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Research article

Criteria of existence and stability of an n-coupled system of generalized Sturm-Liouville equations with a modified ABC fractional derivative and an application to the SEIR influenza epidemic model

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Abstract: The primary objective of this study was to explore the behavior of an n-coupled system of generalized Sturm-Liouville (GSL) and Langevin equations under a modified ABC fractional derivative. We aimed to analyze the dynamics of the system and gain insights into how this operator influences the conditions for the existence and uniqueness of solutions. We established the existence and uniqueness of solutions by employing the Banach contraction principle and Leray-Schauder's alternative fixed-point theorem. We also investigated the Hyers-Ulam stability of the system. This analysis allows us to understand the stability properties of the solutions and evaluate their sensitivity to perturbations. Furthermore, we employed Lagrange's interpolation polynomials to produce a numerical scheme for the influenza epidemic model. By combining theoretical analysis, mathematical principles, and numerical simulations, this study contributes to enriching our understanding of the behavior of the system and offers insights into its dynamics and practical applications in epidemiology.

Keywords: MABC fractional model; generalized Sturm-Liouville equation; stability analysis; Lagrange polynomials; graphical representation
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1. Introduction

Fractional calculus is a mathematical field that deals with derivatives and integrals of non-integer orders. It extends traditional calculus and enables more precise modeling of complex systems with memory and hereditary characteristics [3, 18, 26, 29, 32, 33]. Various fractional operators have been proposed to describe different types of fractional derivatives. For instance, Atangana and Baleanu [10] introduced the concept of the AB-fractional derivative and explored its applications. The ABC fractional derivative exhibits improved behavior for functions with singularities or non-smooth features, making it particularly suited for capturing complex dynamics in practical systems [21]. Its significance extends to a wide range of applications, including signal processing, control systems, and epidemiological modeling [4–6, 28]. Al-Refai and Baleanu [8] proposed a modification to ABC fractional operator (MABC), which allows for easier initialization of fractional models. This modification has been shown to enable the discovery of new solution types, expanding the range of possible solutions for these equations [7]. The significance of the MABC fractional operator lies in its ability to accurately represent the behavior of real-world systems and facilitate the analysis and control of complex nonlinear phenomena. For example, Khan et al. [22] studied the MABC fractional order smoking model. Khan et al. [23] studied the dynamics of a piecewise MABC fractional-order leukemia model with symmetric numerical simulations. Rahman [31] studied some theoretical and numerical investigations of the MABC fractional operator for the spread of polio under the effect of vaccination. Eiman et al. [17] studied the rotavirus infectious disease model using a piecewise MABC fractional derivative. Khan et al. [24,25] investigated the existence of solutions for the hybrid class of the MABC fractional operator with an application to a waterborne disease model.

These studies demonstrate the versatility and effectiveness of the MABC fractional operator in elucidating the dynamics and mechanisms of infectious diseases, thereby paving the way for improved disease understanding, control, and prevention strategies. The local and global stability analyses are important in understanding the behavior and performance of dynamical systems. For example, the authors [36] investigated less conservative stability conditions for the discrete-time hidden semi-Markov jump linear systems emission probability matrix, using the eliminating matrix product technique. They obtained numerically testable conditions for the existence of an observed mode-dependent control. The authors [27] studied the global dynamics and optimal control of malicious signal transmission in wireless sensor networks using Pontryagin's maximum principle. They examined the local and global stability of two types of steady states in the discontinuous system through the analysis of the characteristic equation and comparison arguments method, respectively.

On the other hand, the importance of generalized Sturm-Liouville equations [32] lies in their ability to model a wide range of practical phenomena, including heat conduction, diffusion processes, and quantum mechanics [26]. By capturing the complex dynamics of such systems, these equations provide valuable insights that aid in the design and optimization of engineering structures, the understanding of physical processes, and the development of mathematical models for real-life applications [18].

There are some researchers who have studied the Sturm-Liouville (SL) and generalized Sturm-Liouville (GSL) equations with different fractional derivatives and various conditions. For example, Batiha et al. [12] studied the GSL equations with a Hadamard fractional derivative under three nonlocal Hadamard fractional integral boundary conditions. Boutiara et al. [14] investigated the existence and uniqueness of solutions to a generalized quantum fractional SL difference problem with terminal boundary conditions, utilizing Mönch's fixed-point theorem, the Kuratowski measure of noncompactness, and the Banach contraction principle. Boutiara et al. [13] undertook a meticulous examination of the SL equation, featuring the intriguing p-Laplacian operator, in the context of a generalized Caputo-type derivative. Ercan in [15] presented a novel methodology known as the Laplace-Adomian decomposition method, specifically tailored to address two distinct types of nonlinear fractional SL problems, employing both the Caputo and ABC derivatives. Furthermore, Baleanu et al. [11] delved into the study of a coupled system comprising GSL problems and Langevin fractional differential equations, elucidating their behavior under the influence of the ABC fractional operator. Berhail et al. [16] explored a boundary value problem associated with the system of GSL and Langevin, employing the Hadamard fractional derivative as a pivotal mathematical tool.

Motivated by the importance of both GSL equations and the MABC fractional derivative, this research paper aims to extend the works in [11, 12] by investigating the criteria for the existence and stability of the following n-coupled system of GSL equations utilizing the MABC fractional derivative

$$\begin{cases} {}^{MABC}\mathbb{D}_{\iota}^{\eta}\left[\varsigma_i(\iota){}^{MABC}\mathbb{D}_{\iota}^{\varrho}\hbar_i(\iota)+\rho_i(\iota)\hbar_i(\iota)\right]=g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n),\iota\in\Omega:=[0,b], \\ \hbar_i(0)=0,\varsigma_i(0){}^{MABC}\mathbb{D}_{0}^{\varrho}\hbar_i(\iota)+\rho_i(0)\hbar_i(0)=0, i=1,2,\cdots,n, \end{cases} \tag{1.1}$$

where, ${}^{MABC}\mathbb{D}_{\iota}^{\eta}$ and ${}^{MABC}\mathbb{D}_{\iota}^{\varrho}$ are MABC fractional derivatives of order η and ϱ, respectively, such that $0<\eta,\ \varrho<1$, the functions $\rho_i:\Omega\to\mathbb{R}_+$ are continuous such that $\rho_i^*=\max_{\iota\in\Omega}|\rho_i(\iota)|$, and $\varsigma_i:\Omega\to\mathbb{R}_+-\{0\}$ are continuous functions with $\varsigma_i^*=\min_{\iota\in\Omega}|\varsigma_i(\iota)|$, for all $\iota\in\Omega, i=1,2,\cdots,n$. The functions $g_i:\Omega\times\mathbb{R}^n\to\mathbb{R}$ are continuous such that $(g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n))_{\iota=0}=0$. System (1.1) has some special cases depending on the functions $\varsigma_i(t)$ and $\rho_i(t)$. For example:

- If $\rho_i(t)=0,\forall\iota\in\Omega$, then the n-coupled MABC-GSL system (1.1) is reduced to the following n-coupled MABC-SL fractional system

$$\begin{cases} {}^{MABC}\mathbb{D}_{\iota}^{\eta}\left[\varsigma_i(\iota){}^{MABC}\mathbb{D}_{\iota}^{\varrho}\hbar_i(\iota)\right]=g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n),\iota\in\Omega:=[0,b], \\ \hbar_i(0)=0,\varsigma_i(0){}^{MABC}\mathbb{D}_{0}^{\varrho}\hbar_i(\iota)=0, i=1,2,\cdots,n. \end{cases} \tag{1.2}$$

- If $\varsigma_i(\iota)=1$, and $\rho_i(t)=\rho_i\in\mathbb{R},\forall\iota\in\Omega$, then the n-coupled MABC-GSL system (1.1) is reduced to the following n-coupled MABC-Langevin fractional system

$$\begin{cases} {}^{MABC}\mathbb{D}_{\iota}^{\eta}\left[{}^{MABC}\mathbb{D}_{\iota}^{\varrho}\hbar_i(t)+\rho_i\hbar_i(t)\right]=g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n),\iota\in\Omega:=[0,b], \\ \hbar_i(0)=0,{}^{MABC}\mathbb{D}_{0}^{\varrho}\hbar_i(t)+\rho_i\hbar_i(0)=0, i=1,2,\cdots,n. \end{cases} \tag{1.3}$$

Our contributions to this work are given as:

- We utilize theoretical analysis to establish the foundations of our research. We employ mathematical principles, such as MABC fractional operators, an n-coupled system of generalized Sturm-Liouville (GSL) and Langevin equations, and fixed-point techniques to formulate sufficient conditions for the existence, uniqueness, and stability of the system (1.1).
- To the best of our knowledge, no previous studies have examined the n-coupled MABC systems represented by Eq (1.1). Therefore, our work aims to fill this gap in the existing literature by investigating the behavior of the n-coupled system of GSL equations under the MABC fractional operator. Through the establishment of rigorous criteria for existence and stability, our research contributes to a better understanding of the dynamics of n-coupled systems.

- We introduce a numerical algorithm for the nonlinear MABC-GSL fractional equations by using Lagrange's interpolation method and apply this method to produce a numerical scheme for the influenza epidemic model. By extending the application to infectious disease modeling, we not only broaden the scope of our research but also demonstrate the practical significance of the MABC operator. We seek to deepen knowledge and provide broader insights into the implications of the MABC fractional operator in understanding complex systems and their potential applications in the field of infectious diseases.
- We discuss numerical simulations of the influenza epidemic model to enhance our understanding of the model's behavior. These numerical simulations allow us to investigate the model's dynamics, explore parameter spaces, and analyze the effects of various scenarios or interventions.

The construction of the paper is as follows: In Section 2, we provide a comprehensive review of the basic definitions and lemmas related to the MABC fractional operator. In Section 3, we present our main results, including converting the nonlinear fractional n-coupled system into an equivalent integral equation. We then discuss the sufficient conditions for the existence and uniqueness of solutions of system (1.1) by employing the Banach contraction principle and Leray-Schauder's alternative fixed-point theorem. Furthermore, we discuss the system's stability conditions (1.1). In Section 4, we introduce a numerical scheme based on Lagrange's interpolation method for efficient computation and analysis of the system. In Section 5, we present an application of the numerical scheme to study the influenza epidemic model. Through this organization, we aim to present a clear and logical progression of the research, ensuring that readers can easily comprehend and appreciate the significance of our findings.

2. Auxiliary results

Throughout this paper, we will frequently rely on the fundamental definitions of the MABC operator, which are presented in this section.

Definition 2.1. *[7, 8] For $\varrho \in (0,1)$ and $\hbar \in L^1(0,b)$, the MABC-fractional derivative is given by*

$$^{MABC}\mathbb{D}_{\iota}^{\varrho}\hbar\,(\iota) = \frac{\Delta(\varrho)}{1-\varrho}[\hbar(\iota) - E_{\varrho}(-\mu_{\varrho}\iota^{\varrho})\hbar(0) - \mu_{\varrho}\int_0^{\iota}(\iota-s)^{\varrho-1}E_{\varrho,\varrho}(-\mu_{\varrho}(\iota-s)^{\varrho})\hbar(s)ds],$$

where, $\mu_{\varrho} = \frac{\varrho}{1-\varrho}$. Given this definition, one can easily verify that $^{MABC}D_{\iota}^{\varrho}C = 0$.

Definition 2.2. *[7, 8] For $\varrho \in (0,1)$ and $\hbar \in L^1(0,b)$, the MAB-fractional integral is given by*

$$\begin{aligned} ^{mAB}\mathbb{I}_{\iota}^{\varrho}\hbar\,(\iota) &= \frac{1-\varrho}{\Delta(\varrho)}\left[\hbar(\iota) + \mu_{\varrho}{}^{RL}\mathbb{I}_{\iota}^{\varrho}[\hbar] - \hbar(0)\left(1+\mu_{\varrho}\frac{\iota^{\varrho}}{\Gamma(\varrho+1)}\right)\right] \\ &= \frac{1-\varrho}{\Delta(\varrho)}[\hbar(\iota)-\hbar(0)] + \frac{\varrho}{\Delta(\varrho)}{}^{RL}\mathbb{I}_{\iota}^{\varrho}[\hbar(\iota)-\hbar(0)]. \end{aligned}$$

Lemma 2.3. *[8] For $\hbar' \in L^1(0,\infty)$ and $\varrho \in (0,1)$, we have*

$$^{mAB}\mathbb{I}_{\iota}^{\varrho\,MABC}\mathbb{D}_{\iota}^{\varrho}\hbar\,(\iota) = \hbar(\iota) - \hbar(0).$$

Lemma 2.4. *[34] Let β_{φ} be the closed ball of radius $\varphi > 0$, centred at zero in a Banach space $\mathcal{A}$ and $\Psi\colon \beta_{\varphi} \to \mathcal{A}$ is a contraction mapping such that $\Psi\big(\partial\beta_{\varphi}\big) \subset \beta_{\varphi}$. Then, Ψ has a unique fixed point in β_{φ}.*

Lemma 2.5. *[19] Let β_φ be a non-empty, closed, and convex subset of a Banach space $\mathcal{A}$. If Ψ: $\beta_\varphi \to \mathcal{A}$ is a completely continuous operator and $\Phi(\Psi) = \{(\hbar \in \beta_\varphi, \hbar = \xi\Psi(\hbar), 0 < \xi < 1\}$, then, either $\Phi(\Psi)$ is unbounded or Ψ has a fixed point.*

3. Main results

In this section, we will examine the sufficient conditions that guarantee the existence and uniqueness of the system solution (1.1), as well as special cases (1.2) and (1.3). Additionally, we will discuss four types of stability of systems (1.1), (1.2), and (1.3). We will start by converting the nonlinear fractional n-coupled system (1.1) into an equivalent integral equation to apply fixed-point techniques.

3.1. Nonlinear fractional n-coupled system

Lemma 3.1. *For $0 < \varrho, \eta < 1$, let $\varsigma_i, \rho_i : \Omega \to \mathbb{R}$ be continuous functions such that $\varsigma_i(\iota) \neq 0$ for all $\iota \in \Omega$ and $g_i : \Omega \times \mathbb{R} \to \mathbb{R}$ be continuous functions with $g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n)_{\iota=0} = 0$. Then, the n-coupled MABC-GSL system (1.1) is equivalent to the following integral equation:*

$$\hbar_i(\iota) = {}^{MAB}\mathbb{I}_\iota^\varrho \left[\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) - \rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)} \right]. \tag{3.1}$$

Or

$$\begin{aligned} \hbar_i(\iota) &= \frac{1-\varrho}{\Delta(\varrho)} \left(\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) - \rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)} \right) \\ &\quad + \frac{\varrho}{\Delta(\varrho)\Gamma(\varrho)} \int_0^\iota (\iota - s)^{\varrho-1} \left(\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(s, \hbar_1, \hbar_2, \cdots, \hbar_n) - \rho_i(s)\hbar_i(s)}{\varsigma_i(s)} \right) ds. \end{aligned}$$

Proof. Let $\hbar_i$ be a solution of system (1.1). Applying the operator ${}^{MAB}\mathbb{I}_\iota^\eta$ on both sides of system (1.1) and using Lemma 2.3, we get

$$\left[\varsigma_i(\iota)^{MABC}\mathbb{D}_\iota^\varrho \hbar_i(\iota) + \rho_i(\iota)\hbar_i(\iota)\right] = c_1 + {}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n).$$

Thus, we have

$${}^{MABC}\mathbb{D}_\iota^\varrho \hbar_i(\iota) = \frac{c_1 + {}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) - \rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)}. \tag{3.2}$$

By the second condition $\varsigma_i(0)^{MABC}\mathbb{D}_0^\varrho \hbar_i(\iota) + \rho_i(0)\hbar_i(0) = 0$, we have $c_1 = 0$. Substituting the value of c_1 in (3.2), we get

$${}^{MABC}\mathbb{D}_\iota^\varrho \hbar_i(\iota) = \frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) - \rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)}. \tag{3.3}$$

Applying the operator ${}^{MAB}\mathbb{I}_\iota^\varrho$ on both sides of Eq (3.3) and using Lemma 2.3, we get

$$\hbar_i(\iota) = c_2 + {}^{MAB}\mathbb{I}_\iota^\varrho \frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) - \rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)}. \tag{3.4}$$

By assumption $g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n)_{\iota=0} = 0$, $\hbar_i(0) = 0$, and by Definition 2.2 in [7], we get $c_2 = 0$. Substituting the value of c_2 in Eq (3.4), we get

$$\hbar_i(\iota) = {}^{MAB}\mathbb{I}_\iota^\varrho \frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) - \rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)}.$$

By Definition 2.2, we can write the solution as

$$\begin{aligned}\hbar_i(\iota) &= \frac{1-\varrho}{\Delta(\varrho)}\left(\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n)-\rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)}\right)\\ &+\frac{\varrho}{\Delta(\varrho)\Gamma(\varrho)}\int_0^\iota(\iota-s)^{\varrho-1}\left(\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(s,\hbar_1,\hbar_2,\cdots,\hbar_n)-\rho_i(s)\hbar_i(s)}{\varsigma_i(s)}\right)ds\\ &-\frac{1-\varrho}{\Delta(\varrho)}\left(\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n)-\rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)}\right)_{\iota=0}\left(1+\mu_\varrho\frac{\iota^\varrho}{\Gamma(\varrho+1)}\right).\end{aligned}$$

Also, by the assumption $g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n)_{\iota=0}=0$, $\hbar_i(0)=0$, and by Definition 2.2 in [7], we have

$$\left(\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n)-\rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)}\right)_{\iota=0}=0.$$

Thus, the solution of the n-coupled MABC-GSL system (1.1) is given by

$$\begin{aligned}\hbar_i(\iota) &= \frac{1-\varrho}{\Delta(\varrho)}\left(\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n)-\rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)}\right)\\ &+\frac{\varrho}{\Delta(\varrho)\Gamma(\varrho)}\int_0^\iota(\iota-s)^{\varrho-1}\left(\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(s,\hbar_1,\hbar_2,\cdots\hbar_n)-\rho_i(s)\hbar_i(s)}{\varsigma_i(s)}\right)ds.\end{aligned}$$

Conversely, it can be readily demonstrated through direct computation that the integral equation (3.1) fulfills the requirements of the n-coupled MABC-GSL system (1.1). The proof is complete. □

For our analysis, we shall define the Banach space. Consider an interval $\Omega=[0,b]\subset\mathbb{R}$, and the Banach space $\mathcal{A}=C(\Omega,\mathbb{R})$ which consists of all continuous functions $\hbar:\Omega\to\mathbb{R}$. The norm $\|\hbar\|$ of a function $\hbar$ in this space is defined as $\|\hbar\|=\max_{\iota\in\Omega}|\hbar(\iota)|$. Therefore, $(\mathcal{A},\|\cdot\|)$ forms a Banach space. Define the n-product space $\mathcal{H}=\underbrace{\mathcal{A}\times\mathcal{A}\times\cdots\times\mathcal{A}}_{n\text{ times}}$ with the norm

$$\|\hbar_1,\hbar_2,\cdots,\hbar_n\|=\sum_{i=1}^n\|\hbar_i\|.$$

Clearly, $(\mathcal{H},\|\hbar_1,\hbar_2,\cdots,\hbar_n\|)$ is a Banach space. For $i=1,2,\cdots,n$, we define an operator $\Psi:\mathcal{H}\to\mathcal{H}$ by

$$\Psi(\hbar_1,\hbar_2,\cdots,\hbar_n)(\iota)=(\Psi_1,\Psi_2\cdots,\Psi_n),\tag{3.5}$$

with the norm

$$\|\Psi(\hbar_1,\hbar_2,\cdots,\hbar_n)\|=\sum_{i=1}^n\|\Psi_i(\hbar_i)\|,\tag{3.6}$$

where

$$\Psi_i(\hbar_i(\iota))={}^{MAB}\mathbb{I}_\iota^\varrho\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n)-\rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)},\ i=1,2,\cdots,n.\tag{3.7}$$

So, all the fixed points of (3.7) are the solutions of the system (1.1). In this paper, a closed ball $\mathcal{B}_\varphi$ with radius φ centered on the zero function in product Banach space $\mathcal{H}$ is defined by

$$\mathcal{B}_\varphi=\{(\hbar_1,\hbar_2,\cdots,\hbar_n)\in\mathcal{H}:\|(\hbar_1,\hbar_2,\cdots\hbar_n)\|\le\varphi\}.\tag{3.8}$$

For further analysis, the following conditions must be satisfied:

(C_1) For all $\iota \in \Omega$, there exists a positive constant Υ_i^*, such that

$$|g_i(\iota, \hbar_i(\iota))| \leq \Upsilon_i^*.$$

(C_2) For each $\iota \in \Omega$ and $\hbar_i, \widehat{\hbar_i} \in \mathcal{H}$, there exists constant numbers $\mathcal{K}_i > 1, i = 1, 2, \cdots, n$ such that

$$\left| g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) - g_i(\iota, \widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n) \right| \leq \mathcal{K}_i \sum_{i=1}^{n} \left| \hbar_i - \widehat{\hbar}_i \right|.$$

For $(\hbar_1, \hbar_2, \cdots, \hbar_n) \in \mathcal{B}_\varphi, \iota \in \Omega$, then by condition ($C_2$), we have

$$\begin{aligned} |g_i(\iota, (\hbar_1, \hbar_2, \cdots, \hbar_n))| &= |g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) - g_i(\iota, 0, \cdots, 0) + g_i(\iota, 0, \cdots, 0)| \\ &\leq |g_i(\iota, \hbar_1, \hbar_2, \cdots \hbar_n) - g_i(\iota, 0, \cdots, 0)| + |g_i(\iota, 0, \cdots, 0)| \\ &\leq \mathcal{K}_i \sum_{i=1}^{n} |\hbar_i(\iota)| + |g_i(\iota, 0, \cdots, 0)| \\ &\leq \mathcal{K}_i \sum_{i=1}^{n} |\hbar_i(\iota)| + B_i \\ &\leq \mathcal{K}_i \varphi + B_i, \end{aligned} \tag{3.9}$$

where

$$B_i = \max_{\iota \in \Omega} |g_i(\iota, 0, \cdots, 0)| < \infty, i = 1, 2, \cdots, n.$$

For $i = 1, 2, \cdots, n$, to simplify the analysis, we fix the following notations:

$$\begin{aligned} \varsigma_i^* &= \min_{\iota \in \Omega} |\varsigma_i(\iota)| \neq 0, \\ \rho_i^* &= \max_{\iota \in \Omega} |\rho_i(\iota)|, \\ \Pi_\varrho &= \left(\frac{1 - \varrho}{\Delta(\varrho)} + \frac{b^\varrho}{\Delta(\varrho)\Gamma(\varrho)} \right), \\ \Pi_\eta &= \left(\frac{1 - \eta}{\Delta(\eta)} + \frac{b^\eta}{\Delta(\eta)\Gamma(\eta)} \right). \end{aligned} \tag{3.10}$$

3.2. Uniqueness of solution

In this subsection, we examine the sufficient conditions that guarantee the uniqueness of the solution for system (1.1), as well as special cases (1.2) and (1.3) by using the Banach fixed-point Lemma 2.4.

Theorem 3.2. *Under the condition* (C_2) *and* $0 < \Pi_\varrho \sum_{i=1}^{n} \frac{1}{\varsigma_i^*} \left(\Pi_\eta \mathcal{K}_i + \max_{i=1}^{n} \left\{ \rho_i^* \right\} \right) < 1, i = 1, 2, \cdots, n$, *if we choose*

$$\varphi \geq \left\{ \frac{\Pi_\varrho \sum_{i=1}^{n} \frac{B_i}{\varsigma_i^*}}{1 - \Pi_\varrho \sum_{i=1}^{n} \frac{1}{\varsigma_i^*} \left(\Pi_\eta \mathcal{K}_i + \max_{i=1}^{n} \left\{ \rho_i^* \right\} \right)} \right\},$$

then, the n-coupled MABC-GSL system (1.1) has a unique solution.

Proof. Consider the operator Ψ defined by (3.5) and closed ball $\mathcal{B}_\varphi$ defined by (3.8). To apply Banach fixed-point Lemma 2.4, we divided the proof into the following steps:

Step 1: In the first step, we prove that $\Psi(\hbar_1, \hbar_2, \cdots, \hbar_n) \subset \mathcal{B}_\varphi$.

For $i = 1, 2 \cdots, n$ and $(\hbar_1, \hbar_2, \cdots, \hbar_n) \in \mathcal{B}_\varphi$, $\iota \in \Omega$, we have

$$\begin{aligned} |\Psi_i(\hbar_i(\iota))| &= \left| {}^{MAB}\mathbb{I}_\iota^\varrho \frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots \hbar_n) - \rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)} \right| \\ &\leq \frac{1}{\varsigma_i^*}\left[{}^{MAB}\mathbb{I}_\iota^\varrho \left({}^{MAB}\mathbb{I}_\iota^\eta |g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n)| \right) + {}^{MAB}\mathbb{I}_\iota^\varrho \rho_i^* |\hbar_i(\iota)| \right]. \end{aligned} \tag{3.11}$$

Now, we compute the two integrals in (3.11) separately. By Definition 2.2, with $g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n)_{\iota=0} = 0$, we obtain

$$\begin{aligned} \left| {}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) \right| &\leq \frac{1-\eta}{\Delta(\eta)} |g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n)| \\ &\quad + \frac{\eta}{\Delta(\eta)\Gamma(\eta)} \int_0^\iota (\iota - s)^{\eta-1} |g_i(s, \hbar_1, \hbar_2, \cdots, \hbar_n)| \, ds. \end{aligned} \tag{3.12}$$

Putting (3.9) in (3.12), we get

$$\begin{aligned} \left| {}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) \right| &\leq \left(\frac{1-\eta}{\Delta(\eta)} + \frac{b^\eta}{\Delta(\eta)\Gamma(\eta)} \right) \mathcal{K}_i \varphi + \left(\frac{1-\eta}{\Delta(\eta)} + \frac{b^\eta}{\Delta(\eta)\Gamma(\eta)} \right) B_i \\ &\leq \Pi_\eta (\mathcal{K}_i\varphi + B_i), \end{aligned} \tag{3.13}$$

where Π_η is defined by (3.10). By (3.13), we have

$$\begin{aligned} \left| {}^{MAB}\mathbb{I}_\iota^\varrho \left[{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) \right] \right| &\leq \frac{1-\varrho}{\Delta(\varrho)} \Pi_\eta (\mathcal{K}_i\varphi + B_i) + \frac{\Pi_\eta (\mathcal{K}_i\varphi + B_i)\varrho}{\Delta(\varrho)} \frac{1}{\Gamma(\varrho)} \int_0^\iota (\iota - s)^{\varrho-1} ds \\ &\leq \Pi_\varrho \Pi_\eta (\mathcal{K}_i\varphi + B_i). \end{aligned} \tag{3.14}$$

In the same manner, we obtain

$$\begin{aligned} {}^{MAB}\mathbb{I}_\iota^\varrho [|\rho_i \hbar_i|] &\leq \frac{1-\varrho}{\Delta(\varrho)} \rho_i^* |\hbar_i(\iota)| + \frac{\rho_i^* \varrho}{\Delta(\varrho)\Gamma(\varrho)} \int_0^\iota (\iota - s)^{\varrho-1} |\hbar_i(s)| \, ds \\ &\leq \frac{1-\varrho}{\Delta(\varrho)} \rho_i^* \|\hbar_i\| + \frac{\rho_i^* b^\varrho}{\Delta(\varrho)\Gamma(\varrho)} \|\hbar_i\| \leq \rho_i^* \Pi_\varrho \|\hbar_i\|. \end{aligned} \tag{3.15}$$

Putting (3.14) and (3.15) in (3.11) and taking the maximum value of both sides, we get

$$\|\Psi_i(\hbar_i)\| \leq \frac{1}{\varsigma_i^*} \left[\Pi_\varrho \Pi_\eta (\mathcal{K}_i\varphi + B_i) + \rho_i^* \Pi_\varrho \|\hbar_i\| \right].$$

Thus, by (3.6), we have

$$\begin{aligned} \|\Psi(\hbar_1, \hbar_2, \cdots, \hbar_n)\| = \sum_{i=1}^n \|\Psi_i(\hbar_i)\| &\leq \Pi_\varrho \varphi \sum_{i=1}^n \frac{1}{\varsigma_i^*} \left(\Pi_\eta \mathcal{K}_i + \max_{i=1}^n \{\rho_i^*\} \right) + \Pi_\varrho \sum_{i=1}^n \frac{B_i}{\varsigma_i^*} \\ &\leq \varphi. \end{aligned}$$

This implies that $\Psi(\hbar_1,\hbar_2,\cdots,\hbar_n)\subset\mathcal{B}_\varphi$. Next, let $(\hbar_1,\hbar_2,\cdots,\hbar_n),\left(\widehat{\hbar}_1,\widehat{\hbar}_2,\cdots,\widehat{\hbar}_n\right)\in\mathcal{B}_\varphi,\iota\in\Omega$, then, by (3.12), we get

$$ {}^{MAB}\mathbb{I}_\iota^\eta\left|g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n)-g_i\left(\iota,\widehat{\hbar}_1,\widehat{\hbar}_2,\cdots,\widehat{\hbar}_n\right)\right|\le\Pi_\eta\mathcal{K}_i\sum_{i=1}^n\left|\hbar_i(\iota)-\widehat{\hbar}_i(\iota)\right|. \tag{3.16}$$

In the same manner as in (3.14) and (3.15), we get

$$ {}^{MAB}\mathbb{I}_\iota^\varrho\left[{}^{MAB}\mathbb{I}_t^\eta\left|g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n)-g_i\left(\iota,\widehat{\hbar}_1,\widehat{\hbar}_2,\cdots,\widehat{\hbar}_n\right)\right|\right]\le\Pi_\varrho\Pi_\eta\mathcal{K}_i\sum_{i=1}^n\left|\hbar_i(\iota)-\widehat{\hbar}_i(\iota)\right|, \tag{3.17}$$

and

$$ {}^{MAB}\mathbb{I}_\iota^\varrho\left[\rho_i\left|\hbar_i(\iota)-\widehat{\hbar}_i(\iota)\right|\right]\le\rho_i^*\Pi_\varrho\left|\hbar_i(\iota)-\widehat{\hbar}_i(\iota)\right|. \tag{3.18}$$

Thus, by (3.17) and (3.18), and taking the maximum value of both sides, we have

$$\begin{aligned}&\left\|\Psi_i(\hbar_1,\hbar_2,\cdots,\hbar_n)-\Psi_i\left(\widehat{\hbar}_1,\widehat{\hbar}_2,\cdots,\widehat{\hbar}_n\right)\right\|\\ \le\ &\frac{\Pi_\varrho}{\varsigma_i^*}\Pi_\eta\mathcal{K}_i\left\|(\hbar_1,\hbar_2,\cdots,\hbar_n)-\left(\widehat{\hbar}_1,\widehat{\hbar}_2,\cdots,\widehat{\hbar}_n\right)\right\|+\frac{\rho_i^*}{\varsigma_i^*}\Pi_\varrho\left\|\hbar_i-\widehat{\hbar}_i\right\|.\end{aligned}$$

Thus, by (3.6), we have

$$\begin{aligned}&\left\|\Psi(\hbar_1,\hbar_2,\cdots,\hbar_n)-\Psi\left(\widehat{\hbar}_1,\widehat{\hbar}_2,\cdots,\widehat{\hbar}_n\right)\right\|\\ \le\ &\Pi_\varrho\sum_{i=1}^n\frac{1}{\varsigma_i^*}\left(\Pi_\eta\mathcal{K}_i+\max_{i=1}^n\{\rho_i^*\}\right)\left\|(\hbar_1,\hbar_2,\cdots,\hbar_n)-\left(\widehat{\hbar}_1,\widehat{\hbar}_2,\cdots,\widehat{\hbar}_n\right)\right\|.\end{aligned}$$

Since, $\Pi_\varrho\sum_{i=1}^n\frac{1}{\varsigma_i^*}\left(\Pi_\eta\mathcal{K}_i+\max_{i=1}^n\left\{\rho_i^*\right\}\right)<1, i=1,2,\cdots,n$, then, we conclude that Ψ is a contraction. Consequently, the n-coupled MABC-GSL system (1.1) has a unique solution. □

3.3. Uniqueness of solutions for systems (1.2) and (1.3)

As a result of the above uniqueness Theorem 3.2, we have the following two corollaries.

Corollary 3.3. *Under the condition* (C_2) *and* $0<\Pi_\varrho\sum_{i=1}^n\frac{1}{\varsigma_i^*}\left(\Pi_\eta\mathcal{K}_i\right)<1, i=1,2,\cdots,n$, *if we choose*

$$\varphi\ge\left\{\frac{\Pi_\varrho\sum_{i=1}^n\frac{B_i}{\varsigma_i^*}}{1-\Pi_\varrho\sum_{i=1}^n\frac{1}{\varsigma_i^*}\left(\Pi_\eta\mathcal{K}_i\right)}\right\},$$

then, the n-coupled MABC-SL system (1.2) has a unique solution.

Corollary 3.4. *Under the condition* (C_2) *and* $0<\Pi_\varrho\sum_{i=1}^n\left(\Pi_\eta\mathcal{K}_i+\max_{i=1}^n\{\rho_i\}\right)<1, i=1,2,\cdots,n$, *if we choose*

$$\varphi\ge\left\{\frac{\Pi_\varrho\sum_{i=1}^n B_i}{1-\Pi_\varrho\sum_{i=1}^n\left(\Pi_\eta\mathcal{K}_i+\max_{i=1}^n\{\rho_i\}\right)}\right\},$$

then, the n-coupled MABC- Langevin system (1.3) has a unique solution.

3.4. Existence of solution

In this subsection, we developed the sufficient conditions for the existence of a solution of system (1.1) by using Leray-Schauder's alternative Lemma 2.5.

Theorem 3.5. *Under the condition (C_1), if* $0 < \Pi_\varrho \max_{i=1}^{n}\left\{\frac{\rho_i^*}{\varsigma_i^*}\right\} < 1$, *then the n-coupled MABC-GSL system (1.1) has a solution.*

Proof. Consider the operator Ψ defined by (3.5) and closed ball $\mathcal{B}_{\varphi_i}$ defined by (3.8). To apply the Leray-Schauder's alternative theorem, we divided the proof into the following steps.

Step 1: We shall prove that Ψ is completely continuous (continuous, bounded, and equicontinuous). The continuity of g_i implies that Ψ is continuous too. Let $\iota \in \Omega$. Then, for $i = 1, 2, \cdots, n$ and $(\hbar_1, \hbar_2, \cdots, \hbar_n) \in \mathcal{B}_\varphi$, we have

$$\begin{aligned} |\Psi_i(\hbar_i(\iota))| &= \left| {}^{MAB}\mathbb{I}_\iota^\varrho \left[\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) - \rho_i \hbar_i}{\varsigma_i} \right] \right| \\ &\leq \frac{1}{\varsigma_i^*} \left[{}^{MAB}\mathbb{I}_\iota^\varrho \left({}^{MAB}\mathbb{I}_\iota^\eta |g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n)| \right) + {}^{MAB}\mathbb{I}_\iota^\varrho |\rho_i \hbar_i| \right]. \end{aligned} \tag{3.19}$$

By condition (C_1), we have

$$\left| {}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) \right| \leq \Pi_\eta \Upsilon_i^*. \tag{3.20}$$

Thus, by (3.19) and (3.20), we have

$$\begin{aligned} \|\Psi_i(\hbar_i)\| &\leq \frac{1-\varrho}{\Delta(\varrho)\varsigma_i^*}\Pi_\eta\Upsilon_i^* + \frac{b^\varrho}{\Delta(\varrho)\varsigma_i^*\Gamma(\varrho)}\Pi_\eta\Upsilon_i^* + \frac{1-\varrho}{\Delta(\varrho)}\frac{\rho_i^*}{\varsigma_i^*}\|\hbar_i\| + \frac{\rho_i^*}{\varsigma_i^*}\frac{b^\varrho\|\hbar_i\|}{\Delta(\varrho)\Gamma(\varrho)} \\ &\leq \frac{\Pi_\varrho}{\varsigma_i^*}\left[\Pi_\eta\Upsilon_i^* + \frac{\rho_i^*\|\hbar_i\|}{\varsigma_i^*}\right]. \end{aligned}$$

Hence,

$$\|\Psi(\hbar_1, \hbar_2, \cdots, \hbar_n)\| = \sum_{i=1}^{n}\|\Psi_i(\hbar_i)\| \leq \sum_{i=1}^{n}\frac{\Pi_\varrho}{\varsigma_i^*}\left[\Pi_\eta\Upsilon_i^* + \frac{\rho_i^*\|\hbar_i\|}{\varsigma_i^*}\right] < \infty.$$

Thus, Ψ is bounded by $\sum_{i=1}^{n}\frac{\Pi_\varrho}{\varsigma_i^*}\left[\Pi_\eta\Upsilon_i^* + \frac{\rho_i^*\|\hbar_i\|}{\varsigma_i^*}\right]$. Next, for $i = 1, 2, \cdots, n$, let $\iota_1, \iota_2 \in \Omega$, with $0 < \iota_1 < \iota_2 < b$. Then, for $(\hbar_1, \hbar_2, \cdots, \hbar_n) \in \mathcal{B}_\varphi$, we have

$$\begin{aligned} &|\Psi_i(\hbar_i(\iota_2)) - \Psi_i(\hbar_i(\iota_1))| \\ \leq\; & \frac{\Pi_\eta\Upsilon_i^*\varrho}{\Delta(\varrho)\varsigma_i^*\Gamma(\varrho)}\int_0^{\iota_1}\left[(\iota_2-s)^{\varrho-1} - (\iota_1-s)^{\varrho-1}\right]ds + \frac{\Pi_\eta\Upsilon_i^*\varrho}{\Delta(\varrho)\varsigma_i^*\Gamma(\varrho)}\int_{\iota_1}^{\iota_2}(\iota_2-s)^{\varrho-1}ds \\ &+\frac{\rho_i^*\varrho}{\Delta(\varrho)\varsigma_i^*\Gamma(\varrho)}\int_0^{\iota_1}\left[(\iota_2-s)^{\varrho-1} - (\iota_1-s)^{\varrho-1}\right]|\hbar_i(s)|ds + \frac{\rho_i^*\varrho}{\Delta(\varrho)\varsigma_i^*\Gamma(\varrho)}\int_{\iota_1}^{\iota_2}(\iota_2-s)^{\varrho-1}|\hbar_i(s)|ds \\ \leq\; & \frac{\Pi_\eta\Upsilon_i^*}{\Delta(\varrho)\varsigma_i^*\Gamma(\varrho)}\left[\iota_2^\varrho - \iota_1^\varrho\right] + \frac{\rho_i^*\|\hbar_i\|}{\Delta(\varrho)\varsigma_i^*\Gamma(\varrho)}\left[\iota_2^\varrho - \iota_1^\varrho\right]. \end{aligned}$$

So, as $\iota_2 \to \iota_1$ we conclude that $\Psi_i(\hbar_i(\iota_2)) \to \Psi_i(\hbar_i(\iota_1)), i = 1, 2$. Consequently, Ψ is an equicontinuous operator. Thus, by the above obtained with the Arzelà–Ascoli theorem, we conclude that Ψ is completely continuous.

Step 2: We shall prove that the set Φ is bounded, where

$$\Phi = \{(\hbar_1, \hbar_2, \cdots, \hbar_n) \in \mathcal{H}; \hbar_i = \alpha\Psi_i(\hbar_i(\iota)), \alpha \in (0,1)\}.$$

By (3.20), we have

$$\left|{}^{MAB}\mathbb{I}_\iota^\varrho \left[{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n)\right]\right| \le \Pi_\varrho \Pi_\eta \Upsilon_i^*. \tag{3.21}$$

In the same manner, we obtain

$${}^{MAB}\mathbb{I}_\iota^\varrho |\rho_i \hbar_i| \le \rho_i^* \Pi_\varrho \|\hbar_i\|. \tag{3.22}$$

Thus, by (3.21) and (3.22) with (3.7), we get

$$\begin{aligned} \|\hbar_i\| &= |\alpha\Psi_i(\hbar_i(\iota))| \\ &\le \frac{1}{\varsigma_i^*}\left({}^{MAB}\mathbb{I}_\iota^\varrho \left[{}^{MAB}\mathbb{I}_\iota^\eta |g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n)|\right] + {}^{MAB}\mathbb{I}_\iota^\varrho |\rho_i \hbar_i|\right) \\ &\le \frac{\rho_i^*}{\varsigma_i^*}\Pi_\varrho \|\hbar_i\| + \frac{\Pi_\varrho}{\varsigma_i^*}\Pi_\eta \Upsilon_i^*. \end{aligned}$$

Thus, we have

$$\|(\hbar_1, \hbar_2, \cdots, \hbar_2)\| \le \frac{\sum_{i=1}^n \frac{\Pi_\varrho}{\varsigma_i^*}\Pi_\eta \Upsilon_i^*}{1 - \Pi_\varrho \max_{i=1}^n \left\{\frac{\rho_i^*}{\varsigma_i^*}\right\}}.$$

Hence, Φ is bounded. Thus, by Leray-Schauder's alternative theorem, we conclude that the n-coupled MABC-GSL system (1.1) has a solution. □

3.5. Hyers-Ulam stability

Definition 3.6. *[35] The n-coupled MABC-GSL system (1.1) is Hyers-Ulam stable if there exists constant $\psi > 0$ such that, for each $\sigma = \max_{i=1}^n\{\sigma_i\} > 0$, and for each $(\hbar_1, \hbar_2, \cdots, \hbar_n) \in \mathcal{H}$ satisfying the following inequality*

$$\|\hbar_i - \Psi_i(\hbar_i(\iota))\| \le \sigma_i, i = 1, 2, \cdots, n, \tag{3.23}$$

there is a unique solution $\left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)$ of n-coupled MABC-GSL system (1.1) with $\widehat{\hbar}_i(\iota) = \Psi_i\left(\widehat{\hbar}_i(\iota)\right)$ and

$$\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)\right\| \le \sigma\psi. \tag{3.24}$$

The following definition of generalized Hyers-Ulam stability is obtained from Definition 3.6.

Definition 3.7. *[35] The n-coupled MABC-GSL system (1.1) is generalized Hyers-Ulam stable, if there exists a function $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\psi(0) = 0$ such that, for each $\sigma = \max_{i=1}^n\{\sigma_i\} > 0$, and for each $(\hbar_1, \hbar_2, \cdots, \hbar_n) \in \mathcal{H}$ satisfying (3.23), there is a unique solution $\left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)$ of n-coupled MABC-GSL system (1.1) with $\widehat{\hbar}_i(\iota) = \Psi_i\left(\widehat{\hbar}_i(\iota)\right)$ and*

$$\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)\right\| \le \psi(\sigma).$$

Theorem 3.8. *Under the condition (C_2), if $0 < \Pi_\varrho \sum_{i=1}^n \frac{1}{\varsigma_i^*}\left(\Pi_\eta \mathcal{K}_i + \max_{i=1}^n\left\{\rho_i^*\right\}\right) < 1$, then, the n-coupled MABC-GSL system (1.1) is Hyers-Ulam stable and is also generalized Hyers-Ulam stable.*

Proof. Let $(\hbar_1, \hbar_2, \cdots, \hbar_n) \in \mathcal{H}$ such that $\|\hbar_i - \Psi_i(\hbar_i(\iota))\| \le \sigma_i, i = 1, 2, \cdots, n$ and let $\widehat{\hbar_i}$ be a solution of the n-coupled MABC-GSL system (1.1). Then, we have

$$\begin{aligned}
&\left|\Psi_i(\hbar_i(\iota)) - \Psi_i\left(\widehat{\hbar_i}(\iota)\right)\right| \\
= &\left|{}^{MAB}\mathbb{I}_\iota^{\varrho}\frac{{}^{MAB}\mathbb{I}_\iota^{\eta} g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) - \rho_i\hbar_i}{\varsigma_i} - {}^{MAB}\mathbb{I}_\iota^{\varrho}\frac{{}^{MAB}\mathbb{I}_\iota^{\eta} g_i\left(\iota, \widehat{\hbar_1}, \widehat{\hbar_2}, \cdots, \widehat{\hbar_n}\right) - \rho_i\widehat{\hbar_i}}{\varsigma_i}\right| \\
\le &\frac{1}{\varsigma_i^*}{}^{MAB}\mathbb{I}_\iota^{\varrho}\left[{}^{MAB}\mathbb{I}_\iota^{\eta}\left|g_i(\iota, \hbar_1, \hbar_2, \cdots, \hbar_n) - g_i\left(\iota, \widehat{\hbar_1}, \widehat{\hbar_2}, \cdots, \widehat{\hbar_n}\right)\right|\right] + \frac{1}{\varsigma_i^*}{}^{MAB}\mathbb{I}_\iota^{\varrho}\left[\rho_i\left|\hbar_i(\iota) - \widehat{\hbar_i}(\iota)\right|\right].
\end{aligned}$$

By condition (C_2), we have

$$\left\|\Psi_i(\hbar_i(\iota)) - \Psi_i\left(\widehat{\hbar_i}(\iota)\right)\right\| \le \frac{1}{\varsigma_i^*}\Pi_\varrho\Pi_\eta\mathcal{K}_i\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar_1}, \widehat{\hbar_2}, \cdots, \widehat{\hbar_n}\right)\right\| + \frac{\rho_i^*\Pi_\varrho}{\varsigma_i^*}\left\|\hbar_i - \widehat{\hbar_i}\right\|. \quad (3.25)$$

Thus, by (3.23), (3.24), and (3.25), via triangle inequality, we have

$$\begin{aligned}
\left\|\hbar_i - \widehat{\hbar_i}\right\| &= \left\|\hbar_i - \Psi_i(\hbar_i(\iota)) + \Psi_i(\hbar_i(\iota)) - \widehat{\hbar_i}\right\| \\
&\le \|\hbar_i - \Psi_i(\hbar_i(\iota))\| + \left\|\Psi_i(\hbar_i(\iota)) - \Psi_i\left(\widehat{\hbar_i}(\iota)\right)\right\| \\
&\le \sigma_i + \Pi_\varrho\Pi_\eta\frac{\mathcal{K}_i}{\varsigma_i^*}\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar_1}, \widehat{\hbar_2}, \cdots, \widehat{\hbar_n}\right)\right\| + \frac{\rho_i^*\Pi_\varrho}{\varsigma_i^*}\left\|\hbar_i - \widehat{\hbar_i}\right\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar_1}, \widehat{\hbar_2}, \cdots, \widehat{\hbar_n}\right)\right\| &= \sum_{i=1}^{n}\left\|\hbar_i - \widehat{\hbar_i}\right\| \\
&\le \frac{\sigma_i}{1 - \Pi_\varrho\sum_{i=1}^{n}\frac{1}{\varsigma_i^*}\left(\Pi_\eta\mathcal{K}_i + \max_{i=1}^{n}\left\{\rho_i^*\right\}\right)}. \quad (3.26)
\end{aligned}$$

We put $\psi = \frac{1}{1-\Pi_\varrho\sum_{i=1}^{n}\frac{1}{\varsigma_i^*}\left(\Pi_\eta\mathcal{K}_i+\max_{i=1}^{n}\{\rho_i^*\}\right)} > 0$ and $\sigma = \max_{i=1}^{n}\{\sigma_i\}$ in (3.26). Then, we get

$$\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar_1}, \widehat{\hbar_2}, \cdots, \widehat{\hbar_n}\right)\right\| \le \psi\sigma.$$

Thus, the n-coupled MABC-GSL system (1.1) is Hyers-Ulam stable.

Now, by assuming

$$\psi(\sigma_i) = \frac{\sigma_i}{1 - \Pi_\varrho\sum_{i=1}^{n}\frac{1}{\varsigma_i^*}\left(\Pi_\eta\mathcal{K}_i + \max_{i=1}^{n}\left\{\rho_i^*\right\}\right)},$$

with $\psi(0) = 0$, then the n-coupled MABC-GSL system (1.1) is generalized Hyers-Ulam stable. □

3.6. Hyers-Ulam-Rassias stability

To prove the Hyers-Ulam-Rassias stability, we assume that there is an increasing function $\chi \in C(\Omega, \mathbb{R}^+)$ and there is $\Delta_\chi > 0$ such that ${}^{MAB}\mathbb{I}_\iota^{\varrho}\chi(\iota) = \Delta_\chi\chi(\iota)$, for all $\iota \in \Omega$.

Definition 3.9. *[35] The n-coupled MABC-GSL system (1.1) is Hyers-Ulam-Rassias stable with respect to an increasing function $\chi(\iota)$, if there exists constant $\psi > 0$ such that, for each $\sigma = \max_{i=1}^{n}\{\sigma_i\} > 0$, and for each $(\hbar_1, \hbar_2, \cdots, \hbar_n) \in \mathcal{H}$ satisfying the following inequality*

$$\|\hbar_i - \Psi_i(\hbar_i(\iota))\| \leq \sigma_i \Delta_\chi \chi(\iota), i = 1, 2, \cdots, n, \tag{3.27}$$

there is a unique solution $\left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)$ of n-coupled MABC-GSL system (1.1) with $\widehat{\hbar}_i(\iota) = \Psi_i\left(\widehat{\hbar}_i(\iota)\right)$ and

$$\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)\right\| \leq \psi\sigma\chi(\iota). \tag{3.28}$$

Definition 3.10. *[35] The n-coupled MABC-GSL system (1.1) is generalized Hyers-Ulam-Rassias stable with respect to an increasing function $\chi(\iota)$ if there exists constant $\psi > 0$ such that, for each $(\hbar_1, \hbar_2, \cdots, \hbar_n) \in \mathcal{H}$ satisfying (3.27), there is a unique solution $\left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)$ of n-coupled MABC-GSL system (1.1) with*

$$\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)\right\| \leq \psi\chi(\iota).$$

The Ulam-Hyers-Rassias and generalized Hyers-Ulam-Rassias stability for n-coupled MABC-GSL system (1.1) is discussed in the following theorem.

Theorem 3.11. *Under the conditions in Theorem 3.2, the n-coupled MABC-GSL system (1.1) is Hyers-Ulam-Rassias stable and is also generalized Hyers-Ulam-Rassias stable.*

Proof. Let $(\hbar_1, \hbar_2, \cdots, \hbar_n) \in \mathcal{H}$ satisfying (3.27) and let $\widehat{\hbar}_i$ be a solution of the n-coupled MABC-GSL system (1.1). Then, by (3.7), we have

$$\left\|\Psi_i(\hbar_i(\iota)) - \Psi_i\left(\widehat{\hbar}_i(\iota)\right)\right\| \leq \frac{1}{\varsigma_i^*}\Pi_\varrho\Pi_\eta\mathcal{K}_i\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)\right\| + \frac{\rho_i^*\Pi_\varrho}{\varsigma_i^*}\left\|\hbar_i - \widehat{\hbar}_i\right\|.$$

Thus, by (3.27) and (3.28), via triangle inequality, we have

$$\begin{aligned}\left\|\hbar_i - \widehat{\hbar}_i\right\| &\leq \|\hbar_i - \Psi_i(\hbar_i(\iota))\| + \left\|\Psi_i(\hbar_i(\iota)) - \Psi_i\left(\widehat{\hbar}_i(\iota)\right)\right\| \\ &\leq \sigma_i\Delta_\chi\chi(\iota) + \Pi_\varrho\Pi_\eta\frac{\mathcal{K}_i}{\varsigma_i^*}\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)\right\| + \frac{\rho_i^*\Pi_\varrho}{\varsigma_i^*}\left\|\hbar_i - \widehat{\hbar}_i\right\|.\end{aligned}$$

Consequently,

$$\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)\right\| \leq \frac{\sigma_i\Delta_\chi\chi(\iota)}{1 - \Pi_\varrho\sum_{i=1}^{n}\frac{1}{\varsigma_i^*}\left(\Pi_\eta\mathcal{K}_i + \max_{i=1}^{n}\left\{\rho_i^*\right\}\right)}.$$

Putting $\psi = \frac{\Delta_\chi}{1-\Pi_\varrho\sum_{i=1}^{n}\frac{1}{\varsigma_i^*}\left(\Pi_\eta\mathcal{K}_i+\max_{i=1}^{n}\{\rho_i^*\}\right)} > 0$ and $\sigma = \max_{i=1}^{n}\{\sigma_i\}$, then (3.26) becomes

$$\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)\right\| \leq \psi\sigma\chi(\iota).$$

Thus, the n-coupled MABC-GSL system (1.1) is Hyers-Ulam-Rassias stable. Now, let $\sigma = 1$. Then

$$\left\|(\hbar_1, \hbar_2, \cdots, \hbar_n) - \left(\widehat{\hbar}_1, \widehat{\hbar}_2, \cdots, \widehat{\hbar}_n\right)\right\| \leq \psi\chi(\iota).$$

Then, the n-coupled MABC-GSL system (1.1) is generalized Hyers-Ulam-Rassias stable. □

4. Numerical algorithm for nonlinear MABC-GSL

A numerical algorithm is a step-by-step procedure or a sequence of computational operations designed to solve a numerical problem or perform numerical computations. These algorithms often involve mathematical operations such as arithmetic calculations, numerical approximations, iterative methods, and numerical techniques for solving equations or systems of equations. They are implemented using programming languages and are executed on computers or other computational devices. Numerical algorithms aim to provide efficient and accurate solutions to numerical problems, taking into consideration factors such as computational complexity, numerical stability, convergence, and precision. They play a crucial role in various areas of scientific research, engineering design, data analysis, and computer simulations, enabling researchers and practitioners to solve complex mathematical problems and make quantitative predictions. For more information, see [1, 2, 20, 30]. In this work, we apply Lagrange's interpolation technique to obtain a numerical algorithm for nonlinear MABC-GSL. Let us consider the MABC-GSL system (1.1) with fixed point $\hbar_i$ such that

$$\begin{aligned}\hbar_i(\iota) &= \frac{1-\varrho}{\Delta(\varrho)}\left(\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n)-\rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)}\right)\\ &\quad+\frac{\varrho}{\Delta(\varrho)\Gamma(\varrho)}\int_0^\iota(\iota-s)^{\varrho-1}\left(\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(s,\hbar_1,\hbar_2,\cdots,\hbar_n)-\rho_i(s)\hbar_i(s)}{\varsigma_i(s)}\right)ds. \qquad (4.1)\end{aligned}$$

Define the nonlinear function $\mathbb{F}_i(\iota,\hbar_i(\iota)), i=1,2,\cdots,n$ as

$$\mathbb{F}_i(\iota,\hbar_i(\iota))=\left(\frac{{}^{MAB}\mathbb{I}_\iota^\eta g_i(\iota,\hbar_1,\hbar_2,\cdots,\hbar_n)-\rho_i(\iota)\hbar_i(\iota)}{\varsigma_i(\iota)}\right),$$

where $\mathbb{F}_i(\iota,\hbar_i(\iota))|_{\iota=0}=0$. Thus (4.1) becomes

$$\hbar_i(\iota)=\frac{1-\varrho}{\Delta(\varrho)}\mathbb{F}_i(\iota,\hbar_i(\iota))+\frac{\varrho}{\Delta(\varrho)\Gamma(\varrho)}\int_0^\iota(\iota-s)^{\varrho-1}\mathbb{F}_i(s,\hbar_i(s))\,ds. \qquad (4.2)$$

By discretizing Eq (4.2) at $\iota=\iota_{m+1}=(m+1)h$, where h represents the time step size, we obtain the following discrete equations:

$$\hbar_i(\iota_{m+1})=\frac{1-\varrho}{\Delta(\varrho)}\mathbb{F}_i(\iota_m,\hbar_i(\iota_m))+\frac{\varrho}{\Delta(\varrho)\Gamma(\varrho)}\int_0^{\iota_{m+1}}(\iota_{m+1}-s)^{\varrho-1}\mathbb{F}_i(s,\hbar_i(s))\,ds. \qquad (4.3)$$

By the Lagrange's interpolation, we have

$$\begin{aligned}\mathbb{F}_i(\iota,\hbar_i(\iota)) &= \frac{\mathbb{F}_i(\iota_n,\hbar_i(\iota_n)(\iota-\iota_{n-1})}{\iota_k-\iota_{k-1}}-\frac{\mathbb{F}_i(\iota_{n-1},\hbar_i(\iota_{n-1}))(\iota-\iota_n)}{\iota_k-\iota_{k-1}}\\ &= \frac{\mathbb{F}_i(\iota_n,\hbar_i(\iota_n))(\iota-\iota_{n-1})}{h}-\frac{\mathbb{F}_i(\iota_{n-1},\hbar_i(\iota_{n-1})(\iota-\iota_n)}{h}. \qquad (4.4)\end{aligned}$$

By the help of (4.3) and (4.4), we have

$$
\begin{aligned}
\hbar_i(\iota_{m+1}) &= \frac{1-\varrho}{\Delta(\varrho)}\mathbb{F}_i\left(\iota_m, \hbar_i(\iota_m)\right) + \frac{\varrho}{\Delta(\varrho)\Gamma(\varrho)}\sum_{n=1}^{m}\int_{\iota_n}^{\iota_{n+1}}(\iota_{m+1}-s)^{\varrho-1}\frac{\mathbb{F}_i(\iota_n, \hbar_i(\iota_n)(s-\iota_{n-1})}{h}ds \\
&\quad -\frac{\varrho}{\Delta(\varrho)\Gamma(\varrho)}\sum_{n=1}^{m}\int_{\iota_n}^{\iota_{n+1}}(\iota_{m+1}-s)^{\varrho-1}\frac{\mathbb{F}_i(\iota_{n-1}, \hbar_i(\iota_{n-1})(s-\iota_n)}{h}ds.
\end{aligned}
$$

Now, by computing the above two integrals separately, we get

$$
\begin{aligned}
\hbar_i(\iota_{m+1}) &= \frac{1-\varrho}{\Delta(\varrho)}\mathbb{F}_i\left(\iota_m, \hbar_i(\iota_m)\right) + \frac{\varrho h^{\varrho}}{\Delta(\varrho)\Gamma(\varrho+2)}\sum_{n=1}^{m}\mathbb{F}_i(\iota_n, \hbar_i(\iota_n) \\
&\quad \times\left[(m-n+1)^{\varrho}(2+m-n+\varrho)-(m-n)^{\varrho}(m-n+2+2\varrho)\right] \\
&\quad -\frac{\varrho h^{\varrho}}{\Delta(\varrho)\Gamma(\varrho+2)}\sum_{n=1}^{m}\mathbb{F}_i(\iota_{n-1}, \hbar_i(\iota_{n-1}) \\
&\quad \times\Big[(m-n+1)^{\varrho+1}-(m-n+1+\varrho)(m-n)^{\varrho}\Big]. \qquad (4.5)
\end{aligned}
$$

4.1. Application of the numerical scheme to the influenza epidemic model

In this subsection, we present the influenza epidemic model (SEIR) under the MABC fractional operator

$$
\begin{aligned}
{}^{MABC}\mathbb{D}_{\iota}^{\varrho}S(\iota) &= -\frac{\alpha_2}{N}I(\iota)S(\iota) + \alpha_0 N - \alpha_1 S(\iota), \qquad (4.6)\\
{}^{MABC}\mathbb{D}_{\iota}^{\varrho}E(\iota) &= -(\alpha_1+\alpha_3)E(\iota) + \frac{\alpha_2}{N}I(\iota)S(\iota), \qquad (4.7)\\
{}^{MABC}\mathbb{D}_{\iota}^{\varrho}I(\iota) &= -(\alpha_1+\alpha_4)I(\iota) + \alpha_3 E(\iota), \qquad (4.8)\\
{}^{MABC}\mathbb{D}_{\iota}^{\varrho}R(\iota) &= -\alpha_1 R(\iota) + \alpha_4 I(\iota). \qquad (4.9)
\end{aligned}
$$

In the above (SEIR) model, the susceptible S compartment transitions to the exposed E compartment through effective contact transmission from infected individuals I at a rate of α_2. In Eq (4.6): The term $-\frac{\alpha_2}{N}I(\iota)S(\iota)$ represents the rate at which susceptible individuals get infected by coming into contact with infected individuals. The parameter α_2 represents the transmission rate and N is the total population size. The term $\alpha_0 N - \alpha_1 S(\iota)$ represents the natural birth and death rates affecting the susceptible population. α_0 is the natural birth rate, and α_1 is the natural death rate. In Eq (4.7): The term $-(\alpha_1+\alpha_3)E(\iota)$ represents the rate at which individuals in the exposed compartment transition to other compartments. The parameter α_1 represents the natural death rate, and α_3 represents the incubation rate. The term $\frac{\alpha_2}{N}I(\iota)S(\iota)$ represents the rate at which susceptible individuals become exposed by coming into contact with infected individuals. In Eq (4.8): The term $-(\alpha_1+\alpha_4)I(\iota)$ represents the rate at which infected individuals recover or die. The parameter α_1 is the natural death rate, and α_4 is the recovery rate. The term $\alpha_3 E(\iota)$ represents the rate at which individuals in the exposed compartment transition to the infected compartment. In Eq (4.9): The term $-\alpha_1 R(\iota)$ represents the rate at which recovered individuals experience natural death. α_1 is the natural death rate. The term $\alpha_4 I(\iota)$ represents the rate at which infected individuals recover. The model parameters are estimated using a fitting technique that makes use of the non-linear least squares algorithm [9], which are defined in Table 1.

Table 1. The parameters and their descriptions for the model under consideration.

Parameter	Description	Numerical estimation	Ref
α_0	The natural birth rate	0.001	Estimated
α_1	The natural death rate	0.001	Estimated
α_2	The transmission rate	0.98	Estimated
α_3	The incubation rate	0.78	Estimated
α_4	Recovery rate	0.62	Estimated

We present the sensitivity indices of the parameters of the model given in Figure 1.

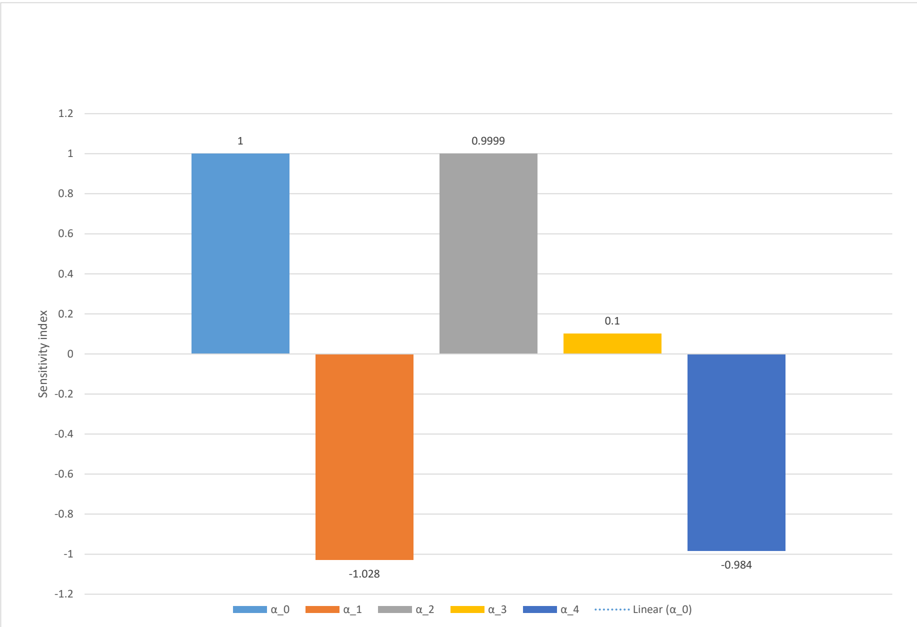

Figure 1. Presentation of sensitivity indices involved in the computation of R_0.

By (4.5), the numerical scheme of the (SEIR) model is given as follows:

$$\begin{aligned} S(\iota_{m+1}) - S(0) &= \frac{1-\varrho}{\Delta(\varrho)}\left(-\frac{\alpha_2}{N} I(\iota_m) S(\iota_m) + \alpha_0 N - \alpha_1 S(\iota_m)\right) \\ &+ \frac{\varrho h^{\varrho}}{\Delta(\varrho)\Gamma(\varrho+2)} \sum_{n=1}^{m} \left(-\frac{\alpha_2}{N} I(\iota_n) S(\iota_n) + \alpha_0 N - \alpha_1 S(\iota_n)\right) \\ &\times [(m-n+1)^{\varrho}(2+m-n+\varrho) - (m-n)^{\varrho}(m-n+2+2\varrho)] \\ &- \frac{\varrho h^{\varrho}}{\Delta(\varrho)\Gamma(\varrho+2)} \sum_{n=1}^{m} \left(-\frac{\alpha_2}{N} I(\iota_{n-1}) S(\iota_{n-1}) + \alpha_0 N - \alpha_1 S(\iota_{n-1})\right) \\ &\times \left[(m-n+1)^{\varrho+1} - (m-n+1+\varrho)(m-n)^{\varrho}\right]. \end{aligned} \tag{4.10}$$

$$
\begin{aligned}
E(\iota_{m+1})-E(0) \;=\;& \frac{1-\varrho}{\Delta(\varrho)}\Big(-(\alpha_1+\alpha_3)\,E(\iota_m)+\frac{\alpha_2}{N}I(\iota_m)\,S(\iota_m)\Big)\\
&+\frac{\varrho h^{\varrho}}{\Delta(\varrho)\Gamma(\varrho+2)}\sum_{n=1}^{m}\Big(-(\alpha_1+\alpha_3)\,E(\iota_n)+\frac{\alpha_2}{N}I(\iota_n)\,S(\iota_n)\Big)\\
&\times[(m-n+1)^{\varrho}(2+m-n+\varrho)-(m-n)^{\varrho}(m-n+2+2\varrho)]\\
&-\frac{\varrho h^{\varrho}}{\Delta(\varrho)\Gamma(\varrho+2)}\sum_{n=1}^{m}\Big(-(\alpha_1+\alpha_3)\,E(\iota_{n-1})+\frac{\alpha_2}{N}I(\iota_{n-1})\,S(\iota_{n-1})\Big)\\
&\times\Big[(m-n+1)^{\varrho+1}-(m-n+1+\varrho)(m-n)^{\varrho}\Big]. \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
I(\iota_{m+1})-I(0) \;=\;& \frac{1-\varrho}{\Delta(\varrho)}\left(-(\alpha_1+\alpha_4)\,I(\iota_m)+\alpha_3E(\iota_m)\right)\\
&+\frac{\varrho h^{\varrho}}{\Delta(\varrho)\Gamma(\varrho+2)}\sum_{n=1}^{m}\left(-(\alpha_1+\alpha_4)\,I(\iota_n)+\alpha_3E(\iota_n)\right)\\
&\times[(m-n+1)^{\varrho}(2+m-n+\varrho)-(m-n)^{\varrho}(m-n+2+2\varrho)]\\
&-\frac{\varrho h^{\varrho}}{\Delta(\varrho)\Gamma(\varrho+2)}\sum_{n=1}^{m}\left(-(\alpha_1+\alpha_4)\,I(\iota_{n-1})+\alpha_3E(\iota_{n-1})\right)\\
&\times\Big[(m-n+1)^{\varrho+1}-(m-n+1+\varrho)(m-n)^{\varrho}\Big]. \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
R(\iota_{m+1})-R(0) \;=\;& \frac{1-\varrho}{\Delta(\varrho)}\left(-\alpha_1R(\iota_m)+\alpha_4I(\iota_m)\right)\\
&+\frac{\varrho h^{\varrho}}{\Delta(\varrho)\Gamma(\varrho+2)}\sum_{n=1}^{m}\left(-\alpha_1R(\iota_n)+\alpha_4I(\iota_n)\right)\\
&\times[(m-n+1)^{\varrho}(2+m-n+\varrho)-(m-n)^{\varrho}(m-n+2+2\varrho)]\\
&-\frac{\varrho h^{\varrho}}{\Delta(\varrho)\Gamma(\varrho+2)}\sum_{n=1}^{m}\left(-\alpha_1R(\iota_{n-1})+\alpha_4I(\iota_{n-1})\right)\\
&\times\Big[(m-n+1)^{\varrho+1}-(m-n+1+\varrho)(m-n)^{\varrho}\Big]. \qquad (4.13)
\end{aligned}
$$

For the initial conditions, we initialize the system as follows: $N = 48{,}000{,}000$, $S(0) = 47{,}999{,}990$, $E(0) = 3$, $I(0) = 7$, and $R(0) = 0$.

We have presented the numerical results for different compartments of the proposed model in Figures 2–13, respectively, by using three different sets of fractional orders. We see a decline in the susceptible class as shown in three different figures (Figures 2, 6, and 10) using various fractional order values. In the same way, we have shown the numerical results for an exposed class using three different sets of fractional order values in Figures 3, 7, and 11. Proceeding with the same procedures, we have presented numerically the graphical presentations of infected and recovered classes for various fractional order values in Figures 4, 8, and 12, and 5, 9, and 13, respectively.

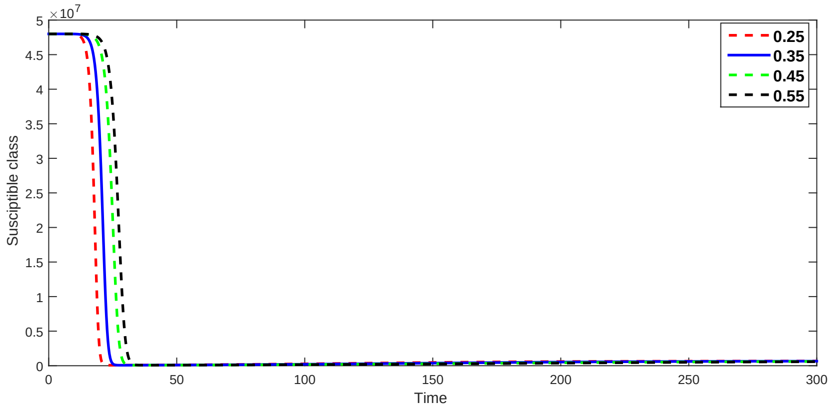

Figure 2. Susceptible populations S for different values of fractional orders.

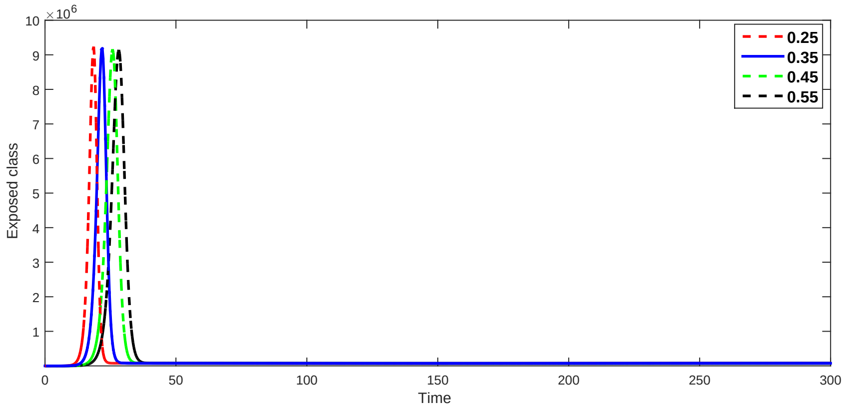

Figure 3. Exposed populations E for different values of fractional orders.

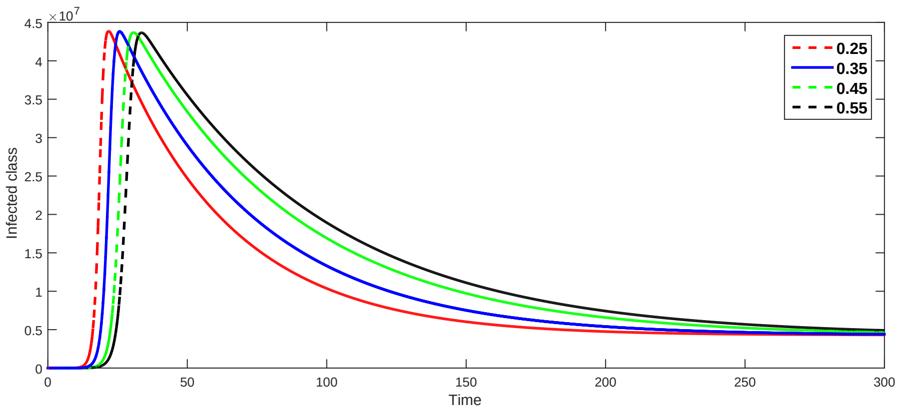

Figure 4. Infected populations I for different values of fractional orders.

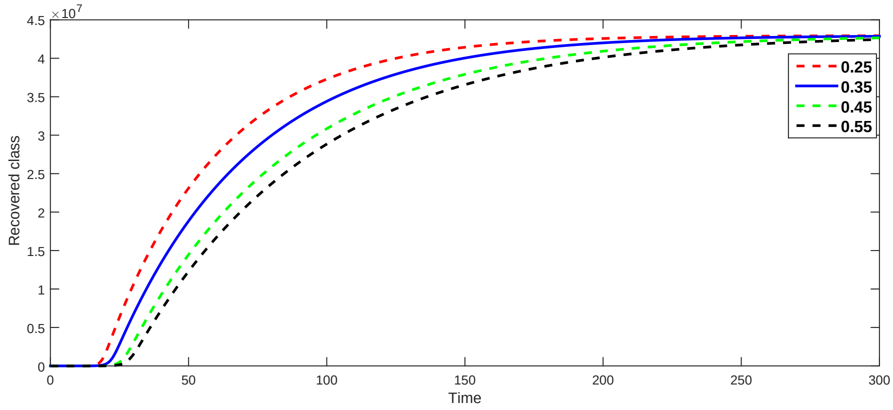

Figure 5. Recovered populations R for different values of fractional orders.

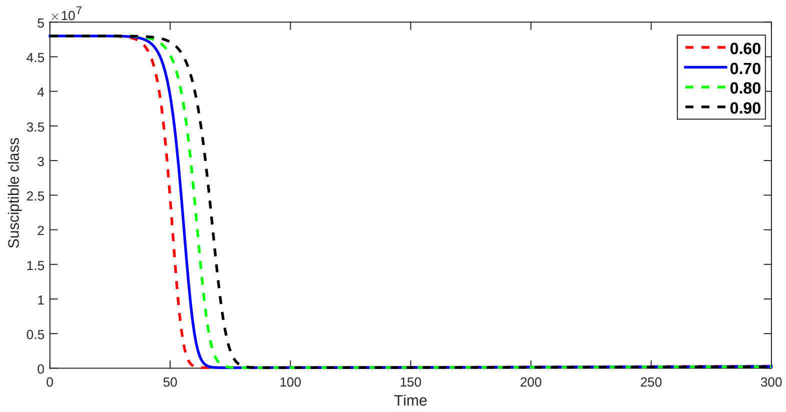

Figure 6. Susceptible populations S for different values of fractional orders.

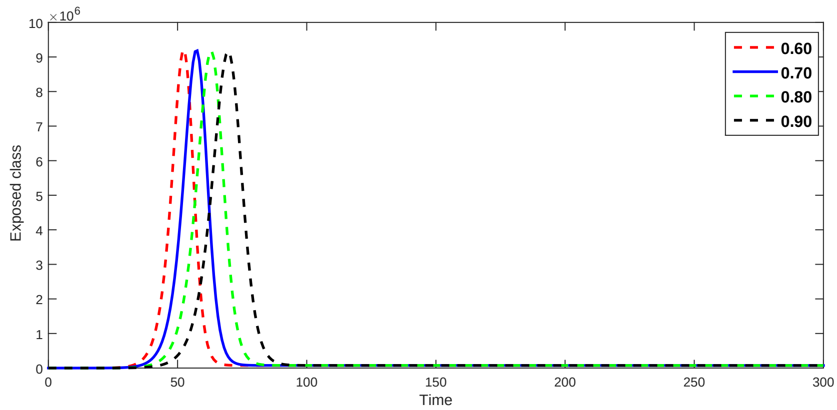

Figure 7. Exposed populations E for different values of fractional orders.

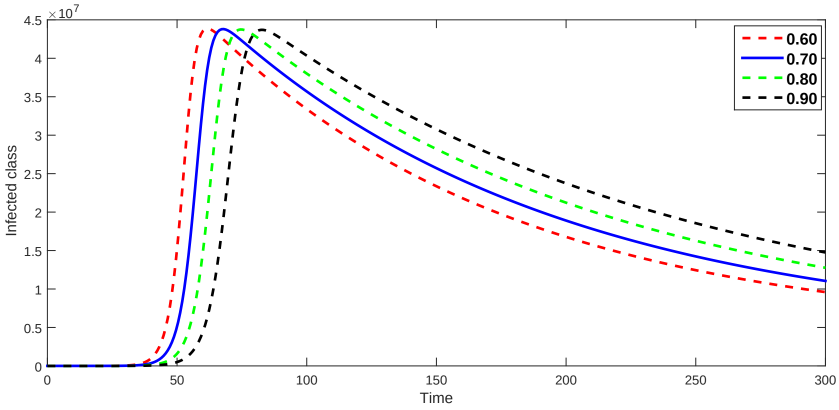

Figure 8. Infected populations I for different values of fractional orders.

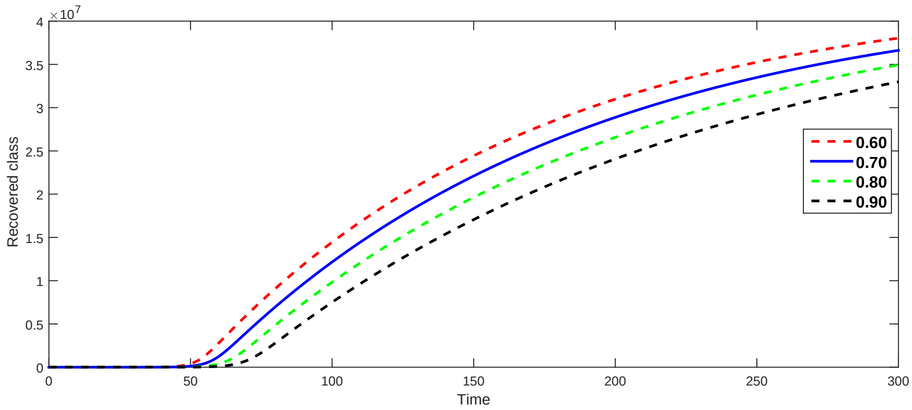

Figure 9. Recovered populations R for different values of fractional orders.

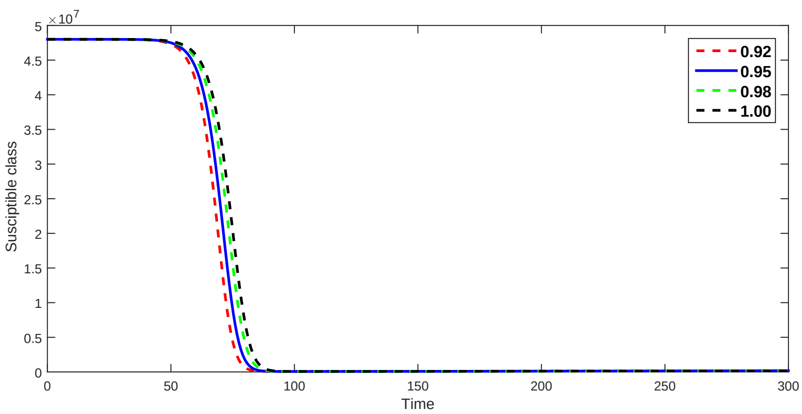

Figure 10. Susceptible populations S for different values of fractional orders.

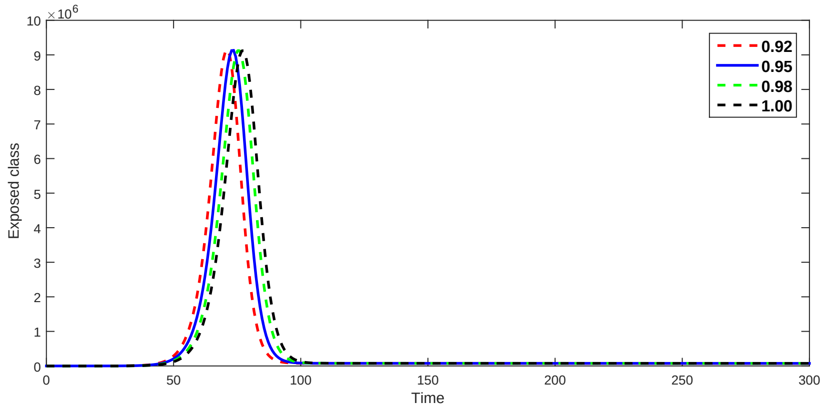

Figure 11. Exposed populations E for different values of fractional orders.

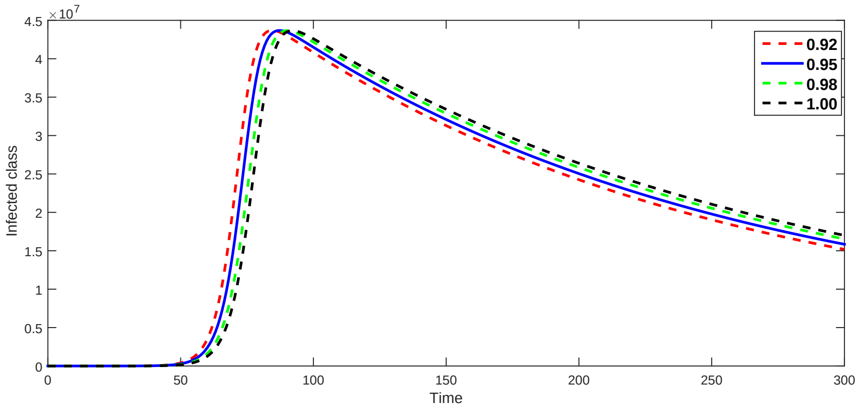

Figure 12. Infected populations I for different values of fractional orders.

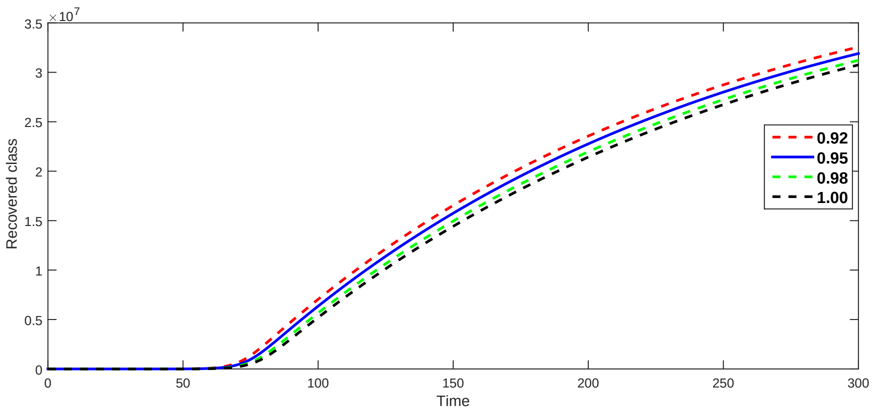

Figure 13. Recovered populations R for different values of fractional orders.

The findings from the sensitivity analysis of the fractional MABC-SEIR model provide valuable insights into the factors that influence the dynamics of influenza transmission. The identified sensitive parameters, including the natural birth rate, transmission rate, and incubation rate, play crucial roles in shaping the prevalence of influenza. It is important to accurately estimate these parameters, as even small changes in their values can have a substantial impact on the quantitative outcomes of the model. By identifying the sensitive parameters, this analysis highlights the importance of collecting accurate data and conducting precise estimations for these key factors. This information can aid in developing more effective strategies for controlling and managing the spread of influenza. Public health interventions can be designed to target these sensitive parameters, focusing on reducing the transmission rate, implementing preventive measures during the incubation period, and considering the demographic factors related to the birth rate.

On the other hand, the insensitive parameters identified in the sensitivity analysis do not require precise estimation. Fluctuations in these parameters have minimal impact on the target variable, and therefore, their values can be approximated or assumed within a reasonable range without significantly affecting the model's predictive capabilities.

Overall, the fractional MABC-SEIR model, coupled with sensitivity analysis, provides a promising approach for predicting confirmed influenza cases. It allows for a better understanding of the complex dynamics of influenza transmission and can assist in decision-making processes related to public health interventions, resource allocation, and outbreak management. However, it is important to note that the model's effectiveness relies heavily on the accuracy of parameter estimation and the availability of reliable data.

5. Conclusions

In this work, we have conducted a comprehensive investigation into a system consisting of n-coupled equations, specifically generalized Sturm-Liouville and Langevin equations. The primary focus was on understanding the behavior of this system when subjected to a MABC fractional derivative. Through rigorous analysis, we have gained valuable insights into the dynamics of the system and the influence of the modified operator on the existence and uniqueness of solutions. To establish the uniqueness of solutions, we employed the Banach contraction principle, a powerful mathematical tool. Additionally, Leray-Schauder's alternative fixed-point theorem was utilized to determine the existence of solutions. This theorem provided a robust framework for examining the properties of the solutions within the system. Furthermore, we discussed the Hyers-Ulam stability of the system, enabling us to assess the stability properties of the solutions and evaluate their sensitivity to perturbations. By understanding the stability characteristics, we gained a deeper understanding of the overall behavior of the system and its response to external influences. In addition to the theoretical analysis, we applied the numerical scheme of MABC to model and simulate the influenza epidemic. This real-world application demonstrated the versatility and effectiveness of the MABC method. By successfully applying the MABC approach, we showcased its potential for capturing and predicting the dynamics of infectious diseases, offering valuable insights into the field of epidemiology. By combining these approaches, our study has made significant contributions to the understanding of the behavior of n-coupled systems under a MABC fractional derivative. We have provided insights into the dynamics of the system, the uniqueness of solutions, the stability properties, and the practical applications in epidemiology. This research expands the current knowledge in the field and serves as a foundation for further investigations and advancements in related areas. In future research, a natural extension of this work would be to analyze systems of delay equations under the influence of the piecewise MABC fractional operator.

This extension would allow for a deeper exploration of the behavior of systems with time delays, providing further insights into complex dynamical phenomena.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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