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Research article

Utilizing Schaefer's fixed point theorem in nonlinear Caputo sequential fractional differential equation systems

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Abstract: In the present study, established fixed-point theories are utilized to explore the requisite conditions for the existence and uniqueness of solutions within the realm of sequential fractional differential equations, incorporating both Caputo fractional operators and nonlocal boundary conditions. Subsequently, the stability of these solutions is assessed through the Ulam-Hyers stability method. The research findings are validated with a practical example that corroborate and reinforce the theoretical results.

Keywords: Caputo fractional derivative; existence and uniqueness; sequential derivatives; fixed point theorem; boundary conditions
Mathematics Subject Classification: 34A08, 34B15, 45G15

1. Introduction

In recent decades, FDEs have become increasingly important for modeling a variety of phenomena, especially those that are anomalous or involve complex natural processes not fully described by classical calculus. FDEs are applied across various domains, including mathematical biology, fluid mechanics, nonlinear optics, image processing, and plasma physics [1–4]. In the literature, numerous complex differential systems in fractional order defy analytical solutions, posing a considerable challenge for researchers. In such cases, qualitative properties play a crucial role in tracing solutions. These properties provide valuable insights and understanding of the system's behavior without explicitly solving it. In the realm of FDEs, two prominent research areas have garnered significant attention from researchers. The first area focuses on the existence theory of solutions, exploring the existence and uniqueness of solutions to FDEs. Various fixed-point theorems are fundamental tools for establishing the existence and uniqueness of solutions for diverse classes of FDEs. For instance, one may refer to [2, 5–8]. The second area, which has recently gained increasing interest, revolves around the stability analysis of differential equations, both for classical and fractional orders [9–11]. These investigations are crucial for understanding the behavior and properties of solutions in various dynamic systems.

In the literature, there are several definitions of derivatives and integrals of arbitrary orders. Over the past few decades, researchers have extensively investigated FDEs with diverse boundary conditions. Notably, a significant area of interest has emerged in the study of nonlocal nonlinear fractional-order boundary value problems. Ahmad and Nieto [12] investigated the existence and uniqueness of solutions for an anti-periodic fractional boundary value problem. The same investigations have been explored for FDEs with fractional non-separated boundary conditions [13]. Subsequently, the properties of solutions (existence and uniqueness results) have been established for a nonlinear coupled system of Caputo-type fractional differential equations accompanied by non-separated coupled boundary conditions [14].

Moreover, a more comprehensive form of nonlocal-integral boundary conditions have been introduced [9, 15–17]. As an illustration, in [18] the authors investigated a system with boundary conditions of the following form, which turned out to be the inspiration for the current effort:

$$\begin{cases} {}^{C}\mathcal{D}^{\varsigma_1}\mathcal{Q}(\mathrm{t}) = \mathcal{L}_1(\mathrm{t},\mathcal{Q}(\mathrm{t}),\Upsilon(\mathrm{t})), \mathrm{t}\in\mathcal{G} := [0,\mathfrak{T}],\\ {}^{C}\mathcal{D}^{\varsigma_2}\mathcal{Q}(\mathrm{t}) = \mathcal{J}_1(\mathrm{t},\mathcal{Q}(\mathrm{t}),\Upsilon(\mathrm{t})),\\ (\mathcal{Q}+\Upsilon)(0) = -(\mathcal{Q}+\Upsilon)(\mathfrak{T}),\\ \int_{\eta}^{\xi}(\mathcal{Q}-\Upsilon)(s)ds = A. \end{cases}$$

Moreover, in [12] the existence of solutions for the following system of fractional derivative equations involving Liouville-Caputo type with multi-point and integral boundary conditions was investigated:

$$\begin{cases} {}^{\mathcal{LC}}\mathcal{D}^{\varsigma_1}\mathcal{Q}(\mathrm{t}) = \mathcal{L}_1(\mathrm{t},\mathcal{Q}(\mathrm{t}),\Upsilon(\mathrm{t})), \mathrm{t}\in\mathcal{G} := [0,\mathfrak{T}],\\ {}^{\mathcal{LC}}\mathcal{D}^{\varsigma_2}\mathcal{Q}(\mathrm{t}) = \mathcal{J}_1(\mathrm{t},\mathcal{Q}(\mathrm{t}),\Upsilon(\mathrm{t})),\\ (\mathcal{Q}+\Upsilon)(0) = -(\mathcal{Q}+\Upsilon)(\mathfrak{T}),\\ \int_{\varrho}^{\zeta}(\mathcal{Q}-\Upsilon)(\varkappa)d\varkappa - \sum_{i=1}^{m}\mathfrak{q}_i(\mathcal{Q}-\Upsilon)(\varphi_i) - \sum_{j=1}^{n}\mathfrak{u}_j(\mathcal{Q}-\Upsilon)(\delta_j) = \mathcal{P}_1, \end{cases}$$

where ${}^{\mathcal{LC}}\mathcal{D}^{\varsigma_i}$ denotes the Liouville-Caputo fractional derivative operator of order $\varsigma_i; i = 1, 2$. $\varsigma_1, \varsigma_2 \in (1, 2]$, $0 < \varphi_i < \varrho < \zeta < \delta_j < \mathfrak{T}, i = 1, \cdots, m, j = 1, \cdots, n$, and $\mathcal{L}_1, \mathcal{J}_1 : \mathcal{G}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions.

In addition, in past decades, the stability of FDEs has emerged as a prominent and essential research area in mathematical analysis [19–21]. The origins of this stability concept can be traced back to Ulam's work in functional equations [22], which was further developed by Hyers in the context of Banach spaces [23], leading to the renowned Hyers-Ulam (HU) stability. Subsequently, Rassias generalized this concept, resulting in Hyers-Ulam-Rassias (HUR) stability [24]. Researchers have extended the idea of HU and HUR stability to various classes of functional equations [13, 25, 26], contributing to a comprehensive understanding of stability phenomena. For example, in [27] the authors investigated the existence and UHR stability of solutions for the Hilfer-Hadamard FDEs. On the other hand, efforts were made to analyze the existence, uniqueness, and Ulam stability of solutions for a coupled system of FDEs with integral boundary conditions [28, 29]. Additionally, the UH stability of a nonlinear fractional Volterra integro-differential equation was studied [30]. Motivated by these advancements in stability theory, the focus of our current research is to explore the stability properties of the derived solutions using the UH stability approach. By employing this stability concept, we aim to gain valuable insights into the behavior and robustness of the solutions of the considered system.

To gain a deeper understanding of the importance and applications of sequential coupled systems of fractional-order differential equations, one can refer to [8, 31–33]. These works provide detailed information and examples that showcase the wide spectrum of problems where such systems find relevance and applicability. While the study of fractional-order boundary value problems for individual equations and inclusions has been well-explored, the exploration of coupled SFDEs has gained contemporary interest in recent decades [34–36]. Motivated by these developments, substantial progress has been made to investigate the existence, uniqueness, and stability of the solutions for the coupled SFD system with different boundary conditions [31, 37]. For instance, in [38] the authors investigated the existence of solutions for the nonlinear SFD system with coupled boundary conditions of the type

$$\begin{cases} ({}^{C}\mathcal{D}^{\varsigma_1} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_1-1})\mathcal{Q}(\mathrm{t}) = \mathcal{L}_1(\mathrm{t}, \mathcal{Q}(\mathrm{t}), \Upsilon(\mathrm{t})), & 0 < \mathrm{t} < 1, \\ ({}^{C}\mathcal{D}^{\varsigma_1} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_1-1})\Upsilon(\mathrm{t}) = \mathcal{L}_2(\mathrm{t}, \mathcal{Q}(\mathrm{t}), \Upsilon(\mathrm{t})), & 0 < \mathrm{t} < 1, \\ \mathcal{Q}(0) = \mathcal{Q}'(0) = 0, \qquad \mathcal{Q}(1) = a\Upsilon(\xi), \\ \Upsilon(0) = \Upsilon'(0) = 0, \qquad \Upsilon(1) = b\mathcal{Q}(\eta), \end{cases}$$

where $\mathcal{K}_1$ is a parameter, $2 < \varsigma_1, \varsigma_2 \leq 3$, $({}^{C}\mathcal{D}^{\varsigma_1})$, $({}^{C}\mathcal{D}^{\varsigma_2})$ are the Caputo fractional derivatives, ξ, η satisfy $\xi, \eta \in (0, 1)$, and the non linearity terms $\mathcal{L}_1, \mathcal{J}_1 : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are the given continuous functions.

In [34], the authors applied the tools of fixed point theory to study the existence and uniqueness of solutions for a boundary value problem of Caputo-SFD equations and inclusions

$$\begin{cases} ({}^{C}\mathcal{D}^{\varsigma_1} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_1-1})\mathcal{Q}(\mathrm{t}) = \mathcal{L}_1(\mathrm{t}, \mathcal{Q}(\mathrm{t}), \Upsilon(\mathrm{t})), \mathrm{t} \in \mathcal{G} := [0, \mathfrak{T}], \\ ({}^{C}\mathcal{D}^{\varsigma_2} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_2-1})\Upsilon(\mathrm{t}) = \mathcal{J}_1(\mathrm{t}, \mathcal{Q}(\mathrm{t}), \Upsilon(\mathrm{t})), \mathrm{t} \in \mathcal{G} := [0, \mathfrak{T}], \\ (\mathcal{Q} + \Upsilon)(0) = -(\mathcal{Q} + \Upsilon)(\mathfrak{T}), \quad \displaystyle\int_{\eta}^{\xi} (\mathcal{Q} - \Upsilon)(s)ds = A, \end{cases}$$

where $\mathcal{K}_1 \in \mathbb{R}^+$ and $\mathcal{L}_1, \mathcal{J}_1 : [0, \mathfrak{T}] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions.

Inspired by the above-mentioned developments, in this work we explore a system of nonlinear coupled SFDEs of the Caputo type accompanied by a novel set of boundary conditions [12], namely

$$\begin{cases} ({}^{C}\mathcal{D}^{\varsigma_1} + \mathcal{K}_1 {}^{C}\mathcal{D}^{\varsigma_1 - 1})Q(\mathrm{t}) = \mathcal{L}_1(\mathrm{t}, Q(\mathrm{t}), \Upsilon(\mathrm{t})), \mathrm{t} \in \mathcal{G} := [0, \mathfrak{T}], \\ ({}^{C}\mathcal{D}^{\varsigma_2} + \mathcal{K}_1 {}^{C}\mathcal{D}^{\varsigma_2 - 1})\Upsilon(\mathrm{t}) = \mathcal{J}_1(\mathrm{t}, Q(\mathrm{t}), \Upsilon(\mathrm{t})), \mathrm{t} \in \mathcal{G} := [0, \mathfrak{T}], \\ (Q + \Upsilon)(0) = -(Q + \Upsilon)(\mathfrak{T}), \\ \int_{\varrho}^{\zeta} (Q - \Upsilon)(\varkappa) d\varkappa - \sum_{i=1}^{m} \mathfrak{q}_i (Q - \Upsilon)(\varphi_i) - \sum_{j=1}^{n} \mathfrak{u}_j (Q - \Upsilon)(\delta_j) = \mathcal{P}_1, \end{cases} \tag{1.1}$$

where ${}^{C}\mathcal{D}^{\varsigma_i}$ denotes the Caputo fractional derivative operator of order $\varsigma_i; i = 1, 2$. $\varsigma_1, \varsigma_2 \in (1, 2]$, $0 < \varphi_i < \varrho < \zeta < \delta_j < \mathfrak{T}, i = 1, \cdots, m, j = 1, \cdots, n$, and $\mathcal{L}_1, \mathcal{J}_1 : \mathcal{G} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. The boundary conditions introduced in problem (1.1) can be interpreted as the sum of the unknown functions Q and Υ at the endpoints of the interval $[0, \mathfrak{T}]$ begin zero, while the second condition asserts that the difference between the unknown functions Q and Υ on an arbitrary strip (ϱ, ζ) within the domain $[0, \mathfrak{T}]$ deviates from the sum of similar contributions arising from various positions at $\varphi_i, i = 1, \cdots, m$ and $\delta_j, j = 1, 2, \cdots, n$, with δ_j being a positive constant. The parameters $\mathfrak{q}_i, \mathfrak{u}_j, \mathcal{P}_1$ are non-negative constants. Here, we utilize Schaefer's fixed point theorem and the Banach contraction mapping principle with arbitrary strip and multi-point boundary conditions to investigate the existence and uniqueness of system (1.1). In addition, we also examine the stability of the system under investigation through UH stability.

Our work is organized as follows. Section 2 provides a review of fundamental concepts and presents auxiliary lemmas related to the linear variant of problem (1.1). The main results concerning problem (1.1), obtained using a standard fixed point theorem, are presented in Section 3. The stability properties of the system of nonlinear coupled SFDEs of the Caputo type are examined using UH stability analysis in Section 4. To further illustrate our findings, Section 5 includes proof with examples. We conclude our findings in Section 6.

2. Preliminaries

For the benefit of readers interested in this study's subject matter, we introduce essential concepts and definitions foundational to the main results discussed in subsequent sections.

Definition 2.1. *[7] The definition of the Riemann-Liouville (RL) fractional integral of order r is defined as*

$$\mathcal{I}^r h_1(\mathrm{t}) = \frac{1}{\Gamma(r)} \int_0^{\mathrm{t}} \frac{h_1(u)}{(\mathrm{t} - u)^{1-r}} du, \quad r > 0.$$

Definition 2.2. *[7] For $(n-1)$ absolutely continuous functions $h_1 : [0, \infty) \to \mathbb{R}$, the C-D of fractional order r is defined as*

$${}^{c}\mathcal{D}^r h_1(\mathrm{t}) = \frac{1}{\Gamma(n - r)} \int_0^{\mathrm{t}} (\mathrm{t} - u)^{n-r-1} h_1^{(n)}(u) du, \quad n - 1 < r < n, \ n = [r] + 1.$$

Lemma 2.3. *[39] Let $\mathfrak{r} > 0$ and $\mathfrak{h}_1(\mathrm{t}) \in \mathcal{AC}^{\mathfrak{n}}[0, \infty)$ or $C^{\mathfrak{n}}[0, \infty)$. Then*

$$(\mathcal{I}^{\mathfrak{r}C}\mathcal{D}^{\mathfrak{r}}\mathfrak{h}_1)(\mathrm{t}) = \mathfrak{h}_1(\mathrm{t}) - \sum_{k=0}^{\mathfrak{n}-1} \frac{\mathfrak{h}_1^{(k)}(0)}{k!} \mathrm{t}^k, \ \mathrm{t} > 0, \mathfrak{n} - 1 < \mathfrak{r} < \mathfrak{n}.$$

Lemma 2.4. *Let $\mathcal{X}$ be a Banach space, $\mathcal{E} \subset \mathcal{X}$ be closed, and $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ is a strict contraction,*

$$|\mathcal{F}\mathfrak{x} - \mathcal{F}\mathfrak{y}| \leq \mathcal{K}|\mathfrak{x} - \mathfrak{y}|$$

for some $\mathcal{K} \in (0, 1)$ and all $\mathfrak{x}, \mathfrak{y} \in \mathcal{E}$. Then $\mathcal{F}$ has a unique fixed point in $\mathcal{E}$.

Lemma 2.5. *[Arzela-Ascoli Theorem] [39] A subset $\mathcal{F}$ in $C([\mathfrak{a}, \mathfrak{b}], \mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous on $[\mathfrak{a}, \mathfrak{b}]$.*

Lemma 2.6. *[Schaefer's Fixed Point Theorem] [7] Let $\mathcal{H}_1 : \mathcal{E} \to \mathcal{E}$ be a completely continuous (c.c) operator in the Banach space $\mathcal{E}$, and let the set $\phi = \{\mathfrak{u} \in \mathcal{E} | \mathfrak{u} = \mu\mathcal{H}_1\mathfrak{u}, 0 < \mu < 1\}$ be bounded. Then $\mathcal{H}_1$ has a fixed point in $\mathcal{E}$.*

The lemma presented next focuses on exploring the linear variant of the problem outlined in problem (1.1).

Lemma 2.7. *Let $\mathcal{H}_1, \mathcal{H}_2 \in C(\mathcal{G}, \mathbb{R})$, then the solution of the system*

$$\begin{cases} ({}^{C}\mathcal{D}^{\varsigma_1} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_1-1})\mathcal{Q}(\mathrm{t}) = \mathcal{H}_1(\mathrm{t}), \ \ \mathrm{t} \in \mathcal{G} := [0, \mathfrak{T}], \\ ({}^{C}\mathcal{D}^{\varsigma_2} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_2-1})\Upsilon(\mathrm{t}) = \mathcal{H}_2(\mathrm{t}), \ \ \mathrm{t} \in \mathcal{G} := [0, \mathfrak{T}], \\ (\mathcal{Q} + \Upsilon)(0) = -(\mathcal{Q} + \Upsilon)(\mathfrak{T}), \\ \displaystyle\int_{\varrho}^{\zeta} (\mathcal{Q} - \Upsilon)(\varkappa)d\varkappa - \sum_{i=1}^{m} \mathfrak{q}_i(\mathcal{Q} - \Upsilon)(\varphi_i) - \sum_{j=1}^{n} \mathfrak{u}_j(\mathcal{Q} - \Upsilon)(\delta_j) = \mathcal{P}_1, \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} \mathcal{Q}(\mathrm{t}) = & \frac{e^{-\mathcal{K}_1\mathrm{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\right)d\varkappa \\ & - \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\right)d\varkappa\Bigg) \\ & + \frac{1}{\Delta_2}\Bigg(\mathcal{P}_1 - \int_{\varrho}^{\zeta}\left(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau} \frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\tau\right)d\varkappa \\ & + \int_{\varrho}^{\zeta}\left(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau} \frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\tau\right)d\varkappa \\ & + \sum_{i=1}^{m} \mathfrak{q}_i \int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\right)d\varkappa \\ & - \sum_{i=1}^{m} \mathfrak{q}_i \int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\right)d\varkappa \\ & + \sum_{j=1}^{n} \mathfrak{u}_j \int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\right)d\varkappa \\ & - \sum_{j=1}^{n} \mathfrak{u}_j \int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\right)d\varkappa\Bigg)\Bigg] \\ & + \int_0^{\mathrm{t}} e^{-\mathcal{K}_1(\mathrm{t}-\varkappa)}\left(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\right)d\varkappa, \end{aligned} \tag{2.2}$$

$$\begin{aligned}
\Upsilon(\mathrm{t}) =& \frac{e^{-\mathcal{K}_1 \mathrm{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\Bigg)d\varkappa \\
&-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_2(\tau)d\tau\Bigg)d\varkappa\Bigg) \\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa \\
&+\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa \\
&+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\Bigg)d\varkappa \\
&-\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\Bigg)d\varkappa+ \\
&\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\Bigg)d\varkappa \\
&-\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\Bigg)d\varkappa\Bigg)\Bigg] \\
&+\int_0^{\mathrm{t}} e^{-\mathcal{K}_1(\mathrm{t}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\Bigg)d\varkappa, \qquad (2.3)
\end{aligned}$$

where

$$\Delta_1 = (1+e^{-\mathcal{K}_1\mathfrak{T}}), \quad \Delta_2 = \int_{\varrho}^{\zeta} e^{-\mathcal{K}_1\varkappa}d\varkappa - \sum_{i=1}^{m}\mathfrak{q}_i e^{-\mathcal{K}_1(\varphi_i)} - \sum_{j=1}^{n}\mathfrak{u}_j e^{-\mathcal{K}_1(\delta_j)} \neq 0, \qquad (2.4)$$

and

$$\begin{aligned}
\mathcal{I}_1 =& \Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\Bigg)d\varkappa \\
&-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_2(\tau)d\tau\Bigg)d\varkappa\Bigg)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_2 =& \Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa \\
&+\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa \\
&+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\Bigg)d\varkappa \\
&-\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\Bigg)d\varkappa
\end{aligned}$$

$$+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\right)d\varkappa$$
$$-\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\right)d\varkappa\Bigg).$$

Proof. Equation (2.1) can be equivalently written as

$$({}^C\mathcal{D}^{\varsigma_1}+\mathcal{K}_1{}^C\mathcal{D}^{\varsigma_1-1})\mathcal{Q}(\mathfrak{t})=\mathcal{H}_1(\mathfrak{t}),$$
$$({}^C\mathcal{D}^{\varsigma_2}+\mathcal{K}_1{}^C\mathcal{D}^{\varsigma_2-1})\Upsilon(\mathfrak{t})=\mathcal{H}_2(\mathfrak{t}).$$

Rewriting the equation in (2.1) as ${}^C\mathcal{D}^{\varsigma_1}(1+\mathcal{K}_1{}^C\mathcal{D}^{-1})\mathcal{Q}(\mathfrak{t})=\mathcal{H}_1(\mathfrak{t})$ and ${}^C\mathcal{D}^{\varsigma_2}(1+\mathcal{K}_2{}^C\mathcal{D}^{-1})\Upsilon(\mathfrak{t})=\mathcal{H}_2(\mathfrak{t})$, then applying the integral operator $\mathcal{I}_{0^+}^{\varsigma_1}$ and $\mathcal{I}_{0^+}^{\varsigma_2}$ to it, we get

$$\mathcal{Q}(\mathfrak{t})=\mathfrak{c}_0e^{-\mathcal{K}_1\mathfrak{t}}+\int_0^{\mathfrak{t}}e^{-\mathcal{K}_1(\mathfrak{t}-\varkappa)}\left(\int_0^{\mathfrak{t}}\frac{(s-\mathfrak{a})^{(\varsigma_1-2)}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\mathfrak{q})d\mathfrak{q}\right)d\varkappa, \tag{2.5}$$
$$\Upsilon(\mathfrak{t})=\mathfrak{d}_0e^{-\mathcal{K}_1\mathfrak{t}}+\int_0^{\mathfrak{t}}e^{-\mathcal{K}_1(\mathfrak{t}-\varkappa)}\left(\int_0^{\mathfrak{t}}\frac{(s-\mathfrak{a})^{(\varsigma_2-2)}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\mathfrak{q})d\mathfrak{q}\right)d\varkappa, \tag{2.6}$$

where $\mathfrak{c}_0,\mathfrak{d}_0$ are arbitrary constants. Using BCs (2.1) in (2.5) and (2.6), we obtain

$$\mathfrak{c}_0+\mathfrak{d}_0=\mathcal{I}_1 \tag{2.7}$$
$$\mathfrak{c}_0-\mathfrak{d}_0=\mathcal{I}_2. \tag{2.8}$$

Solving (2.7) and (2.8) for $\mathfrak{c}_0$ and $\mathfrak{d}_0$ yields

$$\mathfrak{c}_0=\frac{1}{2}\Bigg\{\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\right)d\varkappa$$
$$-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_2(\tau)d\tau\right)d\varkappa\Bigg)$$
$$+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\right)d\tau\right)d\varkappa$$
$$+\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\right)d\tau\right)d\varkappa$$
$$+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\right)d\varkappa$$
$$-\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\right)d\varkappa$$
$$+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\right)d\varkappa$$
$$-\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\right)d\varkappa\Bigg)\Bigg\},$$

and

$$\begin{aligned}\mathfrak{d}_0 =& \frac{1}{2}\Bigg\{\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\Bigg)d\varkappa\\
&-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_2(\tau)d\tau\Bigg)d\varkappa\Bigg)\\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\mathfrak{m})d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa\\
&+\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\mathfrak{m})d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa\\
&+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\Bigg)d\varkappa\\
&-\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\Bigg)d\varkappa\\
&+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{H}_1(\tau)d\tau\Bigg)d\varkappa\\
&-\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{H}_2(\tau)d\tau\Bigg)d\varkappa\Bigg)\Bigg\}.\end{aligned}$$

The solutions (2.2) and (2.3) are obtained by substituting the values of $\mathfrak{c}_0$ and $\mathfrak{d}_0$ in (2.5), respectively. □

3. Main results

Define $\mathcal{E} = C(\mathcal{G},\mathbb{R})\times C(\mathcal{G},\mathbb{R})$ as the Banach space endowed with norm $\|(\mathcal{Q},\Upsilon)\| = \sup_{\mathrm{t}\in\mathcal{G}}|\mathcal{Q}(\mathrm{t})| + \sup_{\mathrm{t}\in\mathcal{G}}|\Upsilon(\mathrm{t})|$, for $(\mathcal{Q},\Upsilon)\in\mathcal{E}$. In view of Lemma 2.7, We define the operator $\Lambda:\mathcal{E}\to\mathcal{E}$ associated with system (1.1) as

$$\Lambda(\mathcal{Q},\Upsilon)(\mathrm{t}) = (\Lambda_1(\mathcal{Q},\Upsilon)(\mathrm{t}),\Lambda_2(\mathcal{Q},\Upsilon)(\mathrm{t})),$$

where

$$\begin{aligned}\Lambda_1(\mathcal{Q},\Upsilon)(\mathrm{t}) =& \frac{e^{-\mathcal{K}_1\mathrm{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau))d\tau\Bigg)d\varkappa\\
&-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}\mathcal{J}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)d\tau\Bigg)d\varkappa\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\mathfrak{m},\mathcal{Q}(\mathfrak{m}),\Upsilon(\mathfrak{m})d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa\\
&+\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(\mathfrak{m},\mathcal{Q}(\mathfrak{m}),\Upsilon(\mathfrak{m})d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa\\
&+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)d\tau\Bigg)d\varkappa\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(\tau,Q(\tau),\Upsilon(\tau)d\tau\right)d\varkappa\\
&+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\tau,Q(\tau),\Upsilon(\tau)d\tau\right)d\varkappa\\
&-\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(\tau,Q(\tau),\Upsilon(\tau)d\tau\right)d\varkappa\Bigg)\Bigg]\\
&+\int_0^{\mathrm{t}}e^{-\mathcal{K}_1(\mathrm{t}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\tau,Q(\tau),\Upsilon(\tau)d\tau\right)d\varkappa, \qquad (3.1)
\end{aligned}
$$

and

$$
\begin{aligned}
\Lambda_2(Q,\Upsilon)(\mathrm{t})=&\frac{e^{-\mathcal{K}_1\mathrm{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\tau,Q(\tau),\Upsilon(\tau))d\tau\right)d\varkappa\\
&-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}\mathcal{J}_1(\tau,Q(\tau),\Upsilon(\tau)d\tau\right)d\varkappa\Bigg)\\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\mathfrak{m},Q(\mathfrak{m}),\Upsilon(\mathfrak{m})d\mathfrak{m}\right)d\tau\right)d\varkappa\\
&+\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(\mathfrak{m},Q(\mathfrak{m}),\Upsilon(\mathfrak{m})d\mathfrak{m}\right)d\tau\right)d\varkappa\\
&+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\tau,Q(\tau),\Upsilon(\tau)d\tau\right)d\varkappa\\
&-\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(\tau,Q(\tau),\Upsilon(\tau)d\tau\right)d\varkappa\\
&+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\tau,Q(\tau),\Upsilon(\tau)d\tau\right)d\varkappa\\
&-\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(\tau,Q(\tau),\Upsilon(\tau)d\tau\right)d\varkappa\Bigg)\Bigg]\\
&+\int_0^{\mathrm{t}}e^{-\mathcal{K}_1(\mathrm{t}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{L}_1(\tau,\mathcal{J}_1(\tau),\Upsilon(\tau)d\tau\right)d\varkappa. \qquad (3.2)
\end{aligned}
$$

Next, we move forward by establishing a set of hypotheses, which are meticulously designed to serve as the foundational pillars for the demonstration of the primary findings of our research. This step is crucial in laying the groundwork for the subsequent analytical and empirical validation of our study's core assertions.

Consider the continuous functions $\mathcal{L}_1$ and $\mathcal{J}_1$, defined as mappings from $\mathcal{G}\times\mathbb{R}^2$ to $\mathbb{R}$.
($\mathcal{W}_1$) There exist continuous positive functions $\Psi_i, k_i \in C(\mathcal{G},\mathbb{R}^+)$, $i=1,2,3$, such that

$$
\begin{aligned}
|\mathcal{L}_1(\mathrm{t},Q,\Upsilon)| &\le \Psi_1(\mathrm{t})+\Psi_2(\mathrm{t})|Q|+\Psi_3|\Upsilon| \text{ forall } (\mathrm{t},Q,\Upsilon)\in\mathcal{G}\times\mathbb{R}^2,\\
|\mathcal{J}_1(\mathrm{t},Q,\Upsilon)| &\le k_1(\mathrm{t})+k_2(\mathrm{t})|Q|+k_3|\Upsilon| \text{ forall } (\mathrm{t},Q,\Upsilon)\in\mathcal{G}\times\mathbb{R}^2.
\end{aligned}
$$

($\mathcal{W}_2$) There exist non-negative constants $\mathcal{Y}_1, \mathcal{Y}_2$, $\mathcal{U}_1$, and $\mathcal{U}_2$ such that, for all $\mathfrak{t} \in \mathcal{G}$, $\mathcal{Q}_i, \Upsilon_i \in \mathbb{R}, i = 1, 2$.

$$|\mathcal{L}_1(\mathfrak{t}, \mathcal{Q}_1, \Upsilon_1) - \mathcal{L}_1(\mathfrak{t}, \mathcal{Q}_2, \Upsilon_2)| \leq (\mathcal{Y}_1|\mathcal{Q}_1 - \mathcal{Q}_2| + \mathcal{Y}_2|\Upsilon_1 - \Upsilon_2|), \text{ for all } \mathfrak{t} \in \mathcal{G},$$
$$|\mathcal{J}_1(\mathfrak{t}, \mathcal{Q}_1, \Upsilon_1) - \mathcal{J}_1(\mathfrak{t}, \mathcal{Q}_2, \Upsilon_2)| \leq (\mathcal{U}_1|\mathcal{Q}_1 - \mathcal{Q}_2| + \mathcal{U}_2|\Upsilon_1 - \Upsilon_2|), \text{ for all } \mathfrak{t} \in \mathcal{G}.$$

To enhance the efficiency of the computational process, we introduce the following notation, which simplifies complex calculations and facilitates a more streamlined analysis:

$$\begin{aligned}\Omega_1 =& \frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\frac{1}{(1+e^{-\mathcal{K}_1\mathfrak{T}})}\left(\frac{\mathfrak{T}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{T}})\right) + \frac{1}{\Delta_2}\Bigg\{\left(\frac{\zeta^{\varsigma_1-1}-\varrho^{\varsigma_1-1}}{\mathcal{K}_1^2\Gamma(\varsigma_1)}\right)(\zeta\mathcal{K}_1 + e^{-\mathcal{K}_1\zeta} - \mathcal{K}_1\varrho - e^{-\mathcal{K}_1\varrho}) \\ &+ \sum_{i=1}^{m}\mathfrak{q}_i\left(\frac{\varphi_i^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\varphi_i})\right) + \sum_{j=1}^{n}\mathfrak{u}_j\left(\frac{\delta_i^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\delta_i})\right)\Bigg\}\Bigg],\end{aligned} \tag{3.3}$$

$$\begin{aligned}\Omega_2 =& \frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\frac{1}{(1+e^{-\mathcal{K}_1\mathfrak{T}})}\left(\frac{\mathfrak{T}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{T}})\right) + \frac{1}{\Delta_2}\Bigg\{\left(\frac{\zeta^{\varsigma_2-1}-\varrho^{\varsigma_2-1}}{\mathcal{K}_1^2\Gamma(\varsigma_2)}\right)(\zeta\mathcal{K}_1 + e^{-\mathcal{K}_1\zeta} - \mathcal{K}_1\varrho - e^{-\mathcal{K}_1\varrho}) \\ &+ \sum_{i=1}^{m}\mathfrak{q}_i\left(\frac{\varphi_i^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\varphi_i})\right) + \sum_{j=1}^{n}\mathfrak{u}_j\left(\frac{\delta_i^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\delta_i})\right)\Bigg\}\Bigg],\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\Phi = \min\Bigg\{&1 - \left[\|\Psi_2\|\left(2\Omega_1 + \frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right) + \|k_2\|\left(2\Omega_2 + \frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)\right], \\ &1 - \left[\|\Psi_3\|\left(2\Omega_1 + \frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right) + \|k_3\|\left(2\Omega_2 + \frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)\right]\Bigg\}.\end{aligned}$$

Subsequently, we aim to delineate the implications arising from the existence of the BVP (1.1), employing Schaefer's fixed point theorem as outlined in [7].

Theorem 3.1. *Suppose that ($\mathcal{W}_1$) holds. Furthermore, the assumption is that*

$$\begin{aligned}&\|\Psi_2\|\left(2\Omega_1 + \frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right) + \|k_2\|\left(2\Omega_2 + \frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right) < 1, \\ &\|\Psi_3\|\left(2\Omega_1 + \frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right) + \|k_3\|\left(2\Omega_2 + \frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right) < 1,\end{aligned} \tag{3.5}$$

where Ω_1, Ω_2 are defined by (3.3) and (3.4). Then the problem (1.1) has at least one solution on $\mathcal{G}$.

Proof. The operator $\Lambda : \mathcal{E} \to \mathcal{E}$ is continuous as a result of continuity of $\mathcal{L}_1$ and $\mathcal{J}_1$. Next, consider the bounded set $\iota_{\bar{\mathfrak{r}}} \subset \mathcal{E}$. Consequently, there exist positive constants $\mathcal{Y}_{\mathcal{L}_1}$ and $\mathcal{Y}_{\mathcal{J}_1}$ such that the following inequalities hold:

$$|\mathcal{L}_1(\mathfrak{t}, \mathcal{Q}(\mathfrak{t}), \Upsilon(\mathfrak{t}))| \leq \mathcal{Y}_{\mathcal{L}_1},$$

$$|\mathcal{J}_1(\mathrm{t}, \mathcal{Q}(\mathrm{t}), \Upsilon(\mathrm{t}))| \leq \mathcal{Y}_{\mathcal{J}_1},$$

and for all $(\mathcal{Q}, \Upsilon) \in \iota_{\bar{\mathrm{r}}}$, $\mathrm{t} \in \mathcal{G}$, we have

$$\begin{aligned}
|\Lambda_1(\mathcal{Q}, \Upsilon)(\mathrm{t})| \leq & \frac{e^{-\mathcal{K}_1 \mathrm{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau, \mathcal{Q}(\tau), \Upsilon(\tau))| d\tau\Bigg) d\varkappa \\
& + \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}|\mathcal{J}_1(\tau, \mathcal{Q}(\tau), \Upsilon(\tau)| d\tau\Bigg) d\varkappa\Bigg) \\
& + \frac{1}{\Delta_2}\Bigg(\mathcal{P}_1 - \int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau} \frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\mathfrak{m}, \mathcal{Q}(\mathfrak{m}), \Upsilon(\mathfrak{m}) d\mathfrak{m}\Bigg) d\tau\Bigg) d\varkappa \\
& + \int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau} \frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\mathfrak{m}, \mathcal{Q}(\mathfrak{m}), \Upsilon(\mathfrak{m})| d\mathfrak{m}\Bigg) d\tau\Bigg) d\varkappa \\
& + \sum_{i=1}^{m} \mathfrak{q}_i \int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau, \mathcal{Q}(\tau), \Upsilon(\tau)| d\tau\Bigg) d\varkappa \\
& + \sum_{i=1}^{m} \mathfrak{q}_i \int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\tau, \mathcal{Q}(\tau), \Upsilon(\tau)| d\tau\Bigg) d\varkappa \\
& + \sum_{j=1}^{n} \mathfrak{u}_j \int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\Bigg(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau, \mathcal{Q}(\tau), \Upsilon(\tau)| d\tau\Bigg) d\varkappa \\
& + \sum_{j=1}^{n} \mathfrak{u}_j \int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\Bigg(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\tau, \mathcal{Q}(\tau), \Upsilon(\tau)| d\tau\Bigg) d\varkappa\Bigg)\Bigg] \\
& + \int_0^{\mathrm{t}} e^{-\mathcal{K}_1(\mathrm{t}-\varkappa)}\Bigg(\int_0^{\varkappa} \frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau, \mathcal{Q}(\tau), \Upsilon(\tau)| d\tau\Bigg) d\varkappa, \\
\leq & \mathcal{Y}_{\mathcal{L}_1}\Bigg\{\frac{e^{-\mathcal{K}_1 \mathrm{t}}}{2}\Bigg[\frac{1}{(1+e^{-\mathcal{K}_1 \mathfrak{T}})}\Bigg(\frac{\mathfrak{T}^{\varsigma_1-1}}{\mathcal{K}_1 \Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1 \mathfrak{T}})\Bigg) \\
& + \frac{1}{\Delta_2}\Bigg\{\Bigg(\frac{\zeta^{\varsigma_1-1}-\varrho^{\varsigma_1-1}}{\mathcal{K}_1^2 \Gamma(\varsigma_1)}\Bigg)(\zeta \mathcal{K}_1 + e^{-\mathcal{K}_1 \zeta} - \mathcal{K}_1 \varrho - e^{-\mathcal{K}_1 \varrho}) \\
& + \sum_{i=1}^{m} \mathfrak{q}_i \Bigg(\frac{\varphi_i^{\varsigma_1-1}}{\mathcal{K}_1 \Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1 \varphi_i})\Bigg) + \sum_{j=1}^{n} \mathfrak{u}_j \Bigg(\frac{\delta_i^{\varsigma_1-1}}{\mathcal{K}_1 \Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1 \delta_i})\Bigg)\Bigg\}\Bigg]\Bigg\} \\
& + \Bigg\{\frac{\mathrm{t}^{\varsigma_1-1}}{\mathcal{K}_1 \Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1 \mathrm{t}})\Bigg\}\mathcal{Y}_{\mathcal{L}_1} + \mathcal{Y}_{\mathcal{J}_1}\Bigg\{\frac{e^{-\mathcal{K}_1 \mathrm{t}}}{2}\Bigg[\frac{1}{(1+e^{-\mathcal{K}_1 \mathfrak{T}})}\Bigg(\frac{\mathfrak{T}^{\varsigma_2-1}}{\mathcal{K}_1 \Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1 \mathfrak{T}})\Bigg) \\
& + \frac{1}{\Delta_2}\Bigg\{\Bigg(\frac{\zeta^{\varsigma_2-1}-\varrho^{\varsigma_2-1}}{\mathcal{K}_1^2 \Gamma(\varsigma_2)}\Bigg)(\zeta \mathcal{K}_1 + e^{-\mathcal{K}_1 \zeta} - \mathcal{K}_1 \varrho - e^{-\mathcal{K}_1 \varrho}) \\
& + \sum_{i=1}^{m} \mathfrak{q}_i \Bigg(\frac{\varphi_i^{\varsigma_2-1}}{\mathcal{K}_1 \Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1 \varphi_i})\Bigg) + \sum_{j=1}^{n} \mathfrak{u}_j \Bigg(\frac{\delta_i^{\varsigma_2-1}}{\mathcal{K}_1 \Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1 \delta_i})\Bigg)\Bigg\}\Bigg]\Bigg\} + \frac{\mathcal{P}_1}{\Delta_2}, \\
\leq & \mathcal{Y}_{\mathcal{L}_1}\Bigg(\Omega_1 + \frac{\mathrm{t}^{\varsigma_1-1}}{\mathcal{K}_1 \Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1 \mathrm{t}})\Bigg) + \mathcal{Y}_{\mathcal{J}_1}\Omega_2 + \frac{\mathcal{P}_1}{2\Delta_2}.
\end{aligned}$$

Hence,

$$\|\Lambda_1(\mathcal{Q},\Upsilon)\| \leq \mathcal{Y}_{\mathcal{L}_1}\left(\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{Y}_{\mathcal{J}_1}\Omega_2+\frac{\mathcal{P}_1}{2\Delta_2}, \tag{3.6}$$

and

$$\begin{aligned}
|\Lambda_2(\mathcal{Q},\Upsilon)(\mathfrak{t})| \leq &\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\frac{1}{\Delta_1}\left(\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau))|d\tau\right)d\varkappa\right.\\
&\left.+\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}|\mathcal{J}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)|d\tau\right)d\varkappa\right)\\
&+\frac{1}{\Delta_2}\left(\mathcal{P}_1-\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\mathfrak{m},\mathcal{Q}(\mathfrak{m}),\Upsilon(\mathfrak{m})d\mathfrak{m}\right)d\tau\right)d\varkappa\right.\\
&+\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\mathfrak{m},\mathcal{Q}(\mathfrak{m}),\Upsilon(\mathfrak{m})|d\mathfrak{m}\right)d\tau\right)d\varkappa\\
&+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)|d\tau\right)d\varkappa\\
&+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)|d\tau\right)d\varkappa\\
&+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)|d\tau\right)d\varkappa\\
&\left.+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)|d\tau\right)d\varkappa\right)\Bigg]\\
&+\int_0^{\mathfrak{t}} e^{-\mathcal{K}_1(\mathfrak{t}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)|d\tau\right)d\varkappa,\\
\leq &\mathcal{Y}_{\mathcal{L}_1}\Omega_1+\mathcal{Y}_{\mathcal{J}_1}\left(\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{2\Delta_2}.
\end{aligned}$$

Hence,

$$\|\Lambda_2(\mathcal{Q},\Upsilon)\| \leq \mathcal{Y}_{\mathcal{L}_1}\Omega_1+\mathcal{Y}_{\mathcal{J}_1}\left(\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{2\Delta_2}. \tag{3.7}$$

Thus, from Eqs (3.6) and (3.7), we obtain

$$\begin{aligned}
\|\Lambda(\mathcal{Q},\Upsilon)\| = \|\Lambda_1(\mathcal{Q},\Upsilon)\|+\|\Lambda_2(\mathcal{Q},\Upsilon)\| \leq &\mathcal{Y}_{\mathcal{L}_1}\left(2\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)\\
&+\mathcal{Y}_{\mathcal{J}_1}\left(2\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{\Delta_2}.
\end{aligned}$$

Thus, the operator Λ is uniformly bounded, we show that the operator Λ is equicontinuous on $\mathcal{E}$. For this we let $\mathfrak{t}_1,\mathfrak{t}_2\in[0,\mathfrak{T}]$, $\mathfrak{t}_1<\mathfrak{t}_2$, and $(\mathcal{Q},\Upsilon\in\iota_{\bar{\mathfrak{r}}})$. Then

$$
\begin{aligned}
|\Lambda_1(\mathcal{Q},\Upsilon)(\mathrm{t}_2)-\Lambda_1(\mathcal{Q},\Upsilon)(\mathrm{t}_1)| \leq & \Bigg|\frac{e^{-\mathcal{K}_1\mathrm{t}_2}-e^{-\mathcal{K}_1\mathrm{t}_1}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{L}_1}d\tau\Bigg)d\varkappa \\
& -\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{J}_1}d\tau\Bigg)d\varkappa\Bigg) \\
& +\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{L}_1}d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa \\
& +\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{Y}_{\mathcal{J}_1}d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa \\
& +\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{L}_1}d\tau\Bigg)d\varkappa \\
& -\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{Y}_{\mathcal{J}_1}d\tau\Bigg)d\varkappa \\
& +\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{L}_1}d\tau\Bigg)d\varkappa \\
& -\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{Y}_{\mathcal{J}_1}d\tau\Bigg)d\varkappa\Bigg)\Bigg]\Bigg| \\
& +\Bigg|\int_0^{\mathrm{t}_1}(e^{-\mathcal{K}_1(\mathrm{t}_2-\varkappa)}-e^{-\mathcal{K}_1(\mathrm{t}_1-\varkappa)})\Big(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{L}_1}d\tau\Big)d\varkappa \\
& +\int_{\mathrm{t}_1}^{\mathrm{t}_2} e^{-\mathcal{K}_1(\mathrm{t}_2-\varkappa)}\Big(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{L}_1}d\tau\Big)d\varkappa\Bigg|, \\
\leq & \frac{\mathcal{Y}_{\mathcal{L}_1}}{\mathcal{K}_1\Gamma(\varsigma_1+1)}(|\mathrm{t}_1^{\varsigma_1}-\mathrm{t}_2^{\varsigma_1}|+|\mathrm{t}_1^{\varsigma_1}e^{-\mathcal{K}_1\mathrm{t}_1}-\mathrm{t}_2^{\varsigma_1}e^{-\mathcal{K}_1\mathrm{t}_2}|)\to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
|\Lambda_2(\mathcal{Q},\Upsilon)(\mathrm{t}_2)-\Lambda_2(\mathcal{Q},\Upsilon)(\mathrm{t}_1)| \leq & \Bigg|\frac{e^{-\mathcal{K}_1\mathrm{t}_2}-e^{-\mathcal{K}_1\mathrm{t}_1}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{L}_1}d\tau\Bigg)d\varkappa \\
& -\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{J}_1}d\tau\Bigg)d\varkappa\Bigg) \\
& -\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{L}_1}d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa \\
& +\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{Y}_{\mathcal{J}_1}d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa \\
& +\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{L}_1}d\tau\Bigg)d\varkappa \\
& -\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{Y}_{\mathcal{J}_1}d\tau\Bigg)d\varkappa
\end{aligned}
$$

$$+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{Y}_{\mathcal{L}_1}d\tau\right)d\varkappa$$

$$-\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{Y}_{\mathcal{J}_1}d\tau\right)d\varkappa\Bigg)\Bigg]\Bigg|$$

$$+\Bigg|\int_0^{\mathfrak{t}_1}(e^{-\mathcal{K}_1(\mathfrak{t}_2-\varkappa)}-e^{-\mathcal{K}_1(\mathfrak{t}_1-\varkappa)})\Big(\int_0^{\varkappa}\frac{(\varkappa-\mathfrak{q})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{Y}_{\mathcal{J}_1}d\tau\Big)d\varkappa$$

$$+\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}e^{-\mathcal{K}_1(\mathfrak{t}_2-\varkappa)}\Big(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{Y}_{\mathcal{J}_1}d\tau\Big)d\varkappa\Bigg|$$

$$\leq\frac{\mathcal{Y}_{\mathcal{J}_1}}{\mathcal{K}_1\Gamma(\varsigma_2+1)}(|\mathfrak{t}_1^{\varsigma_2}-\mathfrak{t}_2^{\varsigma_2}|+|\mathfrak{t}_1^{\varsigma_2}e^{-\mathcal{K}_1\mathfrak{t}_1}-\mathfrak{t}_2^{\varsigma_2}e^{-\mathcal{K}_1\mathfrak{t}_2}|)\to 0.$$

Note that, in the limit case as $\mathfrak{t}_1\to\mathfrak{t}_2$, the right-hand side of the previous inequalities approach zero, regardless of $(\mathcal{Q},\Upsilon)\in\iota_{\bar{\mathfrak{r}}}$. This implies that the operator $\Lambda(\mathcal{Q},\Upsilon)$ is equicontinuous. By applying the Arzela-Ascoli theorem, we can conclude that $\Lambda(\mathcal{Q},\Upsilon)$ is completely continuous. Furthermore, we will now demonstrate that the set $\Theta_1=(\mathcal{Q},\Upsilon)\in\mathcal{E}|(\mathcal{Q},\Upsilon)=\Pi\Lambda(\mathcal{Q},\Upsilon),0<\Pi<1$ is bounded. For any $\mathfrak{t}\in\mathcal{G}$, one can get

$$\mathcal{Q}(\mathfrak{t})=\Pi\Lambda_1(\mathcal{Q},\Upsilon)(\mathfrak{t}),\quad\Upsilon(\mathfrak{t})=\Pi\Lambda_2(\mathcal{Q},\Upsilon)(\mathfrak{t}).$$

Exploiting the definitions of Ω_1 and Ω_2 given by (3.3) and (3.4), we obtain

$$|\mathcal{Q}(\mathfrak{t})|=\Pi|\Lambda_1(\mathcal{Q},\Upsilon)(\mathfrak{t})|$$

$$\leq\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(|\Psi_1|+\Psi_2|\mathcal{Q}(\tau)|+\Psi_3|\Upsilon(\tau)|)d\tau\right)d\varkappa$$

$$+\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}\mathcal{J}_1(|k_1|+k_2|\mathcal{Q}(\tau)|+k_3|\Upsilon(\tau)|)d\tau\right)d\varkappa\Bigg)$$

$$+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(|\Psi_1|+\Psi_2|\mathcal{Q}(\mathfrak{m})|+\Psi_3|\Upsilon(\mathfrak{m})|)d\mathfrak{m}\right)d\tau\Bigg)d\varkappa$$

$$+\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(|k_1|+k_2|\mathcal{Q}(\mathfrak{m})|+k_3|\Upsilon(\mathfrak{m})|)d\mathfrak{m}\right)d\tau\Bigg)d\varkappa$$

$$+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(|\Psi_1|+\Psi_2|\mathcal{Q}(\tau)|+\Psi_3|\Upsilon(\tau)|)d\tau\right)d\varkappa$$

$$+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(|k_1|+k_2|\mathcal{Q}(\tau)|+k_3|\Upsilon(\tau)|)d\tau\right)d\varkappa$$

$$+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(|\Psi_1|+\Psi_2|\mathcal{Q}(\tau)|+\Psi_3|\Upsilon(\tau)|)d\tau\right)d\varkappa$$

$$+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(|k_1|+k_2|\mathcal{Q}(\tau)|+k_3|\Upsilon(\tau)|)d\tau\right)d\varkappa\Bigg)\Bigg]$$

$$+\int_0^{\mathfrak{t}}e^{-\mathcal{K}_1(\mathfrak{t}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(|\Psi_1|+\Psi_2|\mathcal{Q}(\tau)|+\Psi_3|\Upsilon(\tau)|)d\tau\right)d\varkappa$$

$$
\begin{aligned}
&\leq (\|\Psi_1\| + \|\Psi_2\|\|Q\| + \|\Psi_3\|\|\Upsilon\|)\left\{\frac{e^{-\mathcal{K}_1 \mathfrak{t}}}{2}\left[\frac{1}{(1+e^{-\mathcal{K}_1\mathfrak{T}})}\left(\frac{\mathfrak{T}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{T}})\right)\right.\right.\\
&\quad +\frac{1}{\Delta_2}\left\{\left(\frac{\zeta^{\varsigma_1-1}-\varrho^{\varsigma_1-1}}{\mathcal{K}_1^2\Gamma(\varsigma_1)}\right)(\zeta\mathcal{K}_1 + e^{-\mathcal{K}_1\zeta} - \mathcal{K}_1\varrho - e^{-\mathcal{K}_1\varrho})\right.\\
&\quad \left.\left. + \sum_{i=1}^{m}\mathfrak{q}_i\left(\frac{\varphi_i^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\varphi_i})\right) + \sum_{j=1}^{n}\mathfrak{u}_j\left(\frac{\delta_i^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\delta_i})\right)\right\}\right]\\
&\quad \left. + \left\{\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right\}\right\}\\
&\quad + (\|k_1\| + \|k_2\|\|Q\| + \|k_3\|\|\Upsilon\|)\left\{\frac{e^{-\mathcal{K}_1 \mathfrak{t}}}{2}\left[\frac{1}{(1+e^{-\mathcal{K}_1\mathfrak{T}})}\left(\frac{\mathfrak{T}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{T}})\right)\right.\right.\\
&\quad +\frac{1}{\Delta_2}\left\{\left(\frac{\zeta^{\varsigma_2-1}-\varrho^{\varsigma_2-1}}{\mathcal{K}_1^2\Gamma(\varsigma_2)}\right)(\zeta\mathcal{K}_1 + e^{-\mathcal{K}_1\zeta} - \mathcal{K}_1\varrho - e^{-\mathcal{K}_1\varrho})\right.\\
&\quad \left.\left.\left. + \sum_{i=1}^{m}\mathfrak{q}_i\left(\frac{\varphi_i^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\varphi_i})\right) + \sum_{j=1}^{n}\mathfrak{u}_j\left(\frac{\delta_i^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\delta_i})\right)\right\}\right]\right\} + \frac{\mathcal{P}_1}{2\Delta_2},\\
&\leq (\|\Psi_1\| + \|\Psi_2\|\|Q\| + \|\Psi_3\|\|\Upsilon\|)\left(\Omega_1 + \frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)\\
&\quad + (\|k_1\| + \|k_2\|\|Q\| + \|k_3\|\|\Upsilon\|)\,\Omega_2 + \frac{\mathcal{P}_1}{2\Delta_2},
\end{aligned}
$$

and

$$
\begin{aligned}
|\Upsilon(\mathfrak{t})| =&\Pi|\Lambda_2(Q,\Upsilon)(\mathfrak{t})|\\
\leq&\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\left[\frac{1}{\Delta_1}\left(\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(|\Psi_1| + \Psi_2|Q(\tau)| + \Psi_3|\Upsilon(\tau)|)d\tau\right)d\varkappa\right.\right.\\
&+\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left.\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}\mathcal{J}_1(|k_1| + k_2|Q(\tau)| + k_3|\Upsilon(\tau)|)d\tau\right)d\varkappa\right)\\
&+\frac{1}{\Delta_2}\left(\mathcal{P}_1 - \int_{\varrho}^{\zeta}\left(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(|\Psi_1| + \Psi_2|Q(\mathfrak{m})| + \Psi_3|\Upsilon(\mathfrak{m})|)d\mathfrak{m}\right)d\tau\right)d\varkappa\right.\\
&+\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(|k_1| + k_2|Q(\mathfrak{m})| + k_3|\Upsilon(\mathfrak{m})|)d\mathfrak{m}\right)d\tau\right)d\varkappa\\
&+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(|\Psi_1| + \Psi_2|Q(\tau)| + \Psi_3|\Upsilon(\tau)|)d\tau\right)d\varkappa\\
&+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i} e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(|k_1| + k_2|Q(\tau)| + k_3|\Upsilon(\tau)|)d\tau\right)d\varkappa\\
&+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(|\Psi_1| + \Psi_2|Q(\tau)| + \Psi_3|\Upsilon(\tau)|)d\tau\right)d\varkappa\\
&\left.\left.+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j} e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(|k_1| + k_2|Q(\tau)| + k_3|\Upsilon(\tau)|)d\tau\right)d\varkappa\right)\right]
\end{aligned}
$$

$$+\int_0^{\mathfrak{t}} e^{-\mathcal{K}_1(\mathfrak{t}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(|k_1|+k_2|Q(\tau)|+k_3|\Upsilon(\tau)|)d\tau\right)d\varkappa$$
$$\leq (\|\Psi_1\|+\|\Psi_2\|\|Q\|+\|\Psi_3\|\|\Upsilon\|)\,\Omega_1$$
$$+(\|k_1\|+\|k_2\|\|Q\|+\|k_3\|\|\Upsilon\|)\left(\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{2\Delta_2}.$$

As a result, we obtain

$$\|Q\|+\|\Upsilon\| \leq \|\Psi_1\|\left(2\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\|k_1\|\left(2\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{\Delta_2}$$
$$+\left[\|\Psi_2\|\left(2\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\|k_2\|\left(2\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)\right]\|Q\|$$
$$+\left[\|\Psi_3\|\left(2\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\|k_3\|\left(2\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)\right]\|\Upsilon\|.$$

By employing Eq (3.5), we can deduce that

$$\|(Q,\Upsilon)\| \leq \frac{\|\Psi_1\|\left(2\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\|k_1\|\left(2\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{\Delta_2}}{\Phi}.$$

It is shows that $\|(Q,\Upsilon)\|$ is bounded for $\mathfrak{t}\in[0,\mathfrak{T}]$. The set Θ_1 is bounded. Consequently, by applying Schaefer's fixed point theorem, we can conclude that a solution of Eq (1.1) exists. □

The statement of Theorem 3.1 reduces to the following special form by fixing $\Psi_2(\mathfrak{t})=\Psi_3(\mathfrak{t})=0$ and $k_2(\mathfrak{t})=k_2(\mathfrak{t})=0$ in it.

Remark 3.1. *There exist positive functions $\Psi_1, k_1 \in C(\mathfrak{G},\mathbb{R}^+)$ and $\mathcal{L}_1,\mathcal{J}_1 : \mathfrak{G}\times\mathbb{R}^2\to\mathbb{R}$, which are continuous functions such that*

$$|\mathcal{L}_1(\mathfrak{t},Q,\Upsilon)|\leq\Psi_1(\mathfrak{t}),\quad |\mathcal{J}_1(\mathfrak{t},Q,\Upsilon)|\leq k_1(\mathfrak{t})\quad \textit{for all } (\mathfrak{t},Q,\Upsilon)\in\mathfrak{G}\times\mathbb{R}^2.$$

Then system (1.1) has at least one solution on $\mathfrak{G}$.

Remark 3.2. *According to the assumptions of Theorem 3.1, if $\Psi_i(\mathfrak{t})=\iota_i, k_i(\mathfrak{t})=\varepsilon_i, i=1,2,3(\varepsilon_i$ and $\iota_i)$ non-negative constants, and the criteria of the functions $\mathcal{L}_1,\mathcal{J}_1$ have the following form:*

(Q_1') *there are real constants $\iota_i,\varepsilon_i>0, i=1,2,3,$ so*

$$|\mathcal{L}_1(\mathfrak{t},Q,\Upsilon)|\leq\iota_1+\iota_2|Q|+\iota_3|\Upsilon|\quad \textit{for all } (\mathfrak{t},Q,\Upsilon)\in\mathfrak{G}\times\mathbb{R}^2,$$
$$|\mathcal{J}_1(\mathfrak{t},Q,\Upsilon)|\leq\varepsilon_1+\varepsilon_2|Q|+\varepsilon_3|\Upsilon|\quad \textit{for all } (\mathfrak{t},Q,\Upsilon)\in\mathfrak{G}\times\mathbb{R}^2,$$

and (3.5) becomes

$$\iota_2\left(2\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\varepsilon_2\left(2\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)<1,$$
$$\iota_3\left(2\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\varepsilon_3\left(2\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)<1.$$

Now, we proceed to present our second result, which builds upon Banach's fixed point theorem and provides evidence for the existence of a unique solution to (1.1).

Theorem 3.2. *Suppose that* $(\mathcal{W}_2)$ *holds. Then problem 1.1 has a unique solution on* $\mathcal{E}$ *if*

$$\mathcal{Y}\left(2\Omega_1+\frac{\mathrm{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)+\mathcal{U}\left(2\Omega_2+\frac{\mathrm{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)<1, \tag{3.8}$$

where $\mathcal{Y}=\max\{\mathcal{Y}_1,\mathcal{Y}_2\}$*,* $\mathcal{U}=\max\{\mathcal{U}_1,\mathcal{U}_2\}$ *and* $\Omega_i, i=1,2$ *are defined by (3.3) and (3.4).*

Proof. Let us take $\mathcal{M}_1=\sup_{\mathrm{t}\in[0,\mathfrak{T}]}|\mathcal{L}_1(\mathrm{t},0,0)|$ and $\mathcal{M}_2=\sup_{\mathrm{t}\in[0,\mathfrak{T}]}|\mathcal{J}_1(\mathrm{t},0,0)|$ and fix

$$\mathrm{r}>\frac{\mathcal{M}_1\left(2\Omega_1+\frac{\mathrm{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)+\mathcal{M}_2\left(2\Omega_2+\frac{\mathrm{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)}{1-\left(\mathcal{Y}\left(2\Omega_1+\frac{\mathrm{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)+\mathcal{U}\left(2\Omega_2+\frac{\mathrm{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)\right)}.$$

Now, we show that $\Lambda\mathcal{B}_\mathrm{r}\subset\mathcal{B}_\mathrm{r}$, where $\mathcal{B}_\mathrm{r}=\{(Q,\Upsilon)\in\mathcal{E}:\|(Q,\Upsilon)\|\le\mathrm{r}\}$. Then

$$\begin{aligned}
&(\Lambda_1(Q,\Upsilon))\\
\le&\frac{e^{-\mathcal{K}_1\mathrm{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}[|\mathcal{L}_1(\tau,Q(\tau),\Upsilon(\tau))-\mathcal{L}_1(\tau,0,0)|+\mathcal{M}_1]d\tau\right)d\varkappa\\
&-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}[|\mathcal{J}_1(\tau,Q(\tau),\Upsilon(\tau))-\mathcal{J}_1(\tau,0,0)|+\mathcal{M}_2]d\tau\right)d\varkappa\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathrm{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}[|\mathcal{L}_1(\mathrm{m},Q(\mathrm{m}),\Upsilon(\mathrm{m}))-\mathcal{L}_1(\mathrm{m},0,0)|+\mathcal{M}_1]d\mathrm{m}\right)d\tau\right)d\varkappa\\
&+\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathrm{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}[|\mathcal{J}_1(\mathrm{m},Q(\mathrm{m}),\Upsilon(\mathrm{m}))-\mathcal{J}_1(\mathrm{m},0,0)|+\mathcal{M}_2]d\mathrm{m}\right)d\tau\right)d\varkappa\\
&+\sum_{i=1}^{m}\mathrm{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}[|\mathcal{L}_1(\tau,Q(\tau),\Upsilon(\tau))-\mathcal{L}_1(\tau,0,0)|+\mathcal{M}_1]d\tau\right)d\varkappa\\
&-\sum_{i=1}^{m}\mathrm{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}[|\mathcal{J}_1(\tau,Q(\tau),\Upsilon(\tau))-\mathcal{J}_1(\tau,0,0)|+\mathcal{M}_2|]d\tau\right)d\varkappa\\
&+\sum_{j=1}^{n}\mathrm{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}[|\mathcal{L}_1(\tau,Q(\tau),\Upsilon(\tau))-\mathcal{L}_1(\tau,0,0)|+\mathcal{M}_1|]d\tau\right)d\varkappa\\
&-\sum_{j=1}^{n}\mathrm{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}[|\mathcal{J}_1(\tau,Q(\tau),\Upsilon(\tau))-\mathcal{J}_1(\tau,0,0)|+\mathcal{M}_2|]d\tau\right)d\varkappa\Bigg)\Bigg]\\
&+\int_0^{\mathrm{t}}e^{-\mathcal{K}_1(\mathrm{t}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}[|\mathcal{L}_1(\tau,Q(\tau),\Upsilon(\tau))-\mathcal{L}_1(\tau,0,0)|+\mathcal{M}_1|]d\tau\right)d\varkappa\\
\le&\left(\mathcal{Y}\left(\Omega_1+\frac{\mathrm{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)+\mathcal{U}\Omega_2\right)(\|Q\|+\|\Upsilon\|)+\mathcal{M}_1\left(\Omega_1+\frac{\mathrm{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)+\mathcal{M}_2\Omega_2,
\end{aligned}$$

which turns out to be of the form

$$\begin{aligned}
&(\Lambda_1(Q,\Upsilon))\\
&\le\left(\mathcal{Y}\left(\Omega_1+\frac{\mathrm{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)+\mathcal{U}\Omega_2\right)(\|Q\|+\|\Upsilon\|)+\mathcal{M}_1\left(\Omega_1+\frac{\mathrm{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)+\mathcal{M}_2\Omega_2.
\end{aligned}$$

Similarly, we get

$$
\begin{aligned}
&(\Lambda_2(\mathcal{Q},\Upsilon))\\
&\leq\left(\mathcal{U}\left(\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{Y}\Omega_1\right)(\|\mathcal{Q}\|+\|\Upsilon\|)+\mathcal{M}_2\left(\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{M}_1\Omega_1.
\end{aligned}
$$

Consequently, for any $(\mathcal{Q},\Upsilon)\in\mathcal{B}_{\mathfrak{r}}$, we find

$$
\begin{aligned}
&\|(\Lambda(\mathcal{Q},\Upsilon))\|=\|(\Lambda_1(\mathcal{Q},\Upsilon))\|+\|(\Lambda_2(\mathcal{Q},\Upsilon))\|\\
&\leq\left(\mathcal{Y}\left(2\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{U}\left(2\Omega_2+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)\right)(\|\mathcal{Q}\|+\|\Upsilon\|)\\
&\quad+\mathcal{M}_1\left(2\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{M}_2\left(2\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)<\mathfrak{r}.
\end{aligned}
$$

Next, we show that Λ is a contraction, and it is necessary to demonstrate that it maps $\mathcal{B}_{\mathfrak{r}}$ into itself. Let $(\mathcal{Q}_1,\Upsilon_1),(\mathcal{Q}_2,\Upsilon_2)\in\mathcal{E},\mathfrak{t}\in\mathcal{G}$. By considering the relation $(\mathcal{W}_2)$, we can derive the following expression:

$$
\begin{aligned}
&|(\Lambda_1(\mathcal{Q}_1,\Upsilon_1))-(\Lambda_1(\mathcal{Q}_2,\Upsilon_2))|\\
&\leq\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau,\mathcal{Q}_1(\tau),\Upsilon_1(\tau))-\mathcal{L}_1(\tau,\mathcal{Q}_2(\tau),\Upsilon_2(\tau))|d\tau\right)d\varkappa\\
&\quad-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}|\mathcal{J}_1(\tau,\mathcal{Q}_1(\tau),\Upsilon_1(\tau))-\mathcal{J}_1(\tau,\mathcal{Q}_2(\tau),\Upsilon_2(\tau))|d\tau\right)d\varkappa\Bigg)\\
&\quad+\frac{1}{\Delta_2}\left(\mathcal{P}_1-\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\mathfrak{m},\mathcal{Q}_1(\tau),\Upsilon_1(\mathfrak{m}))-\mathcal{L}_1(\mathfrak{m},\mathcal{Q}_2(\mathfrak{m}),\Upsilon_2(\mathfrak{m}))|d\mathfrak{m}\right)d\tau\right)d\varkappa\right.\\
&\quad+\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\mathfrak{m},\mathcal{Q}_1(\mathfrak{m}),\Upsilon_1(\mathfrak{m}))-\mathcal{J}_1(\mathfrak{m},\mathcal{Q}_2(\mathfrak{m}),\Upsilon_2(\mathfrak{m}))|d\mathfrak{m}\right)d\tau\right)d\varkappa\\
&\quad+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau,\mathcal{Q}_1(\tau),\Upsilon_1(\tau))-\mathcal{L}_1(\tau,\mathcal{Q}_2(\tau),\Upsilon_2(\tau))|d\tau\right)d\varkappa\\
&\quad-\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\tau,\mathcal{Q}_1(\tau),\Upsilon_1(\tau))-\mathcal{J}_1(\tau,\mathcal{Q}_2(\tau),\Upsilon_2(\tau))|d\tau\right)d\varkappa\\
&\quad+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau,\mathcal{Q}_1(\tau),\Upsilon_1(\tau))-\mathcal{L}_1(\tau,\mathcal{Q}_2(\tau),\Upsilon_2(\tau))|d\tau\right)d\varkappa\\
&\quad-\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\tau,\mathcal{Q}_1(\tau),\Upsilon_1(\tau))-\mathcal{J}_1(\tau,\mathcal{Q}_2(\tau),\Upsilon_2(\tau))|d\tau\right)d\varkappa\Bigg)\Bigg]\\
&\quad+\int_0^{\mathfrak{t}}e^{-\mathcal{K}_1(\mathfrak{t}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau,\mathcal{Q}_1(\tau),\Upsilon_1(\tau))-\mathcal{L}_1(\tau,\mathcal{Q}_2(\tau),\Upsilon_2(\tau))|d\tau\right)d\varkappa\\
&\leq\left(\mathcal{Y}\left(\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{U}\Omega_2\right)(\|\mathcal{Q}\|+\|\Upsilon\|),
\end{aligned}
$$

and we have

$$
|(\Lambda_1(\mathcal{Q}_1,\Upsilon_1))-(\Lambda_1(\mathcal{Q}_2,\Upsilon_2))|\leq\left(\mathcal{Y}\left(\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{U}\Omega_2\right)(\|\mathcal{Q}\|+\|\Upsilon\|),
$$

and

$$
\begin{aligned}
&|(\Lambda_2(Q_1,\Upsilon_1))-(\Lambda_2(Q_2,\Upsilon_2))|\\
\leq&\Bigg[\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau,Q_1(\tau),\Upsilon_1(\tau))-\mathcal{L}_1(\tau,Q_2(\tau),\Upsilon_2(\tau))|d\tau\Bigg)d\varkappa\\
&-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}|\mathcal{J}_1(\tau,Q_1(\tau),\Upsilon_1(\tau))-\mathcal{J}_1(\tau,Q_2(\tau),\Upsilon_2(\tau))|d\tau\Bigg)d\varkappa\Bigg)\\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\mathfrak{m},Q_1(\mathfrak{m}),\Upsilon_1(\mathfrak{m}))-\mathcal{L}_1(\mathfrak{m},Q_2(\mathfrak{m}),\Upsilon_2(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa\\
&+\int_{\varrho}^{\zeta}\Bigg(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\mathfrak{m},Q_1(\mathfrak{m}),\Upsilon_1(\mathfrak{m}))-\mathcal{J}_1(\mathfrak{m},Q_2(\mathfrak{m}),\Upsilon_2(\mathfrak{m}))|d\mathfrak{m}\Bigg)d\tau\Bigg)d\varkappa\\
&+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau,Q_1(\tau),\Upsilon_1(\tau))-\mathcal{L}_1(\tau,Q_2(\tau),\Upsilon_2(\tau))|d\tau\Bigg)d\varkappa\\
&-\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\tau,Q_1(\tau),\Upsilon_1(\tau))-\mathcal{J}_1(\tau,Q_2(\tau),\Upsilon_2(\tau))|d\tau\Bigg)d\varkappa\\
&+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{L}_1(\tau,Q_1(\tau),\Upsilon_1(\tau))-\mathcal{L}_1(\tau,Q_2(\tau),\Upsilon_2(\tau))|d\tau\Bigg)d\varkappa\\
&-\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\tau,Q_1(\tau),\Upsilon_1(\tau))-\mathcal{J}_1(\tau,Q_2(\tau),\Upsilon_2(\tau))|d\tau\Bigg)d\varkappa\Bigg)\Bigg]\\
&+\int_0^{\mathfrak{t}}e^{-\mathcal{K}_1(\mathfrak{t}-\varkappa)}\Bigg(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{J}_1(\tau,Q_1(\tau),\Upsilon_1(\tau))-\mathcal{J}_1(\tau,Q_2(\tau),\Upsilon_2(\tau))|d\tau\Bigg)d\varkappa\Bigg]\\
\leq&\Bigg(\mathcal{U}\Big(\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\Big)+\mathcal{Y}\Omega_1\Bigg)(\|Q\|+\|\Upsilon\|).
\end{aligned}
$$

In a like manner, we have

$$
|(\Lambda_2(Q_1,\Upsilon_1))-(\Lambda_2(Q_2,\Upsilon_2))|\leq\Bigg(\mathcal{U}\Big(\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\Big)+\mathcal{Y}\Omega_1\Bigg)(\|Q\|+\|\Upsilon\|).
$$

Explicitly, the preceding inequalities indicate that

$$
\begin{aligned}
&\|(\Lambda(Q_1,\Upsilon_1))-(\Lambda(Q_2,\Upsilon_2))\|=|(\Lambda_1(Q_1,\Upsilon_1))-(\Lambda_1(Q_2,\Upsilon_2))|+|(\Lambda_2(Q_1,\Upsilon_1))-(\Lambda_2(Q_2,\Upsilon_2))|\\
\leq&\Bigg(\mathcal{Y}\Bigg(2\Omega_1+\frac{\mathfrak{t}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\Bigg)+\mathcal{U}\Bigg(\Omega_2+\frac{\mathfrak{t}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\Bigg)\Bigg)\|(Q_1-Q_2,\Upsilon_1-\Upsilon_2)\|.
\end{aligned}
$$

Assuming condition (3.8), it follows that the operator Λ demonstrates contraction characteristics. Therefore, through the application of Banach's contraction mapping theorem, we ascertain the presence of a singular fixed point for Λ. This finding definitively verifies that system (1.1) has a unique solution within the domain $\mathcal{G}$. □

4. Stability results

In this section, our attention is directed towards examining the U-H stability of the coupled (SFDEs) represented by (1.1). To grasp the stability characteristics of the system, we delve into the analysis of the following inequality:

$$\begin{cases} |({}^{C}\mathcal{D}^{\varsigma_1} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_1-1})Q(t) - \mathcal{L}_1(t, Q(t), \Upsilon(t))| \leq \epsilon_1 & t \in \mathcal{G} := [0, \mathfrak{T}], \\ |({}^{C}\mathcal{D}^{\varsigma_2} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_2-1})\Upsilon(t) - \mathcal{J}_1(t, Q(t), \Upsilon(t))| \leq \epsilon_2 & t \in \mathcal{G} := [0, \mathfrak{T}]. \end{cases} \tag{4.1}$$

Here $\varepsilon_1, \varepsilon_2$ are given two positive numbers.

Definition 4.1. *[22] Problem (1.1) is Hyers-Ulam stable if there exist $\Omega_i > 0$, where $i = 1, 2$, such that for given $\varepsilon_1, \varepsilon_2 > 0$, and for each solution $(Q, \Upsilon) \in \mathcal{E}([0, \mathfrak{T}] \times \mathbb{R}^2, \mathbb{R})$ of inequality problem (4.1), there exists a solution $(Q^*, \Upsilon^*) \in \mathcal{E}([0, \mathfrak{T}] \times \mathbb{R}^2, \mathbb{R})$ of system 1.1 with*

$$\begin{cases} |Q(t) - Q^*(t)| \leq \Omega_1\varepsilon_1 + \Omega_2\varepsilon_2, & t \in [0, \mathfrak{T}], \\ |\Upsilon(t) - \Upsilon^*(t)| \leq \Omega_1\varepsilon_1 + \Omega_2\varepsilon_2, & t \in [0, \mathfrak{T}]. \end{cases} \tag{4.2}$$

Remark 4.1. *The pair (Q, Υ) is considered a solution of inequality (4.1) if there exist functions $\mathcal{A}_i \in ([0, \mathfrak{T}], \mathbb{R})$, where $i = 1, 2$, which are dependent on (Q, Υ) respectively, such that the following conditions hold:*

$$|\mathcal{A}_1(t)| \leq \varepsilon_1, \qquad |\mathcal{A}_2(t)| \leq \varepsilon_2, \qquad t \in [0, \mathfrak{T}]. \tag{4.3}$$

$$\begin{cases} |({}^{C}\mathcal{D}^{\varsigma_1} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_1-1})Q(t) = \mathcal{L}_1(t, Q(t), \Upsilon(t))| + \mathcal{A}_1(t) & t \in \mathcal{G} := [0, \mathfrak{T}], \\ |({}^{C}\mathcal{D}^{\varsigma_2} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_2-1})\Upsilon(t) = \mathcal{J}_1(t, Q(t), \Upsilon(t))| + \mathcal{A}_2(t) & t \in \mathcal{G} := [0, \mathfrak{T}]. \end{cases} \tag{4.4}$$

Remark 4.2. *If (Q, Υ) represent a solution of inequality (4.1), then (Q, Υ) is a solution of the inequalities*

$$\begin{cases} |Q(t) - Q^*(t)| \leq \Omega_1\varepsilon_1 + \Omega_2\varepsilon_2, & t \in [0, \mathfrak{T}], \\ |\Upsilon(t) - \Upsilon^*(t)| \leq \Omega_1\varepsilon_1 + \Omega_2\varepsilon_2, & t \in [0, \mathfrak{T}], \end{cases}$$

for all $(Q, \Upsilon) \in \mathcal{E}([0, \mathfrak{T}], \mathbb{R})$.

Theorem 4.2. *Assume that $(\mathcal{W}_2)$ holds. Then problem (1.1) is Ulam-Hyers stable.*

Proof. With the help of Definitions 4.1 and Remark 4.1 we get

$$\begin{cases} ({}^{C}\mathcal{D}^{\varsigma_1} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_1-1})Q(t) = \mathcal{L}_1(t, Q(t), \Upsilon(t)) + \mathcal{A}_1(t) & t \in \mathcal{G} := [0, \mathfrak{T}], \\ ({}^{C}\mathcal{D}^{\varsigma_2} + \mathcal{K}_1{}^{C}\mathcal{D}^{\varsigma_2-1})\Upsilon(t) = \mathcal{J}_1(t, Q(t), \Upsilon(t)) + \mathcal{A}_2(t) & t \in \mathcal{G} := [0, \mathfrak{T}]. \end{cases}$$

This implies,

$$\begin{aligned} Q(t) =& Q^*(t) + \left| \frac{e^{-\mathcal{K}_1 t}}{2} \left[\frac{1}{\Delta_1} \left(- \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)} \left(\int_0^{\varkappa} \frac{(\varkappa - \tau)^{\varsigma_1 - 2}}{\Gamma(\varsigma_1 - 1)} \mathcal{L}_1(\tau, Q(\tau), \Upsilon(\tau)) d\tau \right) d\varkappa \right. \right. \right. \\ & - \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)} \left(\int_0^{\varkappa} \frac{(\varkappa - \tau)^{\varsigma_2 - 2}}{\Gamma(\varsigma_1 - 1)} \mathcal{J}_1(\tau, Q(\tau), \Upsilon(\tau) d\tau \right) d\varkappa \Bigg) \\ & + \frac{1}{\Delta_2} \left(\mathcal{P}_1 - \int_{\varrho}^{\zeta} \left(\int_0^{\varkappa} e^{-\mathcal{K}_1(\varkappa - \tau)} \left(\int_0^{\tau} \frac{(\tau - \mathfrak{m})^{\varsigma_1 - 2}}{\Gamma(\varsigma_1 - 1)} \mathcal{L}_1(\mathfrak{m}, Q(\mathfrak{m}), \Upsilon(\mathfrak{m}) d\mathfrak{m} \right) d\tau \right) d\varkappa \right. \end{aligned}$$

$$+\int_{\varrho}^{\zeta}\left(\int_{0}^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_{0}^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(\mathfrak{m},\mathcal{Q}(\mathfrak{m}),\Upsilon(\mathfrak{m})d\mathfrak{m}\right)d\tau\right)d\varkappa$$
$$+\sum_{i=1}^{m}\mathfrak{q}_i\int_{0}^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)d\tau\right)d\varkappa$$
$$-\sum_{i=1}^{m}\mathfrak{q}_i\int_{0}^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)d\tau\right)d\varkappa$$
$$+\sum_{j=1}^{n}\mathfrak{u}_j\int_{0}^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)d\tau\right)d\varkappa$$
$$-\sum_{j=1}^{n}\mathfrak{u}_j\int_{0}^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}\mathcal{J}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)d\tau\right)d\varkappa\Bigg)\Bigg]$$
$$+\int_{0}^{\mathfrak{t}}e^{-\mathcal{K}_1(\mathfrak{t}-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}\mathcal{L}_1(\tau,\mathcal{Q}(\tau),\Upsilon(\tau)d\tau\right)d\varkappa\Bigg|.$$

It follows that

$$|\mathcal{Q}(\mathfrak{t})-\mathcal{Q}^*(\mathfrak{t})|\leq\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\left[\frac{1}{\Delta_1}\left(-\int_{0}^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{A}_1(\mathfrak{t})|)d\tau\right)d\varkappa\right.\right.$$
$$+\int_{0}^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}|\mathcal{A}_2(\mathfrak{t})|d\tau\right)d\varkappa\Bigg)$$
$$+\frac{1}{\Delta_2}\left(\mathcal{P}_1-\int_{\varrho}^{\zeta}\left(\int_{0}^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_{0}^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{A}_1(\mathfrak{t})|d\mathfrak{m}\right)d\tau\right)d\varkappa\right.$$
$$+\int_{\varrho}^{\zeta}\left(\int_{0}^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_{0}^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{A}_2(\mathfrak{t})|d\mathfrak{m}\right)d\tau\right)d\varkappa$$
$$+\sum_{i=1}^{m}\mathfrak{q}_i\int_{0}^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{A}_1(\mathfrak{t})|d\tau\right)d\varkappa$$
$$+\sum_{i=1}^{m}\mathfrak{q}_i\int_{0}^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{A}_2(\mathfrak{t})|d\tau\right)d\varkappa$$
$$+\sum_{j=1}^{n}\mathfrak{u}_j\int_{0}^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{A}_1(\mathfrak{t})|d\tau\right)d\varkappa$$
$$+\sum_{j=1}^{n}\mathfrak{u}_j\int_{0}^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{A}_2(\mathfrak{t})|d\tau\right)d\varkappa\Bigg)\Bigg]$$
$$+\int_{0}^{\mathfrak{t}}e^{-\mathcal{K}_1(\mathfrak{t}-\varkappa)}\left(\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{A}_1(\mathfrak{t})|d\tau\right)d\varkappa$$
$$\leq\epsilon_1\left[\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\left[\frac{1}{(1+e^{-\mathcal{K}_1\mathfrak{T}})}\left(\frac{\mathfrak{T}^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\mathfrak{T}})\right)\right.\right.$$
$$+\frac{1}{\Delta_2}\left\{\left(\frac{\zeta^{\varsigma_1-1}-\varrho^{\varsigma_1-1}}{\mathcal{K}_1^2\Gamma(\varsigma_1)}\right)(\zeta\mathcal{K}_1+e^{-\mathcal{K}_1\zeta}-\mathcal{K}_1\varrho-e^{-\mathcal{K}_1\varrho})\right.$$

$$+\sum_{i=1}^{m}\mathfrak{q}_i\left(\frac{\varphi_i^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\varphi_i})\right)+\sum_{j=1}^{n}\mathfrak{u}_j\left(\frac{\delta_i^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1-e^{-\mathcal{K}_1\delta_i})\right)\Bigg\}\Bigg]\Bigg]$$
$$+\epsilon_2\Bigg[\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\frac{1}{(1+e^{-\mathcal{K}_1\mathfrak{T}})}\left(\frac{\mathfrak{T}^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\mathfrak{T}})\right)$$
$$+\frac{1}{\Delta_2}\Bigg\{\left(\frac{\zeta^{\varsigma_2-1}-\varrho^{\varsigma_2-1}}{\mathcal{K}_1^2\Gamma(\varsigma_2)}\right)(\zeta\mathcal{K}_1+e^{-\mathcal{K}_1\zeta}-\mathcal{K}_1\varrho-e^{-\mathcal{K}_1\varrho})$$
$$+\sum_{i=1}^{m}\mathfrak{q}_i\left(\frac{\varphi_i^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\varphi_i})\right)+\sum_{j=1}^{n}\mathfrak{u}_j\left(\frac{\delta_i^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1-e^{-\mathcal{K}_1\delta_i})\right)\Bigg\}\Bigg]\Bigg]$$
$$\leq\Omega_1\varepsilon_1+\Omega_2\varepsilon_2. \tag{4.5}$$

Similarly, we obtain

$$|\Upsilon(\zeta)-\Upsilon^*(\zeta)|\leq\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{A}_1(\mathfrak{t})|)d\tau\right)d\varkappa$$
$$+\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_1-1)}|\mathcal{A}_2(\mathfrak{t})|d\tau\right)d\varkappa\Bigg)$$
$$+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{A}_1(\mathfrak{t})|d\mathfrak{m}\right)d\tau\right)d\varkappa$$
$$+\int_{\varrho}^{\zeta}\left(\int_0^{\varkappa}e^{-\mathcal{K}_1(\varkappa-\tau)}\left(\int_0^{\tau}\frac{(\tau-\mathfrak{m})^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{A}_2(\mathfrak{t})|d\mathfrak{m}\right)d\tau\right)d\varkappa$$
$$+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{A}_1(\mathfrak{t})|d\tau\right)d\varkappa$$
$$+\sum_{i=1}^{m}\mathfrak{q}_i\int_0^{\varphi_i}e^{-\mathcal{K}_1(\varphi_i-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{A}_2(\mathfrak{t})|d\tau\right)d\varkappa$$
$$+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_1-2}}{\Gamma(\varsigma_1-1)}|\mathcal{A}_1(\mathfrak{t})|d\tau\right)d\varkappa$$
$$+\sum_{j=1}^{n}\mathfrak{u}_j\int_0^{\delta_j}e^{-\mathcal{K}_1(\delta_j-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{A}_2(\mathfrak{t})|d\tau\right)d\varkappa\Bigg)\Bigg]$$
$$+\int_0^{\mathfrak{t}}e^{-\mathcal{K}_1(\mathfrak{t}-\varkappa)}\left(\int_0^{\varkappa}\frac{(\varkappa-\tau)^{\varsigma_2-2}}{\Gamma(\varsigma_2-1)}|\mathcal{A}_2(\mathfrak{t})|d\tau\right)d\varkappa$$
$$\leq\Omega_1\varepsilon_1+\Omega_2\varepsilon_2, \tag{4.6}$$

where Ω_1 and Ω_2 are defined in (3.3)–(3.8), respectively. Hence, problem (1.1) is U-H stable. □

5. Examples

Example 5.1. *Consider the system*

$$\begin{cases} ({}^{C}\mathcal{D}^{\varsigma_1} + \mathcal{K}_1 {}^{C}\mathcal{D}^{\varsigma_1 - 1})Q(\mathrm{t}) = \mathcal{L}_1(\mathrm{t}, Q(\mathrm{t}), \Upsilon(\mathrm{t})), \mathrm{t} \in \mathcal{G} := [0, \mathfrak{T}], \\ ({}^{C}\mathcal{D}^{\varsigma_2} + \mathcal{K}_1 {}^{C}\mathcal{D}^{\varsigma_2 - 1})\Upsilon(\mathrm{t}) = \mathcal{J}_1(\mathrm{t}, Q(\mathrm{t}), \Upsilon(\mathrm{t})), \mathrm{t} \in \mathcal{G} := [0, \mathfrak{T}], \\ (Q + \Upsilon)(0) = -(Q + \Upsilon)(\mathfrak{T}), \\ \int_{\varrho}^{\zeta} (Q - \Upsilon)(\varkappa) d\varkappa - \sum_{i=1}^{m} \mathfrak{q}_i (Q - \Upsilon)(\varphi_i) - \sum_{j=1}^{n} \mathfrak{u}_j (Q - \Upsilon)(\delta_j) = \mathcal{P}_1, \end{cases} \tag{5.1}$$

where $\varsigma_1 = 3/2, \varsigma_2 = 4/3, \varrho = 3/4,$ $\zeta = 3/2, \mathcal{T} = 2, \mathcal{P}_1 = 1,$ $\mathfrak{q}_1 = 1, \mathfrak{q}_2 = 1/5, \mathfrak{u}_1 = 26/100, \mathfrak{u}_2 = 6/25, \varpi_1 = 1/4, \varpi_2 = 1/3, \delta_1 = 7/4, \delta_2 = 47/25.$ *Utilizing the above data, we get* $\Omega_1 = 0.200876$ *and* $\Omega_2 = 0.156334$, *where* Ω_1 *and* Ω_2 *are respectively given by (3.3) and (3.4). To illustrate Theorem 3.1, we will use*

$$\mathcal{L}_1(\mathrm{t}, Q(\mathrm{t}), \Upsilon(\mathrm{t})) = \frac{e^{-\mathrm{t}}}{2\sqrt{900 + \mathrm{t}^2}} (Q\mathrm{t} + \sin \Upsilon + \cos \mathrm{t}), \tag{5.2}$$

$$\mathcal{J}_1(\mathrm{t}, Q(\mathrm{t}), \Upsilon(\mathrm{t})) = \frac{1}{(3 + \mathrm{t})^2} \left(\sin Q + \frac{\Upsilon}{2} + e^{-\mathrm{t}} \right). \tag{5.3}$$

Next, $\mathcal{L}_1$ *and* $\mathcal{J}_1$ *are continuous and accomplish the hypothesis* ($\mathcal{W}_1$) *with*

$$\Psi_1(\mathrm{t}) = \frac{e^{-\mathrm{t}} \cos \mathrm{t}}{2\sqrt{900 + \mathrm{t}^2}}, \Psi_2(\mathrm{t}) = \frac{\mathrm{t} e^{-\mathrm{t}}}{2\sqrt{900 + \mathrm{t}^2}}, \Psi_3(\mathrm{t}) = \frac{e^{-\mathrm{t}}}{2\sqrt{900 + \mathrm{t}^2}}, \tag{5.4}$$

$$k_1 = \frac{1}{(3 + \mathrm{t})^2}, k_2 = \frac{e^{-\mathrm{t}}}{(3 + \mathrm{t})^2}, \text{ and } k_3 = \frac{1}{2(3 + \mathrm{t})^2}. \tag{5.5}$$

Also,

$$\|\Psi_2\| \left(2\Omega_1 + \frac{\mathrm{t}^{\varsigma_1 - 1}}{\mathcal{K}_1 \Gamma(\varsigma_1)} (1 - e^{-\mathcal{K}_1 \mathrm{t}}) \right) + \|k_2\| \left(2\Omega_2 + \frac{\mathrm{t}^{\varsigma_2 - 1}}{\mathcal{K}_1 \Gamma(\varsigma_2)} (1 - e^{-\mathcal{K}_1 \mathrm{t}}) \right) \approx 0.084034843 \tag{5.6}$$

and

$$\|\Psi_3\| \left(2\Omega_1 + \frac{\mathrm{t}^{\varsigma_1 - 1}}{\mathcal{K}_1 \Gamma(\varsigma_1)} (1 - e^{-\mathcal{K}_1 \mathrm{t}}) \right) + \|k_3\| \left(2\Omega_2 + \frac{\mathrm{t}^{\varsigma_2 - 1}}{\mathcal{K}_1 \Gamma(\varsigma_2)} (1 - e^{-\mathcal{K}_1 \mathrm{t}}) \right) \approx 0.01951157668. \tag{5.7}$$

Thus, by Theorem 3.1, $\exists$ *a solution to system (5.1) on* $[0, 2]$.

Example 5.2. *For the application of Theorem 3.2, we take into account*

$$\mathcal{L}_1(\mathrm{t}, Q(\mathrm{t}), \Upsilon(\mathrm{t})) = \frac{1}{40(1 + \mathrm{t}^2)} \left(\frac{|Q||}{1 + |Q|} + \tan^{-1} \Upsilon \right),$$

$$\mathcal{J}_1(\mathrm{t}, Q(\mathrm{t}), \Upsilon(\mathrm{t})) = \frac{1}{\sqrt{900 + \mathrm{t}^2}} \left(\sin \Upsilon + 2 \tan^{-1} Q \right), \tag{5.8}$$

where $\mathcal{L}_1$ *and* $\mathcal{J}_1$ *are continuous and fulfill the hypothesis* ($\mathcal{W}_2$) *with* $\mathcal{Y}_1 = \mathcal{Y}_2 = 1/40 = \mathcal{Y}$ *and* $\mathcal{U}_1 = 1/15, \mathcal{U}_2 = 1/30,$ *and* $\mathcal{U} = 1/15.$

Further, we acquire

$$\left(\mathcal{Y}\left(2\Omega_1 + \frac{t^{\varsigma_1-1}}{\mathcal{K}_1\Gamma(\varsigma_1)}(1 - e^{-\mathcal{K}_1 t})\right) + \mathcal{U}\left(\Omega_2 + \frac{t^{\varsigma_2-1}}{\mathcal{K}_1\Gamma(\varsigma_2)}(1 - e^{-\mathcal{K}_1 t})\right)\right) \approx 0.08291559097 < 1. \tag{5.9}$$

Thus, all the conditions of Theorem 3.2 are satisfied.

6. Conclusions

In this work, we have explored a comprehensive study on a novel set of boundary value problems involving an arbitrary strip and multi-points. These boundary value problems are associated with a coupled system of nonlinear SFDEs of the Caputo type. We construct the existence and uniqueness of solutions by employing Schaefer's fixed point theorem and the Banach contraction mapping principle. Stability features of the considered system are examined through UH stability analysis. Our findings not only contribute new insights, but also advance the understanding of coupled fractional order boundary value problems. Moreover, the solution forms applicable to the generalized SFDEs with different sets of boundary conditions can be leveraged to conduct a more comprehensive investigation into positive solutions and their associated asymmetries.

Furthermore, our research can be expanded to include the study of various integral boundary conditions for coupled (SFDEs) and inclusions, incorporating different derivatives such as Hadamard, Caputo-Hadamard, and Hilfer. Future work could also seamlessly extend this investigation to assess the existence and uniqueness of solutions under specific boundary conditions, employing Mönch and Darbo's fixed point theorems, and to investigate solution stability using generalized Hyers-Ulam and Ulam-Hyers-Rassias stability criteria. Additionally, forthcoming studies will aim to elucidate the stability and existence of neutral time-delay systems, particularly those with finite delays.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

BVP	Boundary Value Problems
FDEs	Fractional Differential Equations
SFDEs	Sequential Fractional Differential Equations
HU	Hyers-Ulam
HUR	Hyers-Ulam-Rassias
SFD	Sequential Fractional Differential
CFDs	Caputo Fractional Derivatives

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