AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 14213–14227.
DOI: 10.3934/math.2024690
Received: 08 February 2024
Revised: 11 April 2024
Accepted: 12 April 2024
Published: 18 April 2024

Research article

Separation axioms via novel operators in the frame of topological spaces and applications

Mesfer H. Alqahtani[1] **and Alaa M. Abd El-latif**[2,*]

[1] Department of Mathematics, University College of Umluj, University of Tabuk, Tabuk 48322, Saudi Arabia

[2] Mathematics Department, College of Sciences and Arts, Northern Border University, Rafha 91911, Saudi Arabia

* **Correspondence:** Email: Alaa.ali@nbu.edu.sa, Alaa_8560@yahoo.com.

Abstract: In this work, we introduce a very wide category of open sets in topological spaces, called $\aleph$-open sets. We study the category of $\aleph$-open sets that contains β-open sets in addition to β^*-open and e^*-open sets. We present the essential properties of this class and disclose its relationships with many different classes of open sets with the help of concrete counterexamples. In addition, we introduce the $\aleph$-interior and $\aleph$-closure operators. Moreover, we study the concept of $\aleph$-continuity of functions inspired by the classes of $\aleph$-open and $\aleph$-closed sets. Also, we discuss some kinds of separation axioms and some theorems related to the graph of functions.

Keywords: $\aleph$-open set; $\aleph$-continuous function; graph of functions; $\aleph$-T_0-space; $\aleph$-T_1-space; $\aleph$-T_2-space; applications
Mathematics Subject Classification: 54A05, 54C05, 54C08

1. Introduction

Open sets are one of the most important keys to topology. Many researchers have developed different versions of open sets including their weaker and stronger versions. The first development was done by Levine [1] in 1963, where he presented the notions of semi-open (closed) sets, and semi-continuity of functions. After that, in 1965, Njåstad [2] introduced the concepts of α-open sets. The approaches of pre-open sets were presented by Mashhour et al. [3] as a tool to study precontinuous and weak precontinuous functions. In 1983, Abd-El-Monsef et al. [4] suggested the notions of β-open sets to investigate β-continuous functions. The notions of b-open or γ-open sets were studied in detail in [5, 6] including γ-continuous functions, and were further developed in [7] in terms of the convergence of nets. The somewhat openness of sets was defined by Piotrowski [8] to characterize somewhat continuity defined in [9]. Recently, Gao and Khalilthe [10] introduced more concepts of $\mathfrak{D}\alpha$-Closed sets as a tool for developing many fields of mathematics. Also, the concept of somewhat continuity of function was developed and extended in [11–13].

In 1968, Veličko [14] introduced the concepts of the δ-interior and δ-closure of sets. Several authors have obtained interesting classes of open sets via δ-interior and δ-closure, such as a-open [15], Z-open [16], δ-semiopen [17], δ-θ-semiopen [18], δ-preopen [19], β^*-open [20], δ-semiregular [18], and e^*-open [21] sets.

This research aims to find a definition called $\aleph$-open, which includes most different classes of open sets, prove its difference from previous similar definitions, and determine its basic properties.

The motive for drafting this paper is as follows: First, we anticipate that by creating a new category of open sets in the frame of topological spaces, we will simplify the path for many future articles on this topic. Second, researchers can investigate alternative concepts, such as covering characteristics and separation axioms via the proposed class of $\aleph$-open and $\aleph$-closed sets. Finally, it reinforces the importance of the concept of classical topology as it is a strong tributary to other modern concepts such as soft and fuzzy topology.

The structure of this work is presented as follows: In Section 2, We recall the fundamental ideas and conclusions that make this work self-contained. In Section 3, we present the main properties of this class and disclose its relationships with many different classes of open sets with the help of appropriate examples and counterexamples. Then, we prove that any e^*-open set is $\aleph$-open, but that the converse may not always be true. The concepts of $\aleph$-interior and $\aleph$-closure operators via $\aleph$-open and $\aleph$-closed sets, respectively, are introduced. In Section 4, we study the concept of $\aleph$-continuity and prove that it does not give continuity and vice versa, and we provide examples of this. After that, we develop several important main theorems about the $\aleph$-continuity of functions via the classes of $\aleph$-open and $\aleph$-closed sets. In Section 5, we study some types of separation axioms via $\aleph$-open and $\aleph$-closed sets and prove their difference from the separation axioms in Euclidean topology, supporting this with several examples. Finally, we present some theorems related to the graph of functions. In Section 6, we end the work with a brief conclusion and potential lines of future research.

2. Definitions and background

In this paper, spaces imply topological spaces on which no other property is presumed. For a subset F of a space Z, we denote the complement (resp., interior, closure, frontier) of F in Z by $Z \setminus F$ (resp., F°, $\overline{F}$, $fr(F)$). We denote the arbitrary indexing set by Θ.

A subset F of a space Z is called nowhere dense if its closure has empty interior. We also say that F is nowhere dense in Z if F is not dense in any non-empty open subset V of Z. The arbitrary union of nowhere dense sets is a subset of the nowhere dense set of the union. A subset F of a space Z is called δ-open [14] if, for each element z in F, there exists a set K such that $z \in \overline{K}^{\circ} \subseteq F$. Moreover, $Z \setminus F$ is called δ-closed. A point $z \in Z$ is called a δ-accumulation point of F [14] if $\overline{K}^{\circ} \cap F \neq \emptyset$ for each open set K, where $z \in K$. The set of all δ-accumulation points of F is called the δ-closure of F and is denoted by $cl_\delta(F)$ or $\overline{F}^{\delta}$. The δ-interior of F is the union of all δ-open sets contained in F and is denoted by $int_\delta(F)$ or F°_δ}. A subset T of a space Z is called nowhere dense (resp., semi-open [1], somewhere dense [2], α-open [2], pre-open [3], β-open [4], b-open [5] or sp-open [6] or γ-open [22], a-open [15], Z-open [16], δ-semiopen [17], δ-preopen [19], β^*-open [20], e^*-open [21]), F-open [23],

C-open [24] sets if $\overline{T}^{\circ} = \emptyset$ (resp., $T \subseteq \overline{T^{\circ}}$, $\overline{T}^{\circ} \neq \emptyset$, $T \subseteq \overline{T^{\circ}}^{\circ}$, $T \subseteq \overline{T}^{\circ}$, $T \subseteq \overline{\overline{T}^{\circ}}$, $T \subseteq (\overline{T}^{\circ} \cup \overline{T^{\circ}})$, $T \subseteq \overline{T^{\circ\delta}}^{\circ}$, $T \subseteq \overline{T}^{\circ} \cup \overline{T^{\circ\delta}}$, $T \subseteq \overline{T^{\circ\delta}}$, $T \subseteq (\overline{T}^{\delta})^{\circ}$, $T \subseteq \overline{\overline{T}^{\circ}} \cup (\overline{T}^{\delta})^{\circ}$, $T \subseteq \overline{((\overline{T}^{\delta})^{\circ})}$, T is open and $\overline{T} \setminus T$ is finite, T is open and $\overline{T} \setminus T$ is countable).

3. ℵ-open sets and relationships

This section begins by presenting the definitions of ℵ-open and ℵ-closed sets and the characterizations based on them, which will be applied in what follows.

Definition 1. A subset F of space Z is said to be an ℵ-open set when either $F = \emptyset$, or $F \subseteq \overline{F}^{\circ} \cup fr(F)$ when F is somewhere dense, or $F \subseteq fr(F)$ when F is nowhere dense and $fr(F)$ is infinite.

Remark 2. The complement of an ℵ-open subset of space Z, is said to be ℵ-closed. Moreover, the collection of all ℵ-open (resp. ℵ-closed) subsets of space Z is denoted by $\aleph O(Z)$ (resp., $\aleph C(Z)$).

Lemma 3. *[21] The following are equivalent for a subset F of space Z:*

(1) F is e^*-open;
(2) There exists a δ-preopen set V such that $V \subseteq \overline{F}^{\delta} \subseteq \overline{V}^{\delta}$;
(3) $\overline{F}^{\delta} = \overline{F^{\circ}}$.

In [20, Remark 2.1], the authors proved that any e^*-open set need not be open (resp., pre-open, semi-open, α-open, β-open, b-open, and β^*-open) set. In the next proposition, we shall show that, our new notion is generalized to all the above-mentioned concepts.

Proposition 4. *Any e^*-open subset F of space Z is ℵ-open.*

Proof. Suppose that F is an e^*-open subset of Z. If $F = \emptyset$, then F is ℵ-open. Assume that $F \neq \emptyset$. Now, if F is nowhere dense, then F is not e^*-open, because $F \nsubseteq \overline{((\overline{F}^{\delta})^{\circ})} = \overline{((\overline{F^{\circ}})^{\circ})} = \emptyset$. Suppose that F is somewhere dense, then $F \subseteq \overline{((\overline{F^{\circ}})^{\circ})} \subseteq \overline{F} = \overline{F}^{\circ} \cup (\overline{F} \setminus F^{\circ}) = \overline{F}^{\circ} \cup fr(F)$ is ℵ-open. Therefore, F is ℵ-open. □

Here we give an example to explain that the ℵ-open set need not be e^*-open.

Example 5. Let $(\mathbb{R}, \mathcal{U})$ be a usual space, and $A = \{1, 2, 3, ...\} \subseteq \mathbb{R}$. Since A is nowhere dense, $fr(A)$ is infinite, and A is subset of $fr(A)$, then A is ℵ-open. However, A is not e^*-open, because $\{1, 2, 3, ...\} \nsubseteq \overline{((\overline{\{1, 2, 3, ...\}}^{\delta})^{\circ})} = \overline{((\{1, 2, 3, ...\})^{\circ})} = \overline{\emptyset} = \emptyset$.

From all the above results the following diagram is obtained:

$$\begin{array}{ccccc}
 & & F\text{-open} & & \\
 & & \downarrow & & \\
 & & C\text{-open} & & \aleph\text{-open} \\
 & & \downarrow & & \uparrow \\
\delta\text{-open} & \longrightarrow & \text{open} & & e^{*}\text{-open} \\
\downarrow & & \downarrow & & \uparrow \\
\text{a-open} & \longrightarrow & \alpha\text{-open} & & \beta^{*}\text{-open} \\
\Big| & & \downarrow & & \uparrow \\
\Big| & & \text{pre-open} & & \beta\text{-open} \\
\downarrow & & \downarrow & & \uparrow \\
\delta\text{-semiopen} & \longrightarrow & Z\text{-open} & \longrightarrow & b\text{-open}
\end{array}$$

None of these implications are reversible, as shown from Example 5. Other examples concerning other implications are shown in [1–6, 11, 12, 15–17, 19–24].

Theorem 6. *Suppose that F_s is an $\aleph$-open subset of space Z, for each $s \in \Theta$. Then $\bigcup_{s\in\Theta} F_s$ is an $\aleph$-open set.*

Proof. Suppose that F_s is an $\aleph$-open subset of space Z, for each $s \in \Theta$. If $F_s = \emptyset$, for each $s \in \Theta$, then $\bigcup_{s\in\Theta} F_s = \emptyset$ is $\aleph$-open. So, suppose that some members are nonempty. However, for the empty set does not affect any union, without loss of generality, we may suppose that $F_s \neq \emptyset$, for each $s \in \Theta$. If there exists $s_\gamma \in \Theta$ such that $F_{s_\gamma} = Z$, then $\bigcup_{s\in\Theta} F_s = Z$ is $\aleph$-open. So, assume that $F_s \neq Z$ for all $s \in \Theta$. Now, if $\bigcup_{s\in\Theta} F_s$ is nowhere dense and $fr(\bigcup_{s\in\Theta} F_s)$ is infinite, then by the nowhere denseness of $\bigcup_{s\in\Theta} F_s$, we have $fr(\bigcup_{s\in\Theta} F_s) = \overline{\bigcup_{s\in\Theta} F_s} \setminus (\bigcup_{s\in\Theta} F_s)^{\circ} = \overline{\bigcup_{s\in\Theta} F_s}$, which implies $\bigcup_{s\in\Theta} F_s \subseteq fr(\bigcup_{s\in\Theta} F_s)$. So, assume that $\bigcup_{s\in\Theta} F_s$ is somewhere dense. Since $(\bigcup_{s\in\Theta} F_s)^{\circ} \subseteq \overline{\bigcup_{s\in\Theta} F_s}^{\circ} \subseteq \overline{\bigcup_{s\in\Theta} F_s}$, then $\overline{\bigcup_{s\in\Theta} F_s}^{\circ} \cup fr(\bigcup_{s\in\Theta} F_s) = \overline{\bigcup_{s\in\Theta} F_s}^{\circ} \cup (\overline{\bigcup_{s\in\Theta} F_s} \setminus (\bigcup_{s\in\Theta} F_s)^{\circ}) = \overline{\bigcup_{s\in\Theta} F_s}$, hence $\bigcup_{s\in\Theta} F_s \subseteq \overline{\bigcup_{s\in\Theta} F_s}^{\circ} \cup fr(\bigcup_{s\in\Theta} F_s)$. Therefore, $\bigcup_{s\in\Theta} F_s$ is an $\aleph$-open set.

□

Theorem 7. *Suppose that F_s is an $\aleph$-closed subset of space Z, for each $s \in \Theta$. Then $\bigcap_{s\in\Theta} F_s$ is an $\aleph$-closed set.*

Proof. It follows from Theorem 6.

□

In general, the finite intersection of $\aleph$-open sets is not $\aleph$-open. For example, $(0, 5]$ and $[5, 7)$ are $\aleph$-open sets in the usual space $\mathbb{R}$, and $(0, 5] \cap [5, 7) = \{5\}$ is not $\aleph$-open, because $\{5\}$ is nowhere dense and has a finite frontier.

Definition 8. Let F be subset of space Z. Then:

(1) $\aleph$-interior F denoted by $F^{\circ_\aleph}$ is $F^{\circ_\aleph} = \cup\{K \in \aleph O(Z) : K \subseteq F\}$;
(2) $\aleph$-closure F denoted by $\overline{F}^{\aleph}$ is the intersection of all $\aleph$-closed sets of Z containing F.

The following three propositions are easy to prove and therefore the proofs are omitted, as we need them in the rest of the paper.

Proposition 9. *Let F_1 and F_2 be subsets of space Z. Then we have the following properties:*

(1) $\overline{Z}^{\aleph} = Z$ and $\overline{\emptyset}^{\aleph} = \emptyset$;

(2) $F_1 \subseteq \overline{F_1}^{\aleph}$;
(3) If $F_1 \subseteq F_2$, then $\overline{F_1}^{\aleph} \subseteq \overline{F_2}^{\aleph}$;
(4) $a \in \overline{F_1}^{\aleph}$ if and only if for each an $\aleph$-open set K containing a, $K \cap F_1 \neq \emptyset$;
(5) F_1 is $\aleph$-closed set if and only if $F_1 = \overline{F_1}^{\aleph}$;
(6) $\overline{\overline{F_1}^{\aleph}}^{\aleph} = \overline{F_1}^{\aleph}$;
(7) $(\overline{F_1}^{\aleph} \cup \overline{F_2}^{\aleph}) \subseteq \overline{F_1 \cup F_2}^{\aleph}$;
(8) $\overline{F_1 \cap F_2}^{\aleph} \subseteq (\overline{F_1}^{\aleph} \cap \overline{F_2}^{\aleph})$.

The inclusion relation in parts 7 and 8, cannot be replaced by the equality, for example:

Example 10. Let $Z = \{\alpha, \beta, \gamma, \delta\}$ and $\mathcal{T} = \{\emptyset, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}, Z\}$. Let $F_1 = \{\alpha, \gamma\}$, $F_2 = \{\beta, \gamma\}$, and $F_3 = \{\alpha, \beta\}$. Then $\overline{F_1}^{\aleph} = F_1$, $\overline{F_2}^{\aleph} = F_2$, and $\overline{F_1 \cup F_2}^{\aleph} = Z$. Hence, $(\overline{F_1}^{\aleph} \cup \overline{F_2}^{\aleph}) \not\supseteq \overline{F_1 \cup F_2}^{\aleph}$. Also, $\overline{F_1}^{\aleph} = F_1$, $\overline{F_3}^{\aleph} = Z$, and $\overline{F_1 \cap F_2}^{\aleph} = \{\alpha\}$. Hence, $\overline{F_1 \cap F_2}^{\aleph} \not\supseteq (\overline{F_1}^{\aleph} \cap \overline{F_2}^{\aleph})$.

Proposition 11. *Let F_1, F_2 be subsets of space Z. Then we have the following properties:*

(1) $Z^{\circ\aleph} = Z$ and $\emptyset^{\circ\aleph} = \emptyset$;
(2) $F_1^{\circ\aleph} \subseteq F_1$;
(3) If $F_1 \subseteq F_2$, then $F_1^{\circ\aleph} \subseteq F_2^{\circ\aleph}$;
(4) $a \in F_1^{\circ\aleph}$ if and only if there exists $\aleph$-open K such that $a \in K \subseteq F_1$;
(5) F_1 is an $\aleph$-open set if and only if $F_1 = F_1^{\circ\aleph}$;
(6) $(F_1^{\circ\aleph})^{\circ\aleph} = F_1^{\circ\aleph}$;
(7) $(F_1 \cap F_2)^{\circ\aleph} \subseteq (F_1^{\circ\aleph} \cap F_2^{\circ\aleph})$;
(8) $F_1^{\circ\aleph} \cup F_2^{\circ\aleph} \subseteq (F_1 \cup F_2)^{\circ\aleph}$.

Proposition 12. *For a subset F of space Z, the following statements are true:*

(1) $\overline{(Z \setminus F)}^{\aleph} = Z \setminus F^{\circ\aleph}$;
(2) $(Z \setminus F)^{\circ\aleph} = Z \setminus \overline{F}^{\aleph}$.

Theorem 13. *Let F be subset of space Z. Then,*

$$F^{\circ\aleph} = \begin{cases} \emptyset & if \quad \overline{F}^{\circ} = \emptyset \text{ and } fr(F) \text{ is finite} \\ F \cap fr(F) & if \quad \overline{F}^{\circ} = \emptyset \text{ and } fr(F) \text{ is infinite} \\ F & if \quad F \text{ is somewhere dense} \end{cases} .$$

Proof. There are three cases:
Case 1: If $\overline{F}^{\circ} = \emptyset$ and $fr(F)$ is finite. Suppose that there exists a non-empty element $x \in F^{\circ\aleph}$. Then there exists an $\aleph$-open set U such that $x \in U \subseteq F$. From the definition of $\aleph$-openness, $x \in U \subseteq (\overline{U}^{\circ} \cup fr(U)) \subseteq (\overline{F}^{\circ} \cup fr(F))$. Since $\overline{F}^{\circ} = \emptyset$, then $\overline{U}^{\circ} = \emptyset$, and $x \in U \subseteq fr(U) \subseteq fr(F)$. However, if $fr(U)$ is infinite, then $fr(F)$ is infinite, and thus we have a contradiction. Therefore, $F^{\circ\aleph} = \emptyset$.
Case 2: If $\overline{F}^{\circ} = \emptyset$ and $fr(F)$ is infinite. Let $x \in F^{\circ\aleph}$. Then there exists an $\aleph$-open set U such that $x \in U \subseteq F$. From the definition of an $\aleph$-open set, we have $x \in U \subseteq \overline{U}^{\circ} \cup fr(U)$. Since $\overline{F}^{\circ} = \emptyset$, then $\overline{U}^{\circ} = \emptyset$, hence $x \in U \subseteq fr(U) \subseteq fr(F)$. Then $x \in F \cap fr(U)$. On the other hand, let $x \in F \cap fr(F)$.

Since F is nowhere dense and $fr(F)$ is infinite, then $F \subseteq fr(F)$, and by the definition of $\aleph$-openness, F is $\aleph$-open. From Proposition 11, part 5, we have $x \in \overline{F}^{\circ}$. Therefore, $F^{\circ\aleph} = F \cap fr(F)$.
Case 3: If F is somewhere dense, then

$$F \subseteq \overline{F}^{\circ} \cup fr(F) = \overline{F},$$

hence F is $\aleph$-open. By Proposition 11 part 5, we have $F^{\circ\aleph} = F$.

□

Example 14. Let $F = \{1, 2, 3\}$, $G = \{\frac{1}{n} : n \in \mathbb{N}\}$, and $H = (0, 5)$ be subsets of the usual space $(\mathbb{R}, \mathcal{U})$. Then:

(1) Since F is nowhere dense, then any non-empty subset K of F is nowhere dense, hence $fr(F)$ and $fr(K)$ are finite. According to the definition of $\aleph$-openness, $\emptyset$ is the only $\aleph$-open subset of F. Hence, $F^{\circ\aleph} = \emptyset$;
(2) $G^{\circ\aleph} = \{\frac{1}{n} : n \in \mathbb{N}\} \cap fr(\{\frac{1}{n} : n \in \mathbb{N}\}) = \{\frac{1}{n} : n \in \mathbb{N}\} \cap (\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}) = G$, because G is nowhere dense and $fr(G)$ is infinite, hence G is $\aleph$-open;
(3) $H^{\circ\aleph} = H$, because H is somewhere dense, then $H \subseteq \overline{H}^{\circ} \cup fr(H)$, hence H is $\aleph$-open.

Theorem 15. *Let F be a subset of space Z. Then,*

$$\overline{F}^{\aleph} = \begin{cases} Z & if \quad \overline{Z \setminus F}^{\circ} = \emptyset \text{ and } fr(Z \setminus F) \text{ is finite} \\ F & if \quad \overline{Z \setminus F}^{\circ} = \emptyset \text{ and } fr(Z \setminus F) \text{ is infinite} \\ F & if \quad Z \setminus F \text{ is somewhere dense} \end{cases};$$

Proof. There are three cases:
Case 1: If $\overline{Z \setminus F}^{\circ} = \emptyset$ and $fr(Z \setminus F)$ is finite. Since $F \subseteq Z$, then $\overline{F}^{\aleph} \subseteq \overline{Z}^{\aleph} = Z$. On the other hand, suppose that there exists $x \in Z$ such that $x \notin \overline{F}^{\aleph}$. Then there exists an $\aleph$-open set U such that $x \in U$ and $U \cap F = \emptyset$. Since U is $\aleph$-open, then $x \in U \subseteq (\overline{U}^{\circ} \cup fr(U))$. However, $U \cap F = \emptyset$, and then $U \subseteq Z \setminus F$. Since $\overline{Z \setminus F}^{\circ} = \emptyset$, then $\overline{U}^{\circ} = \emptyset$, hence, from the definition of $\aleph$-openness, $fr(U)$ is infinite. Since $\overline{Z \setminus F}^{\circ} = \emptyset$, $\overline{U}^{\circ} = \emptyset$, and $U \subseteq Z \setminus F$, then $fr(U) \subseteq fr(Z \setminus F)$, hence $fr(Z \setminus F)$ is infinite, and thus there is a contradiction. Then we have $\overline{F}^{\aleph} \subseteq Z$. Therefore, $\overline{F}^{\aleph} = Z$.
Case 2: Let $\overline{Z \setminus F}^{\circ} = \emptyset$ and $fr(Z \setminus F)$ is infinite, then $(Z \setminus F) \subseteq (\overline{Z \setminus F}^{\circ} \cup fr(Z \setminus F)) \subseteq fr(Z \setminus F)$, hence $Z \setminus F$ is $\aleph$-open. Thus, F is $\aleph$-closed. Therefore, $F = \overline{F}^{\aleph}$.
Case 3: Let $Z \setminus F$ be somewhere dense. Then $(Z \setminus F) \subseteq (\overline{Z \setminus F}^{\circ} \cup fr(Z \setminus F))$, hence $Z \setminus F$ is $\aleph$-open. Thus, F is $\aleph$-closed. Therefore, $F = \overline{F}^{\aleph}$.

□

Proposition 16. *Any nonempty finite closed nowhere dense set is not $\aleph$-open.*

Proof. Let F be a nonempty finite closed set of a space Z and $\overline{F}^{\circ} = \emptyset$. Since F is closed and $\overline{F}^{\circ} = \emptyset$, then $fr(F) = F$ is finite. Hence, by the definition of $\aleph$-open sets, we obtain that F is not $\aleph$-open. □

Proposition 17. *Any infinite nowhere dense set is $\aleph$-open.*

Proof. Obviously, we use the same method that was used in Proposition 16. □

Proposition 18. *For space Z, the singleton set $\{z\}$ is either* ℵ*-open or* $\overline{\{z\}}^{\circ} = \emptyset$.

Proof. Let $\overline{\{z\}}^{\circ} \neq \emptyset$. Then $\{z\} \subseteq \overline{\{z\}}^{\circ}$, hence $\{z\} \subseteq (\overline{\{z\}}^{\circ} \cup fr\{z\})$, which is ℵ-open. If $\{z\}$ is not ℵ-open, then $\{z\} \nsubseteq (\overline{\{z\}}^{\circ} \cup fr\{z\})$, that is $\{z\} \nsubseteq \overline{\{z\}}^{\circ}$. Therefore, $\overline{\{z\}}^{\circ} = \emptyset$. □

4. Continuous functions via ℵ-open sets

In this section, we study the notion of ℵ-continuity of functions, using ℵ-open and ℵ-closed sets.

Definition 19. A function $h : Z_1 \rightarrow Z_2$ is said to be ℵ-*continuous* if $h^{-1}(F) \in \aleph O(Z_1)$, for each $F \in \aleph O(Z_2)$.

In general, ℵ-continuity neither implies continuity nor is implied by continuity, as shown in the following examples:

Example 20. Let $(\{1,2\}, \mathcal{D})$ be a discrete space and $(\mathbb{R}, \mathcal{U})$ be a Euclidean space. Then the function $h : (\mathbb{R}, \mathcal{U}) \rightarrow (\{1,2\}, \mathcal{D})$ defined by:

$$h(r) = \begin{cases} 1 & if \quad r \in [0, \infty) \\ 2 & if \quad r \in (-\infty, 0) \end{cases} ;$$

is ℵ-continuous, but not continuous, as the inverse image of an open set $\{1\}$ is not open.

Example 21. Let $(\mathbb{R}, \mathcal{I})$ be an indiscrete space and $(\mathbb{R}, \mathcal{U})$ be a usual space. Then the identity function $id : (\mathbb{R}, \mathcal{U}) \rightarrow (\mathbb{R}, \mathcal{I})$ is continuous, but not ℵ-continuous, because $\{1,2\}$ is an ℵ-open set in $(\mathbb{R}, \mathcal{I})$ and $id^{-1}(\{1,2\})$ is not ℵ-open in $(\mathbb{R}, \mathcal{U})$.

Definition 22. Let $(Z, \mathcal{F})$ be a topological space. Then:

(1) A collection $\aleph\text{-}\mathcal{B} \subseteq \mathcal{F}$ is called an ℵ-*base* for $(Z, \mathcal{F})$ if every non-empty ℵ-open subset of Z can be represented as a union of a subfamily of $\aleph\text{-}\mathcal{B}$;

(2) A collection $\aleph\text{-}\mathcal{P} \subseteq \mathcal{F}$ is called an ℵ-*subbase* for a topological space $(Z, \mathcal{F})$ if the family $\{F_1 \cap F_2 \cap ... \cap F_m : m \in \mathbb{N}, F_j \in \aleph\text{-}\mathcal{P}, \forall j \in \{1, 2, ..., m\}\}$ is an ℵ-base for $(Z, \mathcal{F})$.

The following theory is easy to prove and therefore it is presented it without proof, as we will need it in the rest of the research.

Theorem 23. *Let Z_1 and Z_2 be two spaces and $h : Z_1 \rightarrow Z_2$ be a function. Then the following statements are equivalent:*

(1) h is ℵ-continuous ;

(2) For each $z_1 \in Z_1$ and $F_1 \in \aleph O(Z_2)$, where $h(z_1) \in F_1$, there exists $F_2 \in \aleph O(Z_1)$ where $z_1 \in F_2$ and $h(F_2) \subseteq F_1$;

(3) The inverse images of all ℵ-closed subsets of Z_2 are ℵ-closed in Z_1;

(4) The inverse images of all members of an ℵ-subbase $\aleph\text{-}\mathcal{P}$ for Z_2 are ℵ-open in Z_1;

(5) The inverse images of all members of an ℵ-base $\aleph\text{-}\mathcal{B}$ for Z_2 are ℵ-open in Z_1;

(6) For every $F \subseteq Z_1$ we have $h(\overline{F}^{\aleph}) \subseteq \overline{h(F)}^{\aleph}$;

(7) For every $K \subseteq Z_2$ we have $\overline{h^{-1}(K)}^{\aleph} \subseteq h^{-1}(\overline{F}^{\aleph})$;

(8) *For every $K \subseteq Z_2$ we have $h^{-1}(K^{\circ\aleph}) \subseteq (h^{-1}(K))^{\circ\aleph}$.*

In the following results, if $Z = \prod_{s\in\Theta} Z_s$ is a product space and $F_s \subseteq Z_s$ for each $s \in \Theta$, we denote $F_{s_1} \times F_{s_2} \times ... \times F_{s_n} \times \prod_{s\notin B} Z_s$, by $< F_{s_1}, F_{s_2}, ..., F_{s_n} >$, where $B = \{s_1, s_2, ..., s_n\}$.

Lemma 24. *Let $Z = \prod_{s=1}^{n} Z_s$ be a finite product space and F_s be $\aleph$-open in Z_s for each $s \in \{1, 2, ..., n\}$. Then, $\langle F_1, F_2, ..., F_n\rangle$ is $\aleph$-open in Z.*

Proof. Let $F_s \in \aleph O(Z_s)$ for each $s \in \{1, 2, ..., n\}$. If there exists $F_{s_\gamma} = \emptyset$ for some $s_\gamma \in \{1, 2, ..., n\}$, then $\langle F_1, F_2, ..., F_n\rangle = \emptyset \in \aleph O(Z)$. Suppose that $F_s \neq \emptyset$ for each $s \in \{1, 2, ..., n\}$. If there exists F_{s_γ} is nowhere dense in Z_{s_γ} for some $s_\gamma \in \{1, 2, ..., n\}$, then $fr(F_{s_\gamma})$ is infinite and also $\langle F_1, F_2, ..., F_n\rangle$ is nowhere dense. Hence,

$$\begin{aligned} \langle F_1, F_2, ..., F_n\rangle &\subseteq \overline{\langle F_1, F_2, ..., F_n\rangle}^{\circ} \cup fr\langle F_1, F_2, ..., F_n\rangle \\ &= fr\langle F_1, F_2, ..., F_n\rangle, \end{aligned}$$

where $fr\langle F_1, F_2, ..., F_n\rangle$ is infinite, because

$$\langle fr(F_1), fr(F_2), ..., fr(F_n)\rangle \subseteq fr\langle F_1, F_2, ..., F_n\rangle$$

and, for some $s_\gamma \in \{1, 2, ..., n\}$, $fr(F_{s_\gamma})$ is infinite. Hence, $\langle F_1, F_2, ..., F_n\rangle \in \aleph O(Z)$. Assume that F_s is somewhere dense in Z_s for each $s \in \{1, 2, ..., n\}$. Since $\langle F_1, F_2, ..., F_n\rangle$ is somewhere dense in Z. then, by Remark 2,

$$\langle F_1, F_2, ..., F_n\rangle \subseteq \overline{\langle F_1, F_2, ..., F_n\rangle}.$$

Therefore, $\langle F_1, F_2, ..., F_n\rangle \in \aleph O(Z)$. □

If at least one coordinate is nowhere dense or equal to the empty set, then the converse of Lemma 24 is not true. For example:

Example 25. Consider $(\mathbb{R}, \mathcal{U})$, where $\mathcal{U}$ is the usual topology on $\mathbb{R}$ [25]. Then $\emptyset \times \{1, 2\} = \emptyset$ is an $\aleph$-open set in $\mathbb{R}^2$. By Theorem 16, $\{1, 2\}$ is not $\aleph$-open in $\mathbb{R}$. Also, $\mathbb{Z} \times \{1, 2\} \times (0, 1)$ is an $\aleph$-open set in $\mathbb{R}^3$ and $\{1, 2\}$ is finite nowhere dense, which is not $\aleph$-open in $\mathbb{R}$.

Corollary 26. *Let $Z = \prod_{s=1}^{n} Z_s$ be a finite product space and $F_s \subseteq Z_s$ for each $s \in \{1, 2, ..., n\}$. Then*

(1) $\overline{\langle F_1, F_2, ..., F_n\rangle}^{\aleph} \subseteq \langle \overline{F_1}^{\aleph}, \overline{F_2}^{\aleph}, ..., \overline{F_n}^{\aleph}\rangle$;
(2) $\langle F_1^{\circ\aleph}, F_2^{\circ\aleph}, ..., F_n^{\circ\aleph}\rangle \subseteq \langle F_1, F_2, ..., F_n\rangle^{\circ\aleph}$.

Since the closure and interior operators are preserved in a finite product, see [26, p. 99–100], then we have the following theorem.

Theorem 27. *Let $Z = \prod_{s=1}^{n} Z_s$ be a finite product space and $F_s \subseteq Z_s$ is not finite nowhere dense in Z_s for each $s \in \{1, 2, ..., n\}$. Then $F_s \in \aleph O(Z_s)$ for each $s \in \{1, 2, ..., n\}$ if and only if $\langle F_1, F_2, ..., F_n\rangle \in \aleph O(Z)$.*

Proof. Necessity. It follows from Lemma 24.
Sufficiency. Suppose that $F_s \subseteq Z_s$ is not finite nowhere dense in Z_s for each $s \in \{1, 2, ..., n\}$ and $\langle F_1, F_2, ..., F_n\rangle \in \aleph O(Z)$. If F_s is somewhere dense in Z_s for each $s \in \{1, 2, ..., n\}$. Then

$$\langle F_1, F_2, ..., F_n\rangle \subseteq \overline{\langle F_1, F_2, ..., F_n\rangle}^{\circ} \cup fr\langle F_1, F_2, ..., F_n\rangle$$

$$
\begin{aligned}
&= \overline{\langle F_1, F_2, ..., F_n \rangle} \\
&= \langle \overline{F_1}, \overline{F_2}, ..., \overline{F_n} \rangle,
\end{aligned}
$$

hence $F_s \subseteq \overline{F_s} = \overline{F_s}^{\circ} \cup fr(F_s)$ for each $s \in \{1, 2, ..., n\}$. Then, $F_s \in \aleph O(Z_s)$ for each $s \in \{1, 2, ..., n\}$. Assume that there exists F_{s_γ} is infinite nowhere dense in Z_{s_γ} for some $s_\gamma \in \{1, 2, ..., n\}$. Then

$$
\begin{aligned}
\langle F_1, F_2, ..., F_{s_\gamma}, ..., F_n \rangle &\subseteq \overline{\langle F_1, F_2, ..., F_{s_\gamma}, ..., F_n \rangle}^{\circ} \cup fr\langle F_1, F_2, ..., F_{s_\gamma}, ..., F_n \rangle \\
&= \langle \overline{F_1}^{\circ}, \overline{F_2}^{\circ}, ..., \overline{F_{s_\gamma}}^{\circ}, ..., \overline{F_n}^{\circ} \rangle \cup fr\langle F_1, F_2, ..., F_{s_\gamma}, ..., F_n \rangle \\
&= fr\langle F_1, F_2, ..., F_{s_\gamma}, ..., F_n \rangle \\
&= \overline{\langle F_1, F_2, ..., F_{s_\gamma}, ..., F_n \rangle} \cap \overline{X \setminus \langle F_1, F_2, ..., F_{s_\gamma}, ..., F_n \rangle} \\
&\subseteq \overline{\langle F_1, F_2, ..., F_{s_\gamma}, ..., F_n \rangle} \\
&= \langle \overline{F_1}, \overline{F_2}, ..., \overline{F_{s_\gamma}}, ..., \overline{F_n} \rangle,
\end{aligned}
$$

hence $F_s \subseteq \overline{F_s} = \overline{F_s}^{\circ} \cup fr(F_s)$ for each $s \in \{1, 2, ..., n\}$. Therefore, $F_s \in \aleph O(Z_s)$ for each $s \in \{1, 2, ..., n\}$. □

Corollary 28. *Let $Z = \prod_{s=1}^{n} Z_s$ be a finite product space and $F_s \subseteq Z_s$ is not finite nowhere dense in Z_s for each $s \in \{1, 2, ..., n\}$. Then,*

(1) $\overline{\langle F_1, F_2, ..., F_n \rangle}^{\aleph} = \langle \overline{F_1}^{\aleph}, \overline{F_2}^{\aleph}, ..., \overline{F_n}^{\aleph} \rangle$;
(2) $\langle F_1^{\circ\aleph}, F_2^{\circ\aleph}, ..., F_n^{\circ\aleph} \rangle = \langle F_1, F_2, ..., F_n \rangle^{\circ\aleph}$.

Lemma 29. *The projection maps $\pi_s : \prod_{\alpha\in\Theta} Z_\alpha \to Z_s$ are $\aleph$-continuous.*

Proof. For $s \in \Theta$, let $V_s \in \aleph O(Z_s)$. Then, $\pi_s^{-1}(V_s) = V_s \times \prod_{s\neq\alpha\in\Theta} Z_\alpha$. By Lemma 24, $\pi_s^{-1}(V_s) \in \aleph O(\prod_{\alpha\in\Theta} Z_\alpha)$. □

Theorem 30. *For space Z and product space $Y = \prod_{s\in\Theta} Y_s$, a function $h : Z \to Y$ is $\aleph$-continuous if and only if $\pi_s \circ h$ is $\aleph$-continuous for each $s \in \Theta$.*

Proof. Let $h : Z \to Y$ be $\aleph$-continuous. Let U_s be arbitrary $\aleph$-open in Y_s, for each $s \in \Theta$. By Lemma 29, $\pi^{-1}(U_s) \in \aleph O(Y)$, for each $s \in \Theta$. Hence, $(\pi_s \circ h)^{-1}(U_s) = h^{-1}(\pi^{-1}(U_s))$ is $\aleph$-open, for each $s \in \Theta$. Therefore, $\pi_s \circ h$ is $\aleph$-continuous for each $s \in \Theta$. Conversely, assume that $\pi_s \circ h$ is $\aleph$-continuous for each $s \in \Theta$. Let V_s be $\aleph$-open in Y_s for each $s \in \Theta$. Then $\langle V_s \rangle$ is an $\aleph$-subbasic $\aleph$-open set in Y. By Theorem 23 and $(\pi_s \circ h)^{-1}(V_s) = h^{-1}(\pi^{-1}(V_s)) = h^{-1}(\langle V_s \rangle)$, we get that $h^{-1}(\langle V_s \rangle)$ is $\aleph$-open in Z. Therefore, h is $\aleph$-continuous. □

Corollary 31. *For a space Z and the product space $Y = \prod_{s\in\Theta} Y_s$, $h_s : Z \to \prod_{s\in\Theta} Y_s$ is a function for each $s \in \Theta$. Let $h : Z \to Y$ be the function defined by $h(z) =< h_s(z) >$. Then h is $\aleph$-continuous if and only if h_s is $\aleph$-continuous for each $s \in \Theta$.*

Theorem 32. *Let $Z = \prod_{s=1}^{n} Z_s$ and $Y = \prod_{s=1}^{n} Y_s$ be finite product spaces. Let $h_s : Z_s \to Y_s$ be an $\aleph$-continuous function for each $s \in \{1, 2, ..., n\}$. Then the function $h : Z \to Y$ defined by $h(\langle z_s \rangle) = \langle h_s(z_s) \rangle$ is $\aleph$-continuous.*

Proof. Let $\langle F_1, F_2, ..., F_n\rangle$ be an $\aleph$-basic $\aleph$-open set in Y. Then

$$h^{-1}\langle F_1, F_2, ..., F_n\rangle = \langle h^{-1}(F_1), h^{-1}(F_2), ..., h^{-1}(F_n)\rangle$$

Since h_s is $\aleph$-continuous for each $s \in \{1, 2, ..., n\}$, then $h_s^{-1}(F_s)$ is $\aleph$-continuous for each $s \in \{1, 2, ..., n\}$. By Lemma 24, we have, $\langle h^{-1}(F_1), h^{-1}(F_2), ..., h^{-1}(F_n)\rangle$ is $\aleph$-open in Z. Hence, from Theorem 23, h is $\aleph$-continuous. □

5. Applications on ℵ-open sets in separation axioms

In this section, we apply our new notions in separation axioms to explore new types of separation axioms and some theorems related to graphs of functions.

Definition 33. A space Z is said to be an

(1) $\aleph$-T_0-space if for each distinct pair $z_1, z_2 \in Z$, then there is either an $\aleph$-open set containing z_1 but not z_2 or an $\aleph$-open set containing z_2 but not z_1.
(2) $\aleph$-T_1-space if for each distinct pair $z_1, z_2 \in Z$, then there are two $\aleph$-open subsets F_1 and F_2 of Z, such that $z_1 \in F_1$, $z_2 \notin F_1$, and $z_1 \notin F_2$, $z_2 \in F_2$.
(3) $\aleph$-T_2-space "$\aleph$-Hausdorff space" if for each distinct pair $z_1, z_2 \in Z$, then there are two disjoint $\aleph$-open subsets F_1 and F_2 of Z, such that $z_1 \in F_1$ and $z_2 \in F_2$.

Theorem 34. *Any topological space is $\aleph$-T_0-space.*

Proof. Let Z be a topological space and let $x \neq y$ for x and y in Z. Then $Z \setminus \{x\}$ is either a nowhere dense or somewhere dense set. If $Z \setminus \{x\}$ is nowhere dense, then $\overline{Z \setminus \{x\}} \neq Z$, hence $Z \setminus \{x\}$ is closed and so $\{x\}$ is an open set. Therefore, $\{x\} \subseteq \overline{\{x\}}^{\circ} \cup fr(\{x\})$ is an $\aleph$-open set containing x, but not y. On the other hand, if $Z \setminus \{x\}$ is somewhere dense, then $Z \setminus \{x\} \subseteq \overline{Z \setminus \{x\}}^{\circ} \cup fr(Z \setminus \{x\})$ is an $\aleph$-open set containing y, but not x. □

Lemma 35. *Any infinite subset of a topological space Z is $\aleph$-open.*

Proof. Let K be any infinite subset of topological space Z. It follows that K is either a nowhere dense or somewhere dense set. If K is nowhere dense, then $K \subseteq \overline{K}^{\circ} \cup fr(K) = fr(K)$ is $\aleph$-open, because $fr(K)$ is infinite. Suppose that K is somewhere dense. Then $K \subseteq \overline{K}^{\circ} \cup fr(K)$ is $\aleph$-open. □

Theorem 36. *Any T_1 topological space is an $\aleph$-T_2-space.*

Proof. Let Z be a T_1 topological space and let $x \neq y$ in Z. If Z is finite, then Z is discrete, hence it is an $\aleph$-T_2-space because any open set is $\aleph$-open. Suppose that Z is infinite. Then there exist two infinite disjoint subsets K and H of Z containing x and y, respectively. By Lemma 35, K and H are $\aleph$-open sets. Therefore, Z is an $\aleph$-T_2-space. □

Theorem 37. *Any infinite topological space is an $\aleph$-T_2-space.*

Proof. This follows from Lemma 35 and the fact that any infinite set contains two disjoint infinite subsets.

□

Theorem 38. *Any $\aleph$-T_1 finite topological space is $\aleph$-T_2-space.*

Proof. Suppose that Z is an $\aleph$-T_1 finite topological space. If Z contains two or fewer elements, then Z is $\aleph$-T_2-space. Assume that Z contains more than two elements. Let $x \neq y$ in Z and, by $\aleph$-T_1-spaceness, there exist two $\aleph$-open sets U and V containing x and y, respectively. If $U \cap V = \emptyset$, we are done. Assume that $U \cap V \neq \emptyset$. Since Z is finite, then U and V are somewhere dense.
Claim: $Z \setminus U$ is $\aleph$-open. If $Z \setminus U$ is somewhere dense, we are done. Assume that $Z \setminus U$ is nowhere dense. Since $(V \cap (Z \setminus U)) \subseteq (Z \setminus U)$ is nowhere dense, and V is somewhere dense, then $U \cap V$ is somewhere dense. If $U \setminus V$ is somewhere dense, we are done. Assume that $U \setminus V$ is nowhere dense, hence $Z \setminus (U \cap V)$ is nowhere dense, which implies that $U \cap V$ is the only subset of Z such that $(U \cap V)^\circ \neq \emptyset$. If $(U \cap V)^\circ$ is an open set, then Z is not $\aleph$-T_1, hence the interior of $U \cap V$ consists of more than one open set. Since Z consists of more than two elements, then there exists an element g such that $g \neq x \neq y$. Let H_i be any $\aleph$-open set containing g, where $i \in \{1, 2, 3, ..., n\}$. Since $Z \setminus (U \cap V)$ is nowhere dense, then $H_i \cap (U \cap V) \neq \emptyset$ for all $i \in \{1, 2, 3, ..., n\}$. If $(\bigcap_{i=1}^{n} H_i) \cap (U \cap V) \neq \emptyset$, then there exists an element $e \in ((\bigcap_{i=1}^{n} H_i) \cap (U \cap V))$ such that any $\aleph$-open set containing x or y must contain e. Thus, contradicts the fact that Z is an $\aleph$-T_1-space. Assume that $(\bigcap_{i=1}^{n} H_i) \cap (U \cap V) = \emptyset$. Then there exist H_{i_1} and H_{i_2} for some $i_1 \neq i_2 \in \{1, 2, 3, ..., n\}$ such that $H_{i_1} \cap H_{i_2} \cap (U \cap V) = \emptyset$, hence $H_{i_1} \cap (U \cap V)$ and $H_{i_2} \cap (U \cap V)$ are $\aleph$-open sets, and $(H_{i_1} \cap (U \cap V)) \cap (H_{i_2} \cap (U \cap V)) = \emptyset$. Hence, $(U \setminus V) \cup (H_{i_1} \cap (U \cap V))$ and $(V \setminus U) \cup (H_{i_2} \cap (U \cap V))$ are two disjoint $\aleph$-open sets containing x and y, respectively. Therefore, Z is an $\aleph$-T_2-space.

□

Remark 39. The following implications hold for a space Z :

$$\begin{array}{ccccc}
T_2\text{-space} & \longrightarrow & T_1\text{-space} & \longrightarrow & T_0\text{-space} \\
\downarrow & & \downarrow & & \downarrow \\
\aleph\text{-}T_2\text{-space} & \longleftrightarrow & \aleph\text{-}T_1\text{-space} & \longrightarrow & \aleph\text{-}T_0\text{-space}
\end{array}$$

The converse of these implications are not true in general. Consider the following examples:

Example 40. Let $Z = \{\alpha, \beta\}$ and $\mathcal{T} = \{\emptyset, \{\alpha\}, Z\}$. Then, $(Z, \mathcal{T})$ is T_0-space, which implies that $(Z, \mathcal{T})$ is an $\aleph$-T_0-space, but not an $\aleph$-T_1-space.

Example 41. Consider $(\mathbb{R}, \mathcal{I})$, where $\mathcal{I}$ is an indiscrete topology on $\mathbb{R}$ [25]. Then $(\mathbb{R}, \mathcal{I})$ is an $\aleph$-T_2-space, but not a T_0-space.

Let $h : Z_1 \to Z_2$ be a function. Recall that the subset $\{(z_1, h(z_1)) : z_1 \in Z_1\}$ of the product space $Z_1 \times Z_2$ is called the graph of h and is denoted by $\Im(h)$.

Theorem 42. *If $h : Z_1 \to Z_2$ is an $\aleph$-continuous function and Z_2 is an $\aleph$-T_2-space, then $\Im(h)$ is $\aleph$-closed in $Z_1 \times Z_2$.*

Proof. Let $(z_1, z_2) \in Z_1 \times Z_2$ where $z_2 \neq h(z_1)$. Since Z_2 is an $\aleph$-T_2-space, then there exist $F_1, F_2 \in \aleph O(Z_2)$ where $z_2 \in F_1$, $h(z_1) \in F_2$ and $F_1 \cap F_2 = \emptyset$. Since h is $\aleph$-continuous and, by Theorem 23 there exists an $\aleph$-open set $h^{-1}(F_2) = K_1$ containing z_1, by Lemma 24 we have that $K_1 \times F_1$ is an $\aleph$-open set in $Z_1 \times Z_2$ such that $(z_1, z_2) \in K_1 \times F_1$. Suppose that there exists $(w, h(w)) \in K_1 \times F_1$, then $h(w) \in F_1$.

However, $w \in K_1 = h^{-1}(F_2)$, which means that $h(w) \in F_2$. Thus, it is contradicts the fact that F_1 and F_2 are disjoint. Hence, $Z_1 \times Z_2 \setminus \Im(h)$ is $\aleph$-open. Therefore, $\Im(h)$ is $\aleph$-closed in $Z_1 \times Z_2$.

□

Theorem 43. *Let $h : Z_1 \to Z_2$ be an $\aleph$-continuous function and Z_2 be an $\aleph$-T_2-space. Then the set $M = \{(z_1, z_2) \in Z_1 \times Z_1 : h(z_1) = h(z_2)\}$ is $\aleph$-closed in $Z_1 \times Z_1$.*

Proof. Let $(z_1, z_2) \notin M$, then $h(z_1) \neq h(z_2)$ in $Z_1 \times Z_1 \setminus M$. Since Z_2 is an $\aleph$-T_2-space, there exist $F_1, F_2 \in \aleph O(Z_2)$ where $h(z_1) \in F_1$, $h(z_2) \in F_2$, and $F_1 \cap F_2 = \emptyset$. However, h is $\aleph$-continuous, and then, from Theorem 23, there exist $\aleph$-open sets $h^{-1}(F_1) = K_1$ and $h^{-1}(F_2) = K_2$ in Z_1 containing z_1 and z_2, respectively, such that $h(K_1) \subseteq F_1$ and $h(K_2) \subseteq F_2$. By Lemma 24 we have that $K_1 \times K_2$ is an $\aleph$-open set in $Z_1 \times Z_1$ such that $(z_1, z_2) \in K_1 \times K_2$. Claim: $K_1 \times K_2 \cap M = \emptyset$. Suppose that $K_1 \times K_2 \cap M \neq \emptyset$. Then there exists $(a, b) \in K_1 \times K_2 \cap M$. Hence, $a \in K_1 = h^{-1}(F_1)$ and $b \in K_2 = h^{-1}(F_2)$, which means that $h(a) \in F_1$ and $h(b) \in F_2$. But, $(a, b) \in M$, and so $h(a) = h(b)$, which contradicts the fact that F_1 and F_2 are disjoint. Hence, $(z_1, z_2) \notin \overline{M}^{\aleph}$. Therefore, M is $\aleph$-closed in $Z_1 \times Z_1$.

□

Theorem 44. *Let $h_1.h_2 : Z_1 \to Z_2$ be an $\aleph$-continuous functions and Z_2 be $\aleph$-T_2-space. Then the set $M = \{(z_1, z_2) : h_1(z_1) = h_2(z_2)\}$ is $\aleph$-closed in $Z_1 \times Z_2$.*

Proof. Let $(z_1, z_2) \notin M$. Then $h_1(z_1) \neq h_2(z_2)$. Since Z_2 is an $\aleph$-T_2-space, there exist $F_1, F_2 \in \aleph O(Z_2)$ where $h_1(z_1) \in F_1$, $h_2(z_2) \in F_2$, and $F_1 \cap F_2 = \emptyset$. Since h_1 is $\aleph$-continuous, and by Theorem 23, there exists an $\aleph$-open set K_1 containing z_1 and $h_1(K_1) \subseteq F_1$, then $h_1(K_1) \cap F_2 = \emptyset$. Since h_2 is $\aleph$-continuous, and by Theorem 23 there exists an $\aleph$-open set K_2 containing z_2 and $h_2(K_2) \subseteq F_2$, then $h_1(K_1) \cap h_2(K_2) = \emptyset$. Claim: $(K_1 \times K_2) \cap M = \emptyset$. Suppose that $(K_1 \times K_2) \cap F \neq \emptyset$. Then there exists $a = (a_1, a_2) \in ((K_1 \times K_2) \cap M)$, where $h_1(a_1) = h_2(a_2)$, $a_1 \in K_1$, and $a_2 \in K_2$. Hence, $h_1(a_1) \in h_1(K_1) \subseteq F_1$ and $h_2(a_2) \in h_2(K_2) \subseteq F_2$. Thus, it contradicts the fact that F_1 and F_2 are disjoint. Hence from Lemma 24, we have that $K_1 \times K_2$ is an $\aleph$-open set in $Z_1 \times Z_2$. This implies that, $(z_1, z_2) \notin \overline{M}^{\aleph}$. Therefore, M is $\aleph$-closed in $Z_1 \times Z_2$.

□

Theorem 45. *Let $h_1.h_2 : Z_1 \to Z_2$ be $\aleph$-continuous functions and Z_2 be an $\aleph$-T_2-space. Then $\{z : h_1(z) = h_2(z)\}$ is an $\aleph$-closed subset of Z_1.*

Proof. It is similar to the proof of Theorem 44. □

Theorem 46. *Let (Z_1, ρ_1) and (Z_2, ρ_2) be two topological spaces and $h : (Z_1, \rho_1) \to (Z_2, \rho_2)$ be an $\aleph$-continuous function. Then $\pi_{z_1}(M \cap \Im(h))$ is $\aleph$-closed in Z_1, where π_{z_1} represents the projection map of $Z_1 \times Z_2$ onto Z_1 and M is $\aleph$-closed in $Z_1 \times Z_2$.*

Proof. Let $z_1 \in \overline{\pi_{z_1}(M \cap \Im(h))}^{\aleph}$ and let F_1 and F_2 be two $\aleph$-open sets in Z_1 and Z_2, respectively, such that $z_1 \in F_1$ and $h(z_1) \in F_2$. Since h is $\aleph$-continuous, and by Theorem 23 there exists an $\aleph$-open $h^{-1}(F_2)$ containing z_1, and since F_2 is $\aleph$-open in Z_2 and h is $\aleph$-continuous, then, by Theorem 23, we have $z_1 \in h^{-1}(F_2) \subseteq (h^{-1}(F_2))^{\circ\aleph}$. Then $F_1 \cap \pi_{z_1}(M \cap \Im(h)) \cap (h^{-1}(F_2))^{\circ\aleph}$ contains some point g in Z_1 which implies that $(g, h(g)) \in M$ and $h(g) \in F_2$. Thus, we have $(F_1 \times F_2) \cap M \neq \emptyset$, and hence $(z_1, h(z_1)) \in \overline{M}^{\aleph}$. Since M is $\aleph$-closed, then $(z_1, h(z_1)) \in M \cap \Im(h)$ and $z_1 \in \pi_{z_1}(M \cap \Im(h))$. Therefore, $\pi_{z_1}(M \cap \Im(h))$ is $\aleph$-closed in Z_1. □

6. Concluding remarks and further work

This study contributes to the literature on novel topological qualities by introducing a new class of open sets. The given results explain that, it is possible to gain new examples and qualities that contribute to a broader understanding of topological spaces by generalizing certain topological properties, such as compression, connectivity, and others.

In this paper, we introduced a novel class of open sets named "$\aleph$-open" sets. We have studied the basic properties of these sets, provide the master properties of $\aleph$-open sets, and disclosed their relationships with many different classes of open sets with the support of appropriate counterexamples. Additionally, we provided the $\aleph$-interior and $\aleph$-closure via $\aleph$-open and $\aleph$-closed sets. In detail, we studied the concept of $\aleph$-continuity of product spaces using $\aleph$-open and $\aleph$-closed sets. Finally, we provided some kinds of separation axioms, some theorems related to graph of functions, and $\aleph$-compact spaces via $\aleph$-open and $\aleph$-closed sets.

This work considers auspicious directions for future work First, we expect that by defining a new class of open sets in topological spaces, we will facilitate a path for future research on the subject. Also, one of the main focuses of future work is to study the relationship between $\aleph$-continuous functions and the other types of continuities such as e^*-continuous functions, β^*-continuous function, α-continuous functions etc. Another direction is to introduce a new framework to be used in the near future to produce soft topological concepts [27] and supra soft topological spaces [28] such as (supra) soft operators and continuity [29–32], which are inspired by classical topologies. Certainly, researchers can explore other notions like covering properties and separation axioms via the proposed class of $\aleph$-open and $\aleph$-closed sets. Finally, it reinforces the importance of the concept of classical topology as it is a strong tributary to other modern concepts such as (supra) soft and fuzzy topology [33] and soft nodec spaces [34].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are appreciative to the anonymous referees for their valuable recommendations, which helped to improve the paper's presentation.

The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number "NBU-FPEJ-2024-2727-04".

Conflict of interest

The authors declare no conflict of interest.

References

1. N. Levine, Semi-open sets and semi-continuity in topological spaces, *Am. Math. Mon.*, **70** (1963), 36–41. https://doi.org/10.1080/00029890.1963.11990039
2. O. Njåstad, On some classes of nearly open sets, *Pac. J. Math.,* **15** (1965), 961–970. https://doi.org/10.2140/pjm.1965.15.961
3. A. S. Mashour, M. E. Abd El-Monsef, S. N. El-Deeb, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt*, **53** (1982), 47–53.
4. M. Abd El-Monsef, S. El-Deeb, R. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Sci. Assuit Univ.*, **12** (1983), 77–90.
5. D. Andrijevic, On b-open sets, *Mat. Vesnik.*, **48** (1996), 59–64.
6. J. Dontchev, M. Przemski, On the various decompositions of continuous and some weakly continuous functions, *Acta Math. Hung.*, **71** (1996), 109–120.
7. A. S. Sulaiman, B. A. Asaad, Z. A. Ameen, On certain types of convergence and γ-continuity, *J. Math. Comput. Sci.*, **11** (2021), 1029–1038. https://doi.org/10.1007/BF00052199
8. Z. Piotrowski, A survey of results concerning generalized continuity on topological spaces, *Acta Math. Univ. Comenian.*, **52** (1987), 91–110.
9. K. R. Gentry, H. B. Hoyle III, Somewhat continuous functions, *Czech. Math. J.*, **21** (1971), 5–12.
10. X. Gao, A. M. Khalil, More on $\mathfrak{D}\alpha$-closed sets in topological spaces, *J. Math.*, 2023, 1–9. https://doi.org/10.21136/CMJ.1971.100999
11. Z. A. Ameen, On somewhat near continuity and some applications, *Selecciones Mat.*, **8** (2021), 360–369. https://doi.org/10.17268/sel.mat.2021.02.12
12. Z. A. Ameen, A note on some forms of continuity, *Divulgaciones Mat.*, **22** (2021), 52–63.
13. Z. A. Ameen, Almost somewhat near continuity and near regularity, *Moroc. J. Pure Appl. Anal.*, **7** (2021), 88–99. https://doi.org/10.2478/mjpaa-2021-0009
14. N. V. Veliko, H-closed topological spaces, *Amer. Math. Soc. Transl.*, **78** (1968), 103–118.
15. E. Ekici, On a-open sets, $A^\star$-sets and decompositions of continuity and super continuity, *Ann. Univ. Sci. Budapest*, **51** (2008), 39–51.
16. A. I. El-Magharabi, A. M. Mubarki, Z-open sets and Z-continuity in topological spaces, *Int. J. Math. Arch.*, **2** (2011), 1819–1827.
17. J. H. Park, B. Y. Lee, M. J. Son, On δ-semiopen sets in topological spaces, *J. Indian Acad. Math.*, **19** (1997), 59–67.
18. Z. A. Ameen, B. A. Asaad, R. A. Muhammed, On superclasses of δ-open sets in topological spaces, *Int. J. Appl. Math.*, **32** (2019), 259–277. https://doi.org/10.12732/ijam.v32i2.8
19. S. Raychaudhuri, M. N. Mukherjee, On δ-almost continuity and δ-preopen sets, *Bull. Inst. Math. Acad.*, **21** (1993), 357–366.
20. A. M. Mubarki, M. M. Al-Rshudi, M. A. Al-Juhani, β^*-open sets and β^*-continuity in topological spaces, *J. Taibah Univ. Sci.*, **8** (2014), 142–148.

21. E. Ekici, On e^*-open sets and $(D, S)^*$-sets and decompositions of continuous functions, *Math. Moravica*, **13** (2009), 29–36.

22. A. El-Atik, *A study of some types of mappings on topological spaces*, Master science thesis, Egypt: Tanta Univ., 1997.

23. M. H. Alqahtani, F-open and F-closed sets in topological spaces, *Eur. J. Pure Appl. Math.*, **16** (2023), 819–832. https://doi.org/10.29020/nybg.ejpam.v16i2.4583

24. M. H. Alqahtani, H. Y. Saleh, A novel class of separation axioms, compactness, and continuity via C-open sets, *Mathematics*, **11** (2023), 4729.

25. L. Steen, J. Seebach, *Counterexamples in topology*, Dover Publications, INC, 1995.

26. J. Dugundji, *Topology*, Prentice-Hall of India Private Ltd., New Delhi, 1975.

27. A. M. Abd El-latif , M. H. Alqahtani, Fatouh A. Gharib, Strictly wider class of soft sets via supra soft δ-closure operator, *Int. J. Anal. Appl.*, **22** (2024), 47. https://doi.org/10.28924/2291-8639-22-2024-47

28. S. A. El-Sheikh, A. M. Abd El-latif, Decompositions of some types of supra soft sets and soft continuity, *Int. J. Math. Tre. Technol.*, **9** (2014), 37–56. https://doi.org/10.14445/22315373/IJMTT-V9P504

29. A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh, A. M. Abd El-latif, γ-operation and decompositions of some forms of soft continuity in soft topological spaces, *Ann. Fuzzy Math. Inform.*, **7** (2014), 181–196.

30. A. M. Abd El-latif, M. H. Alqahtani, New soft operators related to supra soft δ_i-open sets and applications, *AIMS Math.*, **9** (2024), 3076–3096. https://doi.org/10.3934/math.2024150

31. A. M. Abd El-latif, Novel types of supra soft operators via supra soft sd-sets and applications, *AIMS Math.*, **9** (2024), 6586–6602. https://doi.org/10.3934/math.2024321

32. A. M. Abd El-latif, M. H. Alqahtani, Novel categories of supra soft continuous maps via new soft operators, *AIMS Math.*, **9** (2024), 7449–7470. https://doi.org/10.3934/math.2024361

33. A. M. Abd El-latif, Some properties of fuzzy supra soft topological spaces, *Eur. J. Pure Appl. Math.*, **12** (2019), 999–1017. https://doi.org/10.29020/nybg.ejpam.v12i3.3440

34. M. H. Alqahtani, Zanyar A. Ameen, Soft nodec spaces, *AIMS Math.*, **9** (2024), 3289–3302. https://doi.org/10.3934/math.2024160