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Research article

Effectiveness of matrix measure in finding periodic solutions for nonlinear systems of differential and integro-differential equations with delays

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Abstract: In this manuscript, under the matrix measure and some sufficient conditions, we will overcame all difficulties and challenges related to the fundamental matrix for a generalized nonlinear neutral functional differential equations in matrix form with multiple delays. The periodicity of solutions, as well as the uniqueness under the considered conditions has been proved employing the fixed point theory. Our approach expanded and generalized certain previously published findings for example, we studied the uniqueness of a solution that was absent in some literature. Moreover, an example was given to confirm the main results.

Keywords: Arzela-Ascoli's theorem; fixed point theorem; matrix measure; system of differential equations; delay; periodic solutions
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1. Introduction

Many researchers have studied the matrix form of differential systems due to their good applications in biological sciences, engineering, and economics, as well as other branches of mathematics, where the study of stability and the existence of periodic solutions was addressed (see [1–11]).

By employing Krasnoselskii's fixed point theorem, the authors in [12] studied the following two functional neutral differential equations:

$$(x(\iota) - cx(\iota - \gamma(\iota)))' = -a(\iota)x(\iota) + p(\iota, x(\iota - \gamma(\iota))) \tag{1.1}$$

and

$$\frac{d}{d\iota}\left[x(\iota) - c\int_{-\infty}^{0} G(s)x(\iota + s)\,ds\right] = -a(\iota)x(\iota) + \int_{-\infty}^{0} G(s)p(\iota, x(\iota + s))\,ds, \tag{1.2}$$

where the positivity and periodicity of solutions were established, such that x: $\mathbb{R} \to \mathbb{R}$; $a(\iota) \in C(\mathbb{R}, (0, \infty))$; $p \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$; $\gamma(\iota) \in C(\mathbb{R}, \mathbb{R})$ and $a(\iota)$, $b(\iota)$, $\gamma(\iota)$, $p(\iota, x)$ are Υ-periodic functions, $|c| < 1$ and $\Upsilon > 0$ are constants, $G(s) \in C((-\infty, 0], [0, \infty))$, and

$$\int_{-\infty}^{0} G(s)\,ds = 1.$$

However, functional differential Eqs (1.1) and (1.2) appear frequently in applications as models of equations in many mathematical population and ecological models, the models of hematopoiesis (see [13, 14]), Nicholson's blowflies models (see [15, 16]), and blood cell production (see [17]).

In [2], the authors considered the following system:

$$(x(\iota) - cx(\iota - \gamma))' = A(\iota, x(\iota))x(\iota) + p(\iota, x(\iota - \sigma_1(\iota)), ..., x(\iota - \sigma_m(\iota))), \tag{1.3}$$

under the condition on the matrix measure, with periodic coefficients. However, in [9], the author used integrable dichotomy to show that the system (1.3) has periodic solutions where the matrix A depends on ι only.

Motivated by these excellent works, the purpose of this article is to generalize and improve the previous works to become totally nonlinear or in system form by considering the two nonlinear differential systems with multiple delays:

$$(x(\iota) - g(\iota, x(\iota - \gamma(\iota))))' = A(\iota)x(\iota) + p(\iota, x(\iota - \sigma_1(\iota)), ..., x(\iota - \sigma_m(\iota))) \tag{1.4}$$

and

$$\begin{aligned}&\left(x(\iota) - \int_{-\infty}^{0} G(s)g(\iota, x(\iota + \gamma(s)))\,ds\right)' \\ &= A(\iota)x(\iota) + \int_{-\infty}^{0} G(s)p(\iota, x(\iota + \sigma_1(s)), ..., x(\iota + \sigma_m(s)))\,ds,\end{aligned} \tag{1.5}$$

in which x: $\mathbb{R} \to \mathbb{R}^n$, $\gamma(\iota), \sigma_i(\iota), i = 1, ..., m$, are real continuous Υ-periodic functions on $\mathbb{R}$, $\Upsilon > 0$. $A(\iota)$ and $G(s)$ are $n \times n$ real matrices with the continuous Υ-periodic function defined on $\mathbb{R}$ and $(-\infty, 0]$, respectively, with

$$\int_{-\infty}^{0} G(s)\,ds = I.$$

The vector functions $g(\iota, v)$ and $p(\iota, v_1, ..., v_m)$ are real continuous functions defined on $\mathbb{R} \times \mathbb{R}^n$ and $\mathbb{R} \times (\mathbb{R}^n)^m$, respectively, such that

$$p(\iota + \Upsilon, v_1, v_2, ..., v_m) = p(\iota, v_1, v_2, ..., v_m)$$

and

$$g(\iota+\Upsilon,v)=g(\iota,v).$$

The authors in [10] considered the systems (1.4) and (1.5) as a generalization of the work [9] and proved the existence and the uniqueness under the integrable dichotomy condition. So, to the best of our knowledge, no earlier studies have investigated the systems (1.4) and (1.5) by using matrix measure. On the other hand, among the specific advantages and potential impact of our work over existing results in the literature, for example, we can deduce the uniqueness of the solution of (1.1) in [2].

To generalize them to become totally nonlinear and to study the uniqueness of solutions, which has not been studied previously, some other research was also studied in system form.

This article is divided into five parts. Section 2 provides some background information that we will employ to show the existence and uniqueness of solutions of (1.4) and (1.5). Section 3 is concerned with establishing some criteria for the existence and uniqueness of periodic solutions of systems (1.4) and (1.5), respectively. Section 4 includes two examples to demonstrate our findings, followed by a conclusion.

2. Preliminaries

For the sake of convenience, we list some results, definitions of matrix measure, and linear systems, which will be crucial in the proofs of our results. Let the system

$$x'(\iota)=A(\iota)x(\iota), \tag{2.1}$$

in which $A(\iota)$ is a continuous $n\times n$ matrix function. Denote by $\mathcal{G}(\iota,\iota_0)$ the fundamental matrix solution of system (2.1) with $\mathcal{G}(\iota_0,\iota_0)=I$. Recall that

$$\mathcal{G}(\iota,\iota_0)\mathcal{G}(\iota_0,\zeta)=\mathcal{G}(\iota,\zeta),\quad \iota,\iota_0,\zeta\in\mathbb{R},$$

and

$$\mathcal{G}^{-1}(\iota,\zeta)=\mathcal{G}(\zeta,\iota),\quad \iota,\zeta\in\mathbb{R}.$$

Let $|\cdot|_1$ be the 1-norm for the real Euclidean space $\mathbb{R}^n$ and $\|A\|$ the induced matrix norm of A corresponding to the vector norm $|\cdot|$. So, for

$$x=(x_1,x_2,...,x_3)\in\mathbb{R}^n,\quad A=\left(a_{ij}\right)_{n\times n}\in\mathbb{R}^{n\times n},$$

then

$$|x|_1=\sum_{i=1}^{n}|x_i|,\qquad \|A\|=\max_{1\le j\le n}\sum_{i=1}^{n}\left|a_{ij}\right|.$$

The matrix measure of the matrix A is the function (see [18] for more details)

$$\begin{aligned}\mu(A)&=\lim_{\epsilon\to0^+}\frac{\|I+\epsilon A\|-1}{\epsilon}\\&=\max_{1\le j\le n}\left\{a_{jj}+\sum_{i=1,i\neq j}^{n}\left|a_{ij}\right|\right\}.\end{aligned}$$

Lemma 2.1. *[18] Let $x(\iota)$ be a solution of system (2.1), then*

$$|x(\iota_0)|_1 \, e^{\int_{\iota_0}^{\iota} -\mu(-A(\zeta))d\zeta} \leq |x|_1 \leq |x(\iota_0)|_1 \, e^{\int_{\iota_0}^{\iota} \mu(A(\zeta))d\zeta},$$

for $\iota \geq \iota_0$.

Lemma 2.2. *[2] The fundamental matrix of (2.1) satisfies*

$$\|\mathcal{G}(\iota,\zeta)\| \leq e^{\int_{\zeta}^{\iota} \mu(A(s))ds}, \quad \iota \geq \zeta.$$

Lemma 2.3. *[2] If*

$$e^{\int_0^{\Upsilon} \mu(A(\zeta))d\zeta} < 1,$$

then the linear system (2.1) does not have any nontrivial Υ-periodic solution.

Lemma 2.4. *[19] If the linear system (2.1) does not have any nontrivial Υ-periodic solution, then for any Υ-periodic continuous function $p(\iota)$, the nonhomogeneous system*

$$x'(\iota) = A(\iota)\,x(\iota) + p(\iota)$$

has a unique Υ-periodic solution $x(\iota)$ determined by

$$x(\iota) = \mathcal{G}(\iota,\iota_0)\,x(\iota_0) + \int_{\iota_0}^{\iota} \mathcal{G}(\iota,\zeta)\,p(\zeta)\,d\zeta, \quad \iota \in \mathbb{R}.$$

Now, for our study we consider the following lemma.

Lemma 2.5. *Assume that*

$$\theta := e^{\int_0^{\Upsilon} \mu(A(\zeta))d\zeta} < 1 \tag{2.2}$$

holds, then the solutions of the systems (1.4) and (1.5) are equivalent to

$$\begin{aligned} x(\iota) =& g(\iota, x(\iota-\gamma(\iota))) + (I-\mathcal{G}(\iota+\Upsilon,\iota))^{-1} \\ &\times \int_{\iota}^{\iota+\Upsilon} \mathcal{G}(\iota+\Upsilon,\zeta)\,A(\zeta)\,g(\zeta, x(\zeta-\gamma(\zeta)))\,d\zeta + (I-\mathcal{G}(\iota+\Upsilon,\iota))^{-1} \\ &\times \int_{\iota}^{\iota+\Upsilon} \mathcal{G}(\iota+\Upsilon,\zeta)\,p(\zeta, x(\zeta-\sigma_1(\zeta)), ..., x(\zeta-\sigma_m(\zeta)))\,d\zeta \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} x(\iota) =& \int_{-\infty}^{0} G(s)\,g(\iota, x(\iota+\gamma(s)))\,ds + (I-\mathcal{G}(\iota+\Upsilon,\iota))^{-1} \\ &\times \int_{\iota}^{\iota+\Upsilon} \mathcal{G}(\iota+\Upsilon,\zeta)\,A(\zeta) \int_{-\infty}^{0} G(s)\,g(\zeta, x(\zeta+\gamma(s)))\,dsd\zeta + (I-\mathcal{G}(\iota+\Upsilon,\iota))^{-1} \\ &\times \int_{\iota}^{\iota+\Upsilon} \mathcal{G}(\iota+\Upsilon,\zeta) \int_{-\infty}^{0} G(s)\,p(\zeta, x(\zeta+\sigma_1(s)), ..., x(\zeta+\sigma_m(s)))\,dsd\zeta, \end{aligned} \tag{2.4}$$

respectively.

Proof. Let

$$y(\iota) = x(\iota) - g(\iota, x(\iota - \gamma(\iota))),$$

so, the system (1.4) can be written as

$$y'(\iota) = A(\iota)\, y(\iota) + A(\iota)\, g(\iota, x(\iota - \gamma(\iota))) + p(\iota, x(\iota - \sigma_1(\iota)), ..., x(\iota - \sigma_m(\iota))).$$

By Lemma 2.4, for $\iota \in \mathbb{R}$, we have

$$\begin{aligned} y(\iota) =& \mathcal{G}(\iota, \iota_0)\, y(\iota_0) + \int_{\iota_0}^{\iota} \mathcal{G}(\iota, \zeta)\, A(\zeta)\, g(\zeta, x(\zeta - \gamma(\zeta)))\, d\zeta \\ &+ \int_{\iota_0}^{\iota} \mathcal{G}(\iota, \zeta)\, p(\zeta, x(\zeta - \sigma_1(\zeta)), ..., x(\zeta - \sigma_m(\zeta)))\, d\zeta, \\ y(\iota) =& y(\iota + \Upsilon) \\ =& \mathcal{G}(\iota + \Upsilon, \iota_0)\, y(\iota_0) + \int_{\iota_0}^{\iota+\Upsilon} \mathcal{G}(\iota + \Upsilon, \zeta)\, A(\zeta)\, g(\zeta, x(\zeta - \gamma(\zeta)))\, d\zeta \\ &+ \int_{\iota_0}^{\iota+\Upsilon} \mathcal{G}(\iota + \Upsilon, \zeta)\, p(\zeta, x(\zeta - \sigma_1(\zeta)), ..., x(\zeta - \sigma_m(\zeta)))\, d\zeta \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}(\iota + \Upsilon, \iota)\, y(\iota) =& \mathcal{G}(\iota + \Upsilon, \iota)\, \mathcal{G}(\iota, \iota_0)\, y(\iota_0) + \mathcal{G}(\iota + \Upsilon, \iota) \int_{\iota_0}^{\iota} \mathcal{G}(\iota, \zeta)\, A(\zeta)\, g(\zeta, x(\zeta - \gamma(\zeta)))\, d\zeta \\ &+ \mathcal{G}(\iota + \Upsilon, \iota) \int_{\iota_0}^{\iota} \mathcal{G}(\iota, \zeta)\, p(\zeta, x(\zeta - \sigma_1(\zeta)), ..., x(\zeta - \sigma_m(\zeta)))\, d\zeta \\ =& \mathcal{G}(\iota + \Upsilon, \iota_0)\, y(\iota_0) + \int_{\iota_0}^{\iota} \mathcal{G}(\iota + \Upsilon, \zeta)\, A(\zeta)\, g(\zeta, x(\zeta - \gamma(\zeta)))\, d\zeta \\ &+ \int_{\iota_0}^{\iota} \mathcal{G}(\iota + \Upsilon, \zeta)\, p(\zeta, x(\zeta - \sigma_1(\zeta)), ..., x(\zeta - \sigma_m(\zeta)))\, d\zeta. \end{aligned}$$

Thus,

$$\begin{aligned} (I - \mathcal{G}(\iota + \Upsilon, \iota))\, y(\iota) =& \int_{\iota}^{\iota+\Upsilon} \mathcal{G}(\iota + \Upsilon, \zeta)\, A(\zeta)\, g(\zeta, x(\zeta - \gamma(\zeta)))\, d\zeta \\ &+ \int_{\iota}^{\iota+\Upsilon} \mathcal{G}(\iota + \Upsilon, \zeta)\, p(\zeta, x(\zeta - \sigma_1(\zeta)), ..., x(\zeta - \sigma_m(\zeta)))\, d\zeta. \end{aligned}$$

Furthermore, since (2.2) holds, then $(I - \mathcal{G}(\iota + \Upsilon, \iota))^{-1}$ exists for every $\iota \in \mathbb{R}$. Therefore, we have (2.3).
By the same way, we can show that (2.4) holds. □

In our study, the proofs of the existence and uniqueness of the periodic solutions to the systems (1.4) and (1.5) utilize the fixed point theorems. So, we present them below (see [20, 21]).

Theorem 2.1. ***(Banach)*** *Assume that* (S, ρ) *is a complete metric space and* $\Phi\colon S \to S$. *If there is a constant* $\tau < 1$ *such that for* $a, b \in S$,

$$\rho(\Phi a, \Phi b) \le \tau \rho(a, b),$$

then there is one, and only one, point $x \in S$ *with* $\Phi x = x$.

Theorem 2.2. ***(Krasnoselskii)*** *Let Π be a nonempty, convex, closed, bounded subset of a Banach space S. Assume that Φ_1 and Φ_2 map Π into S such that:*
(i) Φ_1 is a contraction mapping on Π;
(ii) Φ_2 is completely continuous on Π;
(iii) $a, b \in \Pi$, implies $\Phi_1 a + \Phi_2 b \in \Pi$.
Thus, there exists $x \in \Pi$ with $x = \Phi_1 x + \Phi_2 x$.

Now, we state our sufficient conditions. So, let $BC(\mathbb{R}, \mathbb{R}^n)$ be the space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$ and assume $E > 0$ is a constant. Denote by

$$\|v\| = \max_{\iota\in[0,\Upsilon]} |v(t)|_1,$$

and set

$$\Pi = \{v \in BC(\mathbb{R}, \mathbb{R}^n) : \|v\| \le E \text{ and } v(\iota + \Upsilon) = v(\iota) \text{ for all } \iota \in \mathbb{R}\}.$$

Clearly, the set Π is a bounded, nonempty, closed, and convex subset of $BC(\mathbb{R}, \mathbb{R}^n)$.

Assume that, for $v, w \in \Pi$, there exists $L_1 \in (0, 1)$ such that

$$|g(\iota, v) - g(\iota, w)|_1 \le L_1 |v - w|_1, \quad \text{for all } \iota \in \mathbb{R}, \tag{2.5}$$

and for $v_1, v_2, ...v_m, w_1, w_2, ..., w_m \in \Pi$, there exists $L_2 > 0$ such that

$$|p(\iota, v_1, v_2, ..., v_m) - p(\iota, w_1, w_2, ..., w_m)|_1 \le L_2 (|v_1 - w_1|_1 + ... + |v_m - w_m|_1), \quad \text{for all } \iota \in \mathbb{R}.$$

Denote

$$\sup_{\iota\in[0,\Upsilon]} |g(\iota, 0)|_1 = \alpha, \quad \sup_{\iota\in[0,\Upsilon]} |p(\iota, 0, ..., 0)|_1 = \beta, \quad \sup_{\iota\in[0,\Upsilon]} \|A(\iota)\| = \lambda,$$

and we assume there exists a Υ-periodic continuous function $\kappa(\iota)$ such that for $\iota \in [0, \Upsilon]$, $\mu(A(\iota)) \le \kappa(\iota)$ satisfies

$$\sup_{\zeta\le s\le\iota\in[0,\Upsilon]} \int_\iota^{\iota+\Upsilon} e^{\int_\zeta^{\iota+\Upsilon} \kappa(s)ds} d\zeta = \mu_0 \tag{2.6}$$

and

$$\frac{\lambda(L_1 E + \alpha) + L_2 m E + \beta}{1 - \theta} \mu_0 \le E. \tag{2.7}$$

3. Main results

We start this section by studying the periodicity of the system (1.4). For $v \in BC(\mathbb{R}, \mathbb{R}^n)$, we define the operators Φ_1 and Φ_2 by

$$(\Phi_1 v)(\iota) = g(\iota, v(\iota - \gamma(\iota))) \tag{3.1}$$

and

$$\begin{aligned}(\Phi_2 v)(\iota) = &(I - \mathcal{G}(\iota + \Upsilon, \iota))^{-1} \int_\iota^{\iota+\Upsilon} \mathcal{G}(\iota + \Upsilon, \zeta) A(\zeta) g(\zeta, v(\zeta - \gamma(\zeta))) d\zeta \\ &+ (I - \mathcal{G}(\iota + \Upsilon, \iota))^{-1} \int_\iota^{\iota+\Upsilon} \mathcal{G}(\iota + \Upsilon, \zeta) p(\zeta, v(\zeta - \sigma_1(\zeta)), ..., v(\zeta - \sigma_m(\zeta))) d\zeta.\end{aligned} \tag{3.2}$$

Note that if the operator $\Phi_1 + \Phi_2$ has a fixed point, then this fixed point is a periodic solution of (1.4).

Lemma 3.1. *If (2.2), (2.5) and (2.6) hold, then the operators Φ_1 and Φ_2 defined by (3.1) and (3.2), respectively, from Π turn into $BC(\mathbb{R},\mathbb{R}^n)$, that is, Φ_1,Φ_2: $\Pi \to BC(\mathbb{R},\mathbb{R}^n)$.*

Proof. Let $v \in \Pi$,. By (2.5), we have

$$\begin{aligned}|(\Phi_1 v)(\iota)|_1 &= |g(\iota, v(\iota-\gamma(\iota)))|_1\\ &\le L_1|v(\iota-\gamma(\iota))|_1 + |g(\iota,0)|_1\\ &\le L_1\|v\| + \sup_{\iota\in[0,\Upsilon]}|g(\iota,0)|_1\\ &\le L_1E+\alpha.\end{aligned}$$

Second, for $v \in \Pi$, by (2.5) and (2.6), we get

$$\begin{aligned}|(\Phi_2 v)(\iota)|_1 \le{}& \left\|(I-\mathcal{G}(\iota+\Upsilon,\iota))^{-1}\right\| \int_\iota^{\iota+\Upsilon} \|\mathcal{G}(\iota+\Upsilon,\zeta)\|\,\|A(\zeta)\|\,|g(\zeta, v(\zeta-\gamma(\zeta)))|_1\,d\zeta\\ &+\left\|(I-\mathcal{G}(\iota+\Upsilon,\iota))^{-1}\right\| \int_\iota^{\iota+\Upsilon} \|\mathcal{G}(\iota+\Upsilon,\zeta)\|\,|p(\zeta, v(\zeta-\sigma_1(\zeta)),\dots,v(\zeta-\sigma_m(\zeta)))|_1\,d\zeta.\end{aligned}$$

Since $\theta < 1$, then

$$\begin{aligned}\left\|(I-\mathcal{G}(\iota+\Upsilon,\iota))^{-1}\right\| &= \left\|\sum_{n=0}^{\infty}(\mathcal{G}(\iota+\Upsilon,\iota))^n\right\|\\ &\le \left\|\sum_{n=0}^{\infty}\mathcal{G}(\iota+\Upsilon,\iota)\right\|^n\\ &\le \sum_{n=0}^{\infty}\theta^n\\ &= \frac{1}{1-\theta},\end{aligned}$$

and by Lemma 2.2, we have

$$\begin{aligned}\|\mathcal{G}(\iota+\Upsilon,\zeta)\| &\le e^{\int_\zeta^{\iota+\Upsilon}\mu(A(\zeta))d\zeta}\\ &\le e^{\int_\zeta^{\iota+\Upsilon}\kappa(\zeta)d\zeta},\end{aligned}$$

so

$$\begin{aligned}|(\Phi_2 v)(\iota)| &\le \frac{\lambda(L_1E+\alpha)+L_2mE+\beta}{1-\theta}\int_\iota^{\iota+\Upsilon} e^{\int_\zeta^{\iota+\Upsilon}\kappa(s)ds}d\zeta\\ &\le \frac{\lambda(L_1E+\alpha)+L_2mE+\beta}{1-\theta}\mu_0.\end{aligned}$$

Due to Lemma 2.5, Φ_1 and Φ_2 are periodic, then Φ_1,Φ_2: $\Pi \to BC(\mathbb{R},\mathbb{R}^n)$. □

Lemma 3.2. *If (2.5) holds, then the operator Φ_1: $\Pi \to BC(\mathbb{R},\mathbb{R}^n)$ defined by (3.1) is a contraction.*

Proof. Let $v, w \in \Pi$. By (2.5), we get

$$\begin{aligned}|(\Phi_1 v)(\iota) - (\Phi_1 w)(\iota)|_1 &= |g(\iota, v(\iota-\gamma(\iota))) - g(\iota, w(\iota-\gamma(\iota)))|_1\\ &\le L_1 |v(\iota-\gamma(\iota)) - w(\iota-\gamma(\iota))|_1\\ &\le L_1 \|v-w\|,\end{aligned}$$

then

$$\|\Phi_1 v - \Phi_1 w\| \le L_1 \|v-w\|.$$

Therefore, Φ_1 is a contraction because $L_1 \in (0,1)$. □

Lemma 3.3. *If (2.2), (2.5) and (2.6) hold, then the operator* Φ_2*:* $\Pi \to BC(\mathbb{R}, \mathbb{R}^n)$ *defined by (3.2) is completely continuous.*

Proof. To show that the operator Φ_2: $\Pi \to BC(\mathbb{R}, \mathbb{R}^n)$ is completely continuous, first we must prove the continuity of Φ_2. So, for $n \in \mathbb{N}$, let $v_n \in \Pi$, such that $v_n \to v$ as $n \to \infty$, then

$$\begin{aligned}|(\Phi_2 v_n)(\iota) - (\Phi_2 v)(\iota)|_1 \le& (I - \mathcal{G}(\iota+\Upsilon,\iota))^{-1} \int_\iota^{\iota+\Upsilon} \|\mathcal{G}(\iota+\Upsilon,\zeta)\|\\ &\times \|A(\zeta)\| \, |g(\zeta, v_n(\zeta-\gamma(\zeta))) - g(\zeta, v(\zeta-\gamma(\zeta)))|_1 \, d\zeta\\ &+ (I - \mathcal{G}(\iota+\Upsilon,\iota))^{-1} \int_\iota^{\iota+\Upsilon} \|\mathcal{G}(\iota+\Upsilon,\zeta)\|\\ &\times |p(\zeta, v_n(\zeta-\sigma_1(\zeta)), ..., v_n(\zeta-\sigma_m(\zeta)))\\ &- p(\zeta, v(\zeta-\sigma_1(\zeta)), ..., v(\zeta-\sigma_m(\zeta)))|_1 \, d\zeta\\ \le& \frac{\lambda L_1 + L_2 m}{1-\theta} \mu_0 \|v_n - v\|,\end{aligned}$$

which implies

$$\lim_{n\to\infty} |(\Phi_2 v_n)(\iota) - (\Phi_2 v)(\iota)|_1 = 0,$$

hence, Φ_2 is continuous. Now, let $v_n \in \Pi$, where n is a positive integer, then we have

$$\|\Phi_2 v_n\| \le \mu_0 \frac{[\lambda(L_1 E + \alpha) + L_2 m E + \beta]}{1-\theta}.$$

Second

$$\begin{aligned}(\Phi_2 v_n)'(\iota) &= v_n'(\iota) - g'(\iota, v_n(\iota-\gamma(\iota)))\\ &= A(\iota) v_n(\iota) + p(\iota, v_n(\iota-\sigma_1(\iota)), ..., v_n(\iota-\sigma_m(\iota))),\end{aligned}$$

then

$$\left\|(\Phi_2 v_n)'\right\| \le (\lambda + L_2 m) E + \beta,$$

hence, $(\Phi_2 v_n)$ is uniformly bounded and equicontinuous. Therefore, by Ascoli-Arzela's theorem, $\Phi_2(\Pi)$ is relatively compact. □

Next, we prove for any $v, w \in \Pi$ that $\Phi_1 v + \Phi_2 w \in \Pi$, in the following lemma.

Lemma 3.4. *If (2.2), (2.5)–(2.7) hold, then for any $v, w \in \Pi$, we have $\Phi_1 v + \Phi_2 w \in \Pi$.*

Proof. Let $v, w \in \Pi$, then $\|v\|, \|w\| \leq E$. By (2.6), we have

$$\begin{aligned}
|(\Phi_1 v)(\iota) + (\Phi_2 w)(\iota)|_1 \leq & |g(\iota, v(\iota - \gamma(\iota)))|_1 + \left\|(I - \mathcal{G}(\iota + \Upsilon, \iota))^{-1}\right\| \\
& \times \int_\iota^{\iota+\Upsilon} \|\mathcal{G}(\iota + \Upsilon, \zeta)\| \|A(\zeta)\| |g(\zeta, w(\zeta - \gamma(\zeta)))|_1 \, d\zeta + \left\|(I - \mathcal{G}(\iota + \Upsilon, \iota))^{-1}\right\| \\
& \times \int_\iota^{\iota+\Upsilon} \|\mathcal{G}(\iota + \Upsilon, \zeta)\| |p(\zeta, w(\zeta - \sigma_1(\zeta)), ..., w(\zeta - \sigma_m(\zeta)))|_1 \, d\zeta \\
\leq & \frac{\lambda(L_1 E + \alpha) + L_2 m E + \beta}{1 - \theta} \mu_0 \\
\leq & E.
\end{aligned}$$

It follows that

$$\|\Phi_1 v + \Phi_2 w\| \leq E,$$

for all $v, w \in \Pi$. Hence, $\Phi_1 v + \Phi_2 w \in \Pi$. □

The following theorem provides the periodicity of solution of (1.4).

Theorem 3.1. *Suppose (2.2), (2.5)–(2.7) hold, then there exists at least one Υ-periodic solution for the system (1.4).*

Proof. Obviously, the requirements of Krasnoselskii's theorem are satisfied due to Lemmas 3.1–3.4. So, there exists a fixed point $x \in \Pi$ such that $x = \Phi_1 x + \Phi_2 x$, and this fixed point is a solution of (1.4). Hence, (1.4) has a Υ-periodic solution. □

Theorem 3.2. *Assume that (2.2), (2.5) and (2.6) hold. If*

$$L_1 + \frac{(\lambda L_1 + L_2 m)}{1 - \theta} \mu_0 < 1, \tag{3.3}$$

then there exists a unique Υ-periodic solution for the system (1.4).

Proof. Let Φ be defined by $\Phi = \Phi_1 + \Phi_2$. For $v_1, v_2 \in BC(\mathbb{R}, \mathbb{R}^n)$, we obtain

$$\begin{aligned}
|(\Phi v_1)(\iota) - (\Phi v_2)(\iota)|_1 \leq & |g(\iota, v_1(\iota - \gamma(\iota))) - g(\iota, v_2(\iota - \gamma(\iota)))|_1 \\
& + \left\|(I - \mathcal{G}(\iota + \Upsilon, \iota))^{-1}\right\| \int_\iota^{\iota+\Upsilon} \|\mathcal{G}(\iota + \Upsilon, \zeta)\| \|A(\zeta)\| |g(\zeta, v_1(\zeta - \gamma(\zeta))) \\
& - g(\zeta, v_2(\zeta - \gamma(\zeta)))|_1 \, d\zeta + \left\|(I - \mathcal{G}(\iota + \Upsilon, \iota))^{-1}\right\| \int_\iota^{\iota+\Upsilon} \|\mathcal{G}(\iota + \Upsilon, \zeta)\| \\
& \times |p(\zeta, v_1(\zeta - \sigma_1(\zeta)), ..., v_1(\zeta - \sigma_m(\zeta))) \\
& - p(\zeta, v_2(\zeta - \sigma_1(\zeta)), ..., v_2(\zeta - \sigma_m(\zeta)))|_1 \, d\zeta \\
= & \left(L_1 + \frac{(\lambda L_1 + L_2 m)}{1 - \theta} \mu_0\right) \|v_1 - v_2\|,
\end{aligned}$$

then

$$\|\Phi v_1 - \Phi v_2\| \leq \left(L_1 + \frac{(\lambda L_1 + L_2 m)}{1 - \theta} \mu_0\right) \|v_1 - v_2\|.$$

Since the condition (3.3) holds, system (1.4) has a unique Υ-periodic solution by the Banach fixed point theorem. □

Now, to study the periodicity of the system (1.5), we define, for $v \in BC(\mathbb{R}, \mathbb{R}^n)$, the operators Ψ_1 and Ψ_2 by

$$(\Psi_1 v)(\iota) = \int_{-\infty}^{0} G(t)\, g(\iota, v(\iota + \gamma(t)))\, dt \tag{3.4}$$

and

$$\begin{aligned}
(\Psi_2 v)(\iota) = &\int_{-\infty}^{0} G(s)\, g(\iota, v(\iota + \gamma(s)))\, ds \\
&+ (I - \mathcal{G}(\iota + \Upsilon, \iota))^{-1} \int_{\iota}^{\iota+\Upsilon} \mathcal{G}(\iota + \Upsilon, \zeta) A(\zeta) \int_{-\infty}^{0} G(s)\, g(\zeta, v(\zeta + \gamma(s)))\, ds d\zeta \\
&+ (I - \mathcal{G}(\iota + \Upsilon, \iota))^{-1} \\
&\times \int_{\iota}^{\iota+\Upsilon} \mathcal{G}(\iota + \Upsilon, \zeta) \int_{-\infty}^{0} G(s)\, p(\zeta, v(\zeta + \sigma_1(s)), ..., v(\zeta + \sigma_m(s)))\, ds d\zeta.
\end{aligned} \tag{3.5}$$

We will study the fixed point of the operator $\Psi_1 + \Psi_2$.

Lemma 3.5. *If (2.2), (2.5) and (2.6) hold, then the operators Ψ_1 and Ψ_2 defined above are operators from Π into $BC(\mathbb{R}, \mathbb{R}^n)$, that is, $\Psi_1, \Psi_2\colon \Pi \to BC(\mathbb{R}, \mathbb{R}^n)$.*

Proof. Let $v \in \Pi$. By (2.5), we get

$$\begin{aligned}
|(\Psi_1 v)(\iota)|_1 &= \left| \int_{-\infty}^{0} G(t)\, g(\iota, v(\iota + \gamma(t)))\, dt \right|_1 \\
&\le (L_1 |v(\iota - \gamma(\iota))|_1 + |g(\iota, 0)|_1) \left\| \int_{-\infty}^{0} G(t)\, dt \right\| \\
&\le \left(L_1 \|v\| + \sup_{\iota \in [0,\iota]} |g(\iota, 0)|_1 \right) \|I\| \\
&\le L_1 E + \alpha.
\end{aligned}$$

Next, for $v \in \Pi$, by Lemma 2.2 and the conditions (2.5) and (2.6), we get

$$\begin{aligned}
|(\Psi_2 v)(\iota)| &\le \frac{\lambda(L_1 E + \alpha) + L_2 m E + \beta}{1 - \theta} \int_{\iota}^{\iota+\Upsilon} e^{\int_{\zeta}^{\iota+\Upsilon} \kappa(s) ds} \left\| \int_{-\infty}^{0} G(s)\, ds \right\| d\zeta \\
&\le \mu_0 \frac{\lambda(L_1 E + \alpha) + L_2 m E + \beta}{1 - \theta}.
\end{aligned} \tag{3.6}$$

Due to Lemma 2.5, Ψ_1 and Ψ_2 are periodic, then $\Psi_1, \Psi_2\colon \Pi \to BC(\mathbb{R}, \mathbb{R}^n)$. □

By the same technique, the proof of the following lemmas is similar to that of Lemmas 3.2–3.4.

Lemma 3.6. *If (2.5) holds, then the operator $\Psi_1\colon \Pi \to BC(\mathbb{R}, \mathbb{R}^n)$ defined by (3.4) is a contraction.*

Lemma 3.7. *If (2.2), (2.5) and (2.6) hold, then the operator $\Psi_2\colon \Pi \to BC(\mathbb{R}, \mathbb{R}^n)$ defined by (3.5) is completely continuous.*

Lemma 3.8. *If (2.2), (2.5)–(2.7), then for any $v, w \in \Pi$, we have $\Psi_1 v + \Psi_2 w \in \Pi$.*

The following theorem provides the periodicity of solution of (1.5).

Theorem 3.3. *Suppose (2.2), (2.5)–(2.7) hold, then there exists at least one Υ-periodic solution for the system (1.5).*

Proof. The requirements in Krasnoselskii's theorem are satisfied due to Lemmas 3.5–3.8. Hence, (1.5) has a Υ-periodic solution. □

Theorem 3.4. *Suppose the conditions (2.2), (2.5), (2.6) and (3.3) hold, then there exists a unique Υ-periodic solution for the system (1.5).*

Proof. Le $\Psi = \Psi_1 + \Psi_2$. For $v_1, v_2 \in BC(\mathbb{R}, \mathbb{R}^n)$; it is easy to see that,

$$\|\Psi v_1 - \Psi v_2\| \leq \left(L_1 + \frac{(\lambda L_1 + L_2 m)}{1-\theta}\mu_0\right)\|v_1 - v_2\|.$$

The condition (3.3) confirms to us that the system (1.5) has one Υ-periodic solution. □

4. Examples

We provide in this section two examples to confirm and strengthen the previous results.

Example 4.1. *Let $v = (v_1, v_2)^t$, $m = 2$, and $\Upsilon = 2\pi$ in the system (1.4), where*

$$v(\iota) = \begin{pmatrix} x_1(\iota) \\ x_2(\iota) \end{pmatrix}, \quad A(\iota) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4}\sin(\iota) \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

$$g(\iota, v(\iota - \gamma(\iota))) = 10^{-4}\sin(\iota)\begin{pmatrix} v_2(\iota - \cos(\iota)) + 1 \\ v_1(\iota - \cos(\iota)) \end{pmatrix},$$

$$p(\iota, v(\iota - \sigma_1(\iota)), v(\iota - \sigma_2(\iota))) = 10^{-5}\cos(\iota)\begin{pmatrix} v_1\left(\iota - 10^{-2}\right) + v_2(\iota - \sin(\iota)) \\ v_2\left(\iota - 10^{-2}\right) + v_1(\iota - \sin(\iota)) \end{pmatrix}.$$

Note that $L_1 = 10^{-4}$, $L_2 = 10^{-5}$, $\alpha = 10^{-4}$, $\beta = 0$, $\lambda = 1$, $\mu(A(\iota)) \leq -\frac{1}{4}$, and

$$\begin{aligned}
\theta &= e^{\int_0^{2\pi}\mu(A(\zeta))d\zeta} \\
&= e^{\int_0^{2\pi}\left(-\frac{1}{2}+\frac{1}{4}|\sin(\zeta)|\right)d\zeta} \\
&= e^{-\pi} < 1, \\
\mu_0 &= \sup_{\zeta \leq \iota \in [0,2\pi]} \int_\iota^{\iota+2\pi} e^{-\int_\zeta^{\iota+2\pi}\frac{1}{4}ds} d\zeta \\
&= 4\left(1 - e^{-\frac{\pi}{2}}\right) \approx 3.16.
\end{aligned}$$

So, the conditions (2.2), (2.5)–(2.7) are held; hence, by Theorem 3.1, the system (1.4) has a 2π-periodic solution. Since $g(\iota, x(\iota - \gamma(\iota)))$ is not identically zero, this solution is also nontrivial.

Example 4.2. *Let $v = (v_1, v_2)^t$, $m = 2$, and $\Upsilon = 2\pi$ in the system (1.5), where*

$$g(\iota, v(\iota + \gamma(\iota))) = 10^{-3}\cos(\iota)\begin{pmatrix} v_2(\iota + \sin(\iota)) + 3 \\ v_1(\iota + \sin(\iota)) \end{pmatrix}, \quad v(\iota) = \begin{pmatrix} v_1(\iota) \\ v_2(\iota) \end{pmatrix},$$

$$A(\iota) = \begin{pmatrix} -\frac{1}{4} & \frac{1}{8}\cos(\iota) \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}, \quad G(\iota) = \begin{pmatrix} e^{\iota} & e^{2\iota} - \frac{1}{2}e^{\iota} \\ e^{3\iota} - \frac{1}{3}e^{\iota} & e^{\iota} \end{pmatrix},$$

$$p(\iota, v(\iota - \sigma_1(\iota)), v(\iota - \sigma_2(\iota))) = 10^{-7}\sin(\iota)\begin{pmatrix} v_1\left(\iota + 10^{-3}\right) + v_2(\iota + \sin(\iota)) \\ v_2\left(\iota + 10^{-3}\right) + v_1(\iota + \sin(\iota)) \end{pmatrix}.$$

Note that $L_1 = 10^{-3}$, $L_2 = 10^{-5}$, $\alpha = 3 \times 10^{-3}$, $\beta = 0$, $\lambda = \frac{1}{2}$, $\mu(A(\iota)) \leq -\frac{1}{8}$, and

$$\theta = e^{\int_0^{2\pi}\mu(A(\zeta))d\zeta} = e^{\int_0^{2\pi}\left(-\frac{1}{4}+\frac{1}{8}|\cos(\zeta)|\right)d\zeta} = e^{\frac{1}{2}-\frac{1}{2}\pi} < 1,$$

$$\int_{-\infty}^{0} G(\iota)\,d\iota = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,$$

$$\mu_0 = \sup_{\zeta \leq \iota \in [0,2\pi]} \int_{\iota}^{\iota+2\pi} e^{-\int_{\zeta}^{\iota+2\pi}\frac{1}{8}ds}d\zeta = 8\left(1 - e^{-\frac{\pi}{4}}\right) \approx 4.35.$$

So, the conditions (2.2), (2.5) and (2.6) are held, and since

$$L_1 + \frac{(\lambda L_1 + L_2 m)}{1-\theta}\mu_0 \approx 4.44 \times 10^{-3} < 1,$$

then the condition (3.3) is held too. Hence, by Theorem 3.4, the system (1.5) has a unique 2π-periodic solution. Since $g(\iota, x(\iota + \gamma(\iota)))$ is not identically zero, then this solution is also nontrivial.

5. Conclusions and aspirations

This research addressed more broadly the study of the kinds of neutral equations represented in nonlinear systems with multiple delays by using matrix measure and the fixed point technique to prove existence and uniqueness.

The purpose of this study is to enhance and generalize various well-known studies such as [2, 12]. Indeed, our study is in the space C^0; however, [2] is in the space C^1. Furthermore, if

$$g(\iota, v(\iota - \gamma(\iota))) = cv(\iota - \gamma),$$

then our findings will be applicable to system (1.3) of [2]. Additionally, the periodic solutions of (1.1) and (1.2) in [12] can be found by our systems (1.4) and (1.5) in n-dimensional case.

Finally, some of the concrete goals and explicit directions in the future study of this field are the studies of periodic solutions in both advanced and delayed differential systems or impulsive systems which means with piecewise constant arguments. For example, we can rely on the articles [22,23] in the future works, since they provide some useful insights about both numerical and theoretical aspects in the context of piecewise constant arguments and irregularities in delay differential equations. Another very important area in which the study of our current systems can be considered, is the regularizations of systems by showing the uniqueness, stability, and periodicity of the orbit like in [24].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

There are no conflicts of interest.

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