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Research article

Local linear estimation for the censored functional regression

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Abstract: This work considers the Local Linear Estimation (LLE) of the conditional functional mean. This regression model is used when the independent variable is functional , and the dependent one is a censored scalar variable. Under standard postulates, we establish the asymptotic distribution of the LLE by proving its asymptotic normality. The obtained results show the superiority of the LLE approach over the functional local constant one. The feasibility of the studied model is demonstrated using artificial data. Finally, the usefulness of the obtained asymptotic distribution in incomplete functional data is highlighted through a real data application.

Keywords: regression function; functional data; asymptotic normality; local linear estimation; Kaplan-Meier estimator
Mathematics Subject Classification: 62R20, 62G05, 62G08

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1. Introduction

Evaluating the correlation between two random variables is a fundamental issue in mathematical statistics. Often this issue is analyzed by the regression function. The main objective of this paper is to use the LLE method to investigate the nonparametric conditional mean with regard to censored response variables when the explanatory variable takes values in a semi-metric space. Furthermore, the LLE approach has been used to interpolate many functional models (see [12, 13, 24] for some examples). This approach achieves better results than the kernel estimator. For an overview on the kernel method, the reader can refer to [16, 18, 23]. We return to [1] for more recent advances in the kernel method.

In reliability and survival analysis, the study of censored data is particularly useful because it permits the coverage of cases when the variable is incompletely observed. This latter has received the attention of many academics (see [7]). We refer also to [26] for the estimation of the conditional survival function in the nonparametric framework. In the case of multivariate data, many studies have explored the estimation of the conditional mean function (see [9] for some references). The authors of this cited work have used the LLE approach for the relative error mean. They established the uniform almost sure convergence of the constructed estimator. Meanwhile, the authors of [10] have examined the asymptotic normality in the dependent case. In the context of data with a functional nature, this topic has received much attention. The point-wise and the almost complete convergence (ACC) of the estimator of the conditional mean was studied in [22]. It considers the case involving a functional explanatory variable with a censored response. Rahmani and Bouanani [25] obtained the asymptotic normality of the conditional cumulative distribution function in the i.i.d. case. Recently, [6] described the convergence of the conditional density estimation. In the dependence case , [15] obtained the rate of the point-wise ACC of the regression estimate. We return to [27] for the conditional density function, where the authors have studied the asymptotic normality of the local linear constant.

This paper studies the LLE of the conditional expectation in the incomplete functional data case. We suppose that the input variable belongs to an infinite-dimensional space and the output exhibits a censoring feature. We state the limit distribution of the estimator by proving its asymptotic normality of the constructed estimator. We point out that the challenging issue of this subject is the fact that the LLE is defined with a sum of double index , unlike the classical kernel case , where the estimator is defined with only one index. Additionally to this feature , the incomplete functional observation make s the establishment of the normality asymptotic more difficult. Thus, the statement of the limit distribution of our LLE in the incomplete functional data case requires a special decomposition of the estimator , allowing us to characterize a specific leading term that satisfies Lindeberg's central limit. Finally, the impact of the censoring phenomena on the estimation quality is evaluated via simulated data and real data examples to show that the LLE estimator has more advantages over the local constant one in this incomplete functional data situation.

The remaining sections of this paper are structured as follows: In Section 2, we define the LLE regression estimator for our model. In Section 3, we give the needed conditions and the main results. In Section 4, we give an application to build a confidence interval. The demonstrations are detailed in the last Section.

2. The model and the estimator

We define a random couple (X, Y) where Y is valued in $\mathbb{R}$ and X has values in a space of infinite dimensional $\mathcal{F}$ with a semi-metric d. We point out that in practice the choice of the semi-metric is closely linked to the basis functions used to generate the functional space $\mathcal{F}$. Of course the choice of the basis functions is also based on the degree of smoothing property of the functional data. In the sense that if the functional data is very smooth we use the semi-metric associated with the Fourier or the spline basis functions. However, if the functional is discontinuous the semi-metric based on the principal component analysis is more adequate. Now, to construct the LLE estimator we consider $(X_i, Y_i)_{i=1,\dots,n}$ a sequence of independent and identically distributed as (X, Y).

For the complete data, the LLE method's estimator for the regression [2] of Y given $X = x$ is constructed by assuming that the conditional mean $m(\cdot)$ is approximate by a linear function in the neighborhood of x

$$m(.) = A_x + B_x\delta(X, x) + o(\ell(\cdot, x)).$$

The functions $\delta(.,.)$ and $\ell(.,.)$ are known from $\mathcal{F}\times\mathcal{F}$ into $\mathbb{R}$, where $d(.,.) = |\ell(.,.)|$. Next the estimation of A_x and B_x are solutions of

$$\hat{A}_x, \hat{B}_x = \arg\min_{a,b} \sum_{i=1}^{n} (Y - a - b\delta(X_i, x))^2\, L(h^{-1}\ell(X_i, x)).$$

$L(.)$ is the kernel function and h is the bandwidth. We suppose that $\delta(x, x) = 0$ which permit s to deduce that $A_x = m(x)$ and the LLE of $m(x)$ is

$$\hat{A}_x = m_{LL}(x) = \frac{\sum_{1\le j\le n} \Omega_j Y_j L(h^{-1}\ell(X_j, x))}{\sum_{1\le j\le n} \Omega_j L(h^{-1}\ell(X_j, x))}, \qquad \forall y \in \mathbb{R},$$

with

$$\Omega_j = \sum_{1\le i\le n} \delta^2(X_i, x)L(h^{-1}\ell(X_i, x)) - \left(\sum_{1\le i\le n} \delta(X_i, x)L(h^{-1}\ell(X_i, x))\right)\delta(X_j, x).$$

Censorship model:

We define a sequence of random variables $Y_1, Y_2, \dots Y_n$ of unidentified and continuously distributed function F. Next, we generate $C_1, C_2, ..., C_n$ n-observations of C called censoring time. We assume that the sample $C_1, C_2, \dots, C_n$ have also an unknown cumulative $\mathcal{G}$. The censoring phenomena means that Y is observed only when $Y \le C$. In this situation, we proceed with a sample $(X_i, T_i, \ell_i)_{i\in\mathbb{N}}$ of $(X, T = Y \wedge C, \ell)$, $\wedge$ denotes the minimum. The censoring status is controlled by the quantity $\ell = 1_{\{Y\le C\}}$. Moreover, by simple analytical argument, we prove that

$$E\left[\frac{\ell}{\bar{\mathcal{G}}(T)}T|X\right] = E[Y|X].$$

Thus it suffices to replace Y with $\frac{\ell}{\bar{\mathcal{G}}}(Y)Y$ to construct the LLE nonparametric pseudo estimator of $m(.)$ in the censored case:

$$\widetilde{S}(x) = \frac{\sum_{1\le j\le n} \Omega_j T_j L_j \ell_j \bar{\mathcal{G}}^{-1}(T_j)}{\sum_{1\le j\le n} \Omega_j L_j},$$

with

$$\delta_i = \delta(X_i, x) \text{ and } \quad L_i = L(h^{-1}\ell(X_i, x))\ quad i = 1, \dots n.$$

The latter can be rewritten as follows

$$\widetilde{S}(x) = \frac{\frac{1}{n(n-1)\mathbb{E}[\Omega_1 L_1]} \sum_{1\le j\le n} \Omega_j T_j L_j \ell_j \bar{\mathcal{G}}^{-1}(T_j)}{\frac{1}{n(n-1)\mathbb{E}[\Omega_1 L_1]} \sum_{1\le j\le n} \Omega_j L_j} =: \frac{\widetilde{m}_N^x}{\widehat{S}_D^x}.$$

Recall that in practice, $\mathcal{G}$ is typically unidentified. So, we replace it by the Kaplan and Meier [19] estimator $\mathcal{G}_n(.)$ defined by

$$\bar{\mathcal{G}}_n(t) = 1 - \mathcal{G}_n(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \frac{1-\ell_i}{n-i+1}\right)^{1_{\{T_i \le t\}}} & \text{if } t < T_{(n)} \\ 0 & \text{if } t \ge T_{(n)}, \end{cases}$$

where $T_{(1)} \le T_{(2)} \le ... \le T_{(n)}$ are the statistics order of T.

Therefore,

$$\widehat{S}(x) = \frac{\dfrac{1}{n(n-1)\mathbb{E}[\Omega_1 L_1]} \displaystyle\sum_{1 \le j \le n} \Omega_j T_j L_j \ell_j \bar{\mathcal{G}}_n^{-1}(T_j)}{\widehat{S}_D^x} =: \frac{\widehat{S}_N^x}{\widehat{S}_D^x}.$$

We specify the extremities of F and $\mathcal{G}$ by

$$\tau_G = \sup\{y, \bar{\mathcal{G}}(y) > 0\}, \quad \text{and} \quad \tau_F = \sup\{y, \bar{F}(y) > 0\},$$

and we assume that $\bar{\mathcal{G}}(\tau_F) > 0$ (this implies $\tau_F < \tau_G < +\infty$).

3. Assumptions

For any fixed $x \in \mathcal{F}$ and S is a compact subset of $]-\infty, \tau]$. Let $h \ge 0$, $\psi_x(h) = \mathbb{P}(x' \in B(x,h)) = \mathbb{P}(x' \in \mathcal{F}, 0 < d(x', x) < h)$, we use the notation for C and C' positive numeric values. We establish the following conditions for our main result.

(A1) $\forall h > 0$, $\psi_x(h) > 0$.

$$\forall t \in [-1, 1], \lim_{h \to 0} \frac{\psi_x(-h, th)}{\psi_x(h)} = \psi(t).$$

(A2) The regression satisfies the Lipschitz condition with respect to x and y such that

$$\exists C, a_1 > 0,\ \forall (x_1, x_2) \in S_{\mathcal{F}}^2,\ \forall (y_1, y_2) \in S^2, \quad |m(x_1) - m(x_2)| \le C d(x_1, x_2)^{a_1}.$$

(A3) The bi-function $\delta(.,.)$ satisfies

(i) $\forall x' \in \mathcal{F},\ C'|\ell(x, x')| \le |\delta(x, x')| \le C|\ell(x, x')|$.

(ii) $\sup_{s \in B(x,v)} |\delta(x, s) - \ell(x, s)| = o(v)$.

(A4) L is a positive kernel and bounded with support $[-1, 1]$.

(A5) The bandwidth h satisfies

(i) There is a positive integer n_0 for which

$$\frac{1}{\psi_x(h)} \int_{-1}^{1} \psi_x(zh, h) \frac{\mathrm{d}}{\mathrm{d}z} \left(z^2 L(z)\right) \mathrm{d}z > C_3 > 0, \quad \text{for } n > n_0,$$

and

$$h \int_{B(x,h)} \delta(u, x) \mathrm{d}\mathbb{P}(u) = o\left(\int_{B(x,h)} \delta^2(u, x) \mathrm{d}\mathbb{P}(u)\right),$$

where $\mathbb{P}(u)$ is the cumulative distribution of X.

(ii) $\lim_{n\to\infty} h = 0, \quad \lim_{n\to\infty} \frac{\log n}{n\psi(h)} = 0, \quad \lim_{n\to\infty} n\psi(h) = \infty.$

(A6) (i) The moment $V(X_1) = \mathbb{E}\left(\left(\left[\bar{\mathcal{G}}(Y_1)\right]^{-1}\right) Y_1^2 \mid X_1\right) < +\infty$ exists.

(ii) $(C_n)_{n\geq 1}$ and $(X_n, Y_n)_{n\geq 1}$ are independent .

(iii) $\exists \tau < \tau_F, \ T_i \leqslant \tau, \quad \forall i \in n.$

4. The main result

We need to introduce specific notations to derive the asymptotic normality result.

$$M_j = L^j(1) - \int_{-1}^{1} \left(L^j(u)\right)' \psi(u)\mathrm{d}u, \quad \text{where } j = 1, 2.$$

$$N(a, b) = L^a(1) - \int_{-1}^{1} \left(u^b L^a(u)\right)' \psi(u)\mathrm{d}u, \quad \text{for all } a > 0 \text{ and } b = 2, 4.$$

Asymptotic normality:

Here is our main result, where the theorem below deals with the asymptotic normal distribution of the LLE regression.

Theorem 1. *Assuming that* (A1)–(A6) *hold, we have for n large enough:*

$$\left(\frac{n\psi_x(h)}{\sigma^2(x)}\right)^{1/2} \left(\widehat{S}(x) - m(x) - B(x)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1),$$

where $\xrightarrow{\mathcal{D}}$ *indicates the convergence in terms of distribution, and*

$$\sigma(x) = \frac{M_2}{M_1^2}\left(v(x) - m(x)^2\right) \text{ and } B(x) = \frac{\mathbb{E}\left(\widetilde{m}_N^x\right)}{\mathbb{E}\left(\widehat{S}_D^x\right)} - m(x).$$

Proof of Theorem 1. The proof utilizes the following decomposition and subsequent results:

$$\begin{aligned} \widehat{S}(x) - m(x) - B(x) &= \frac{\widetilde{m}_N^x - m(x)\widehat{S}_D^x - \mathbb{E}\left(\widetilde{m}_N^x - m(x)\widehat{S}_D^x\right)}{\widehat{S}_D^x} + \frac{\widehat{S}_N^x - \widetilde{m}_N^x + \mathbb{E}\left(\widetilde{m}_N^x\right) - m(x)}{\widehat{S}_D^x} - B(x) \\ &= \frac{M_{n,1} + M_{n,2}}{\widehat{S}_D^x}, \end{aligned} \tag{4.1}$$

where

$$M_{n,1} = \widetilde{m}_N^x - m^x\widehat{S}_D^x - \mathbb{E}\left(\widetilde{m}_N^x - m(x)\widehat{S}_D^x\right)$$

and

$$M_{n,2} = \widehat{S}_N^x - \widetilde{m}_N^x + B(x)(1 - \widehat{S}_D^x).$$

Following this, the proof is a direct result of Lemmas 1 and 2. □

Lemma 1. *In accordance with the postulates of Theorem 1, we find*

$$\sqrt{n\psi_x(h)}(M_{n,1}) \xrightarrow{D} \mathcal{N}(0, \sigma(x)), \quad \text{as } n \longrightarrow +\infty.$$

Lemma 2. *In accordance with the postulates of Theorem 1, we find*

$$M_{n,2} \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \longrightarrow +\infty, \tag{4.2}$$

where $\xrightarrow{\mathbb{P}} 0$ *indicates convergence in probability.*

5. The model's application to confidence intervals

If we use the method of plug-in and replace ψ_x, M_j and σ by their estimates $\widehat{\psi_x}, \widehat{S}_j$ and $\widehat{\sigma}$, where $j = 1, 2$,

$$\widehat{\psi_x}(h) = \frac{\sharp\,(i : |\ell\,(X_i, x)| \leq h)}{n}, \quad \widehat{S}_j = \frac{1}{n\widehat{\psi_x}\,(h)} \sum_{i=1}^{n} L^j \left(\frac{|\ell\,(X_i, x)|}{h} \right), \quad \widehat{\sigma} = \frac{\widehat{S}_2}{\widehat{S}_1^2} (\widehat{\nu} - m(x)^2),$$

where $\sharp\{.\}$ is the cardinal number and $\widehat{\nu} = \left(\mathbb{E} \left(\left(\left[\bar{\mathcal{G}}_n\,(Y_1) \right]^{-1} \right) Y_1^2 \mid X_1 \right) \right)$. When we put $\lim_{n\to\infty} \sqrt{n\psi_x(h)} B(x) = 0$, then, the bias term can be removed. Therefore, we obtain the following Corollary:

Corollary 1. *Under postulates* $(A1)$–$(A6)$, *we obtain*

$$\sqrt{n\widehat{\psi_x}(h)}\,(\hat{m}(x) - m(x))\,\frac{\widehat{S}_1}{\widehat{\sigma}(x)\sqrt{\widehat{S}_2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

From the corollary, it follows that the following confidence interval

$$\widehat{S}(x) \pm t_{1-\eta/2} \times \frac{\widehat{\sigma}(x)}{\sqrt{n\widehat{\psi_x}(h)}},$$

where $t_{1-\eta/2}$ signified the $1 - \eta/2$ standard normal quantile (for each fixed $\eta \in (0, 1)$).

6. Artificial data analysis

The goal of this section is to examine the feasibility and the efficiency of the constructed estimator through a simulation study. More precisely, we aim to:

(1) Examine the easy computation-ability of the constructed estimator.
(2) Control the effect of the censorship feature on the performance of the estimator.

For these issues we generate a functional variable

$$X_i(t) = log \left(\frac{(1 + asin^2(t))}{1 + bcos^2(t)} \right), \quad t \in (0, 2\Pi)$$

where a (resp. b) is a normal random variable $N(0, 1)$ (resp. $N(0, 1)$). The shape of the curves regressor is shown in the following figure:

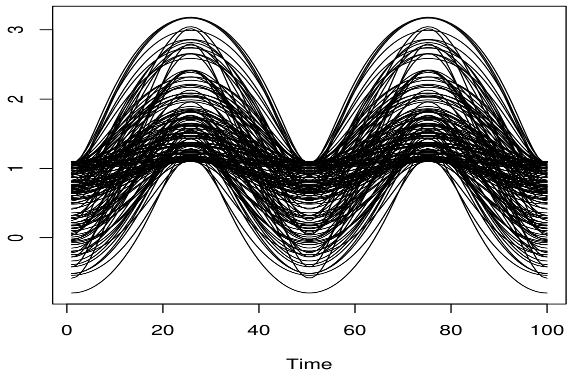

Figure 1. The functional explanatory variable.

For the response variables, we drown through the nonparametric regression model equation that is

$$Y_i = m(X_i) + \epsilon_i, \qquad i = 1, \dots n,$$

where r is the operator defined by $m(X) = \frac{1}{10}\int_0^{2\pi} \exp(X(t))dt$. and (ϵ) is a normal distribution $N(0, 0.5)$. After generating the response and the explanatory variable, we proceed to examine the easy implementation of the estimator and to quantify the impact of the censoring phenomena in the estimation. To do that, we generate a censoring variable C distributed as an exponential distribution $Exp(\lambda)$. Thus the censoring rate of this artificial study is checked by the parameter λ. Thus, we simulate with three values of λ that are $\lambda = 0.01, 0.05, 0.1$. Such values generate data with three censoring percentage s 6%, 40%, and 70%.

Furthermore, the implementation of the LLE $\widehat{S}$ is strongly related to the suppleness of the determination of the parameters of the estimator. Typically, the practical use of $\widehat{S}$ is linked to the determination of the metric the locating functions ℓ and δ the bandwidth and the kernel L. As discussed in the second section , the choice of the semi-metric in practice is closely linked to the smoothing degree of the curves. It is clear that in this situation, our curves are sufficiently smooth to use the spline basis function. For this simulation study, we have tested many metrics including the L_2 over the first, second and the tired derivative. It seems that the L_2 over the second derivative gives better results than other metrics. Thus, we choose locating functions ℓ and δ by

$$\ell(X, X') = \left(\int_0^{2\pi} (X^{(2)}(t) - X'^{(2)}(t))^2 dt\right)^{1/2} \text{ and } \delta(X, X') = \int_0^1 \theta(t)(X^{(2)}(t) - X'^{(2)}(t))dt$$

where $X^{(i)}$ denotes the i^{th} derivative of X and θ denotes the eigenfunction associated to the greatest eigenvalue of the covariance matrix

$$\frac{1}{n}\sum_{j=1}^{n}(X_j^{(i)} - \overline{X^{(i)}})^t\,(X_j^{(i)} - \overline{X^{(i)}}).$$

Next, we choose L as the quadratic kernel on $(0, 1)$, which is adequate to the technical assumptions. Now, for the bandwidth we use the cross-validation rule as follows

$$h_{CV} = \arg\min_{h_n \in \mathcal{H}_n} \sum_{i=1}^{n} (Y_i - \widehat{S})^2$$

where $\mathcal{H}_n$ is the subset of positive numbers, h_n, such that the ball centered $B(x, h_n)$ contains exactly L neighbors of x. Typically, L is selected from $\{5, 10, 20, \ldots, 0.5 * n\}$.

For this empirical analysis, we generate $n = 150$ observations of (X_i, Y_i, C_i) and compare the LLE $\widehat{S}(x)$ with the local constant estimator defined by

$$\widetilde{m}(x) = \frac{\sum_{i=1}^{n} T_i L_i \ell_i \bar{\mathcal{G}}^{-1}(T_i)}{\sum_{i=1}^{n} L_i}.$$

Recall that the local constant can be viewed as aparticular case of the LLE (refer to Barrientos-Marin et al. [2]). We compare the behavior of both estimators over different censoring levels. The comparison results are presented in Figures 2–4 where we plot $r(X_i)$ versus its estimation $\widehat{S}(X_i)$ and $\widetilde{m}(X_i)$

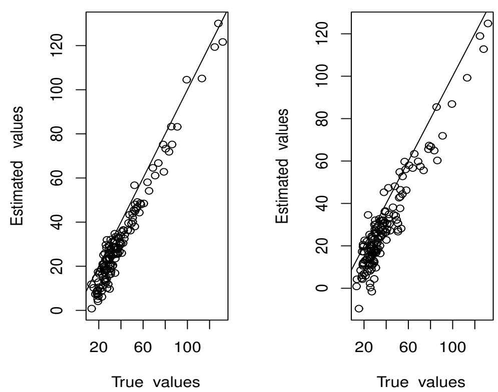

Figure 2. Case 1: The censoring rate is 6%. LLE in the left and the local constant in the right.

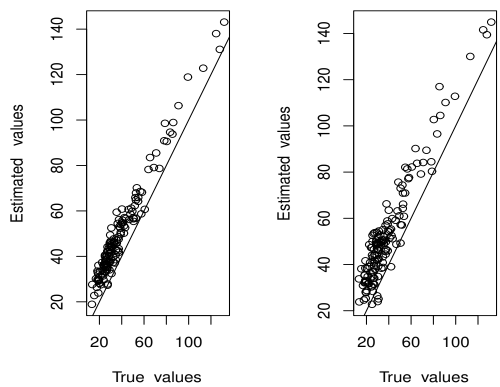

Figure 3. Case 2: The censoring rate is 40%. LLE in the left and the local constant in the right.

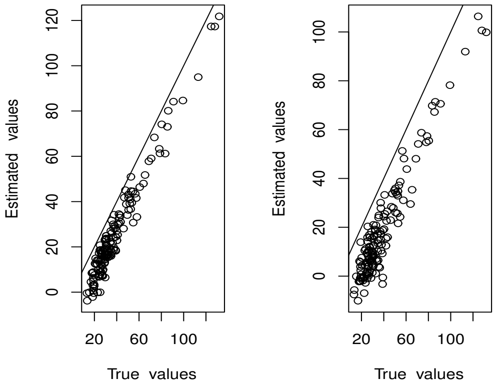

Figure 4. Case 3: The censoring rate is 70%. LLE in the left and the local constant in the right.

It appears, clearly, that the behaviors of both estimators are strongly affected by the percentage of censoring threshold. In a sense that the efficiency of estimation decreases with the level of censoring. Secondly, it is clear that the LLE approach improves upon the local constant approach which confirms the well -known theoretical results concerning the superiority of the local linear over the local constant. To highlight this statement, we report in the following table the MSE error defined by

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(m(X_i) - \bar{m}(X_i))^2$$

where $\bar{m}$ means either $\widehat{S}(X_i)$ and $\widetilde{m}(X_i)$. The latter is calculated for varied values of the sample size n.

Table 1. *MSE*-results.

Estimator	n	$\lambda = 0.01$	$\lambda = 0.05$	$\lambda = 0.1$
Local linear estimator	100	1.04	1.97	2.31
	150	0.89	1.15	2.02
	200	0.83	1.15	1.92
	250	0.73	0.94	1.41
Local constant estimator	100	1.67	2.16	2.76
	150	1.16	1.79	2.33
	200	1.07	1.21	2.09
	250	0.97	1.02	1.81

7. Real data example

Having shown the easy implantation of the LLE $\widehat{S}(X_i)$, we return to this section to apply our model to real life example. Specifically, we aim to incorporate the obtained asymptotic normality through the prediction of a future characteristic of the financial data using the confidence interval approach. For this goal, we consider a sample of financial data that records the intraday return of the stock index Nikkei during the period from October 11, 1983 to October 11, 2022. Of course this kind of time-varying financial data can be analyzed as censored data because often the stock markets are closed whenever they fall below certain index, particularly in the crisis period. In addition we recall that the stock markets were stopped many times during the COVID-19 pandemic period. On the other hand, in order to ensure the independence property of the observations we define the functional variables using the curves of the intraday in some spaced months between October 1983 and October 2022. We point out that the considered data are available from the website "https://fred.stlouisfed.org/series/NIKKEI225". Formally, the functional variable $X_i(d) = -100\log\left(\frac{r(d)}{r(d-1)}\right)$ for a day d in the selected month i. Now, to predict the maximum gain one month ahead, we proceed with a response variable $Y_i = \max_d X_{i+1}(d)$. So, in order to carry out this forecasting issue, we compare our estimator to the standard kernel estimator defined by

$$\widetilde{m}(x) = \frac{\sum_{i=1}^{n} T_i L_i \ell_i \bar{\mathcal{G}}^{-1}(T_i)}{\sum_{i=1}^{n} L_i}.$$

Recall that the asymptotic normality of this estimator can be deduced by the same procedure as a particular case of the present study. It follows that the confidence interval of the kernel regression is

$$\widetilde{m}(x) \pm t_{1-\eta/2} \times \frac{\widehat{\sigma}(x)}{\sqrt{n\widehat{\psi}_x(h)}}.$$

Now, we fix $\alpha = 0.95$, and we compare both estimators by computing the probability coverage of the two confidence interval s. The latter gives the percentage of the response variables that belongs to the confidence intervals. Furthermore, we point out that we have used the same bandwidth selector and the same kernel to calculate the estimators $\widehat{S}(x)$ and $\widetilde{m}(x)$. The shape of the curves implies that the appropriate metric for the distance calculator is the PCA metric. Formally, we assume that $\ell = \delta = d$ and are equal to the distance obtained by the standard Euclidean norm with respect to the basis of the q-eigenfunctions associated with the q largest eigenvalues of the covariance matrix. We simulate with $q = 3$. Finally, for this comparative study, we split the data many times (exactly 70 times), in a random way, into subsets (90% learning sample and 100% testing sample). Then, we compute the coverage probability for each case. In the following figure, we compare the different values of the coverage probability with the horizontal line for $\alpha = 0.95$.

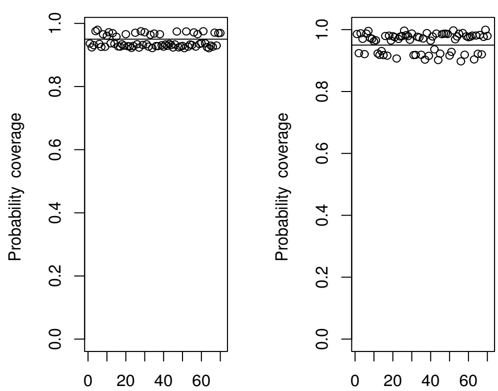

Figure 5. LLE estimator in the left and the kernel estimator in the right.

Without surprise, we can see that the LLE approach is more important than the classical kernel estimator. In the sense that the prediction by $\widehat{S}(x)$ is more accurate than the estimator $\widetilde{m}(x)$. It is clear that the points are closer to the horizontal line in the left graph than the right one.

8. Conclusions

The superiority of the LLE approach over the standard kernel estimation is a principal motivation to investigate the regression model in the incomplete functional data case. The main achievement of the present contribution is to investigate the LLE of the classical regression with a functional regressor and when the response variable is observed with censoring. In the theoretical part, we have demonstrated the asymptotic normality under some mild conditions, which are standard in functional data analysis. Furthermore, the assumed conditions explore the different structures of this topic. In particular, the first condition (A1) combines the functional nature of the data with the measurability space of the random variable. The condition (A2) explores the nonparametric path of the model, and the censoring character is controlled through condition (A6). In the practical study, we show that the LLE approach keeps its superiority over the local constant method in this context of incomplete functional data. Moreover, from the computational part, we deduce that the proposed plugin estimation of the asymptotic variance is very easy to compute, and the different parameter involved in this estimation can be selected using some practical strategy. In addition to this practical and theoretical development, the present project opens some interesting tracks for the future. For example, it will be important to study the LLE of the relative regression in the case of censored functional data. Such a prospect is motivated by the fact that the relative regression is a good alternative predictive model to the classical regression. Secondly, the local linear of the robust regression is a crucial issue in statistical forecasting . Furthermore, extending our results to the dependent case and/or functional time series cases including long memory, ergodic , and associated processes, is a very interesting open question. However, it would require nontrivial mathematical inequalities that are well beyond the scope of this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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Appendix

Proof of Lemma 1. Following the proofs of Lemma 1 of Rahmani and Bounani ([25]), we get

$$\begin{aligned}\sqrt{n\psi_x(h)}M_{n,1} = &\frac{\sqrt{n\psi_x(h)}}{n\mathbb{E}(\Omega_1 L_1)}\sum_{j=1}^{n}\Omega_j L_j\left(\ell_j\left[\bar{\mathcal{G}}\left(T_j\right)\right]^{-1}T_j - m(x)\right)\\ &-\mathbb{E}\left(\sum_{j=1}^{n}\Omega_j L_j\left(\Omega_j\left[\bar{\mathcal{G}}\left(T_j\right)\right]^{-1}T_j - m(x)\right)\right).\end{aligned} \tag{8.1}$$

It is obvious that

$$\begin{aligned}\sqrt{n\psi_x(h)}M_{n,1} = &\underbrace{\left(\frac{1}{n\mathbb{E}\left(\delta_1^2 L_1\right)}\sum_{i=1}^{n}\delta_i^2 L_i - 1\right)\frac{\sqrt{n\psi_x(h)}\mathbb{E}\left(\delta_1^2 L_1\right)}{\mathbb{E}(\Omega_1 L_1)}\sum_{j=1}^{n}L_j\left(\ell_j\left[\bar{\mathcal{G}}\left(T_j\right)\right]^{-1}T_j - m(x)\right)}_{\gamma_1}\\ &-\underbrace{\frac{\sqrt{n\psi_x(h)}\mathbb{E}\left(\delta_1^2 L_1\right)}{\mathbb{E}(\Omega_1 L_1)}\sum_{j=1}^{n}L_j\left(\ell_j\left[\bar{\mathcal{G}}\left(T_j\right)\right]^{-1}T_j - m(x)\right)}_{\gamma_2}\\ &-\left(\underbrace{\frac{1}{n\mathbb{E}(\delta_1 L_1)}\sum_{i=1}^{n}\delta_i L_i\frac{\sqrt{n\psi_x(h)}\mathbb{E}(\delta_1 L_1)}{\mathbb{E}(\Omega_1 L_1)}\sum_{j=1}^{n}\delta_j L_j\left(\ell_j\left[\bar{\mathcal{G}}\left(T_j\right)\right]^{-1}T_j - m(x)\right)}_{\gamma_3}\right)\\ &-\mathbb{E}\left[\left(\frac{1}{n\mathbb{E}\left(\delta_1^2 L_1\right)}\sum_{i=1}^{n}\delta_i^2 L_i - 1\right)\frac{\sqrt{n\psi_x(h)}\mathbb{E}\left(\delta_1^2 L_1\right)}{\mathbb{E}(\Omega_1 L_1)}\sum_{j=1}^{n}L_j\left(\ell_j\left[\bar{\mathcal{G}}\left(T_j\right)\right]^{-1}T_j - m(x)\right)\right]\\ &+\mathbb{E}\left[\frac{\sqrt{n\psi_x(h)}\mathbb{E}\left(\delta_1^2 L_1\right)}{\mathbb{E}(\Omega_1 L_1)}\sum_{j=1}^{n}L_j\left(\ell_j\left[\bar{\mathcal{G}}\left(T_j\right)\right]^{-1}T_j - m(x)\right)\right]\\ &+\mathbb{E}\left[\frac{1}{n\mathbb{E}(\delta_1 L_1)}\sum_{i=1}^{n}\delta_i L_i\frac{\sqrt{n\psi_x(h)}\mathbb{E}(\delta_1 L_1)}{\mathbb{E}(\Omega_1 L_1)}\sum_{j=1}^{n}\delta_j L_j\left(\ell_j\left[\bar{\mathcal{G}}\left(T_j\right)\right]^{-1}T_j - m(x)\right)\right].\end{aligned} \tag{8.2}$$

Let

$$\Gamma_k = \gamma_k - \mathbb{E}[\gamma_k], \quad \text{for } k \in \{1,2,3\},$$

and we put

$$\sqrt{n\psi_x(h)}M_{n,1} = \Gamma_1 + \Gamma_2 - \Gamma_3.$$

Now, we need to show the following claim

$$\Gamma_2 \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2(x)). \tag{8.3}$$

$$\Gamma_1 \xrightarrow{\mathbb{P}} 0. \tag{8.4}$$

$$\Gamma_3 \xrightarrow{\mathbb{P}} 0. \tag{8.5}$$

- First, let us calculate the variance of Γ_2

$$\begin{aligned}Var(\Gamma_2) =& \frac{n^2\psi_x(h)\mathbb{E}^2\left(\delta_1^2 L_1\right)}{\mathbb{E}^2(\Omega_1 L_1)}\left(\mathbb{E}\left(L_1^2\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1 - m(x)\right)^2\right)\right)\\&- \frac{n^2\psi_x(h)\mathbb{E}^2\left(\delta_1^2 L_1\right)}{\mathbb{E}^2(\Omega_1 L_1)}\mathbb{E}^2\left(L_1\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1 - m(x)\right)\right)\\=:&\kappa_{n,1}+\kappa_{n,2}.\end{aligned} \tag{8.6}$$

  - Discussing the term $\kappa_{n,2}$

$$\kappa_{n,2} = \mathbb{E}^2\left(L_1\left(\mathbb{E}\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1|X_1\right) - m(x)\right)\right). \tag{8.7}$$

By making use of the properties of conditional expectation and considering that $1_{Y_1\le C_1}\psi(T_1) = 1_{Y_1\le C_1}\psi(Y_1)$, where $\psi(.)$ is a measurable function and assumption $(A6_{(ii)})$, we get

$$\begin{aligned}\mathbb{E}\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1 \mid X_1\right) &= \mathbb{E}\left(1_{Y_1\le C_1}\left[\bar{\mathcal{G}}(Y_1)\right]^{-1}Y_1 \mid X_1\right)\\&= \mathbb{E}\left(\left[\bar{\mathcal{G}}(Y_1)\right]^{-1}Y_1\mathbb{E}(1_{Y_1\le C_1} \mid (Y_1,X_1)) \mid X_1\right)\\&= \mathbb{E}(Y_1 \mid X_1) = m(X_1)\end{aligned} \tag{8.8}$$

and by assumption $(A2)$,

$$\mathbb{E}\left(\ell_1\left[\bar{\mathcal{G}}(Y_1)\right]^{-1}Y_1|X_1\right) - m(x) \longrightarrow 0, \quad \text{as } n\to+\infty.$$

Then, by assumption $(A3)$ and $(A4)$ we deduce that

$$\frac{n^2\psi_x(h)\mathbb{E}^2\left(\delta_1^2 L_1\right)}{\mathbb{E}^2(\Omega_1 L_1)}\left(\mathbb{E}^2\left(L_1\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1 - m(x)\right)\right)\right) \longrightarrow 0, \quad \text{as } n\to+\infty.$$

  - Discussing the term $\kappa_{n,1}$. After a simple calculation, we get

$$\begin{aligned}\kappa_{n,1} =& \frac{n^2\psi_x(h)\mathbb{E}^2\left(\delta_1^2 L_1\right)}{\mathbb{E}^2(\Omega_1 L_1)}\left(\mathbb{E}\left(L_1^2\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1 - m(x)\right)^2\right)\right)\\=& \frac{n^2\psi_x(h)\mathbb{E}^2\left(\delta_1^2 L_1\right)}{\mathbb{E}^2(\Omega_1 L_1)}\mathbb{E}\left(L_1^2 Var\left(\ell_1\left[G(T_1)\right]^{-1}T_1|X_1\right)\right)\\&+ \frac{n^2\psi_x(h)\mathbb{E}^2\left(\delta_1^2 L_1\right)}{\mathbb{E}^2(\Omega_1 L_1)}\mathbb{E}\left(L_1^2\left(\mathbb{E}\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1|X_1\right) - m(x)\right)^2\right).\end{aligned} \tag{8.9}$$

    * About the first term of equation (8.9), we observe that

$$\mathrm{Var}\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1 \mid X_1\right) = \mathbb{E}\left(\left(\ell_1\left(\left[\bar{\mathcal{G}}(T_1)\right]^{-1}\right)Y_1\right)^2 \mid X_1\right) - \mathbb{E}\left[\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1 \mid X_1\right)\right]^2$$

we have

$$\mathbb{E}\left[\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1 \mid X_1\right)\right]^2 = \mathbb{E}\left[Y_1|X_1\right]^2 = (m(x))^2, \tag{8.10}$$

however,

$$\begin{aligned}\mathbb{E}\left(\left(\ell_1\left(\left[\bar{\mathcal{G}}(T_1)\right]^{-1}\right)T_1\right)^2 \mid X_1\right) &= \mathbb{E}\left(\ell_1\left(\left[\bar{\mathcal{G}}(Y_1)\right]^{-1}\right)^2 Y_1^2 \mid X_1\right)\\ &= \mathbb{E}\left(\mathbb{E}\left(1_{Y_1\le C_1}|(X_1|Y_1)\right)\left(\left[G(Y_1)\right]^{-1}\right)^2 Y_1^2\middle|X_1\right)\\ &= \mathbb{E}\left(\left(\left[\bar{\mathcal{G}}(Y_1)\right]^{-1}\right)Y_1^2 \mid X_1\right) = \nu(x)\end{aligned}$$

and from Lemma A.1 in Zhou and Lin ([28]), we have

$$\begin{aligned}\mathbb{E}\left(L_1^j\right) &= M_j\psi_x(h) + o\left(\psi_x(h)\right), for j = 1,2;\\ \mathbb{E}\left(L_1^a\delta_1^b\right) &= N(a,b)h^b\psi_x(h) + o\left(h^b\psi_x(h)\right),\ for\ all\ a>0\ and\ b=2,4;\\ \mathbb{E}\left(\Omega_1L_1\right) &= (n-1)\mathbb{E}\left(\Omega_{12}\right) = (n-1)N(1,2)M_1h^2\psi_x^2(h)(1+o(1)),\end{aligned} \tag{8.11}$$

so, for n large enough and assumption $(A6_{(i)})$, we obtain

$$\frac{n^2\psi_x(h)\,\mathbb{E}^2\left(\delta_1^2L_1\right)}{\mathbb{E}^2\left(\Omega_1L_1\right)}\mathbb{E}\left(L_1^2Var\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1|X_1\right)\right) \longrightarrow \frac{M_2}{M_1^2}\left(\nu(x) - m(x)^2\right).$$

* About the second term of Eq (8.9), and by the Lemma A.1 in Zhou and Lin ([28]) and from Eq (8.10), we get

$$\frac{n^2\psi_x(h)\,\mathbb{E}^2\left(\delta_1^2L_1\right)}{\mathbb{E}^2\left(\Omega_1L_1\right)}\mathbb{E}\left(L_1^2\left(\mathbb{E}\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1|X_1\right) - m(x)\right)^2\right) \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty.$$

To complete the proof of (8.3), we emphasize the importance of the theorem on the central limit and we use the Lindeberg's central limit condition on Σ_n, where

$$\Sigma_n = \frac{\sqrt{n\psi_x(h)}\mathbb{E}\left(\delta_1^2L_1\right)}{\mathbb{E}\left(\Omega_1L_1\right)}\left(L_1\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1\right) - \mathbb{E}\left(L_1\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}T_1\right)\right)$$

are i.i.d random variables with 0 average. For all $\varepsilon > 0$ and assumptions $(A4)$ and $(A5)$, we have

$$\frac{1}{\sigma(x)}\mathbb{E}\left(\left(\sqrt{n}\Sigma_{1n}\right)^2\mathbb{I}_{\left|\sqrt{n}\Sigma_{1n}\right|>\varepsilon\sqrt{n\sigma(x)}}\right) \longrightarrow 0,$$

where $\mathbb{E}\left(\left(\sqrt{n}\Sigma_{n1}\right)^2\right) \longrightarrow \sigma(x)$, and $\{\left|\sqrt{n}\Sigma_{1n}\right| > \varepsilon\sqrt{n\sigma(x)}\}$ is empty set, because that $\left|\frac{\Sigma_{1n}}{\sqrt{n}}\right| \le \frac{C\sqrt{n}}{(n-1)^2\psi_x(h)\,\Sigma_{1n}} \longrightarrow 0$, **as** $n \longrightarrow \infty$.

- Secondly, to compute the asymptotic term Γ_1 we must to show that by Cauchy-Schwarz's inequality,

$$\mathbb{E} \mid (\Gamma_1 - \mathbb{E}(\Gamma_1)) \mid \leq 2\sqrt{\mathbb{E}\left(\frac{1}{n\mathbb{E}\left(\delta_1^2 L_1\right)}\sum_{i=1}^{n}\delta_i^2 L_i - 1\right)^2} \times \sqrt{\mathbb{E}\left(\frac{\sqrt{n\psi_x(h)}\mathbb{E}\left(\delta_1^2 L_1\right)}{\mathbb{E}(\Omega_1 L_1)}\sum_{j=1}^{n} L_j\left(\ell_j\left[\bar{\mathcal{G}}\left(T_j\right)\right]^{-1} T_j - m(x)\right)\right)^2}. \tag{8.12}$$

  - For the initial component of the portion on the right side of (8.12), we delete the details because they are well-know and by Lemma A.1 of Zhou and Lin [28] (8.11) and hypothesis ($A3$) and ($A4$), we get

$$\mathbb{E}\left(\frac{1}{n\mathbb{E}\left(\delta_1^2 L_1\right)}\sum_{i=1}^{n}\delta_i^2 L_i - 1\right)^2 = O\left(\frac{1}{n\psi_x(h)}\right). \tag{8.13}$$

  - Now, we turn to the second term of (8.12), we have

$$\begin{aligned} &\mathbb{E}\left(\frac{\sqrt{n\psi_x(h)}\mathbb{E}\left(\delta_1^2 L_1\right)}{\mathbb{E}(\Omega_1 L_1)}\sum_{j=1}^{n} L_j\left(\ell_j\left[\bar{\mathcal{G}}\left(T_j\right)\right]^{-1} T_j - m(x)\right)\right)^2 \\ &= \frac{n\psi_x(h)\,\mathbb{E}^2\left(\delta_1^2 L_1\right)}{\mathbb{E}^2(\Omega_1 L_1)}\left(n\mathbb{E}\left(L_1\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1} T_1 - m(x)\right)\right)^2\right) \\ &+ \frac{n\psi_x(h)\,\mathbb{E}^2\left(\delta_1^2 L_1\right)}{\mathbb{E}^2(\Omega_1 L_1)}\left(n(n-1)\mathbb{E}^2\left(L_1\left(\ell_1\left[G(T_1)\right]^{-1} T_1 - m(x)\right)\right)\right) \end{aligned} \tag{8.14}$$

  we using that $\mid T_i\ell_j\left[G(T_i)\right]^{-1} - m(x) \mid \leq \left[\tau(\tau_F)\right]^{-1}$ and Lemma A.1 of Zhou and Lin [28](8.11) and by postulates ($A3$) and ($A4$), we have

$$\mathbb{E}\left(\frac{\sqrt{n\psi_x(h)}\mathbb{E}\left(\delta_1^2 L_1\right)}{\mathbb{E}(\Omega_1 L_1)}\sum_{j=1}^{n} L_j\left(\ell_j\left[\bar{\mathcal{G}}\left(T_j\right)\right]^{-1} T_j - m(x)\right)\right)^2 = O(1) + o\left(n\psi_x(h)\right), \tag{8.15}$$

  and by Eqs (8.13) and (8.15), we get $\mathbb{E} \mid (\Gamma_1 - \mathbb{E}(\Gamma_1)) \mid = o(1)$, then, the Bienayme Tchebychev's inequality $\forall \epsilon > 0$

$$\mathbb{P}\left(\mid (\Gamma_1 - \mathbb{E}(\Gamma_1)) \mid > \epsilon\right) \leq \frac{\mathbb{E} \mid (\Gamma_1 - \mathbb{E}(\Gamma_1)) \mid}{\epsilon} \longrightarrow 0. \tag{8.16}$$

- Thirdly, as in the same steps in Eq (8.5), we obtain the asymptotic term of Γ_3.

Finally, from (8.3)–(8.5) and by Slutsky's theorem, we get the result for Eq (8.1). □

Proof of Lemma 2. We must prove

$$\mathbb{P}\left(\left|\widehat{S}_N(x) - \widetilde{m}_N(x)\right| > \varepsilon\right) \leq \frac{\mathbb{E}\left(\left|\widehat{S}_N(x) - \widetilde{m}_N(x)\right|\right)}{\varepsilon}, \tag{8.17}$$

$$\mathbb{E}\left(\widetilde{m}_N(x)\right) - m(x) \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{8.18}$$

- For inequality (8.17), we have

$$\left|\widehat{S}_N(x) - \widetilde{m}_N(x)\right| \leq \frac{1}{n(n-1)\mathbb{E}(\Omega_{12})} \sum_{i\neq j} \left|T_i\ell_j\Omega_{ij}\left(\left[\bar{\mathcal{G}}_n(T_i)\right]^{-1} - \left[\bar{\mathcal{G}}(T_i)\right]^{-1}\right)\right|$$
$$\leq \frac{C|\tau| \sup_{t\leq\tau}\left|\bar{\mathcal{G}}_n(t) - G(t)\right|}{\bar{\mathcal{G}}_n(\tau)\bar{\mathcal{G}}(\tau)} \left|\frac{1}{n(n-1)\mathbb{E}(\Omega_{12})}\sum_{i\neq j}\Omega_{ij}\right|,$$

using the application of Glivenko-Cantelli Theorem for the censored data and by an analogous proof to that of Lemma 3 in Leulmi ([21]) and assumption $(A6_{(ii)})$ and $(A6_{(iii)})$, we get

$$\widehat{S}_N(x) - \widetilde{m}_N(x) \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

- Clearly, for Eq (8.18), we have

$$\mathbb{E}(\widetilde{m}_N(y)) - m(x) = \frac{1}{\mathbb{E}(\Omega_1 L_1)}\mathbb{E}\left(\Omega_1 L_1 \ell_1 T_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1}\right) - m(x)$$
$$= \frac{1}{\mathbb{E}(\Omega_1 L_1)}\mathbb{E}\left(\Omega_1 L_1 \mathbb{E}\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1} T_1 \mid X_1\right)\right) - m(x),$$

Using that $\mathbb{E}\left(\ell_1\left[\bar{\mathcal{G}}(T_1)\right]^{-1} T_1 \mid X_1\right) = m(X_1)$ and assumption $(A2)$ for Eq (8.18), we conclude the proof. □