwa.2010.01.037
20. M. S. Hafshejani, S. K. Vanani, J. S. Hafshejani, Numerical solution of delay differential equations using Legendre wavelet method, *World Appl. Sci. J.*, **13** (2011), 27–33.
21. O. Tunç, C. Tunç, Solution estimates to Caputo proportional fractional derivative delay integro-differential equations, *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.*, **117** (2023), 12. https://doi.org/10.1007/s13398-022-01345-y
22. J. P. Antoine, Wavelet transforms and their applications, *Phys. Today*, **56** (2003), 68. https://doi.org/10.1063/1.1580056
23. S. Etemad, A. Shikongo, K. M. Owolabi, B. Tellab, İ. Avcı, S. Rezapour, et al., A new fractal-fractional version of giving up smoking model: application of Lagrangian piece-wise interpolation along with asymptotical stability, *Mathematics*, **10** (2022), 4369. https://doi.org/10.3390/math10224369
24. T. Kanwal, A. Hussain, İ. Avcı, S. Etemad, S. Rezapour, D. F. M. Torres, Dynamics of a model of polluted lakes via fractal-fractional operators with two different numerical algorithms, *Chaos Soliton. Fract.*, **181** (2024), 114653. https://doi.org/10.1016/j.chaos.2024.114653
25. Q. Li, W. Zou, Normalized ground states for Sobolev critical nonlinear Schr*ö*dinger equation in the L^2-supercritical case, *Discrete Contin. Dyn. Syst.*, **44** (2024), 205–227. https://doi.org/10.3934/dcds.2023101
26. Q. Li, V. D. R*ă*dulescu, W. Zhang, Normalized ground states for the Sobolev critical Schr*ö*dinger equation with at least mass critical growth, *Nonlinearity*, **37** (2024), 025018. https://doi.org/10.1088/1361-6544/ad1b8b

27. Z. Meng, L. Wang, H. Li, W. Zhang, Legendre wavelets method for solving fractional integro-differential equations, *Int. J. Comput. Math.*, **92** (2014), 1275–1291. https://doi.org/10.1080/00207160.2014.932909
28. F. Mohammadi, M. M. Hossein, A new Legendre wavelet operational matrix of derivative and its applications in solving the singular ordinary differential equations, *J. Franklin Inst.*, **348** (2011), 1787–1796. https://doi.org/10.1016/j.jfranklin.2011.04.017
29. P. Rahimkhani, Y. Ordokhani, E. Babolian, A new operational matrix based on Bernoulli wavelets for solving fractional delay differential equations, *Numer. Algor.*, **74** (2017), 223–245. https://doi.org/10.1007/s11075-016-0146-3
30. M. ur Rehman, R. Ali Khan, The Legendre wavelet method for solving fractional differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **16** (2011), 4163–4173. https://doi.org/10.1016/j.cnsns.2011.01.014
31. M. Yi, L. Wang, J. Huang, Legendre wavelets method for the numerical solution of fractional integro-differential equations with weakly singular kernel, *Appl. Math. Model.*, **40** (2016), 3422–3437. https://doi.org/10.1016/j.apm.2015.10.009
32. D. Abbaszadeh, M. Tavassoli Kajani, M. Momeni, M. Zahraei, M. Maleki, Solving fractional Fredholm integro-differential equations using Legendre wavelets, *Appl. Numer. Math.*, **166** (2021), 168–185. https://doi.org/10.1016/j.apnum.2021.04.008
33. I. A. Bhat, L. N. Mishra, V. N. Mishra, C. Tunç, O. Tunç, Precision and efficiency of an interpolation approach to weakly singular integral equations, *Int. J. Numer. Methods Heat Fluid Flow*, **34** (2024), 1479–1499. https://doi.org/10.1108/HFF-09-2023-0553
34. R. Hilfer, Y. Luchko, Z. Tomovski, Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives, *Fract. Calculus Appl. Anal.*, **12** (2009), 299–318.
35. H. Jafari, S. A. Yousefi, M. A. Firoozjaee, S. Momani, C. M. Khalique, Applications of Legendre wavelets for solving fractional differential equations, *Comput. Math. Appl.*, **62** (2011), 1038–1045. https://doi.org/10.1016/j.camwa.2011.04.024
36. K. Muthuselvan, B. Sundaravadivoo, Analyze existence, uniqueness and controllability of impulsive fractional functional differential equations, *Adv. Stud.: Euro-Tbilisi Math. J.*, **10** (2022), 171–190.
37. K. Muthuselvan, B. Sundaravadivoo, S. Alsaeed, K. S. Nisar, New interpretation of topological degree method of Hilfer fractional neutral functional integro-differential equation with nonlocal condition, *AIMS Math.*, **8** (2023), 17154–17170. https://doi.org/10.3934/math.2023876
38. K. Muthuselvan, B. Sundaravadivoo, K. S. Nisar, S. R. Munjam, W. Albalawi, A. H. Abdel-Aty, Results on nonlocal controllability for impulsive fractional functional integro-differential equations via degree theory, *Results Phys.*, **51** (2023), 106698. https://doi.org/10.1016/j.rinp.2023.106698
39. P. Rahimkhani, Y. Ordokhani, E. Babolian, Numerical solution of fractional pantograph differential equations by using generalized fractional-order Bernoulli wavelet, *J. Comput. Appl. Math.*, **309** (2017), 493–510. https://doi.org/10.1016/j.cam.2016.06.005
40. Y. Zhou, J. R. Wang, L. Zhang, *Basic theory of fractional differential equations*, 2 Eds., World Scientific, 2016. https://doi.org/10.1142/10238
41. K. Yoshida, *Functional analysis*, Berlin: Springer-Verlag, 1966. https://doi.org/10.1007/978-3-642-61859-8

42. N. H. Shah, *Ordinary and partial differential equations: theory and applications*, PHI Learning Private Limited, 2015.
43. B. Yuttanan, M. Razzaghi, T. N. Vo, Legendre wavelet method for fractional delay differential equations, *Appl. Numer. Math.*, **168** (2021), 127–142. https://doi.org/10.1016/j.apnum.2021.05.024
44. M. Razzaghi, S. Yousefi, Legendre wavelets direct method for variational problems, *Math. Comput. Simul.*, **53** (2000), 185–192. https://doi.org/10.1016/S0378-4754(00)00170-1
45. A. Bellen, M. Zennaro, *Numerical methods for delay differential equations*, Numerical Mathematics and Scientific Computation, Oxford: Oxford Academic, 2013. https://doi.org/10.1093/acprof:oso/9780198506546.001.0001

Appendix

This section provides the Matlab code for our numerical computation.

A1: calculating exact solution

```
clc;
clear;
close all;
% The inputs

h=0.001;
t(1)=0;
tfinal=1;
%Initial conditions

y(0)=0;
y'(0)=0;
nu=2;
tfinal=2;
t=t(1):h:tfinal;

N=ceil((tfinal-t(1)/h));
for  t=-0:0.1:1

v=1;

%Exact Solution

%Example 6.1:
Exact=exp(-t)
```

```
%Example 6.2:
Exact=2*t.^2

%example 6.3:
Exact=cos(t)

%Example 6.4:
Exact=t*exp(-t)
V=Exact;

fprintf('\n  %1.5f', V);
end
```

A2: calculating error estimation and approximate solution

```
function [y] =Legendre_wavelets(n,m,t)
%Usage: LEGENDRE_WAVELETS( n , m , t )

% Arguments( input ) :
n : Dilation parameter of the wavelet( integer positive ).

% m : Translation parmeter of the wavelet and the order of the
legendre polynomial( integer non-negative ).

% t : normalized time( real positive).

% Arguments( Output ):
y : Legendre wavelet.

if (n>0) && (m>=0)
%Checking the range of n and m.

for i=0:log2 (n)
if  ((2^i) <= n) && (n <= (2^(i+1)))
k=i+2;

break
end

k=i+2;
% Finding k based on n(n = 1,2,... 2^k-1).
end
```

```
nhat=(2*n)-1;
% Defining nhat based on 'n'.
j0=((nhat - 1)/ (2^k));
j1=((nhat + 1)/(2^k));
% Function defined limits j0 and j1.

for t=0:0.1:1
L=39.0625
L^(1/2)= 6.2500

C_{n*m}=16*L/2^((5*k+1)/2))*(sqrt(4*m^2+4*m+9)/(2*m+1)*((2*m+3)^1/2)
*((2*m+5)^1/2)*((2*m-1)^1/2)*((2*m-3)^1/2))

B=(m,(((2^k).*t)-nhat))).*(heaviside(t-j0)).
 *(heaviside(j1-t));

chi_{n*m}=abs((sqrt(m+(1/2)))*(2^(k/2))
*Legendre_recursive_formula (B);

% Defining the wavelet
Error=C_{n*m}
F =chi_{n*m}
end;

plot(t, F, '*b', 'LineWidth',0.1)
else
disp('Error as the value of n cannot be less than or equal to zero
or value of m cannot be less than zero');
% Returning error for invalid values of 'n' and 'm'
end
```