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Research article

On Cauchy-type problems with weighted R-L fractional derivatives of a function with respect to another function and comparison theorems

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Abstract: The main aim of this paper is to study the Cauchy problem for nonlinear differential equations of fractional order containing the weighted Riemann-Liouville fractional derivative of a function with respect to another function. The equivalence of this problem and a nonlinear Volterra-type integral equation of the second kind have been presented. In addition, the existence and uniqueness of the solution to the considered Cauchy problem are proved using Banach's fixed point theorem and the method of successive approximations. Finally, we obtain a new estimate of the weighted Riemann-Liouville fractional derivative of a function with respect to functions at their extreme points. With the assistance of the estimate obtained, we develop the comparison theorems of fractional differential inequalities, strict as well as nonstrict, involving weighted Riemann-Liouville differential operators of a function with respect to functions of order δ, $0 < \delta < 1$.

Keywords: weighted fractional integrals and derivatives; comparison theorems; fractional differential equations; volterra integral equation; fractional differential inequalities; Cauchy problem; fixed point theorem
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1. Introduction

Fractional calculus is a generalization of ordinary calculus that studies the integration and differentiation of real or complex orders. Even though it is ancient, this calculus has experienced a surge in popularity in recent decades because of the interesting results obtained when it was applied to model some real-world phenomena. The presence of numerous new fractional operators is what distinguishes fractional calculus. This feature enables scientists to enhance the development of models for real-world problems by selecting the operator that best fits the model they are studying. Numerous phenomena across diverse fields such as science, mathematics, engineering, bioengineering, and economics are more precisely elucidated through the application of fractional derivatives, as we refer the readers to [1–4].

Recently, the theory concerning the existence and uniqueness of solutions for the Cauchy-type problems of fractional differential equations has been developing and is increasingly recognized as a valuable tool for modeling real-world problems across various domains, leading to a significantly richer theoretical framework compared to the theory of ordinary differential equations [5–7]. Applications for fractional differential equations are found in many scientific domains, including but not limited to engineering [8], chemistry [9], physics [10], and other fields [11, 12].

The complexity of applications is the main reason that prompted researchers to expand the definitions of fractional derivatives. This leads to the proposal of weighted fractional derivatives to generalize many known fractional operators (see [13–15]). In [16, 17], Agrawal introduced generalized fractional derivatives known as weighted fractional derivatives and outlined some of their properties. Nevertheless, no details were provided regarding the spaces on which these operators are defined. In [18, 19], weighted fractional operators are studied, and many fundamental properties of these operators are studied. In addition, some spaces in which these operators are defined are presented. Recent research has discussed the weighted fractional operators with their applications and studied the mathematical properties in [16, 17, 20, 21], along with a study of the associated differential equations presented in [22–24]. There are two categories of nonlocal fractional derivatives: The classical ones characterized by singular kernels, such as the Riemann-Liouville and Caputo derivatives, and the more recently introduced ones with nonsingular kernels, such as the Atangana-Baleanu and Caputo-Fabrizio derivatives [25, 26]. In recent publications [20, 21], the authors introduced the weighted Caputo-Fabrizio fractional operators. Furthermore, they expanded the scope of their investigation to encompass the weighted Atangana-Baleanu fractional operators and a study of their properties.

The comparison theorem is one of the basic theories of differential equations, playing an important role in the qualitative analysis of research. In [27–29], comparison theorems relative to fractional differential and integral inequalities involving Riemann-Liouville derivatives of order δ, $0 < \delta < 1$, were proved. In [30], the comparison results were presented for the Riemann-Liouville fractional differential equation of order δ, $0 < \delta < 1$. Notably, these results were presented without necessitating the Hölder continuity assumption [29], where the local Hölder continuity assumption is replaced with C_p-continuity of the functions involved in the Riemann-Liouville fractional differential equations; see [31, 32] and references therein for further details.

This paper's principal contribution lies in extending the methodology introduced by Chunhai Kou et al. [5] to tackle fractional Cauchy problems involving weighted Riemann-Liouville operators of a function with respect to another function. Our contributions include not only establishing the existence and uniqueness of solutions but also generalizing the comparison result for these problems introduced by V. Lakshmikantham et al. [29]. As it stands, there has been no initiation of work on comparison theorems for fractional differential equations that incorporate the weighted Riemann-Liouville derivative of a function with respect to another function of order δ, $0 < \delta < 1$. We have confidence that the obtained results will serve as a beneficial addition and extension to the current findings in the literature. In fact, the existence and uniqueness investigations in this work are necessary steps to proceed with proving the comparison theory of the considered weighted fractional system as an essential part of analyzing the qualitative theory of dynamic systems. The used kernel and weight require extra effort to analyze such a general system. Moreover, weighted fractional derivatives include a considerable class of fractional operators. For example, tempered fractional operators can be considered a particular case. Such a general type of fractional operator makes it easier to handle more complex real-world systems in modeling.

The current work is as follows: Section 2 consists of preliminary definitions and results concerning the weighted fractional operators of a function with respect to another function. In Section 3, we outline our problem and discuss the equivalence between an initial value problem and a Volterra integral equation. In Section 4, we substantiate the existence and uniqueness theorem for the initial value problem discussed in the preceding section. Section 5 deals with the estimation of the weighted Riemann-Liouville derivative of a function with respect to another function at extreme points. Based on the obtained results, we derive comparison results for the weighted Riemann-Liouville fractional differential equations involving initial conditions. Lastly, we conclude with a summary, and we talk about future extensions for this work.

2. Preliminaries

In this section, we present some fundamental definitions of weighted fractional calculus of a function with respect to another function, lemmas, theorems, and propositions that are important and needed for our findings in this article (cf. [16, 18, 19]).

Theorem 2.1. *(Banach fixed point theorem [11]) Let (S, σ) be a nonempty, complete metric space, let $0 \leq \lambda < 1$, and let $A : S \to S$ be the map such that, for every $u, v \in S$, the relation*

$$\sigma(Au, Av) \leq \lambda\sigma(u, v) \quad (0 \leq \lambda < 1), \tag{2.1}$$

holds. Then the operator A has a unique fixed point $u^* \in S$.

Furthermore, if $A^i (i \in \mathbb{N})$ is the sequence of operators defined by

$$A^1 = A \quad \text{and} \quad A^i = AA^{i-1} \quad (i \in \mathbb{N}/\{1\}), \tag{2.2}$$

then, for any $u_0 \in S$, the sequence $\{A^i u_0\}_{i=1}^{\infty}$ converges to the above fixed point u^*.

Definition 2.1. *[11] Assume that $\kappa(t, z)$ is defined on the $(\ell, r] \times \Omega(\Omega \subset \mathbb{R})$. $\kappa(t, z)$ is said to satisfy Lipschitzian condition with respect to the second variable, if for all $t \in (\ell, r]$ and for any $z_1, z_2 \in \Omega$ one has*

$$|\kappa(t, z_1) - \kappa(t, z_2)| \leq L|z_1 - z_2|, \tag{2.3}$$

where $L > 0$ does not depend on $t \in [\ell, r]$.

Let $D = [\ell, r]$ $(0 \leq \ell < r < \infty)$ be a finite interval and ϱ be a parameter such that $m-1 \leq \varrho < m$. The space of continuous functions κ on D is denoted by $C[\ell, r]$ and the associated norm is defined by [33]

$$\|\kappa\|_{C[\ell,r]} = \max_{t\in[\ell,r]} |\kappa(t)|,$$

and

$$AC^m[\ell, r] = \{\kappa : [\ell, r] \to \mathbb{R} \quad \text{such that} \quad \kappa^{(m-1)} \in AC[\ell, r]\},$$

be the space of m times absolutely continuous differentiable functions.

The weighted space $C_{\varrho,\theta}[\ell, r]$ of functions κ on $(\ell, r]$ is defined by

$$C_{\varrho,\theta}[\ell, r] = \{\kappa : (\ell, r] \to \mathbb{R}; (\theta(t) - \theta(\ell))^{\varrho}\kappa(t) \in C[\ell, r]\},$$

having norm

$$\|\kappa\|_{C_{\varrho,\theta}[\ell,r]} = \|(\theta(t) - \theta(\ell))^{\varrho}\kappa(t)\|_{C[\ell,r]} = \max_{t\in[\ell,r]} |(\theta(t) - \theta(\ell))^{\varrho}\kappa(t)|.$$

The weighted space $C^m_{\varrho,\theta}[\ell, r]$ of functions κ on $(\ell, r]$ is defined by

$$C^m_{\varrho,\theta}[\ell, r] = \{\kappa : [\ell, r] \to \mathbb{R}; \kappa(t) \in C^{m-1}[\ell, r]; \quad \kappa^{(m)}(t) \in C_{\varrho,\theta}[\ell, r]\},$$

along with the norm

$$\|\kappa\|_{C^m_{\varrho,\theta}[\ell,r]} = \sum_{i=0}^{m-1} \|\kappa^{(i)}\|_{C[\ell,r]} + \|\kappa^{(m)}\|_{C_{\varrho,\theta}[\ell,r]}.$$

The above space satisfies the following properties:

(1) $C^0_{\varrho,\theta}[\ell, r] = C_{\varrho,\theta}[\ell, r]$, for $m = 0$.
(2) For $m - 1 \leq \varrho_1 < \varrho_2 < m$, $C_{\varrho_1,\theta}[\ell, r] \subset C_{\varrho_2,\theta}[\ell, r]$.

The space $AC^m_\pi[\ell, r]$ is defined by [18]

$$AC^m_\pi[\ell, r] = \left\{\kappa : [\ell, r] \to \mathbb{R} \quad \text{such that} \quad \kappa^{(m-1)}_\pi \in AC[\ell, r]\right\}$$

where $AC[\ell, r]$ is the set of absolute continuous functions on the interval $[\ell, r]$, and

$$\kappa^{(m)}_\pi(t) = \frac{1}{\pi(t)}\left(\frac{D_t}{\theta'(t)}\right)^{(m)} (\pi(t)\kappa(t)), \quad m = 0, 1, 2, ...$$

The space $\chi^p_\pi(\ell, r)$, $1 \leq p \leq \infty$ is defined by [18]

$$\chi^p_\pi(\ell, r) = \{\kappa : [\ell, r] \to \mathbb{R}; \quad \kappa \text{ is measurable function} \quad \int_\ell^r |\pi(t)\kappa(t)|^p\theta'(t)dt < \infty\},$$

along with the norm

$$\|\kappa\|_{\chi^p_\pi} = \left(\int_\ell^r |\pi(t)\kappa(t)|^p\theta'(t)dt\right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty.$$

It should be noted that

$$f \in \chi^p_\pi(\ell, r) \Leftrightarrow \pi(t)\kappa(t)(\theta'(t))^{\frac{1}{p}} \in L_p(\ell, r),$$

where θ is a differential strictly increasing function and $\pi(t) \neq 0$ be a weight function on $[\ell, r]$.

2.1. The weighted fractional integrals and derivatives of a function with respect to another function

Definition 2.2. *[16, 18, 19] Let $\theta : [\ell, r] \to \mathbb{R}$ be a strictly increasing C^1 function, so that $\theta' > 0$ everywhere, and let $\pi \in L^\infty(\ell, r)$ be a weight function, $\kappa \in X^1_\pi(\ell, r)$. The π-weighted Riemann-Liouville fractional integral of order $\delta > 0$ of a function $\kappa(t)$ with respect to another function $\theta'(t)$, is defined by*

$$(\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa)(t) = \frac{1}{\pi(t)\Gamma(\alpha)}\int_\ell^t (\theta(t)-\theta(s))^{\delta-1}\pi(s)\theta'(s)\kappa(s)ds, \quad t \in (\ell, r). \tag{2.4}$$

Here, $\Gamma(\alpha) = \int_0^\infty s^{\alpha-1}e^{-s}ds$, $Re(\alpha) > 0$ is the Gamma function extending the factorial notation.

Definition 2.3. *[16, 18, 19] Let $m \in \mathbb{N}$. The π-weighted derivatives of integer order m of a function κ with respact to another function θ is defined as*

$$(D^{m,\theta(t)}_{\pi(t)}\kappa)(t) = \frac{1}{\pi(t)}\left[\left(\frac{\vartheta_t}{\theta'(t)}\right)^m (\pi(t)\kappa(t))\right](t), \tag{2.5}$$

where $\vartheta_t = \frac{d}{dt}$, and the first-order operator $D^{1,\theta(t)}_{\pi(t)}$ is defined by

$$(D^{1,\theta(t)}_{\pi(t)}\kappa)(t) = \frac{1}{\pi(t)}\left[\left(\frac{\vartheta_t}{\theta'(t)}\right)(\pi(t)\kappa(t))\right](t). \tag{2.6}$$

Definition 2.4. *[16, 18, 19] Let $\kappa \in AC^m_{\pi,\theta}[\ell, r]$. Then, the π-weighted Riemann-Liouville fractional derivative of order $\delta > 0$ of the function κ with respect to another function θ is defined as*

$$\begin{aligned}(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa)(t) &= (D^{m,\theta(t)}_{\pi(t)}\Im^{m-\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa)(t)\\ &= \frac{D^{m,\theta(t)}_{\pi(t)}}{\pi(t)\Gamma(m-\delta)}\int_\ell^t (\theta(t)-\theta(s))^{m-\delta-1}\pi(s)\theta'(s)\kappa(s)ds,\end{aligned} \tag{2.7}$$

where $m = [\delta] + 1$ so that $m - 1 < \delta < m$, $[\delta]$ being the integer part of δ.

Property 2.2. *[16, 18, 19]*

1) For $\delta > 0$ and $\mu > 0$, we have

$$\left(\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}\left[\frac{(\theta(t)-\theta(\ell))^{\mu-1}}{\pi(t)}\right]\right)(t) = \frac{\Gamma(\mu)}{\Gamma(\mu+\delta)\pi(t)}(\theta(t)-\theta(\ell))^{\mu+\delta-1}. \tag{2.8}$$

2) For $\delta < m$ ($m \in \mathbb{N}$) and $\mu > 0$, we have

$$\left(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}\left[\frac{(\theta(t)-\theta(\ell))^{\mu-1}}{\pi(t)}\right]\right)(t) = \frac{\Gamma(\mu)}{\Gamma(\mu-\delta)\pi(t)}(\theta(t)-\theta(\ell))^{\mu-\delta-1}. \tag{2.9}$$

On the other hand, for $i = 1, 2, \dots, [\delta] + 1$, we have

$$\left(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}\left[\frac{(\theta(t)-\theta(\ell))^{\delta-i}}{\pi(t)}\right]\right)(t) = 0. \tag{2.10}$$

Theorem 2.3. *[16, 18, 19] Let $\kappa \in \chi^p_\pi(\ell, r)$, $1 \le p \le \infty$, $\delta > 0$ and $\mu > 0$. Then,*

$$\left(\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}\Im^{\mu,\theta(t)}_{\ell^+,\pi(t)}\kappa\right)(t) = \left(\Im^{\delta+\mu,\theta(t)}_{\ell^+,\pi(t)}\kappa\right)(t). \tag{2.11}$$

Property 2.4. *[16, 18, 19] Let $\delta > \mu > 0$ and $\kappa \in \chi_\pi^p(\ell, r)$, $1 \le p < \infty$. Then,*

$$\left(D_{\ell^+,\pi(t)}^{\mu,\theta(t)} \Im_{\ell^+,\pi(t)}^{\delta,\theta(t)} \kappa\right)(t) = \left(\Im_{\ell^+,\pi(t)}^{\delta-\mu,\theta(t)} \kappa\right)(t), \tag{2.12}$$

where $m = [\delta] + 1$.

Theorem 2.5. *[16, 18, 19] Let $\delta > 0$. Then, we have*

$$\left(D_{\ell^+,\pi(t)}^{\delta,\theta(t)} \Im_{\ell^+,\pi(t)}^{\delta,\theta(t)} \kappa\right)(t) = \kappa(t). \tag{2.13}$$

Lemma 2.6. *[16, 18, 19] Let $\delta > 0$, $m = -[-\delta]$, $\kappa \in \chi_\pi^p(\ell, r)$ and $\Im_{\ell^+,\pi(t)}^{\delta,\theta(t)} \kappa \in AC_\pi^m[\ell, r]$. Then*

$$\left(\Im_{\ell^+,\theta(t)}^{\delta,\theta(t)} D_{\pi(t)}^{\delta,\theta(t)} k\right)(t) = \kappa(t) - \pi^{-1}(t) \sum_{i=1}^{m} \frac{(\theta(t) - \theta(\ell))^{\delta-i}}{\Gamma(\delta - i + 1)} \left(\Im_{\ell^+,\pi(t)}^{m-\delta,\theta(t)} \kappa\right)_{m-i} (\ell^+), \tag{2.14}$$

where

$$\left(\Im_{\ell^+,\pi(t)}^{m-\delta,\theta(t)} \kappa\right)_i (\ell^+) = \left(\frac{\vartheta_t}{\theta'(t)}\right)^i (\pi(t) \Im_{\ell^+,\pi(t)}^{m-\delta,\theta(t)} \kappa)(\ell^+), \qquad i = 0, 1, \ldots, m. \tag{2.15}$$

In particular, if $0 < \delta < 1$, then

$$\left(\Im_{\ell^+,\pi(t)}^{\delta,\theta(t)} D_{\pi(t)}^{\delta,\theta(t)} \kappa\right)(t) = \kappa(t) - \frac{\pi(\ell^+)(\Im_{\ell^+,\pi(t)}^{1-\delta,\theta(t)} \kappa)(\ell^+)}{\Gamma(\delta)} (\theta(t) - \theta(\ell))^{\delta-1} \pi^{-1}(t). \tag{2.16}$$

3. Main results

Let $D = [\ell, r]$ be a finite interval and ϱ be a parameter such that $m - 1 < \varrho \le m$, then

1) The weighted space $C_{\varrho,\theta}^{\pi}[a, b]$ of functions κ with respect to θ and weighted π on $[\ell, r]$ is defined by

$$C_{\varrho,\theta}^{\pi}[\ell, r] = \{\kappa : (\ell, r] \to \mathbb{R}; \quad (\theta(t) - \theta(\ell))^{\varrho} \pi(t) \kappa(t) \in C[\ell, r]\}, \tag{3.1}$$

having norm

$$\|\kappa\|_{C_{\varrho,\theta}^{\pi}[\ell,r]} = \|(\theta(t) - \theta(\ell))^{\varrho} \pi(t) \kappa(t)\|_{C[\ell,r]}.$$

The above space satisfies the following properties:

i) $C_{\varrho,\theta}^{\pi}[\ell, r] = C[\ell, r]$, for $\varrho = 0$ and $\pi(t) = 1$.

ii) For $\pi(t) = 1$, $C_{\varrho,\theta}^{\pi}[\ell, r] = C_{\varrho,\theta}[\ell, r]$.

2) The weighted space $C_{\varrho,\theta}^{m,\pi}[\ell, r]$ of functions κ with respect to θ and weighted π on $[\ell, r]$ is defined by

$$C_{\varrho,\theta}^{m,\pi}[\ell, r] = \left\{\kappa : [\ell, r] \to \mathbb{R}; \quad (\pi\kappa)(t) \in C^{m-1}[\ell, r]; \quad (D_{\pi(t)}^{m,\theta(t)} \kappa)(t) \in C_{\varrho,\theta}^{\pi}[\ell, r]\right\}, \tag{3.2}$$

where

$$(D_{\pi(t)}^{m,\theta(t)} \kappa)(t) = \frac{1}{\pi(t)} \left(\frac{\vartheta_t}{\theta'(t)}\right)^m (\pi(t)\kappa(t)), \quad \vartheta_t = \frac{d}{dt}, \tag{3.3}$$

along with the norm

$$\|\kappa\|_{C^{m,\pi}_{\varrho,\theta}[\ell,r]} = \sum_{i=0}^{m-1} \|(\pi\kappa)^{(i)}\|_{C[\ell,r]} + \|D^{m,\theta(t)}_{\pi(t)}\kappa\|_{C^{\pi}_{\varrho,\theta}[\ell,r]}.$$

The above space satisfies the following properties:

p1) $C^{0,\pi}_{\varrho,\theta}[\ell, r] = C^{\pi}_{\varrho,\theta}[\ell, r]$, for $m = 0$.

p2) $C^{\pi}_{\varrho,\theta}[\ell, r] = C_{\varrho,\theta}[\ell, r]$ and $C^{m,\pi}_{\varrho,\theta}[\ell, r] = C^{m}_{\varrho,\theta}[\ell, r]$, for $\pi(t) = 1$.

3) For $m - 1 < \delta \leq m$ ($m \in \mathbb{N}$), we denote by $C^{\delta,\pi}_{\varrho,\theta}[\ell, r]$

$$C^{\delta,\pi}_{\varrho,\theta}[\ell, r] = \left\{z(t) \in C^{\pi}_{\varrho,\theta}[\ell, r] : \quad (D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) \in C^{\pi}_{\varrho,\theta}[\ell, r]\right\}. \tag{3.4}$$

4) The space $C^{\pi}[\ell, r]$ of functions κ with respect to weighted π on $[\ell, r]$ is defined by

$$C^{\pi}[\ell, r] = \{\kappa : (\ell, r] \to \mathbb{R}; \quad \pi(t)\kappa(t) \in C[\ell, r]\}. \tag{3.5}$$

We will study the existence and uniqueness of a Cauchy-type problem with a π-weighted Riemann-Liouville fractional derivative of a function with respect to another function

$$(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) = \kappa(t, z(t)) \quad (\delta > 0, \quad t > \ell), \tag{3.6}$$

with initial conditions

$$(\pi D^{\delta-k,\theta(t)}_{\ell^+\pi(t)}z)(\ell^+) = r_k, \quad r_k \in \mathbb{R} \quad (k = 1, ..., m = -[-\delta]). \tag{3.7}$$

From the above initial condition and by Definition 2.4, it is clear that

$$(\pi D^{\delta-k,\theta(t)}_{\ell^+,\pi(t)}z)(t) = (\mathfrak{I}^{m-\delta,\theta(t)}_{\ell^+,\pi(t)}z)_{m-k}(t) = \left(\frac{\vartheta_t}{\theta'(t)}\right)^{m-k}(\pi\mathfrak{I}^{m-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t), \tag{3.8}$$

where, $(\mathfrak{I}^{\delta,\theta}_{\ell^+,\pi(t)}z(t))$ is the π-weighted Riemann-Liouville fractional integration operator of order δ defined by (2.4).

The notation $(D^{\delta-k,\theta(t)}_{\ell^+,\pi(t)}z)(\ell^+)$ means that the limit is taken at almost all points of the right-sided neighborhood $(\ell, \ell + \varepsilon)$ $(\varepsilon > 0)$ of a as follows:

$$(\pi D^{\delta-k,\theta(t)}_{\ell^+,\pi(t)}z)(\ell^+) = \pi(\ell^+) \lim_{t\to\ell^+}(D^{\delta-k,\theta(t)}_{\ell^+,\pi(t)}z)(t) \quad (1 \leq k \leq m - 1), \tag{3.9}$$

$$(\pi D^{\delta-m,\theta(t)}_{\ell^+,\pi(t)}z)(\ell^+) = \pi(\ell^+) \lim_{t\to\ell^+}(\mathfrak{I}^{m-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t), \quad (\delta \neq m). \tag{3.10}$$

The nonlinear Volterra integral equation of the second kind corresponding to the problems (3.6) – (3.7) takes the form

$$z(t) = \frac{1}{\pi(t)} \sum_{j=1}^{m} \frac{r_j}{\Gamma(\delta - j + 1)}(\theta(t) - \theta(\ell))^{\delta-j} + \frac{1}{\pi(t)\Gamma(\delta)} \int_{\ell}^{t} (\theta(t) - \theta(s))^{\delta-1}\pi(s)\theta'(s)\kappa(s, z(s))ds. \tag{3.11}$$

In particular, if $0 < \delta < 1$, the problems (3.6)–(3.7) takes the form

$$\left\{ \begin{array}{c} (D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) = \kappa(t, z(t)), \quad (0 < \delta < 1), \\ \\ (\pi\Im^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(\ell^+) = r \quad r \in \mathbb{R}. \end{array} \right\} \tag{3.12}$$

and this problem can be rewritten as a weighted Cauchy type problem

$$\left\{ \begin{array}{c} (D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) = \kappa(t, z(t)), \quad (0 < \delta < 1), \\ \\ \lim_{t\to\ell^+}(\theta(t) - \theta(\ell))^{1-\delta}\pi(t)z(t)] = C \quad C \in \mathbb{R}. \end{array} \right\} \tag{3.13}$$

The corresponding integral equation to the problem (3.12) has the form:

$$Z(t) = \frac{r(\theta(t) - \theta(\ell))^{\delta-1}}{\Gamma(\delta)\pi(t)} + \frac{1}{\pi(t)\Gamma(\delta)}\int_\ell^t (\theta(t) - \theta(s))^{\delta-1}\pi(s)\theta'(s)\kappa(s, z(s))ds. \tag{3.14}$$

In this section, we give conditions for a unique solution $z(t)$ to the Cauchy type problems (3.6) − (3.7) in the space $C^{\delta,\pi}_{m-\delta,\theta}[\ell, r]$.

3.1. Equivalence of the Cauchy type problem and the Volterra integral equation

First, we prove that the Cauchy type problems (3.6) − (3.7) and the nonlinear Volterra integral equation (3.11) are equivalent in the space $C^\pi_{m-\delta,\theta}[\ell, r]$, in the sense that, if $z(t) \in C^\pi_{m-\delta,\theta}[\ell, r]$ satisfies one of these relations, then it also satisfies the other one. For that, we need the following lemmas:

Lemma 3.1. *If $\varrho \in \mathbb{R}(0 \le \varrho < 1)$, then the π-weighted Riemann-Liouville fractional integral operator $\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}$ with $\delta \in \mathbb{R}$ $(\delta > 0)$ is bounded from $C^\pi_{\varrho,\theta}[\ell, r]$ into $C^\pi_{\varrho,\theta}[\ell, r]$, and*

$$||\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa||_{C^\pi_{\varrho,\theta}[\ell,r]} \le (\theta(r) - \theta(\ell))^\delta \frac{\Gamma(1 - \varrho)}{\Gamma(1 + \delta - \varrho)}||\kappa||_{C^\pi_{\varrho,\theta}[\ell,r]}. \tag{3.15}$$

Proof. Using the definition of the weighted fractional integral (2.4) and property 2.2, for any $\kappa \in C^\pi_{\varrho,\theta}[\ell, r]$ and $t \in [\ell, r]$, we obtain

$$\begin{aligned} |(\theta(t) - \theta(\ell))^\varrho \pi(t)\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa(t)| &= |(\theta(t) - \theta(\ell))^\varrho \frac{1}{\Gamma(\delta)}\int_\ell^t (\theta(t) - \theta(s))^{\delta-1}\theta'(s)\pi(s)\kappa(s)ds| \\ &\le \frac{|(\theta(t) - \theta(\ell))^\varrho|}{|\Gamma(\delta)|}\int_\ell^t |(\theta(t) - \theta(s))^{\delta-1}(\theta(s) - \theta(\ell))^{-\varrho}\theta'(s)| \\ &\quad \times|(\theta(s) - \theta(\ell))^\varrho\kappa(s)\pi(s)|ds \\ &\le \pi(t)(\theta(t) - \theta(\ell))^\varrho||\kappa||_{C^\pi_{\varrho,\theta}[\ell,r]}\left(\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}(\pi^{-1}(t)(\theta(t) - \theta(\ell))^{-\varrho})\right)(t) \\ &= (\theta(t) - \theta(\ell))^\delta \frac{\Gamma(1 - \varrho)}{\Gamma(1 + \delta - \varrho)}||\kappa||_{C^\pi_{\varrho,\theta}[\ell,r]}. \end{aligned}$$

Now, by the definition of the weighted space $C^\pi_{\varrho,[\ell,r]}$ defined by (3.1), we have

$$||\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa||_{C^\pi_{\varrho,\theta}[\ell,r]} = ||(\theta(t) - \theta(\ell))^\varrho\pi(t)\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa(t)||_{C[\ell,r]} \le (\theta(r) - \theta(\ell))^\delta \frac{\Gamma(1 - \varrho)}{\Gamma(1 + \delta - \varrho)}||\kappa||_{C^\pi_{\varrho,\theta}[\ell,r]}.$$

Hence, the proof of this lemma is complete. □

Lemma 3.2. *Let $\delta > 0$, $\mu > 0$, and $0 \leq \varrho < 1$. The following assertions are then true:*

a) If $\kappa(t) \in C^{\pi}_{\varrho,\theta}[\ell, r]$, then the relation (2.11) holds any point $t \in (\ell, r]$.

b) If $\kappa(t) \in C^{\pi}_{\varrho,\theta(t)}[\ell, r]$, then the equality (2.13) holds any point $t \in (\ell, r]$.

c) Let $\delta > \mu > 0$. If $\kappa(t) \in C^{\pi}_{\varrho,\theta}[\ell, r]$, then the relation (2.12) holds at any point $t \in (\ell, r]$.

d) Let $m = [\delta] + 1$. Also, let $(\Im^{m-\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa)(t)$ be the weighted fractional integral (2.4) of order $m - \delta$. If $\kappa(t) \in C^{\omega}_{\varrho,\theta}[\ell, r]$ and $(\Im^{m-\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa)(t) \in C^{m,\pi}_{\varrho,\theta}[\ell, r]$, then the relation (2.14) holds at any point $t \in (\ell, r]$.
In particular, when $0 < \delta < 1$ and $(\Im^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa)(t) \in C^{1,\pi}_{\varrho,\theta}[\ell, r]$, the equality (2.16) is valid.

Proof. As the proof is similar to the proofs in [18], we deleted it. □

Lemma 3.3. *Let $0 < \ell < r < \infty$, $\delta > 0$ and $m - 1 \leq \varrho < m$ with $m \in \mathbb{N}$ and $\kappa \in C^{\pi}_{\varrho,\theta}[\ell, r]$. If $\delta > \varrho$ and $\pi(t) > 0$, for all $t \in [\ell, r]$, then $\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa$ is continuous on $[\ell, r]$ and*

$$\left(\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa\right)(\ell) = \lim_{t\to\ell^+}\left(\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa\right)(t) = 0. \tag{3.16}$$

Proof. Since $\kappa \in C^{\pi}_{\varrho,\theta}[\ell, r]$, then $(\theta(t) - \theta(\ell))^{\varrho}\pi(t)\kappa(t)$ is continuous on $[\ell, r]$ and hence

$$|(\theta(t) - \theta(\ell))^{\varrho}\pi(t)\kappa(t)| < C,$$

where $t \in [\ell, r]$ and $C > 0$ is a constant.

Therefore,

$$|\left(\Im^{\varrho,\theta(t)}_{\ell^+,\pi(t)}\kappa\right)(t)| < C\left(\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}(\pi^{-1}(t)(\theta(t) - \theta(\ell))^{-\varrho})\right)(t),$$

and by Property 2.2, we can write

$$|\left(\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa\right)(t)| < C\frac{\Gamma(1-\varrho)}{\Gamma(\delta-\varrho+1)}(\pi^{-1}(t)(\theta(t) - \theta(\ell))^{\delta-\varrho})). \tag{3.17}$$

As $\delta > \varrho$, the right-hand side of (3.17) goes to zero when $t \to \ell^+$, we obtain the result. □

Theorem 3.4. *Let $\delta > 0$, $m = -[-\delta]$. Let B be an open set in $\mathbb{R}$ and let $\kappa : (\ell, r] \times B \to \mathbb{R}$ be a function such that $\kappa(t, z(t)) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ for any $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$. If $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$, then $z(t)$ satisfies the relations (3.6) − (3.7) if, and only if, $z(t)$ satisfies the Volterra integral equation (3.11).*

Proof. First, we prove the necessity. Let $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ satisfy the relations (3.6) − (3.7). By hypothesis, $\kappa(t, z) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ and it follows from (3.6) that

$$(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r].$$

According to (2.7)

$$(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) = (D^{m,\theta(t)}_{\pi(t)}\Im^{m-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t), \quad m = -[-\delta], \tag{3.18}$$

and hence, by Lemma 3.1, we have

$$(\mathfrak{I}^{m-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) \in C^{m,\pi(t)}_{m-\delta,\theta(t)}[\ell, r].$$

Thus, we can apply Lemma 3.2 (d), and, in accordance with (2.14), we have

$$\left(\mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)} D^{\delta}_{\pi(t)} z\right)(t) = z(t) - \sum_{j=1}^{m} \frac{(\theta(t) - \psi(\ell))^{\delta-j}\pi^{-1}(t)}{\Gamma(\delta - j + 1)} \left(\mathfrak{I}^{m-\delta,\theta(t)}_{\ell^+,\pi(t)} z\right)_{m-j}(\ell^+), \tag{3.19}$$

where

$$\left(\mathfrak{I}^{m-\delta,\theta(t)}_{\ell^+,\pi(t)} z\right)_{m-j}(\ell^+) = \left(\frac{\vartheta_t}{\theta'(t)}\right)^{m-j} (\pi(t)\mathfrak{I}^{m-\delta,\theta(t)}_{\ell^+,\pi(t)} z)(\ell^+), \quad \vartheta_t = \frac{d}{dt}.$$

By (3.7) and (3.9), we rewrite (3.19) in the form

$$\begin{aligned} \left(\mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)} D^{\delta}_{\pi(t)} z\right)(t) &= z(t) - \sum_{j=1}^{m} \frac{(\theta(t) - \theta(\ell))^{\delta-j}\pi^{-1}(t)}{\Gamma(\delta - j + 1)} \left(\pi D^{\delta-j,\theta(t)}_{\ell^+,\pi(t)} t\right)(\ell^+) \\ &= z(t) - \sum_{j=1}^{m} \frac{r_j}{\Gamma(\delta - j + 1)} (\theta(t) - \theta(\ell))^{\delta-j}\pi^{-1}(t). \end{aligned} \tag{3.20}$$

By Lemma 3.1, the integral $\left(\mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa(t, z(t))\right)(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ exists on $[\ell, r]$. Applying the operator $\mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)}$ to both sides of (3.6) and using (3.20), we obtain Eq (3.11), and hence necessity is proved.

Now, we prove sufficiency. Let $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ satisfy the Eq (3.11). Applying the operator $D^{\delta,\theta(t)}_{\ell^+\pi(t)}$ to both sides of (3.11), we have

$$\left(D^{\delta,\theta(t)}_{\ell^+,\pi(t)} z\right)(t) = \sum_{j=1}^{m} \frac{r_j}{\Gamma(\delta - j + 1)} \left(D^{\delta,\theta}_{\ell^+,\pi(t)}(\pi^{-1}(t)(\theta(t) - \theta(\ell))^{\delta-j})\right)(t) + \left(D^{\delta,\theta(t)}_{\ell^+,\pi(t)} \mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa(t, z(t))\right)(t).$$

From here, in accordance with the formula (2.10) and Lemma 3.2 (b), we arrive at Eq (3.6).

Now, we show that the relation in (3.7) also holds. For this, we apply the operators $D^{\delta-k}_{\ell^+,\pi(t)}$ ($k = 1, ..., m$) to both sides of (3.11).

If $1 \leq k \leq m - 1$, then, in accordance with (2.9) and Lemma 3.2(c), we have

$$\begin{aligned} \left(D^{\delta-k,\theta(t)}_{\ell^+,\pi(t)} z\right)(t) &= \sum_{j=1}^{m} \frac{r_j}{\Gamma(\delta - j + 1)} \left(D^{\delta-k,\theta(t)}_{\ell^+,\pi(t)}((\theta(t) - \theta(\ell))^{\delta-j}\pi^{-1}(t))\right)(t) + \left(D^{\delta-k,\theta(t)}_{\ell^+,\pi(t)} \mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa(t, z(t))\right)(t) \\ &= \sum_{j=1}^{m} \frac{r_j}{\Gamma(k - j + 1)} (\theta(t) - \theta(\ell))^{k-j}\pi^{-1}(t) + \left(\mathfrak{I}^{k,\theta(t)}_{\ell^+,\pi(t)}\kappa(t, (t))\right)(t). \end{aligned}$$

Hence,

$$\left(D^{\delta-k,\theta(t)}_{\ell^+,\pi(t)} z\right)(t) = \sum_{j=1}^{m} \frac{r_j}{(k - j)!} (\theta(t) - \theta(\ell))^{k-j}\pi^{-1}(t) + \frac{\pi^{-1}(t)}{(k - 1)!} \int_{\ell}^{t} (\theta(t) - \theta(s))^{k-1}\pi(s)\kappa(s, z(s))\theta'(s)ds. \tag{3.21}$$

If $k = m$, then, in accordance with (3.10), (2.8) and, Lemma 3.2 (a), and using Lemma 3.3, similarly to (3.21), we obtain

$$\begin{aligned}\left(D^{\delta-m,\theta(t)}_{\ell^+,\pi(t)}z\right)(t) &= \sum_{j=1}^{m}\frac{r_j}{\Gamma(\delta-j+1)}\left(\mathfrak{I}^{m-\delta,\theta(t)}_{\ell^+,\pi(t)}((\theta(t)-\theta(\ell))^{\delta-j}\pi^{-1}(t))\right)(t)+\left(\mathfrak{I}^{m-\delta,\theta(t)}_{\ell^+,\pi(t)}\mathfrak{I}^{\delta,\theta(t)}_{\ell^+\pi}\kappa(t,z(t))\right)(t)\\ &= \sum_{j=1}^{m}\frac{r_j}{\Gamma(m-j+1)}(\theta(t)-\theta(\ell))^{m-j}\pi^{-1}(t)+\left(\mathfrak{I}^{m,\theta(t)}_{\ell^+\pi(t)}\kappa(t,z(t))\right)(t).\end{aligned}$$

Therefore,

$$\left(D^{\delta-m,\theta(t)}_{\ell^+,\pi(t)}(t)\right)(t)=\sum_{j=1}^{m}\frac{r_j}{(m-j)!}(\theta(t)-\theta(\ell))^{m-j}\pi^{-1}(t)+\frac{\pi^{-1}(t)}{(m-1)!}\int_{\ell}^{t}(\theta(t)-\theta(s))^{m-1}\pi(s)\kappa(s,z(s))\psi'(s)ds. \tag{3.22}$$

Multiplying (3.21), (3.22) by $\pi(t)$ and taking the limit as $t \to 0^+$ above, we obtain (3.7). Thus, sufficiency is proven, and the proof is completed. □

Corollary 3.5. *Let* $0 < \delta < 1$*, let* B *be an open set in* $\mathbb{R}$ *and let* $\kappa : (\ell, r] \times B \to \mathbb{R}$ *be a function such that* $\kappa(t, z(t)) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$ *for any* $z(t) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$*. If* $z(t) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$*, then* $z(t)$ *satisfies the relations in* (3.12) *if, and only if,* $z(t)$ *satisfies the integral equation* (3.14).

3.2. Existence and uniqueness of the solution to the Cauchy type problem

Next, we will establish the existence and uniqueness of a solution for the Cauchy-type problems (3.6) − (3.7) in the space $C^{\delta,\omega}_{\varrho,\theta}[\ell, r]$, as defined in (3.4), utilizing the Banach fixed point theorem. This requires the following lemmas:

Lemma 3.6. *Let* $\lambda \in [0, \infty)$*,* $\ell < n < r$*,* $\kappa \in C^{\pi}_{\lambda,\theta}[\ell, n]$ *and* $\kappa \in C^{\pi}[n, r]$*. Then* $\kappa \in C^{\pi}_{\lambda,\theta}[\ell, r]$ *and*

$$\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,r]} \leq \max\{\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,n]}, (\theta(r)-\psi(\ell))^{\lambda}\|\kappa\|_{C^{\pi}[n,r]}\}, \tag{3.23}$$

where the space $C^{\pi}[\ell, r]$ *is the same as defined in* (3.5).

Proof. Since $\kappa \in C^{\pi}_{\lambda,\theta}[\ell, n]$ and $\kappa \in C^{\pi}[n, r]$, then we obtain

$$\kappa \in C^{\pi}(\ell, r] \quad \textit{and} \quad \kappa \in C^{\pi}_{\lambda,\theta}[\ell, r].$$

Now, we prove the estimate. Because $\kappa \in C^{\pi}_{\lambda,\theta}[\ell, r]$, there exists $t^* \in [\ell, r]$ such that

$$\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,r]} = |(\theta(t^*)-\theta(\ell))^{\lambda}\pi(t^*)\kappa(t^*)|. \tag{3.24}$$

Assume that $t^* \in [\ell, n]$, then we have

$$\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,r]} \leq \|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,n]}.$$

Similarly, if we suppose that $t^* \in [n, r]$, then we have

$$\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,r]} \leq (\theta(r)-\theta(\ell))^{\lambda}\|\kappa\|_{C^{\pi}[n,r]}.$$

Now, we can write this result

$$\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,r]} \leq \max\{\|\kappa\|_{C^{\pi}_{\lambda,\theta}[\ell,n]}, (\theta(r)-\theta(\ell))^{\lambda}\|\kappa\|_{C^{\pi}[n,r]}\}.$$

The proof of this lemma is complete. □

Lemma 3.7. *The weighted fractional integration operator* $\mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)}$ *of order* δ $(\delta > 0)$ *is a mapping from* $C^{\pi}[\ell, r]$ *to* $C^{\pi}[\ell, r]$*, and*

$$\|\mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa\|_{C^{\pi}[\ell,r]} \leq \frac{(\theta(r)-\psi(\ell))^{\lambda}}{\delta\Gamma(\delta)}\|\kappa\|_{C^{\pi}[\ell,r]}, \tag{3.25}$$

where $\kappa \in C^{\pi}[\ell, r]$.

Proof. we prove the estimate in (3.25) as follows:

$$\begin{aligned} |\pi(t)\mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa(t)| &= |\frac{1}{\Gamma(\delta)}\int_{\ell}^{t}(\theta(t)-\theta(s))^{\delta-1}\theta'(s)\pi(s)\kappa(s)ds| \\ &\leq \frac{\|\pi\kappa\|_{C[\ell,r]}}{\Gamma(\delta)}\int_{\ell}^{t}(\theta(t)-\theta(s))^{\delta-1}\theta'(s)ds \\ &= \frac{(\theta(t)-\theta(\ell))^{\lambda}}{\delta\Gamma(\delta)}\|\kappa\|_{C^{\pi}[\ell,r]}. \end{aligned}$$

Therefore,

$$\|\mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)}\kappa\|_{C^{\pi}[\ell,r]} \leq \frac{(\theta(r)-\theta(\ell))^{\lambda}}{\delta\Gamma(\delta)}\|\kappa\|_{C^{\pi}[\ell,r]}.$$

Hence, the proof of this lemma is complete. □

Theorem 3.8. *Let* $\delta > 0$ *and* $m = -[-\delta]$*. Let* B *be an open set in* $\mathbb{R}$ *and let* $\kappa : (\ell, r] \times B \to \mathbb{R}$ *be a function such that* $\kappa(t, z(t)) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ *for any* $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ *and the Lipschitzian condition* (2.3) *holds.*

Then there exists a unique solution $z(t)$ *to the Cauchy type problems* (3.6) − (3.7) *in the space* $C^{\pi}_{m-\delta,\theta}[\ell, r]$.

Proof. **Step1.** First we prove the existence of a unique solution $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$. According to the previous Theorem 3.4, it is sufficient to prove the existence of a unique solution $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ to the nonlinear Volterra integral equation (3.11), which, mainly based on Theorem 2.1(Banach fixed point theorem).

Divide the interval $[\ell, r]$ into M subdivisions $[\ell, t_1], [t_1, t_2], ..., [t_{M-1}, r]$ such that $\ell < t_1 < t_2 < ... < t_{M-1} < r$.

I) Choose $t_1 \in (\ell, r]$ such that the following estimate holds

$$\xi_1 = L(\theta(t_1)-\theta(\ell))^{\delta}\frac{\Gamma(\delta-m+1)}{\Gamma(2\delta-m+1)} < 1, \tag{3.26}$$

where L is the Lipschitzian constant. Now we prove that there exists a unique solution $z(t) \in C^{\omega}_{m-\delta,\theta}[\ell, t_1]$ to (3.11) on the interval $(\ell, t_1]$. To do this, we apply the Banach fixed point theorem

(Theorem 2.1) for the space $C^{\pi}_{m-\delta,\theta}[\ell, r]$, which is the complete metric space equipped with the distance given by

$$\sigma(z_1, z_2) = \|z_1 - z_2\|_{C^{\pi}_{m-\delta,\theta}[\ell,t_1]} = \max_{t\in[\ell,t_1]} |(\theta(t) - \theta(\ell))^{m-\delta}\pi(t)[z_1(t) - z_2(t)]|.$$

For any $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, t_1]$, we define the operator A by expressing the integral equation (3.11) as follows:

$$z(t) = (Az)(t),$$

$$(Az)(t) = z_0(t) + \frac{1}{\Gamma(\delta)\pi(t)} \int_{\ell}^{t} (\theta(t) - \theta(s))^{\delta-1}\theta'(s)\omega(s)\kappa(s, z(s))ds, \tag{3.27}$$

with

$$z_0(t) = \frac{1}{\pi(t)} \sum_{j=1}^{m} \frac{r_j}{\Gamma(\delta - j + 1)}(\theta(t) - \theta(\ell))^{\delta-j}. \tag{3.28}$$

Applying the Banach contraction mapping, we shall prove that A has a unique fixed point.

Firstly, we have to show that:

I1) If $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, t_1]$, then $(Az)(t) \in C^{\pi}_{m-\delta,\theta}[\ell, t_1]$.

I2) For any $z_1, z_2 \in C^{\pi}_{m-\delta,\theta}[\ell, t_1]$ the following estimate holds:

$$\|Az_1 - Az_2\|_{C^{\pi}_{m-\delta,\theta}[\ell,t_1]} \le \xi_1\|z_1 - z_2\|_{C^{\pi}_{m-\delta,\theta}[\ell,t_1]}, \quad \xi_1 = L(\theta(t_1) - \theta(\ell))^{\delta}\frac{\Gamma(\delta - m + 1)}{\Gamma(2\delta - m + 1)}. \tag{3.29}$$

It is evident from Eq (3.28) that $z_0(t) \in C^{\pi}_{m-\delta,\theta}[\ell, t_1]$. Since $\kappa(t, z(t)) \in C^{\pi}_{m-\delta,\theta}[\ell, t_1]$ for any $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, t_1]$, then, by Lemma 3.1, the integral in the right-hand side of (3.27) also belongs to $C^{\pi}_{m-\delta,\theta}[\ell, t_1]$. The above implies that $(Az)(t) \in C^{\pi}_{m-\delta,\theta}[\ell, t_1]$.
Let $z_1, z_2 \in C^{\pi}_{m-\delta,\theta}[\ell, t_1]$. Using (3.27), (2.3) and hance by Lemma 3.1, we obtain

$$\begin{aligned}
\|Az_1 - Az_2\|_{C^{\pi}_{m-\delta,\theta}[\ell,t_1]} &= \|\Im^{\delta,\theta(t)}_{\ell^+,\pi(t)}(\kappa(t, z_1(t)) - \kappa(t, z_2(t)))\|_{C^{\pi}_{m-\delta,\theta}[\ell,t_1]} \\
&\le (\theta(t_1) - \theta(\ell))^{\delta}\frac{\Gamma(\delta - m + 1)}{\Gamma(2\delta - m + 1)}\|\kappa(t, z_1(t)) - \kappa(t, z_2(t))\|_{C^{\pi}_{m-\delta,\theta}}[\ell, t_1] \\
&\le L(\theta(t_1) - \theta(\ell))^{\delta}\frac{\Gamma(\delta - m + 1)}{\Gamma(2\delta - m + 1)}\|z_1 - z_2\|_{C^{\pi}_{m-\delta,\theta}}[\ell, t_1] \\
&= \xi_1\|z_1 - z_2\|_{C^{\pi}_{m-\delta,\theta}}[\ell, t_1].
\end{aligned}$$

This results in obtaining the estimate (3.29). By (3.26), $0 < \xi_1 < 1$, and therefore by using the Banach fixed point theorem, there exists a unique solution $z^*(t) \in C^{\pi}_{m-\delta,\theta}[\ell, t_1]$ to (3.11) on the interval $[\ell, t_1]$. This solution $z^*(t)$ is a limit of a convergent sequence $(A^i z_0^*)(t)$:

$$\lim_{i\to\infty} \|A^i z_0^* - z^*\|_{C^{\pi}_{m-\delta,\theta}[\ell,t_1]} = 0, \tag{3.30}$$

where $z_0^*(t)$ is any function in $C^{\pi}_{m-\delta,\theta}[\ell, t_1]$, and

$$(A^i z_0^*)(t) = z_0(t) + \frac{1}{\Gamma(\delta)\pi(t)} \int_{\ell}^{t} (\theta(t) - \theta(s))^{\delta-1}\theta'(s)\pi(s)\kappa(s, (A^{i-1}z_0^*)(s))ds, \quad (i \in \mathbb{N}).$$

If we denote $z_i(t) = (A^i z_0^*)(t)$, $(i \in \mathbb{N})$, then it is clear that

$$\lim_{m\to\infty} \|z_i - z^*\|_{C^{\pi}_{m-\delta,\theta}[\ell,t_1]} = 0. \tag{3.31}$$

If there exists at least one $r_k \neq 0$ in the initial condition (3.7), then we can choose $z_0^*(t) = z_0(t)$, where $z_0(t)$ is defined by (3.28).

E) Next, we prove the existence of a unique solution $z \in C^{\pi}[t_1, r]$ to (3.11) on the interval $[t_1, r]$. Moreover, if we consider the interval $[t_1, r]$, we can express Eq (3.11) in the following manner:

$$z(t) = z_{01}(t) + \frac{1}{\pi(t)\Gamma(\delta)} \int_{t_1}^{t} (\theta(t) - \theta(s))^{\delta-1} \pi(s)\theta'(s) f(s, z(s))ds, \tag{3.32}$$

where $z_{01}(t)$ is defined by

$$z_{01}(t) = \sum_{j=1}^{m} \frac{r_j}{\Gamma(\delta - j + 1)} (\theta(t) - \theta(\ell))^{\delta-j} \pi^{-1}(t) + \frac{1}{\pi(t)\Gamma(\delta)} \int_{\ell}^{t_1} (\theta(t) - \theta(s))^{\delta-1} \pi(s)\theta'(s)\kappa(s, z(s))ds,$$

is a known function.

We note that $z_{01}(t) \in C^{\pi}[t_1, r]$. Because, $z_0(t) \in C^{\pi}[t_1, r]$. Also, by hypothesis $\kappa(t, z(t)) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ for any $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$, then, $\kappa(t, z(t)) \in C^{\pi}[t_1, r]$, therefore, we can apply Lemma 3.7, we have $\mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)} \kappa(t, z(t)) \in C^{\pi}[t_1, r]$. Thus, $z_{01}(t) \in C^{\pi}[t_1, r]$.

We consider the interval $[t_1, t_2]$, where $t_2 = t_1 + \varepsilon_1$ and $\varepsilon_1 > 0$ are such that $t_2 \in (t_1, r]$. We also use Banach fixed point theorem for the space $C^{\pi}[t_1, t_2]$, where t_2 satisfies

$$\xi_2 = \frac{L(\theta(t_2) - \theta(t_1))^{\delta}}{\delta\Gamma(\delta)} < 1. \tag{3.33}$$

The space $C^{\pi}[t_1, t_2]$ is a complete metric space, with the distance given by

$$\delta(z_1, z_2) = \|z_1 - z_2\|_{C^{\pi}[t_1,t_2]} = \max_{t\in[t_1,t_2]} |\pi(t)[z_1(t) - z_2(t)]|.$$

Also, we can rewrite the integral equation (3.32) in the form:

$$z(t) = (Az)(t),$$

where A is the operator given by

$$(Az)(t) = z_{01}(t) + \frac{1}{\Gamma(\delta)\pi(t)} \int_{t_1}^{t} (\theta(t) - \theta(s))^{\delta-1} \theta'(s)\pi(s)\kappa(s, z(s))ds. \tag{3.34}$$

To apply Theorem 2.1, we have to prove the following:

E1) If $z(t) \in C^{\pi}[t_1, t_2]$, then $(Az)(t) \in C^{\pi}[t_1, t_2]$.

E2) For any $z_1, z_2 \in C^{\pi}[t_1, t_2]$ the following estimate holds:

$$\|Az_1 - Az_2\|_{C^{\pi}[t_1,t_2]} \le \xi_2 \|z_1 - z_2\|_{C^{\pi}[t_1,t_2]}, \quad \xi_2 = \frac{L(\theta(t_2) - \theta(t_1))^{\delta}}{\delta\Gamma(\delta)}. \tag{3.35}$$

Similarly, by hypothesis $\kappa(t, z(t)) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ for any $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$, then, by Lemma 3.7, the integral in the right-hand side of (3.34) also belongs to $C^{\pi}[t_1, t_2]$, and hence $(Az)(t) \in C^{\pi}[t_1, t_2]$. Now, we prove the estimate in (3.35), using the Lipschitz condition and applying Lemma 3.7, we find

$$\begin{aligned}
||Az_1 - Az_2||_{C^{\pi}[t_1,t_2]} &= ||\mathfrak{I}^{\delta,\theta(t)}_{\ell^+,\pi(t)}(\kappa(t, z_1(t)) - \kappa(t, z_2(t)))||_{C^{\pi}[t_1,t_2]} \\
&\leq \frac{(\psi(t_2) - \psi(t_1))^{\delta}}{\delta\Gamma(\delta)}||\kappa(t, z_1(t)) - \kappa(t, z_2(t))||_{C^{\pi}[t_1,t_2]} \\
&\leq \frac{L(\theta(t_2) - \theta(t_1))^{\delta}}{\delta\Gamma(\delta)}||z_1 - z_2||_{C^{\pi}[t_1,t_2]} \\
&= \xi_2||z_1 - z_2||_{C^{\pi}[t_1,t_2]},
\end{aligned}$$

which yields the estimate (3.35). This, together with our assumption $0 < \xi_2 < 1$, shows that A is a contraction and therefore from Theorem 2.1, there exists a unique solution $z_1^*(t) \in C^{\pi}[t_1, t_2]$ to (3.11) on the interval $[t_1, t_2]$. Further, Theorem 2.1 guarantees that this solution $z_1^*(t)$ is the limit of the convergent sequence $(A^i z_{01}^*)(t)$:

$$\lim_{i\to\infty} ||A^i z_{01}^* - z_1^*||_{C^{\pi}[t_1,t_2]} = 0, \tag{3.36}$$

where $z_{01}^*(t)$ is any function in $C^{\pi}[t_1, t_2]$.
If $z_0(t) \neq 0$ on $[t_1, t_2]$, then we can take $z_{01}^*(t) = z_0(t)$ with $z_0(t)$ defined by (3.28). The last relation can be rewritten in the form

$$\lim_{i\to\infty} ||z_i - z_1^*||_{C^{\pi}[t_1,t_2]} = 0, \tag{3.37}$$

where

$$z_i(t) = (A^i z_{01}^*)(t) = z_{01}(t) + \frac{1}{\pi(t)\Gamma(\delta)}\int_{t_1}^{t}(\theta(t) - \theta(s))^{\delta-1}\pi(s)\theta'(s)\kappa(s, (A^{i-1}z_{01}^*)(s))ds, \quad (i \in \mathbb{N}). \tag{3.38}$$

E3) Next, if $t_2 \neq r$, we consider the interval $[t_2, t_3]$, where $t_3 = t_2 + \varepsilon_2, \varepsilon_2 > 0$, such that $t_3 \leq r$ and

$$\xi_3 = \frac{L(\theta(t_3) - \theta(t_2))^{\delta}}{\delta\Gamma(\delta)} < 1.$$

By using the same arguments as above, we conclude that there exists a unique solution $z_2^*(t) \in C^{\pi}[t_2, t_3]$ to (3.11) on the interval $[t_2, t_3]$. If $t_3 \neq r$, repeating the above process, then we find that there exists a unique solution $z(t)$ to (3.11), $z(t) = z_k^*(t)$, and $z_k^*(t) \in C^{\pi}[t_{k-1}, t_k]$ for $k = 1, ..., M$, where $a = t_0 < t_1 < ... < t_M = r$ and

$$\xi_k = \frac{L(\theta(t_k) - \theta(t_{k-1}))^{\delta}}{\delta\Gamma(\delta)} < 1.$$

Consequently, there exists a unique solution $z(t) \in C^{\pi}[t_1, r]$ to (3.11) on the interval $[t_1, r]$. Using Lemma 3.6, we can conclude that there exists a unique solution $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ to the Volterra integral equation (3.11) on the whole interval $[\ell, r]$. Therefore, $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ is the unique solution to the Cauchy-type problems (3.6) – (3.7).

Step 2. Finally, it remains to show that such a unique solution is actually in $C^{\delta,\pi}_{m-\delta,\theta}[\ell, r]$. By (3.4), it is sufficient to prove that $(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$. By the above proof, the solution $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ is a limit of the sequence $z_i(t)$, where $z_i(t) = (A^i z^*_0)(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$:

$$\lim_{i\to\infty} \|z_i - z\|_{C^{\pi}_{m-\delta,\theta}[\ell,r]} = 0, \tag{3.39}$$

with the choice of certain z^*_0 on each $[\ell, t_1], ..., [t_{M-1}, r]$.

If $z_0(t) \neq 0$, then we can take $z^*_0(t) = z_0(t)$. Hence, by using (3.6) and (2.3), we have

$$\begin{aligned} \|D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z_i - D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z\|_{C^{\pi}_{m-\delta,\theta}[\ell,r]} &= \|\kappa(t, z_i(t)) - \kappa(t, z(t))\|_{C^{\pi}_{m-\delta,\theta}[\ell,r]} \\ &\leq L\|z_i - z(t)\|_{C^{\pi}_{m-\delta,\theta}[\ell,r]}. \end{aligned} \tag{3.40}$$

In virtue of (3.39) and (3.40), it can be said that

$$\lim_{i\to\infty} \|D^{\delta,\theta}_{\ell^+,\pi(t)}z_i - D^{\delta,\theta}_{\ell^+,\pi(t)}z\|_{C^{\pi}_{m-\delta,\theta}[\ell,r]} = 0.$$

By hypothesis, $(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z_i)(t) = \kappa(t, z_{i-1}(t))$ and $\kappa(t, z(t)) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ for any $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$, we have $(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z_i)(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$. Hence $(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$. Consequently, $z(t) \in C^{\pi}_{m-\delta,\theta}[\ell, r]$ is the unique solution to the problems (3.6) − (3.7). The proof is complete. □

Corollary 3.9. *Let* $0 < \delta < 1$*, let* B *be an open set in* $\mathbb{R}$ *and let* $\kappa : (\ell, r] \times B \to \mathbb{R}$ *be a function such that* $\kappa(t, z(t)) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$ *for any* $z(t) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$ *and* (2.3) *holds. Then there exists a unique solution* $z(t)$ *to the Cauchy type problem* (3.12) *in the space* $C^{\delta,\pi}_{1-\delta,\theta}[\ell, r]$.

3.3. The weighted Cauchy type problem with weighted R-L fractional derivatives of a function with respect to another function

When $0 < \delta < 1$, the result of Corollary 3.9 remains true for the following weighted Cauchy type problem (3.13) with $C \in \mathbb{R}$:

$$(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) = \kappa(t, z(t)); \quad \lim_{t\to\ell^+}[(\theta(t) - \theta(\ell))^{1-\delta}\pi(t)z(t)] = C, \quad (0 < \delta < 1). \tag{3.41}$$

Its proof is based on the following lemma assertion:

Lemma 3.10. *Let* $0 < \delta < 1$ *and let* $z(t) \in C^{\pi}_{1-\delta,\theta}[\ell, r]$.

S1) If there exists a limit

$$\lim_{t\to\ell^+} \left[(\theta(t) - \theta(\ell))^{1-\delta}\pi(t)z(t)\right] = C, \quad C \in \mathbb{R}, \tag{3.42}$$

then there also exists a limit

$$(\pi\Im^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(\ell^+) = \lim_{t\to\ell^+}(\pi\Im^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) = C\Gamma(\delta). \tag{3.43}$$

S2) If there exists a limit

$$\lim_{t\to\ell^+}(\pi\Im^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) = r, \quad r \in \mathbb{R} \tag{3.44}$$

and if there exists the limit $\lim_{t\to\ell^+}\left[(\theta(t) - \theta(\ell))^{1-\delta}\pi(t)z(t)\right]$*, then*

$$\lim_{t\to\ell^+} \left[(\theta(t) - \theta(\ell))^{1-\delta}\pi(t)z(t)\right] = \frac{r}{\Gamma(\delta)}. \tag{3.45}$$

Proof. Choose an arbitrary $\varepsilon > 0$. According to (3.42), there exists $\eta = \eta(\varepsilon) > 0$ such that

$$|(\theta(t) - \theta(\ell))^{1-\delta}\pi(t)z(t) - C| < \frac{\varepsilon}{\Gamma(\delta)}. \tag{3.46}$$

For $\ell < t < \ell + \eta$.By using (2.8), we have

$$\Gamma(\delta) = (\pi\mathfrak{I}^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}(\pi^{-1}(t)(\theta(t) - \theta(\ell))^{\delta-1}))(t), \quad 0 < \delta < 1. \tag{3.47}$$

Using this equality and (2.4), we obtain

$$\begin{aligned}
& |(\omega\mathfrak{I}^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) - C\Gamma(\delta)| \\
= & |(\pi\mathfrak{I}^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) - C(\pi\mathfrak{I}^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}(\pi^{-1}(t)(\theta(t) - \theta(\ell))^{\delta-1}))(t)| \\
\leq & \frac{1}{\Gamma(1-\delta)}\int_\ell^t (\theta(t) - \theta(s))^{-\delta}\theta'(s)|\pi(s)z(s) - C(\theta(s) - \theta(\ell))^{\delta-1}|ds \\
\leq & \frac{1}{\Gamma(1-\delta)}\int_\ell^t (\theta(t) - \theta(s))^{-\delta}\theta'(s)(\theta(s) - \theta(\ell))^{\delta-1}|(\theta(s) - \theta(\ell))^{1-\delta}\pi(s)z(s) - C|ds.
\end{aligned}$$

Now, by making use of (3.46) and the formula (3.47), we have

$$|(\pi\mathfrak{I}^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) - C\Gamma(\delta)| \leq \frac{\varepsilon\pi(t)}{\Gamma(\delta)}(\mathfrak{I}^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}(\pi^{-1}(t)(\theta(t) - \theta(\ell))^{\delta-1}))(t) = \varepsilon, \tag{3.48}$$

which proves the assertion $(S1)$ of Lemma 3.10.

Assume that the limit in (3.45) is equal to C :

$$\lim_{t\to\ell^+}\left[(\theta(t) - \theta(\ell))^{1-\delta}\pi(t)z(t)\right] = C.$$

Consequently, based on $(S1)$, we have

$$(\pi\mathfrak{I}^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(\ell^+) = \lim_{t\to\ell^+}(\pi\mathfrak{I}^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) = C\Gamma(\delta),$$

and hence, by (3.44), $C = \frac{r}{\Gamma(\delta)}$, which proves (3.45). □

Now, by Corollary 3.9 and Lemma 3.10, we deduce the existence and uniqueness result for the weighted Cauchy type problem (3.41).

Theorem 3.11. *Let $0 < \delta < 1$, let B be an open set in $\mathbb{R}$ and let $\kappa : (\ell, r] \times B \to \mathbb{R}$ be a function such that $\kappa(t, z(t)) \in C^\pi_{1-\delta,\theta}[\ell, r]$ for any $z(t) \in C^\pi_{1-\delta,\theta}[\ell, r]$ and the Lipschitzian condition* (2.3) *holds.*

Then there exists a unique solution $z(t)$ to the weighted Cauchy type problem (3.41) *in the space $C^{\delta,\pi}_{1-\delta,\theta}[\ell, r]$.*

Proof. If $z(t)$ fulfills the conditions (3.41), then, according to Lemma 3.10 $(S1)$, $z(t)$ also satisfies the conditions (3.12) with $r = C\Gamma(\delta)$:

$$(D^{\delta,\theta(t)}_{\ell^+,\pi(t)}z)(t) = \kappa(t, z(t)) \quad (0 < \delta < 1), \quad (\pi\mathfrak{I}^{1-\delta,\theta(t)}_{\ell^+,\pi(t)}z)(\ell^+) = C\Gamma(\delta) \in \mathbb{R}. \tag{3.49}$$

By Corollary 3.9, there exists a unique solution $z(t) \in C^{\delta,\pi}_{1-\delta,\theta}[\ell, r]$ to this problem. Furthermore, by Lemma 3.10 $(S2)$, $z(t)$ is also a unique solution to the weighted Cauchy problem (3.41). □

3.4. Fractional Differential Inequalities

We consider the initial value problem (IVP) for the fractional differential equation given by

$$D^{\delta,\theta(t)}_{\pi(t)} z = \kappa(t, z(t)), \quad z(t_0) = z^0 = z(t)\pi(t)(\theta(t) - \theta(t_0))^{1-\delta}|_{t=t_0}, \quad t_0 \le t \le H, \quad H > 0, \tag{3.50}$$

$\kappa \in C([t_0, H] \times \mathbb{R}, \mathbb{R})$, $D^{\delta,\theta(t)}_{\pi(t))} z$ is the weighted Riemann Liouville fractional derivative of order δ of z, such that $0 < \delta < 1$.

Definition 3.1. *Let $0 < \delta < 1$ and $\beta = 1 - \delta$. We denote by $C^{\pi}_{\beta,\theta}([\ell, r], \mathbb{R})$, the function space*

$$C^{\pi}_{\beta,\theta}([t_0, H], \mathbb{R}) = \left\{z \in C^{\pi}((t_0, H], \mathbb{R}), \quad (\theta(t) - \theta(t_0))^{\beta}\pi(t)z(t) \in C([t_0, H], \mathbb{R})\right\}. \tag{3.51}$$

Definition 3.2. *(Locally Hölder continuous with respect to θ)*

Let κ be a real function. We say that κ is locally Hölder continuous with respect to θ at a point t_1, with exponent $\vartheta \in (0, 1]$, if there exist a real number $N > 0$, such that for all $h > 0$, small enough, we have

$$|\kappa(t_1) - \kappa(t)| \le N|\theta(t_1) - \theta(t)|^{\vartheta} \quad \forall t \in]t_1 - h,\ t_1 + h[\cap dom(\kappa), \quad h > 0, \tag{3.52}$$

where θ is a strictly increasing C^1 function.

A function κ is simply said to be locally Hölder continuous with respect to θ, if it is locally Hölder continuous with respect to θ at all points in $dom(\kappa)$.

Lemma 3.12. *Let $0 < \delta,\beta < 1$, let $\theta \in C^1$ is a strictly increasing function and $\pi(t) \neq 0$ for $t \in [\ell, r]$. Consider the function*

$$m(t) = \frac{(\theta(t) - \theta(\ell))^{\beta-1}}{\pi(t)} E_{\delta,\beta}\left[\mu(\theta(t) - \theta(\ell))^{\delta}\right],$$

where $E_{\delta,\mu}(.)$ is the Mttag-Leffler function with two parameters. Then,

$$D^{\delta,\theta}_{\ell^+\pi} m(t) = \mu m(t). \tag{3.53}$$

Proof. Using the definition of the Mittag Leffler function and Property 2.2, we have

$$\begin{aligned}
D^{\delta,\theta(t)}_{\ell^+,\pi(t)} m(t) &= D^{\delta,\theta(t)}_{\ell^+,\pi(t)} \left[\frac{(\theta(t) - \theta(\ell))^{\beta-1}}{\pi(t)} E_{\delta,\beta}\left[\mu(\theta(t) - \theta(\ell))^{\delta}\right]\right] \\
&= D^{\delta,\theta(t)}_{\ell^+,\pi(t)} \left[\frac{(\theta(t) - \theta(\ell))^{\beta-1}}{\pi(t)} \sum_{i=0}^{\infty} \frac{\mu^i(\theta(t) - \theta(\ell))^{\delta i}}{\Gamma(\delta i + \beta)}\right] \\
&= \sum_{i=0}^{\infty} \frac{\mu^i}{\Gamma(\delta i + \beta)} D^{\delta,\theta(t)}_{\ell^+,\pi(t)} \left[\frac{(\theta(t) - \theta(\ell))^{\delta i+\beta-1}}{\pi(t)}\right] \\
&= \mu \frac{(\theta(t) - \theta(\ell))^{\beta-1}}{\pi(t)} \sum_{i=1}^{\infty} \frac{\mu^{i-1}(\theta(t) - \theta(\ell))^{\delta(i-1)}}{\Gamma(\delta(i-1) + \beta)} \\
&= \mu m(t).
\end{aligned}$$

This completes the proof of the lemma. □

3.4.1. Estimates on weighted R-L fractional derivatives at extreme points

Lemma 3.13. *Let $G \in C^{\pi}_{\beta,\theta}([t_0, H], \mathbb{R})$, such that π is a positive function in $L^{\infty}((t_0, H))$. Assume that G is locally Hölder continuous with respect to θ at $t^* \in (t_0, H]$ and exponent $\vartheta > 1-\beta$. If t^* satisfies*

$$G(t^*) = 0 \quad and \quad G(t) \leq 0 \quad for \quad t_0 \leq t \leq t^*, \tag{3.54}$$

then it follows that

$$D^{\delta,\theta(t)}_{\pi(t)} G(t^*) \geq 0, \tag{3.55}$$

where $0 < \delta < 1$ *and* $\beta = 1 - \delta$.

Proof. From (2.6), it is clear that

$$D^{1,\theta(t)}_{\pi(t)} \left[\frac{K(t)}{\pi(t)} \right] = \frac{1}{\theta'(t)\pi(t)} \frac{dK(t)}{dt}. \tag{3.56}$$

Thus, according to (2.7), we find that

$$\begin{aligned} (D^{\delta,\theta(t)}_{\ell^+,\pi(t)} G)(t) &= (D^{1,\theta(t)}_{\pi(t)} \mathfrak{I}^{1-\delta,\theta(t)}_{\ell^+,\pi(t)} G)(t) \\ &= D^{1,\theta(t)}_{\pi(t)} \left[\frac{1}{\Gamma(\beta)\pi(t)} \int_{t_0}^{t} (\theta(t) - \theta(s))^{\beta-1} \theta'(s)\pi(s)G(s)ds \right] \\ &= \frac{1}{\Gamma(\beta)\pi(t)\theta'(t)} \frac{d}{dt} \int_{t_0}^{t} (\theta(t) - \theta(s))^{\beta-1} \theta'(s)\pi(s)G(s)ds, \end{aligned}$$

we set, $K(t) = \int_{t_0}^{t} (\theta(t) - \theta(s))^{\beta-1} \theta'(s)\pi(s)G(s)ds$.

Consider the following for a small $\eta > 0$:

$$\begin{aligned} K(t^*) - K(t^* - \eta) &= \int_{t_0}^{t^*-\eta} [(\theta(t^*) - \theta(s))^{\beta-1} - (\theta(t^* - \eta) - \theta(s))^{\beta-1}] \theta'(s)\pi(s)G(s)ds \\ &+ \int_{t^*-\eta}^{t^*} (\theta(t^*) - \theta(s))^{\beta-1} \theta'(s)\pi(s)G(s)ds. \\ &= \Delta_1 + \Delta_2. \end{aligned}$$

Since $t_0 \leq s \leq t^* - \eta$ and $\beta - 1 < 0$, then from (3.54), we obtain

$$[(\theta(t^*) - \theta(s))^{\beta-1} - (\theta(t^* - \eta) - \theta(s))^{\beta-1}] < 0 \quad and \quad G(s) \leq 0.$$

Thus implying that $\Delta_1 \geq 0$. Therefore,

$$K(t^*) - K(t^* - \eta) \geq \int_{t^*-\eta}^{t^*} (\theta(t^*) - \theta(s))^{\beta-1} \theta'(s)\pi(s)G(s)ds = \Delta_2.$$

Since $G(t)$ is locally Hölder continuous with respect to θ and exponent ϑ, there exists a real number $N(t^*) > 0$ such that for $t^* - \eta \leq s \leq t^* + \eta$

$$-N(t^*)(\theta(t^*) - \theta(s))^{\vartheta} \leq G(t^*) - G(s) \leq N(t^*)(\theta(t^*) - \theta(s))^{\vartheta},$$

where $0 < \vartheta < 1$ is such that $\vartheta > 1-\beta$. Knowing that π is a positive function, then by (3.54) we have

$$\begin{aligned}\Delta_2 &\geq -N(t^*)\|\pi\|_{L^\infty}\int_{t^*-\eta}^{t^*}(\theta(t^*)-\theta(s))^{\beta-1+\vartheta}\theta'(s)ds\\ &= -\frac{N(t^*)\|\pi\|_{L^\infty}(\theta(t^*)-\theta(t^*-\eta))^{\beta+\vartheta}}{\beta+\vartheta}.\end{aligned}$$

Hence, for sufficiently small $\eta > 0$

$$\frac{K(t^*)-K(t^*-\eta)}{\eta} \geq -\frac{N(t^*)\|\pi\|_{L^\infty}}{\beta+\vartheta}\left(\frac{\theta(t^*)-\theta(t^*-\eta)}{\eta}\right)^{\beta+\vartheta}\eta^{\beta+\vartheta-1}.$$

Letting $\eta \to 0$, we obtain $\frac{d}{dt}G(t^*) \geq 0$, which implies $D^{\delta,\theta(t)}_{\pi(t)}G(t^*) \geq 0$, and the proof is complete. □

3.4.2. Comparison theorems

Theorem 3.14. *Let $Y, Z \in C^{\pi}_{\beta,\theta}([t_0,H],\mathbb{R})$, such that π is a positive function in $L^\infty((t_0,H))$ and $\theta \in C^1$ be a strictly increasing function on $[t_0,H]$, $\kappa \in C([t_0,H]\times\mathbb{R},\mathbb{R})$. Assume that Y, Z are locally Hölder continuous with respect to θ for respectively an exponent ϑ_1 and ϑ_2 in $]0,1]$ such that $\min\{\vartheta_1,\vartheta_2\}+\beta > 1$ and*

(D1) $D^{\delta,\theta(t)}_{\pi(t)}Y(t) \leq \kappa(t,Y(t))$,

(D2) $D^{\delta,\theta(t)}_{\pi(t)}Z(t) \geq \kappa(t,Z(t)), \quad t_0 < t \leq H$,

one of the inequalities (D1) or (D2) being strict. Then

$$Y^0 < Z^0, \tag{3.57}$$

where $Y^0 = Y(t)\pi(t)(\theta(t)-\theta(t_0))^{1-\delta}|_{t=t_0}$ and $Z^0 = Z(t)\pi(t)(\theta(t)-\theta(t_0))^{1-\delta}|_{t=t_0}$, implies

$$Y(t) < Z(t), \quad t_0 \leq t \leq H. \tag{3.58}$$

Proof. Assume that the conclusion (3.58) is not true. Then, since $Y^0 < Z^0$ and $Y(t)\pi(t)(\theta(t)-\theta(t_0))^{1-\delta}$, $Z(t)\pi(t)(\theta(t)-\theta(t_0))^{1-\delta}$ are continuous functions, there exists a ξ such that $t_0 < \xi \leq H$

$$Y(\xi) = Z(\xi) \quad \text{and} \quad Y(t) < Z(t) \quad t_0 \leq t < \xi. \tag{3.59}$$

Define $G(t) = Y(t) - Z(t)$, $t \in [t_0,H]$. Then, we find that $G(\xi) = 0$ and $G(t) < 0 \quad t_0 \leq t < \xi$, with $G \in C^{\pi}_{\beta,\theta}([t_0,H],\mathbb{R})$. Hence by Lemma 3.13, we obtain

$$D^{\delta,\theta(t)}_{\pi(t)}G(\xi) \geq 0.$$

This gives

$$D^{\delta,\theta(t)}_{\pi(t)}Y(\xi) \geq D^{\delta,\theta(t)}_{\pi(t)}Z(\xi).$$

Suppose that the inequality $(D2)$ is strict, then we have

$$\kappa(\xi,Y(\xi)) \geq D^{\delta,\theta(t)}_{\pi(t)}Y(\xi) \geq D^{\delta,\theta(t)}_{\pi(t)}Z(\xi) > \kappa(\xi,Z(\xi)),$$

which is a contradiction with $Y(\xi) = Z(\xi)$. Hence, the conclusion (3.58) is valid and the proof is complete. □

The next result is for non-strict fractional differential inequalities, which demand a Lipschitz-type condition.

Theorem 3.15. *Assume that the condition of Theorem 3.14 holds with non-strict inequalities* $(D1)$ *and* $(D2)$*. Further, assume that* κ *satisfies the Lipschitz condition*

$$\kappa(t,U)-\kappa(t,V)\leq\rho(U-V),\quad U\geq V\quad and\quad \rho>0. \tag{3.60}$$

Then, $Y^0\leq Z^0$*, implies*

$$Y(t)\leq Z(t),\quad t_0\leq t\leq H. \tag{3.61}$$

Proof. For small h, we define

$$Z_h(t)=Z(t)+h\Lambda(t), \tag{3.62}$$

where $\Lambda(t)=\pi^{-1}(t)(\theta(t)-\theta(t_0))^{\delta-1}E_{\delta,\delta}[2\rho(\theta(t)-\theta(t_0))^{\delta}]$, with $Z_h\in C^{\pi}_{\beta,\theta}([t_0,H],\mathbb{R})$.

It follows from this

$$Z_h(t)\pi(t)(\theta(t)-\theta(t_0))^{1-\delta}|_{t=t_0}=Z(t)\theta(t)(\theta(t)-\theta(t_0))^{1-\delta}|_{t=t_0}+h\Lambda(t)\pi(t)(\theta(t)-\theta(t_0))^{1-\delta}|_{t=t_0}.$$

So, we obtain, $Z_h^0=Z^0+h\Lambda^0$. This leads to

$$Z_h^0>Z^0\geq Y^0\quad and\quad Z_h(t)>Z(t). \tag{3.63}$$

Next, by applying the Lipschitz condition (3.60) and Lemma 3.12 (with $\mu=2\rho,\beta=\delta$ and $\ell=t_0$), we deduce

$$\begin{aligned} D^{\delta,\theta(t)}_{\pi(t)}Z_h(t) &= D^{\delta,\theta(t)}_{\pi(t)}Z(t)+hD^{\delta,\theta(t)}_{\pi(t)}\Lambda(t)\\ &\geq \kappa(t,Z(t))+2h\rho\Lambda(t)\\ &> \kappa(t,Z_h(t)-\rho h\Lambda(t)+2h\rho\Lambda(t)\\ &> \kappa(t,Z_h(t)),\quad t_0<t\leq H. \end{aligned}$$

Therefore,

$$D^{\delta,\theta(t)}_{\pi(t)}Z_h(t)>\kappa(t,Z_h(t)),\quad t_0\leq t\leq H.$$

In this case, we have made use of the fact that $\Lambda(t)$ is the linear weighted weighted Riemann-Liouville fractional differential equation

$$D^{\delta,\theta(t)}_{\pi(t)}\Lambda(t)=2\rho\Lambda(t),\quad t_0<t_1\leq H\qquad \Lambda(t)\pi(t)(\theta(t)-\theta(t_0))^{1-\delta}|_{t=t_0}=\Lambda^0>0.$$

Utilizing (3.63), we can apply Theorem 3.14 to $Y(t)$ and $Z_h(t)$. As a result, we have

$$Y(t)<Z_h(t),\quad t\in[t_0,H],\quad \varepsilon>0. \tag{3.64}$$

By taking the limit as $h\to 0$, in the above inequality and using (3.62), we deduce that

$$Y(t)\leq Z(t),\quad t\in[t_0,H].$$

Hence, then the proof is complete. □

4. Conclusions

In this paper, we establish the equivalence between a nonlinear initial value problem and a Volterra integral equation. Furthermore, we discussed the existence and uniqueness of the solution for this initial value problem, along with specifying the space in which this solution exists. Finally, we obtained estimates on the weighted Riemann-Liouville fractional derivatives at extreme points, which were used to develop the comparison results. This work has opened new horizons for us to expand the theory of comparison for the weighted Caputo fractional operators with respect to another function. Our forthcoming focus will be directed towards these intriguing aspects in the near future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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