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*Research article*

# Hyper-instability of Banach algebras

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**Abstract:** In this paper, we introduce and study the concept of hyper-instability as a strong version of multiplicative instability. This concept provides a powerful tool to study the multiplicative instability of Banach algebras. It replaces the condition of the iterated limits in the definition of multiplicative instability with conditions that are easier to examine. In particular, special conditions are suggested for Banach algebras that admit bounded approximate identities. Moreover, these conditions are preserved under isomorphisms. This enlarges the class of studied Banach algebras. We prove that many interesting Banach algebras are hyper-unstable, such as $C^*$-algebras, Fourier algebras, and the algebra of compact operators on Banach spaces, each under certain conditions.

**Keywords:** Banach algebra; multiplicative stability; hyper-instability; Fourier algebra; Fourier-Stieltjes algebra; $C^*$-algebra
**Mathematics Subject Classification:** 47L10 , 46L05, 46J10, 46J99

## 1. Introduction

Stability theory has caught the attention of mathematicians in many areas, such as in model theory and functional analysis. In particular, in the early 80's, J.-L. Krivine and B. Maurey introduced the concept of stable Banach spaces. This stability has a significant impact on the geometry of such spaces. They proved that any separable infinite-dimensional stable Banach space contains a copy of $l_p$ for some $p \in [1, \infty)$ almost isometrically [1, Théorème IV.1]. Recently, S. Ferri and M. Neufang introduced the notion of multiplicative stability of Banach algebras as an analogue of stability of Banach spaces in Krivine-Maurey's sense, to which they refer to as additive stability. They studied the multiplicative stability of various well-known Banach algebras, such as Fourier algebras and $C^*$-algebras. As a part of a PhD thesis, we investigated some properties of additive and multiplicative stability of Banach algebras. Further, we introduced and studied the hyper-instability of Banach algebras. Moreover, inspired by function spaces on topological semigroups, we defined weakly almost periodic, almost periodic, and tame algebras. This yields a dynamical hierarchy of Banach algebras, a new classification providing different dividing lines between Banach algebras. In this paper, we present the notion of hyper-instability and some of our results. Other topics will be discussed in a separate paper, and for details, see [2]. Throughout the following, by a locally compact space we mean a locally compact Hausdorff space. We follow the usual notations for the classical sequence spaces $l_\infty$, $c_0$, and $l_p$, $p \in [1,\infty)$, as well as the $L_p$ spaces on locally compact spaces, $p \in [1,\infty]$.

First, recall that an algebra $\mathcal{A}$ over $\mathbb{C}$ is a complex vector space with a multiplication, which turns $\mathcal{A}$ into a ring and satisfies $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for all $\alpha \in \mathbb{C}$ and $a, b \in \mathcal{A}$ . A Banach algebra is an algebra $\mathcal{A}$ over $\mathbb{C}$ with a norm $\|\cdot\|$ that turns $\mathcal{A}$ into a Banach space and is submultiplicative, i.e., $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is unital if it has an identity element, denoted by $e$, and we require that $\|e\| = 1$. A bounded net $(e_i)_{i\in I}$ in a Banach algebra $\mathcal{A}$ is called a bounded left approximate identity (BLAI) if $\lim_i \|e_i a - a\| = 0 \ \forall a \in \mathcal{A}$. A bounded right approximate identity (BRAI) is defined in an obvious way. A bounded net $(e_i)_{i\in I}$ is called a bounded approximate identity (BAI) if it is both a left and a right approximate identity.

Let $X$ be a locally compact space. We denote by $C_b(X)$, $C_0(X)$, and $C_c(X)$ the algebras of all continuous complex-valued functions on $X$ that are bounded, vanish at infinity, and have compact support, respectively. Algebra operations are the usual pointwise addition, multiplication, and scalar multiplication. Equipped with the supremum norm (sup-norm), i.e., $\|f\|_\infty = \sup_{x\in X} |f(x)| \ \forall f \in C_b(X)$, the algebra $C_b(X)$ is a unital commutative Banach algebra with the identity element being the constant function 1. Further, $C_0(X)$ is a closed subalgebra of $C_b(X)$, which is nonunital unless $X$ is compact. However, it has a BAI that consists of compactly supported functions. Moreover, the algebra $C_c(X)$ is complete only if $X$ is compact. In this case we have $C_b(X) = C_0(X) = C_c(X)$ and we denote them by $C(X)$. An important class of Banach algebras is the class of $C^*$-algebras. A Banach algebra $\mathcal{A}$ is called a $C^*$-algebra if it is equipped with an involution function $* : \mathcal{A} \longrightarrow \mathcal{A}$ which satisfies $(a+b)^* = a^* + b^*$, $(\lambda a)^* = \bar{\lambda} a^*$, $(ab)^* = b^* a^*$, $(a^*)^* = a$, $\|a^* a\| = \|a\|^2$ for all $a, b \in \mathcal{A}$, $\lambda \in \mathbb{C}$. For more about the Banach algebra and $C^*$-algebra theory, we refer to [3].

Next, we recall the definitions of multiplicative and additive stability of Banach algebras as given in [4].

**Definition 1.1.** *Let $\mathcal{A}$ be a Banach algebra. We say that $\mathcal{A}$ is* ***additively (multiplicatively) stable*** *if the following condition holds:*

*For any bounded sequences $(a_n), (b_m)$ in $\mathcal{A}$ and any free ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, we have*

$$\lim_{m,\mathcal{V}} \lim_{n,\mathcal{U}} \|a_n + b_m\| = \lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} \|a_n + b_m\|, \tag{1.1}$$

$$(\lim_{m,\mathcal{V}} \lim_{n,\mathcal{U}} \|a_n b_m\| = \lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} \|a_n b_m\|). \tag{1.2}$$

*Otherwise, $\mathcal{A}$ is called* ***additively (multiplicatively) unstable****.*

**Example 1.2.** (i) $l_p$, $1 \leq p < \infty$ is additively and multiplicaively stable. The first stability was obtained in [1, 278], while the latter was proved in [2, Theorem 4.1.4] in a more general setup, as we showed that the $l_p$-direct sum of multiplicatively stable Banach algebras is multiplicatively stable. Another proof is also provided in the same reference on page 54 after Remark 4.1.2.

(ii) Another example of additively and multiplicatively stable Banach algebra is the abstract Segal algebra that was introduced first in [5, p. 4]. Choose $\xi \in l_1$ with $\|\xi\|_2 = 1$. Define a new product on $l_2$ as follows:

$$a.b = \langle a, \xi \rangle b \quad \forall a, b \in l_2,$$

where $\langle ., . \rangle$ denotes the inner product in $l_2$. Let $\mathcal{A}, \mathcal{B}$ denote $l_2, l_1$ with this new product, respectively. Then $(\mathcal{B}, \|.\|_1)$ is an abstract Segal algebra with respect to $\mathcal{A}$. In addition, it is noncommutative and nonunital. Moreover, a direct calculation of the two iterated limits shows it is multiplicatively stable. However, $\mathcal{B}$ cannot contain a subalgebra isomorphic to $l_p$ for any $p \in [1, \infty)$. Indeed, if there would exist a subalgebra $\mathcal{A}_0 \subseteq \mathcal{B}$ which is isomorphic to $l_p$ for some $p \in [1, \infty)$, then $l_p$ would have an identity. The reason behind this is the existence of a left identity in $\mathcal{A}_0$. Indeed, let $\eta \in \mathcal{A}_0$ such that $\langle \eta, \xi \rangle \neq 0$, then

$$\frac{\eta}{\langle \eta, \xi \rangle}.b = b \quad \forall b \in \mathcal{A}_0.$$

Hence, $\dfrac{\eta}{\langle \eta, \xi \rangle}$ is a left identity of $\mathcal{A}_0$. Thus, $l_p$ has an identity - a contradiction. Note that $\mathcal{A}_0$ must have such an element with a nonzero inner product, or otherwise the product on $\mathcal{A}_0$ is the zero product. For details, see [2, Example 4.1.5]. This example shows that there is no analogue of Krivine-Maurey's famous theorem in the case of multiplicatively stable Banach algebras, at least not without additional assumptions.

Note that a model-theoretical version of the stability of Banach algebras has been studied by Farah, Hart, and Sherman in [6, 7]. Stability, in this sense, implies both additive and multiplicative stability of Banach algebras. For example, it is noted in [7] before Proposition 6.2 that $l_p$ for $p \in [1, \infty)$ with pointwise multiplication is stable in the model theory's sense, and so it is multiplicativity stable.

An important example of a multiplicatively unstable Banach algebra is any separable unital infinite-dimensional $C^*$-algebra. It has been proved in [6, Lemma 5.3] that such a $C^*$-algebra is not stable in the model-theoretical sense. In the proof, the authors used the formula $\phi(x, y) = \|xy - y\|$ to witness instability through violation of the double limit criterion. Moreover, it has been shown in [4, Proposition 2.2] that any infinite-dimensional $C^*$-algebra is multiplicatively unstable.

## 2. Hyper-instability of Banach algebras

In this section, we introduce hyper-instability of Banach algebras, which provides a powerful tool to study multiplicative instability of Banach algebras. Studying multiplicative instability of Banach algebras relies on the possibility of constructing two bounded sequences in the Banach algebra such that the two iterated limits are different. However, conditions that are easier to examine may replace the condition of the iterated limits. By carefully studying the proofs of the multiplicative instability of the examples provided in [4], we note some common properties of the constructed sequences, such as using the bounded approximate identity of the Fourier algebra and a sequence that behaves like a bounded approximate identity in $C^*$-algebras. Moreover, one of the iterated limits in both examples equals zero. This gives the motivation to define the hyper-instability of Banach algebras. It turns out that hyper-instability implies multiplicative instability, and it is preserved under isomorphisms. This increases the number of studied Banach algebras. We start with the following proposition.

**Proposition 2.1.** *Let $\mathcal{A}$ be a Banach algebra. Assume that there exist two bounded nets $(e_i)_{i\in I}, (f_j)_{j\in J}$ in $\mathcal{A}$, where $I, J$ are directed sets and two ultrafilters $\mathcal{U}, \mathcal{V}$ which dominate the order filters on $I, J$, respectively, and $C > 0$ such that:*

*(1)* $\forall i \in I, \quad \lim_{j,\mathcal{V}} \|e_i f_j\| = 0$;

*(2)* $\forall j \in J, \quad \lim_{i,\mathcal{U}} \|e_i f_j\| \geq C$.

*Then, $\mathcal{A}$ is multiplicatively unstable.*

*Proof.* First, note that condition (1) implies $\lim_{i,\mathcal{U}} \lim_{j,\mathcal{V}} \|e_i f_j\| = 0$.

On the other hand, we get by (2) that $\lim_{j,\mathcal{V}} \lim_{i,\mathcal{U}} \|e_i f_j\|$ cannot be zero since $C > 0$. Thus, $\mathcal{A}$ is multiplicatively unstable. □

**Definition 2.2.** *A Banach algebra that satisfies the conditions in Proposition 2.1 is called hyper-unstable.*

Note that the conditions in Proposition 2.1 can be simplified if the Banach algebra admits a bounded approximate identity (BAI) or even a right (BRAI) or a left (BLAI) one, as we shall see now.

**Corollary 2.3.** *Let $\mathcal{A}$ be a Banach algebra. Assume that $\mathcal{A}$ has a BLAI (BRAI) $(e_i)_{i\in I}$ and there exist a bounded net $(f_j)_{j\in J}$ and two ultrafilters $\mathcal{U}, \mathcal{V}$ which dominate the order filters on $I, J$, respectively, such that*

*(1)* $\forall i \in I, \quad \lim_{j,\mathcal{V}} \|e_i f_j\| = 0$ $(\lim_{j,\mathcal{V}} \|f_j e_i\| = 0)$;

*(2)* $\exists C > 0$ *such that* $\forall j \in J$, $\|f_j\| \geq C$.

*Then, $\mathcal{A}$ is hyper-unstable.*

*Proof.* We prove the corollary in the case that $(e_i)_{i\in I}$ is a BLAI. The case of a BRAI follows analogously. According to Proposition 2.1, to prove hyper-instability of $\mathcal{A}$, we need to prove that condition (2) holds. Now, since $(e_i)_{i\in I}$ is a BLAI, then $\forall j \in J$ we have

$$\lim_i \|e_i f_j - f_j\| = 0.$$

Moreover, $\forall i \in I$, $j \in J$ we have

$$|\, \|e_i f_j\| - \|f_j\| \,| \leq \|e_i f_j - f_j\|, \text{ so } \lim_i \|e_i f_j\| = \|f_j\| \geq C > 0.$$

As $\mathcal{U}$ dominates the order filter on $I$, the limits along the order filter and along $\mathcal{U}$ are equal. Therefore, $\forall j \in J$ we have

$$\lim_{i,\mathcal{U}} \|e_i f_j\| = \|f_j\| \geq C > 0.$$

□

In fact, all that we need from the existence of a BLAI is the last limit in the previous proof. The next corollary states it formally and gives weaker conditions than Corollary 2.3.

**Corollary 2.4.** *Let $\mathcal{A}$ be a Banach algebra. Assume that there exist two bounded nets $(e_i)_{i\in I}, (f_j)_{j\in J}$ and two ultrafilters $\mathcal{U}, \mathcal{V}$ which dominate the order filters on $I, J$, respectively, such that*

*(1)* $\forall i \in I, \quad \lim_{j,\mathcal{V}} \|e_i f_j\| = 0$*;*

*(2)* $\forall j \in J, \quad \lim_{i,\mathcal{U}} \|e_i f_j\| = \|f_j\|$*;*

*(3)* $\exists C > 0$ *such that* $\forall j \in J,\ \|f_j\| \geq C$.

*Then, $\mathcal{A}$ is hyper-unstable.*

It is not difficult to prove that hyper-instability is preserved under isomorphisms, see [2, Proposition 4.3.7]. Clearly, hyper-instability implies multiplicative instability. But, we do not know if the converse holds.

## 3. Examples of hyper-unstable Banach algebras

Next, we study the hyper-instability of many well-known Banach algebras. Of course, this implies multiplicative instability, but we emphasize hyper-instability because we gain the same for any isomorphic Banach algebra.

### 3.1. $C^*$-algebras

The first example is any infinite-dimensional $C^*$-algebra. Since the two sequences that were constructed in [4, Proposition 2.2] satisfy the conditions of Proposition 2.1, any such $C^*$-algebra is hyper-unstable. Moreover, any $C^*$-algebra admits a BAI, which, in the case of $C_0(X)$, the Banach algebra of all complex-valued continuous functions on a locally compact space $X$ that vanish at infinity can be realized as a net of functions $(f_K)_{K\in\mathfrak{B}}$, where $\mathfrak{B}$ is the set of all compact subsets of $X$; here, using Urysohn's Lemma for locally compact spaces, one defines for each $K \in \mathfrak{B}$ the function $f_K$ as being of compact support, equal to 1 on $K$ and vanishing outside a neighborhood of $K$. Hence, one may construct another net that satisfies the conditions of Corollary 2.3 to obtain another proof of hyper-instability of infinite-dimensional $C^*$-algebras. Thus, we get the following.

**Proposition 3.1.** *Every infinite-dimensional $C^*$-algebra is hyper-unstable.*

### 3.2. Fourier and Fourier-Stieltjes algebras

Next we study Fourier and Fourier–Stieltjes algebras on a locally compact group. Recall that on a locally compact group $G$, the Fourier algebra, $A(G)$, is defined to be the set of all functions of the form $f * \tilde{g}$, where $f, g \in L_2(G)$ and for all $X \in G$ we have $f * \tilde{g}(x) = \int f(xy)\overline{g(y)}dy$. Note that $A(G) \subseteq C_0(G)$. When $A(G)$ is endowed with the norm

$$\|u\| = \inf\{\|f\|_2\|g\|_2 : u = \langle \lambda_G(\cdot)f, g\rangle, f, g \in L_2(G)\}\ \forall u \in A(G),$$

and with pointwise multiplication, it becomes a commutative Banach algebra.

The Fourier–Stieltjes algebra, $B(G)$, consists of all functions of the form $x \mapsto \langle \pi(x)\xi, \eta\rangle$, where $\pi$ is a unitary representation of $G$ on some Hilbert space $\mathcal{H}$ and $\xi, \eta \in \mathcal{H}$. With the norm

$$\|u\| = \sup\{|\int f(x)u(x)dx| : f \in L_1(G), \|f\|_{C^*} \leq 1\},$$

and with pointwise multiplication, $B(G)$ is a unital commutative Banach algebra. Moreover, $B(G)$ contains $A(G)$ as a closed ideal. Note that the Fourier algebra $A(G)$ is unital if and only if $G$ is compact. However, $A(G)$ may admit a BAI. Leptin's theorem shows that this is equivalent to the amenability of the group $G$, that is, the existence of a functional $m \in L_\infty(G)^*$ such that $\langle m, 1\rangle = \|m\| = 1$, and for all $f \in L_\infty(G), g \in G$, $\langle m, {}_gf\rangle = \langle m, f\rangle$, where ${}_gf(x) = f(gx)\ \forall x \in G$. Moreover, a BAI can be chosen to consist of compactly supported functions bounded by 1, see [8, Theorem 2.7.2].

In [4, Theorem 2.3], it was proved that Fourier and Fourier-Stieltjes algebras are multiplicatively unstable on a certain class of locally compact groups. We provide a much similar proof to show that they are, in fact, hyper-unstable on such groups.

**Theorem 3.2.** *Let $G$ be a locally compact group containing a non-compact amenable open subgroup. Then the Fourier algebra $A(G)$ is hyper-unstable.*

*Proof.* Let $H$ be a non-compact amenable open subgroup of $G$. Then, by [8, Proposition 2.4.1], $A(H)$ can be identified with a closed subalgebra of $A(G)$. The aim is to prove the hyper-instability of $A(H)$. Note that

$$H = \bigcup_{K\in\mathfrak{B}} K^\circ, \quad \mathfrak{B} := \{K \subseteq H : K \text{ is compact}, K^\circ \neq \emptyset\}.$$

Then $\mathfrak{B}$ can be directed by inclusion, so $\forall K_1, K_2 \in \mathfrak{B},\ K_1 \leq K_2$ iff $K_1 \subseteq K_2$. Now, for each $K \in \mathfrak{B}$, pick $x_K \in H \setminus K$. This is possible due to the non-compactness of $H$. Choose a neighborhood $V_K$ of $x_K$ such that $V_K \cap K = \emptyset$. As in the proof of Lemma 2.9.5 in [9], we define a net of functions $(f_K)_{K\in\mathfrak{B}}$ as follows. For $K \in \mathfrak{B}$, take a compact symmetric neighborhood $W_K$ of the identity such that $x_K W_K^2 \subseteq V_K$. Define

$$f_K := \frac{1}{|W_K|}\chi_{x_K W_K} * \check{\chi}_{W_K}.$$

Then $f_K \in A_c(H)$, where $A_c(H)$ is the set of functions in $A(H)$ with compact supports. Moreover, $\mathrm{supp} f_K \subseteq x_K W_K^2 \subseteq V_K$. Also, $\|f_K\| = 1$; indeed,

$$1 = |f_K(x_K)| \leq \|f_K\|_\infty \leq \|f_K\| \leq \frac{1}{|W_K|}\|\chi_{x_K W_K}\|_2\|\check{\chi}_{W_K}\|_2 = 1.$$

In particular, $(f_K)_{K\in\mathfrak{B}}$ is a bounded net in $A_c(H)$.

Furthermore, since $H$ is amenable, there exists a BAI $(e_i)_{i\in I}$ in $A(H)$ which consists of compactly supported functions; cf. [8, Theorem 2.7.2]. Fix $i$ in $I$. Since $\mathrm{supp}\, e_i$ is compact, it is contained in $K_0$ for some $K_0 \in \mathfrak{B}$. Now, since for any $K \in \mathfrak{B}$, $\mathrm{supp}\, f_K \subseteq V_K$ and $V_K \cap K = \emptyset$, we have $K_0 \cap V_K = \emptyset\ \forall K \geq K_0$. Thus, $\mathrm{supp}\, f_K \cap \mathrm{supp}\, e_i = \emptyset$, which implies that $e_i f_K = 0$, and

$$\lim_{K\in\mathfrak{B}} \|e_i f_K\| = 0 \quad \forall i \in I.$$

Then, by using Corollary 2.3, we obtain the hyper-instability of $A(H)$, as claimed. □

**Corollary 3.3.** *For a locally compact group $G$ containing a non-compact amenable open subgroup, the Fourier-Stieltjes algebra $B(G)$ is hyper-unstable.*

**Example 3.4.** (i) In view of Theorem 3.2, the Fourier algebra on any non-compact amenable group is hyper-unstable.

(ii) The theorem also covers many non-amenable groups, for instance, any discrete group that contains an infinite amenable subgroup. A notable example of such a group is the free group on two generators $\mathbb{F}_2$, which is non-amenable.

(iii) As any locally compact abelian group is amenable, our theorem also applies to locally compact non-compact abelian groups. Hence, $A(\mathbb{R})$ and $A(\mathbb{Z})$ are hyper-unstable.

Along these lines, we note that, as the next corollary shows, hyper-instability of the group algebra $L_1(G)$ on a large class of groups can be obtained.

**Corollary 3.4.** *For any locally compact non-discrete abelian group $G$, the group algebra $L_1(G)$ is hyper-unstable.*

The proof follows from the fact that $L_1(G)$ is isometrically isomorphic to $A(\widehat{G})$ as Banach algebras via the Fourier transform, where $\widehat{G}$ is the dual group of $G$. Direct examples are $L_1(\mathbb{T})$ and $L_1(\mathbb{R})$. This implies their instability in the model-theoretical sense. For $L_1(\mathbb{R})$, the latter result was established first in [7].

In [4, Theorem 2.5], multiplicative instability of measure algebras was obtained by identifying measure algebras and Fourier-Stieltjes algebras on locally compact abelian groups. In fact, a similar argument gives hyper-instability of these algebras.

**Theorem 3.5.** *Let $G$ be an infinite compact group. Then the measure algebra $M(G)$ is hyper-unstable.*

For proofs, see [4, Theorem 2.5] and [2, Theorem 4.4.6].

### *3.3. Multiplier and completely bounded multiplier algebras of Fourier algebras*

In this section, not going far from the Fourier algebra, we consider related Banach algebras, namely, the algebra of multipliers and the algebra of completely bounded multipliers of the Fourier algebra on a locally compact group $G$, denoted by $M(A(G))$ and $M_{cb}(A(G))$, respectively. We shall prove that weaker assumptions are needed on $G$ than what we assumed in the case of $A(G)$ to prove hyper-instability of $M_{cb}(A(G))$. Recall that the multiplier algebra of the Fourier algebra $A(G)$ on a locally compact group $G$ consists of bounded continuous functions $f$ on $G$ satisfying $fA(G) \subseteq A(G)$. The multiplier norm of $f$ is given by

$$\|f\|_{M(A(G))} = \sup\{\|fu\|_{A(G)} : u \in A(G), \|u\|_{A(G)} \leq 1\}.$$

With this norm and the pointwise product, $M(A(G))$ is a commutative Banach algebra. We can associate with $f \in M(A(G))$ the operator $T_f$ on $A(G)$ and its adjoint $T_f^*$, where $T_f$ is defined by $T_f(g) = fg\ \forall g \in A(G)$, see [8, Proposition 5.1.2]. Moreover, $\|f\|_{M(A(G))} = \|T_f\| = \|T_f^*\|$. One of the subalgebras of $M(A(G))$ of interest is the algebra of completely bounded multipliers, that is, the set of multipliers $f$ of $A(G)$ in which the operator $T_f^*$ is completely bounded. With pointwise multiplication and the norm

$$\|f\|_{M_{cb}(A(G))} = \|T_f^*\|_{cb}\ \forall f \in M_{cb}(A(G)),$$

$M_{cb}(A(G))$ is a commutative Banach algebra. Further, $B(G) \subseteq M_{cb}(A(G)) \subseteq M(A(G))$, and the inclusion maps are contractive, see [10, 509]; in particular, we have

$$\|x\|_{M(A(G))} \leq \|x\|_{M_{cb}(A(G))} \leq \|x\|_{A(G)} \quad \forall x \in A(G).$$

Before stating the theorem, recall the definition of weakly amenable groups. Cowling and Haagerup first introduced this notion in [10]. A locally compact group $G$ is said to be weakly amenable if there exists a net $(u_i)_{i\in I}$ in $A(G)$ such that

$$\|u_i\|_{M_{cb}(A(G))} \leq L \text{ for some } L > 0,$$

$$\lim_i u_i = 1 \text{ uniformly on compacta.}$$

Cowling and Haagerup showed in their paper that weak amenability of a locally compact group $G$ implies the existence of an approximate identity in $A(G)$ with compact supports, which is bounded in the $M_{cb}(A(G))$ norm, see [10, Proposition 1.1].

**Theorem 3.6.** *Let $G$ be a locally compact group containing a non-compact open weakly amenable subgroup. Then $M_{cb}(A(G))$, the Banach algebra of completely bounded multipliers of $A(G)$, is hyper-unstable.*

*Proof.* Let $H$ be a non-compact open weakly amenable subgroup of $G$. Then, by [11, Proposition 4.1], $M_{cb}(A(H))$ is a closed subalgebra of $M_{cb}(A(G))$, and hence it is enough to prove hyper-instability of $M_{cb}(A(H))$. Now, the weak amenability of $H$ ensures the existence of an approximate identity in $A(H)$ with compact supports, which is bounded in the $M_{cb}(A(H))$ norm. Thus, there exists a net $(u_i)_{i\in I}$ in $A_c(H)$ with the properties

$$\|u_i\|_{M_{cb}(A(H))} \leq L \text{ for some } L > 0,$$

$$\lim_i \|vu_i - v\|_{A(H)} = 0 \quad \forall v \in A(H).$$

Since $\|\cdot\|_{M_{cb}(A(H))} \leq \|\cdot\|_{A(H)}$, we have

$$\lim_i \|vu_i - v\|_{M_{cb}(A(H))} = 0 \quad \forall v \in A(H).$$

This implies that

$$\lim_i \|vu_i\|_{M_{cb}(A(H))} = \|v\|_{M_{cb}(A(H))} \quad \forall v \in A(H).$$

Now put $\mathfrak{B} := \{K \subseteq H : K \text{ compact}, K^\circ \neq \emptyset\}$. Then $(\mathfrak{B}, \leq)$ is a directed set, where $\leq$ is the inclusion relation. The second net $(f_K)_{K\in\mathfrak{B}}$ in $A_c(H)$, bounded in $\|\cdot\|_{M_{cb}(A(H))}$, can be chosen by following steps as in the proof of Theorem 3.2 such that

$$\|f_K\|_{A(H)} = \|f_K\|_\infty = 1 \quad \forall K \in \mathfrak{B} \quad \text{and} \quad \lim_{K\in\mathfrak{B}} \|u_i f_K\|_{A(H)} = 0 \quad \forall i \in I.$$

Now, since $A(H) \subseteq M_{cb}(A(H))$, the nets $(u_i)_{i\in I}$ and $(f_K)_{K\in\mathfrak{B}}$ satisfy the conditions of Corollary 2.4. However, we need to prove that $(f_K)_{K\in\mathfrak{B}}$ has a nonzero lower bound in the $M_{cb}(A(H))$ norm. To this end, first recall that

$$\|f\|_{M(A(H))} \leq \|f\|_{M_{cb}(A(H))} \leq \|f\|_{A(H)} \quad \forall f \in A(H).$$

Moreover, we have $\|f\|_{M(A(H))} = \|T_f\|$, where $T_f$ is the map on $A(H)$ such that $T_f(g) = fg, \ \forall g \in A(H)$. Thus, for all $K \in \mathfrak{B}$,

$$\|f_K\|_{M(A(H))} = \|T_{f_K}\| = \sup_{\|g\|_{A(H)} \leq 1} \|f_K g\|_{A(H)}.$$

Since $\|f_K\|_{A(H)} = 1$, we have

$$\|f_K\|_{M(A(H))} \geq \|f_K^2\|_{A(H)} \geq \|f_K^2\|_\infty = 1.$$

So,

$$1 \leq \|f_K\|_{M(A(H))} \leq \|f_K\|_{M_{cb}(A(H))} \leq \|f_K\|_{A(H)} = 1.$$

Thus, for all $K \in \mathfrak{B}$, $\|f_K\|_{M_{cb}(A(H))} = 1$, which proves the claim. □

The argument given above also shows the following.

**Theorem 3.7.** *For any locally compact group $G$ which contains a non-compact open weakly amenable subgroup, the Banach algebra $M(A(G))$ of multipliers of $A(G)$ is hyper-unstable.*

**Example 1.2.** (i) Obviously, by the above, $M_{cb}(A(G))$ and $M(A(G))$ are hyper-unstable on any non-compact locally compact weakly amenable group $G$; in particular, if $G$ is a non-compact locally compact amenable group. Note that the latter result can be deduced from Theorem 3.2 as, in this case, we have $B(G) = M_{cb}(A(G)) = M(A(G))$ isometrically.

(ii) It is known that $\mathbb{F}_n$, the free group of $n$ generators ($n \geq 2$), is non-amenable but weakly amenable; cf. [12, Corollary 3.9]. Hence, $M_{cb}(A(\mathbb{F}_n))$ and $M(A(\mathbb{F}_n))$ are hyper-unstable.

### *3.4. Banach algebras of compact operators on a Banach space*

We explore the multiplicative stability of another important class of Banach algebras, namely the algebra $\mathcal{K}(E)$ of compact operators on a Banach space $E$. We prove that $\mathcal{K}(E)$ is hyper-unstable in two cases. The first case is when $E$ contains a complemented basic sequence, i.e., if there exists a sequence $(e_n)$ in $E$ which is a Schauder basis of $F := \overline{span}\{e_n\}$ and $F$ is complemented in $E$; in particular, any Banach space with a Schauder basis. The second case is when $E$ contains a subspace $F$ such that $F$ has a Schauder basis and $F^{**} = E$. An example of this case is $E = l_\infty$.

**Theorem 3.8.** *Let $E$ be a Banach space with a complemented basic sequence. Then the algebra of compact operators $\mathcal{K}(E)$ is hyper-unstable.*

*Proof.* Let $(e_n)$ be a complemented basic sequence in $E$. Take $F := \overline{span}\{e_n\}$. Since $F$ is complemented in $E$, there exists a bounded projection $P : E \twoheadrightarrow F$. As $(\frac{e_n}{\|e_n\|})$ is also a Schauder basis of $F$, we may assume that $(e_n)$ is normalized. Let $(e_n^*)$ be the biorthogonal functionals associated with $(e_n)$ and $K_b$ be the basis constant. Define $\tilde{e}_n^* : E \longrightarrow \mathbb{C}$ by $\tilde{e}_n^* = e_n^* \circ P$. Then

$$\|\tilde{e}_n^*\| \leq \|e_n^*\|\|P\| \leq 2K_b\|P\|.$$

Further, $P(x) = \sum_{k=1}^{\infty} \langle \tilde{e}_k^*, x \rangle e_k \quad \forall x \in E$. To prove the hyper-instability of $\mathcal{K}(E)$, we construct two sequences that satisfy the hypothesis of Corollary 2.4. First, take $a_n = \sum_{k=1}^{n} e_k \otimes \tilde{e}_k^*$. Then

$$a_n = \sum_{k=1}^{n} e_k \otimes (e_k^* \circ P) = (\sum_{k=1}^{n} e_k \otimes e_k^*) \circ P.$$

Since $\sum_{k=1}^{n} e_k \otimes e_k^*$ is uniformly bounded by $K_b$, we have $\|a_n\| \leq K_b \|P\|$. Hence, $(a_n)$ is a bounded sequence in $\mathcal{K}(E)$. For the other sequence, put $b_m := e_m \otimes \tilde{e}_m^* \ \forall m \in \mathbb{N}$. Then

$$\|b_m\| = \|e_m\| \|\tilde{e}_m^*\| = \|\tilde{e}_m^*\| \leq 2K_b \|P\| \quad \forall m \in \mathbb{N}.$$

Hence, $(b_m)$ is bounded in $\mathcal{K}(E)$. Fix $m$ in $\mathbb{N}$. We calculate $\lim_{n\to\infty} \|a_n b_m\|$. To this end, let $x \in E$. Then

$$\begin{aligned} a_n b_m(x) &= a_n(e_m \otimes \tilde{e}_m^*(x)) = a_n(e_m \otimes e_m^*(P(x))) = a_n(\langle e_m^*, P(x) \rangle e_m) \\ &= \langle e_m^*, P(x) \rangle \sum_{k=1}^{n} \langle \tilde{e}_k^*, e_m \rangle e_k = \langle e_m^*, P(x) \rangle \sum_{k=1}^{n} \langle e_k^*, e_m \rangle e_k. \end{aligned} \tag{3.1}$$

But, as $\langle e_i^*, e_j \rangle = \delta_{i,j}$, if $n \geq m$, we get that

$$a_n b_m(x) = \langle e_m^*, P(x) \rangle e_m = \langle e_m^*, \sum_{k=1}^{\infty} \langle \tilde{e}_k^*, x \rangle e_k \rangle e_m = \langle \tilde{e}_m^*, x \rangle e_m = b_m(x).$$

Thus, for $n \geq m$, $a_n b_m = b_m$ and so $\lim_{n\to\infty} \|a_n b_m\| = \|b_m\|$. Moreover, as $\|e_m\| = 1$, we have

$$\|b_m\| = \|e_m\| \|\tilde{e}_m^*\| = \|\tilde{e}_m^*\| \geq |\langle \tilde{e}_m^*, e_m \rangle| = |\langle e_m^*, e_m \rangle| = 1.$$

On the other hand, fix $n$ and let $m > n$. Let $x \in E$. We have $\langle \tilde{e}_n^*, e_m \rangle = \langle e_n^*, e_m \rangle = 0$. Hence, by applying this to (3.1), we get that $a_n b_m(x) = 0$. Thus,

$$a_n b_m = 0, \text{ so } \lim_{m\to\infty} \|a_n b_m\| = 0.$$

Therefore, by Corollary 2.4, $\mathcal{K}(E)$ is hyper-unstable. □

**Corollary 3.9.** *If $E$ is a Banach space containing a complemented subspace isomorphic to $c_0$ or $l_p$ for any $p \in [1, \infty)$, then $\mathcal{K}(E)$ is hyper-unstable.*

Note that, as $\mathcal{K}(E)$ is a closed subalgebra of $\mathcal{B}(E)$, under the hypothesis of the previous theorem we gain hyper-instability of $\mathcal{B}(E)$ as a bonus. In addition, the constructed sequences, in fact, consist of finite-rank operators. Hence, they lie in the Banach algebra of approximable operators $\mathcal{A}(E)$ on $E$. This leads to the next result.

**Theorem 3.10.** *Let $E$ be a Banach space with a complemented basic sequence, then $\mathcal{B}(E)$ and $\mathcal{A}(E)$ are hyper-unstable.*

As it has been shown, the existence of a complemented basic sequence in $E$ is sufficient for the hyper-instability of $\mathcal{K}(E)$; however, it is not necessary. This can be illustrated by $\mathcal{K}(l_\infty)$. In fact, $l_\infty$ is a prime Banach space, i.e., every infinite-dimensional complemented subspace of $l_\infty$ is isomorphic to $l_\infty$. This well-known result due to Lindenstrauss can be found in [13] as the main theorem of the paper. So, $l_\infty$ does not contain any complemented basic sequence. But, $\mathcal{K}(l_\infty)$ is hyper-unstable. Indeed, $\mathcal{K}(E)$ is hyper-unstable if $E$ contains a subspace that admits a Schauder basis, and whose second dual is $E$.

**Theorem 3.11.** *Let $E$ be a Banach space and $F$ a closed subspace of $E$ such that $F$ has a Schauder basis and $F^{**} = E$. Then $\mathcal{K}(E)$ is hyper-unstable.*

*Proof.* Let $(e_n)$ be a normalized Schauder basis of $F$. Let $P_n = \sum_{k=1}^{n} e_k \otimes e_k^*$ be the canonical projections associated with $(e_n)$, and $K_b$ be the basis constant. Consider the adjoint operators $P_n^* : F^* \longrightarrow F^*$ , $P_n^{**} : E \longrightarrow E$. Let $y \in F, y^* \in F^*$, and $x \in E$. We have

$$P_n(y) = \sum_{k=1}^{n} \langle e_k^*, y\rangle e_k , \quad P_n^*(y^*) = \sum_{k=1}^{n} \langle e_k, y^*\rangle e_k^*.$$

Further, $\langle P_n^{**}(x), y^*\rangle = \sum_{k=1}^{n} \langle x, e_k^*\rangle\langle e_k, y^*\rangle$. Hence, $P_n^{**}(x) = \sum_{k=1}^{n} \langle x, e_k^*\rangle e_k$. So, $P_n^{**}$ may be written as $P_n^{**} = \sum_{k=1}^{n} e_k \otimes e_k^*$, and one may see the operator $e_k \otimes e_k^*$ as a rank one operator on $E$. See the first paragraph of the proof of Proposition 4.14 in [14]. Since $\|P_n^{**}\| = \|P_n^*\| = \|P_n\|$, the sequence $(P_n^{**})$ is uniformly bounded by $K_b$. Now, take

$$a_n = \sum_{k=1}^{n} e_k \otimes e_k^* , \quad b_m = e_m \otimes e_m^*.$$

For $n, m \in \mathbb{N}$, we have $\|a_n\| = \|P_n\| \leq K_b$, and $\|b_m\| = \|e_m\|\|e_m^*\| = \|e_m^*\| \leq 2K_b$. Hence, $(a_n)$ and $(b_m)$ are bounded in $\mathcal{K}(E)$. Next, let $x \in E$, then

$$a_n b_m(x) = a_n(\langle x, e_m^*\rangle e_m) = \langle x, e_m^*\rangle \sum_{k=1}^{n} \langle e_m, e_k^*\rangle e_k.$$

Since $\langle e_m, e_n^*\rangle = 0$ for $m \neq n$ , if $m > n$, we have $a_n b_m(x) = 0 \ \forall x \in E$. Thus,

$$a_n b_m = 0, \text{ so } \lim_{m\to\infty} \|a_n b_m\| = 0.$$

On the other hand, if $m \leq n$, then

$$a_n b_m(x) = \langle x, e_m^*\rangle e_m = e_m \otimes e_m^*(x) = b_m(x).$$

Moreover, $\|b_m\| = \|e_m^*\| \geq |\langle e_m^*, e_m\rangle| = 1$. Thus,

$$\lim_{n\to\infty} \|a_n b_m\| = \|b_m\| \quad \text{and} \quad \|b_m\| \geq 1 \quad \forall m \in \mathbb{N}.$$

Therefore, hyper-instability of $\mathcal{K}(E)$ follows by Corollary 2.4. □

*3.5. The Banach algebras $C_0^k(\mathbb{R})$ and $C_b^k(\mathbb{R})$*

In the present section we consider the Banach algebras $C_b^k(\mathbb{R})$ and $C_0^k(\mathbb{R})$, $k \in \mathbb{N}$, consisting of all complex-valued $k$ times continuously differentiable functions $f$ on $\mathbb{R}$ such that the function $f$ and its derivatives $f^{(j)}$ are bounded, respectively, vanish at infinity, where $1 \leq j \leq k$. With pointwise addition and multiplication, scalar multiplication, and the norm

$$\|f\| = \sum_{j=0}^{k} \frac{1}{j!} \|f^{(j)}\|_\infty \quad \forall f \in C_b^k(\mathbb{R}),$$

$C_b^k(\mathbb{R})$ is a Banach function algebra on $\mathbb{R}$ and $C_0^k(\mathbb{R})$ is a closed ideal in $C_b^k(\mathbb{R})$; cf. [15, Proposition 2, Proposition 3]. It follows from the same article (Theorem 1) that the algebra $C_0^k(\mathbb{R})$ has a bounded approximate identity consisting of functions with compact support. We show that $C_0^1(\mathbb{R})$ is hyper-unstable as we construct a sequence which, along with the BAI, satisfies the conditions of Corollary 2.3. Moreover, we generalize the result to $C_0^k(\mathbb{R})$.

**Theorem 3.12.** *The Banach algebras $C_0^1(\mathbb{R})$ and $C_b^1(\mathbb{R})$ are hyper-unstable.*

*Proof.* It is enough to show the hyper-instability of $C_0^1(\mathbb{R})$. By [15, Theorem 1], $C_0^1(\mathbb{R})$ has a BAI $(e_n)_{n\in\mathbb{N}}$ such that

$$e_n(x) = 1 \quad \forall x \in [-n, n], \quad \text{and} \quad \operatorname{supp} e_n \subseteq [-n-2, n+2],$$

see the first paragraph of the proof of the mentioned theorem. Now, choose a function $f$ in $C_0^1(\mathbb{R})$ such that

- $\operatorname{supp} f \subseteq [4, 5]$;
- $f(4) = f(5) = 0, \quad f'(4) = f'(5) = 0$;
- $f' \not\equiv 0$,

(e.g., put $f(x) = (x-4)^2(x-5)^2 \ \forall x \in [4, 5]$ and 0 otherwise). For all $m \in \mathbb{N}$, define

$$f_m : \mathbb{R} \longrightarrow \mathbb{C}, \ f_m(x) = f(x - m + 1),$$

i.e., $f_m$ is a translation of $f$. Note that $\operatorname{supp} f_m \subseteq [m+3, m+4]$ and

$$f_m(m+3) = f_m(m+4) = 0, \ \ f_m'(m+3) = f_m'(m+4) = 0.$$

Moreover, for all $m \in \mathbb{N}$, we have

$$\|f_m\| = \|f_m\|_\infty + \|f_m'\|_\infty = \|f\|_\infty + \|f'\|_\infty > 0.$$

In particular, $(f_m)$ is a bounded sequence in $C_0^1(\mathbb{R})$, and $\|f_m\| = C > 0 \ \forall m \in \mathbb{N}$, where $C = \|f\|_\infty + \|f'\|_\infty$. Now, fix $n \in \mathbb{N}$. As

$$\operatorname{supp} e_n \subseteq [-n-2, n+2], \quad \text{and} \quad \operatorname{supp} f_m \subseteq [m+3, m+4],$$

we have $\operatorname{supp} e_n \cap \operatorname{supp} f_m = \emptyset$ for all $m \geq n$. Thus, $e_n f_m = 0 \ \ \forall m \geq n$, which implies that $\lim_{m\to\infty} \|e_n f_m\| = 0$. Therefore, the claim holds by Corollary 2.3. □

**Remark 3.13.** Generally, $C_0^k(\mathbb{R})$ for $k \in \mathbb{N}$ is hyper-unstable. This can be proved by using, for example, the sequence of translations by $2n - 2$ of the function

$$f(x) = \begin{cases} x^{k+1}(x-1)^{k+1} & \text{if } x \in [0,1] \\ 0 & \text{if } x \notin [0,1] \end{cases},$$

i.e.,

$$e_n(x) = f(x - 2n + 2) \quad \forall x \in \mathbb{R},\ n \in \mathbb{N},\ \text{and}$$

$$f_m = \sum_{k=1}^{m} e_k \quad \forall m \in \mathbb{N}.$$

## 4. Conclusions

Our work introduces the concept of hyper-instability, which offers an alternative technique to study the multiplicative instability of Banach algebras. The conditions, which do not include iterated limits, are more applicable and straightforward to examine. Furthermore, these conditions can be simplified if the Banach algebra admits a bounded approximate identity (BAI) or even a right or a left one. In addition, hyper-instability implies multiplicative instability and is preserved under isomorphisms between Banach algebras. By examining the conditions of hyper-instability, we proved that infinite-dimensional $C^*$-algebras, Fourier and Fourier-Stieltjes algebras on a locally compact group, the algebra of compact operators on a Banach space, and the Banach algebras $C_0^k(\mathbb{R})$ and $C_b^k(\mathbb{R})$ are hyper-unstable under certain conditions. This new concept opens up numerous ventures for further research problems. Some of these problems are mentioned next.

- Is hyper-instability strictly stronger than multiplicative instability?
- Theorem 3.2 shows hyper-instability of $A(G)$ when $G$ is a locally compact group containing a non-compact, open, amenable subgroup, particularly when G itself is non-compact and amenable. What about other groups?
- Corollary 3.4 shows that $L_1(G)$ is hyper-unstable for any locally compact, non-discrete, abelian group $G$. What about other groups?

## Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The author declares no conflicts of interest in this paper.

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