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Research article

Global co-dynamics of viral infections with saturated incidence

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Abstract: Several mathematical models of two competing viruses (or viral strains) that have been published in the literature assume that the infection rate is determined by bilinear incidence. These models do not show co-existence equilibrium; moreover, they might not be applicable in situations where the virus concentration is high. In this paper, we developed a mathematical model for the co-dynamics of two competing viruses with saturated incidence. The model included the latently infected cells and three types of time delays: discrete (or distributed): (i) The formation time of latently infected cells; (ii) The activation time of latently infected cells; (iii) The maturation time of newly released virions. We established the mathematical well-posedness and biological acceptability of the model by examining the boundedness and nonnegativity of the solutions. Four equilibrium points were identified, and their stability was examined. Through the application of Lyapunov's approach and LaSalle's invariance principle, we demonstrated the global stability of equilibria. The impact of saturation incidence, latently infected cells, and time delay on the viral co-dynamics was examined. We demonstrated that the saturation could result in persistent viral coinfections. We established conditions under which these types of viruses could coexist. The coexistence conditions were formulated in terms of saturation constants. These findings offered new perspectives on the circumstances under which coexisting viruses (or strains) could live in stable viral populations. It was shown that adding the class of latently infected cells and time delay to the coinfection model reduced the basic reproduction number for each virus type. Therefore, fewer treatment efficacies would be needed to keep the system at the infection-free equilibrium and remove the viral coinfection from the body when utilizing a model with latently infected cells and time delay. To demonstrate the associated mathematical outcomes, numerical simulations were conducted for the model with discrete delays.

Keywords: viral coinfection; saturated incidence; time delay; global stability; Lyapunov function; LaSalle's invariance principle
Mathematical Subject Classification: 34D20, 34D23, 37N25, 92B05

1. Introduction

Human viral infections such as human immunodeficiency virus (HIV), hepatitis B virus (HBV), hepatitis C virus (HCV), ebola virus, influenza A virus (IAV), influenza B virus (IBV), chikungunya virus (CHIKV), middle east respiratory syndrome coronavirus (MERS-CoV), human T-cell lymphotropic virus (HTLV), zika virus (ZIKV), dengue virus (DENV), and severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) are a global health concern. It is possible for a person to be infected with two or more types of viruses (or different viral strains) simultaneously or successively. This situation is defined as viral coinfection [1]. Examples of viral coinfections include: HIV and viral hepatitis [2]; HBV and HCV [3]; different strains of SARS-CoV-2 [4]; and SARS-CoV-2 and HBV [5].

Many researchers are interested in mathematical modeling of viral infections within the host. The development of antiviral drug therapies and vaccines, the understanding of the dynamics of viral infection and the immune system's response to viruses, and the identification of the minimum number of variables needed to analyze experimental data and explain biological phenomena are all made possible by mathematical models. In [6], a basic model for viral single-infection within a host has been formulated. A mathematical model that describes competition of two virus types (or virus variants) for uninfected cells can be given as [7]:

$$\dot{H}(t) = \overbrace{\phi}^{\text{production of uninfected cells}} - \overbrace{\eta_H H(t)}^{\text{death}} - \overbrace{\gamma_{HC} H(t)C(t)}^{\text{infection via virus type } C} - \overbrace{\gamma_{HB} H(t)B(t)}^{\text{infection via virus type } B}, \tag{1.1}$$

$$\dot{Y}(t) = \overbrace{\gamma_{HC} H(t)C(t)}^{\text{growth of infected cells by virus type } C} - \overbrace{\eta_Y Y(t)}^{\text{death}}, \tag{1.2}$$

$$\dot{Z}(t) = \overbrace{\gamma_{HB} H(t)B(t)}^{\text{growth of infected cells by virus type } B} - \overbrace{\eta_Z Z(t)}^{\text{death}}, \tag{1.3}$$

$$\dot{C}(t) = \overbrace{\theta_C Y(t)}^{\text{generation of virus type } C} - \overbrace{\eta_C C(t)}^{\text{viral clearance}}, \tag{1.4}$$

$$\dot{B}(t) = \overbrace{\theta_B Z(t)}^{\text{generation of virus type } B} - \overbrace{\eta_B B(t)}^{\text{viral clearance}}, \tag{1.5}$$

where $H(t)$, $Y(t)$, $Z(t)$, $C(t)$, and $B(t)$ are the concentrations of the uninfected cells, infected cells by virus type C, infected cells by virus type B, free virus type C particles, and free virus type B particles at time t, respectively.

Examples of viruses (or virus strains) which compete for the same target cells including:

- Respiratory viruses: Such SARS-CoV-2 and IAV which compete for the epithelial cells in the respiratory tract [1, 8]. Human rhinovirus, respiratory syncytial virus, human enterovirus, human metapneumovirus, influenza A/B viruses, parainfluenza virus, coronavirus, and human bocavirus and adenovirus are among the respiratory viruses that have been found to be able to participate in simultaneous infections [9].
- Chronic viruses: HIV and HTLV infect the $CD4^+$ T cells, often known as "helper" T cells which play a central role in immune system [10]. HBV and HCV target hepatocytes in the human liver [11].

- Victor-born infections: Both DENV and CHIKV infect the monocytes [12].

- Virus strains: As new mutants continue to evolve, the genotypes of the same virus in infected hosts that are wild-type and mutant overlap [13]. A number of recombinant viral strains of worldwide epidemiologic significance have been observed as a result of co-occurring HIV infections, which has significant implications for our knowledge of HIV transmission and the development of the AIDS vaccine [14]. Recent research has shown that coinfection can act as a catalyst for the recombination of distinct SARS-CoV-2 subtypes [15].

Several mathematical models have been developed for viral coinfections including: HIV-1/HTLV-I [16], SARS-CoV-2/IAV [8, 17], SARS-CoV-2/HTLV-I [18], SARS-CoV-2/HIV-1 [19], HIV-1/HBV [20] and HIV-1/HCV [21]. Moreover, coinfection with two viral strains have been modeled in several works (see e.g., [13, 22, 23]).

In the case of HIV infection, there is no medicine to cure acquired immune deficiency syndrome (AIDS) completely to date, but highly active antiretroviral therapy (HAART) has been used for the last two decades to treat HIV patients, and it has been found successful in suppressing HIV replication and reconstituting the immune system in the human body. However, using HAART cannot eradicate the virus completely [24]. An important reason is that HIV provirus can reside in latently infected cells, which live long, but can be activated to produce virus by relevant antigens, [25]. It has been reported in [26] that a coexistence of two HIV strains in the latent reservoirs is possible.

Models (1.1)–(1.5) operate under the premise that, upon entry of the virus into an uninfected cell, the cell becomes infected and produces new mature viruses. It is commonly recognized that the infection of free viruses into uninfected cells and the generation of new mature viruses often do not occur instantly but develop over a period of time [27]. Regarding HIV infection, it is believed that the duration between HIV entry into an uninfected cell and the production of new mature HIV particles is around 0.9 days [28]. Consequently, the delay finds extensive application in several models of viral infection, which are essential for studying biological processes that are more like reality (see e.g., [29–31]).

The infection rate is one important factor influencing the propagation of viruses [32]. A number of mathematical models of two competing viruses (or strains of viruses) that have appeared in the literature (see, e.g., [7, 23]) include the assumption that bilinear incidence determines the infection rate. In this case, the infection rate per target cell and per virus is a constant. This situation implies that the rate of infection is precisely proportional to the product of the concentrations of the viruses B (or C) interacting with uninfected cells (H), a phenomenon known as the mass-action principle. The incidence rate is linear in each variable over the entire range of B (or C) and H. However, as reported in [33], experiments have demonstrated that the infection rate of microparasitic infections generally increases with the parasite dose and typically exhibits a sigmoidal shape. The law of mass action, for instance, will not apply if the concentration of viruses is higher than the concentration of uninfected cells. In such a scenario, an increase in virus concentration will not result in a rise in infection. A sublinear response in virus concentration might arise from saturation at high virus concentrations, where the infectious fraction is high, leading to a high likelihood of exposure [34]. The goal of the saturation incidence function in epidemiology is to characterize the variance in infection force brought on by the crowding impact of infectious [35]. It is important to note that the models of two competing viruses with bilinear incidences shown in [7, 23] do not exhibit the co-existence equilibrium. As a result, these models might not be able to explain situations in which two chronic viruses co-exist, such

as HIV, HTLV, HBV, and HCV.

Papers [27,36,37] studied two strain viral dynamics models with saturated incidence. Nevertheless, these models do not incorporate latently infected cells. Further, the maturation delay of newly released virions was not included in the model given in [27]. Furthermore, it has not been addressed how saturation affects the conditions in which the two strains coexist. Thus, the aim of this study is to construct and analyze mathematical models that characterize the co-dynamics of two competing viruses (or virus variants) with saturated incidence and latently infected cells. The paper's novelty resides in the following aspects:

A1: Two models (one with discrete delay and the other with distributed delay) have been developed to describe the co-dynamics of two competing viruses within a host.

A2: Three kinds of discrete (or distributed) time delays have been incorporated into the model: (i) The formation delay of latently infected cells; (ii) the activation delay of latently infected cells; and (iii) the maturation delay of newly released virions.

A3: The non-negativity and boundedness of the model's solutions are rigorously analyzed, confirming that the presented models are both mathematically well-posed and biologically plausible.

A4: Global stability analysis has been presented for both models.

A5: Conditions for the co-existence of competing viruses that target the same host cells have been established.

A6: The theoretical findings have been validated through numerical simulations.

Our proposed model could be valuable for modeling the competitive transmission dynamics of different strains of COVID-19, such as Omicron and Delta [38, 39].

The paper is organized as follows: Sections 2 and 4 focus on formulating the two models, proving the non-negativity and boundedness of the model's solutions, determining the model's equilibria and threshold parameters, and establishing the global stability of the equilibria. Section 3 contains comparison results. Section 5 presents numerical simulations for the model with discrete delays, while Section 6 summarizes our findings and outlines future perspectives.

2. Model with discrete-time delays

2.1. Model formulation

In this section, we develop a two-virus co-dynamics model with a saturated incidence rate, latently infection cells, and six discrete-time delays as follows:

$$\dot{H}(t) = \phi - \eta_H H(t) - \frac{\gamma_{HC} H(t)C(t)}{1+\psi_C C(t)} - \frac{\gamma_{HB} H(t)B(t)}{1+\psi_B B(t)}, \tag{2.1}$$

$$\dot{L}(t) = e^{-\alpha_1\omega_1}\frac{\gamma_{HC} H(t-\omega_1)C(t-\omega_1)}{1+\psi_C C(t-\omega_1)} - (\eta_L+\delta_L)L(t), \tag{2.2}$$

$$\dot{Y}(t) = e^{-\alpha_2\omega_2}\delta_L L(t-\omega_2) - \eta_Y Y(t), \tag{2.3}$$

$$\dot{E}(t) = e^{-\alpha_4\omega_4}\frac{\gamma_{HB} H(t-\omega_4)B(t-\omega_4)}{1+\psi_B B(t-\omega_4)} - (\eta_E+\delta_E)E(t), \tag{2.4}$$

$$\dot{Z}(t) = e^{-\alpha_5\omega_5}\delta_E E(t-\omega_5) - \eta_Z Z(t), \tag{2.5}$$

$$\dot{C}(t) = e^{-\alpha_3\omega_3}\theta_C Y(t-\omega_3) - \eta_C C(t), \tag{2.6}$$

$$\dot{B}(t) = e^{-\alpha_6\omega_6}\theta_B Z(t-\omega_6) - \eta_B B(t), \tag{2.7}$$

where L and E are the concentrations of the latent cells infected by virus types C and B, respectively. The activation and death rate constants of compartments (L, E) are denoted by (δ_L, δ_E) and (η_L,η_E), respectively. The terms $\frac{\gamma_{HC}HC}{1+\psi_C C}$ and $\frac{\gamma_{HB}HB}{1+\psi_B B}$ represent saturated incidence for virus types C and B, respectively, where $\psi_C \geq 0$ and $\psi_B \geq 0$ are saturation parameters. Saturated incidence leads to bilinear incidence when $\psi_C = \psi_B = 0$. There are three types of time delays:

(i) ω_1 and ω_4, the formation times of latent cells infected by viruses type C and B, respectively;

(ii) ω_2 and ω_5, the activation times of latent cells infected by viruses type C and B, respectively;

(iii) ω_3 and ω_6, the maturation times of newly released virions type C and B, respectively.

The factor $e^{-\alpha_i\omega_i}$, $i = 1, 2, \ldots, 6$ represents the probability of survival of a cell or virion throughout the delay time $[t-\omega_i, t]$, and $\alpha_i > 0$.

The initial conditions for system (2.1)–(2.7) are:

$$\begin{aligned}
&H(u) = \ell_1(u), \quad L(u) = \ell_2(u), \quad Y(u) = \ell_3(u), \quad E(u) = \ell_4(u),\\
&Z(u) = \ell_5(u), \quad C(u) = \ell_6(u), \quad B(u) = \ell_7(u),\\
&\ell_i(u) \geq 0, \qquad u \in [-\omega^*, 0],\\
&\ell_i(u) \in \mathbb{C}([-\omega^*, 0], \mathbb{R}_{\geq 0}), \quad i = 1, 2, \ldots, 7,
\end{aligned} \tag{2.8}$$

where

$$\omega^* = \max\{\omega_1, \omega_2, \ldots, \omega_6\},$$

and $\mathbb{C}$ is the Banach space of continuous functions mapping the interval $[-\omega^*, 0]$ into $\mathbb{R}_{\geq 0}$ with

$$\|\ell_i\| = \sup_{-\omega^* \leq u \leq 0} |\ell_i(u)|$$

for $\ell_i \in \mathbb{C}$. System (2.1)–(2.7), with initial conditions (2.8), has a unique solution [40, 41].

Remark 1. The production rate of uninfected cells has been represented by a variety of functions in virology literature, including: constant, ϕ, saturated, $\frac{\phi V}{1+\epsilon V}$ [42], exponential, $\phi e^{-\epsilon V}$ [43], decreasing, $\frac{\phi\epsilon}{\epsilon+V}$ [44], and general, $H\Xi(H)$ [45], where ϵ is constant and Ξ is a general function.

2.2. Preliminaries

Proposition 1. The solutions of system (2.1)–(2.7) with initial (2.8) are nonnegative and ultimately bounded.

Proof. We have that

$$\dot{H}\,|_{H=0} = \phi > 0.$$

Hence, $H(t) > 0$ for all $t \geq 0$. Moreover, for all $t \in [0, \omega^*]$, we have:

$$L(t) = \ell_2(0)e^{-(\eta_L+\delta_L)t} + e^{-\alpha_1\omega_1}\gamma_{HC}\int_0^t e^{-(\eta_L+\delta_L)(t-r)}\frac{H(r-\omega_1)C(r-\omega_1)}{1+\psi_C C(r-\omega_1)}dr,$$

$$Y(t) = \ell_3(0)e^{-\eta_Y t} + e^{-\alpha_2\omega_2}\delta_L \int_0^t e^{-\eta_Y(t-r)} L(r-\omega_2)dr,$$

$$E(t) = \ell_4(0)e^{-(\eta_E+\delta_E)t} + e^{-\alpha_4\omega_4}\gamma_{HB} \int_0^t e^{-(\eta_E+\delta_E)(t-r)} \frac{H(r-\omega_4)B(r-\omega_4)}{1+\psi_B B(r-\omega_4)} dr,$$

$$Z(t) = \ell_5(0)e^{-\eta_Z t} + e^{-\alpha_5\omega_5}\delta_E \int_0^t e^{-\eta_Z(t-r)} E(r-\omega_5)dr,$$

$$C(t) = \ell_6(0)e^{-\eta_C t} + e^{-\alpha_3\omega_3}\theta_C \int_0^t e^{-\eta_C(t-r)} Y(r-\omega_3)dr,$$

$$B(t) = \ell_7(0)e^{-\eta_B t} + e^{-\alpha_6\omega_6}\theta_B \int_0^t e^{-\eta_B(t-r)} Z(r-\omega_6)dr.$$

Hence, $L(t), Y(t), E(t), Z(t), C(t), B(t) \geq 0$ for all $t \in [0, \omega^*]$. Through recursive argumentation, we get $L(t), Y(t), E(t), Z(t), C(t), B(t)$ for all $t \geq 0$. Therefore, H, L, Y, E, Z, C and B are nonnegative.

The nonnegativity of the system's solution implies that

$$\dot{H}(t) \leq \phi - \eta_H H(t) \implies \limsup_{t\to\infty} H(t) = \frac{\phi}{\eta_H} = a_0.$$

Let us define

$$\Psi_1(t) = e^{-\alpha_1\omega_1} H(t-\omega_1) + L(t),$$

then

$$\begin{aligned}
\dot{\Psi}_1(t) &= e^{-\alpha_1\omega_1}\dot{H}(t-\omega_1) + \dot{L}(t) \\
&= e^{-\alpha_1\omega_1}\left[\phi - \eta_H H(t-\omega_1) - \frac{\gamma_{HC}H(t-\omega_1)C(t-\omega_1)}{1+\psi_C C(t-\omega_1)} - \frac{\gamma_{HB}H(t-\omega_1)B(t-\omega_1)}{1+\psi_B B(t-\omega_1)}\right] \\
&+ e^{-\alpha_1\omega_1}\frac{\gamma_{HC}H(t-\omega_1)C(t-\omega_1)}{1+\psi_C C(t-\omega_1)} - (\eta_L+\delta_L)L(t) \\
&= e^{-\alpha_1\omega_1}\phi - e^{-\alpha_1\omega_1}\eta_H H(t-\omega_1) - e^{-\alpha_1\omega_1}\frac{\gamma_{HB}H(t-\omega_1)B(t-\omega_1)}{1+\psi_B B(t-\omega_1)} - (\eta_L+\delta_L)L(t) \\
&\leq \phi - \sigma_1\left[e^{-\alpha_1\omega_1}H(t-\omega_1) - L(t)\right] \\
&= \phi - \sigma_1\Psi_1(t),
\end{aligned}$$

where

$$\sigma_1 = \min\{\eta_H, \eta_L + \delta_L\}.$$

It follows that

$$\limsup_{t\to\infty} \Psi_1(t) \leq \frac{\phi}{\sigma_1} = a_1 \Longrightarrow \limsup_{t\to\infty} L(t) \leq a_1.$$

Let

$$\Psi_2(t) = e^{-\alpha_4\omega_4} H(t-\omega_4) + E(t),$$

then,

$$\begin{aligned}
\dot{\Psi}_2(t) &= e^{-\alpha_4\omega_4}\dot{H}(t-\omega_4) + \dot{E}(t)\\
&= e^{-\alpha_4\omega_4}\left[\phi - \eta_H H(t-\omega_4) - \frac{\gamma_{HC}H(t-\omega_4)C(t-\omega_4)}{1+\psi_C C(t-\omega_4)} - \frac{\gamma_{HB}H(t-\omega_4)B(t-\omega_4)}{1+\psi_B B(t-\omega_4)}\right]\\
&+ e^{-\alpha_4\omega_4}\frac{\gamma_{HB}H(t-\omega_4)B(t-\omega_4)}{1+\psi_B B(t-\omega_4)} - (\eta_E+\delta_E)E(t)\\
&= e^{-\alpha_4\omega_4}\phi - e^{-\alpha_4\omega_4}\eta_H H(t-\omega_4) - e^{-\alpha_4\omega_4}\frac{\gamma_{HC}H(t-\omega_4)C(t-\omega_4)}{1+\psi_C C(t-\omega_4)} - (\eta_E+\delta_E)E(t)\\
&\leq \phi - \sigma_2\left[e^{-\alpha_4\omega_4}H(t-\omega_4) - E(t)\right]\\
&= \phi - \sigma_2\Psi_2(t),
\end{aligned}$$

where

$$\sigma_2 = \min\{\eta_H, \eta_E + \delta_E\}.$$

Thus

$$\limsup_{t\to\infty}\Psi_2(t) \leq \frac{\phi}{\sigma_2} = a_2 \Longrightarrow \limsup_{t\to\infty} E(t) \leq a_2.$$

From Eq (2.3) we get

$$\dot{Y}(t) = e^{-\alpha_2\omega_2}\delta_L L(t-\omega_2) - \eta_Y Y(t) \leq \delta_L a_1 - \eta_Y Y(t) \Longrightarrow \limsup_{t\to\infty} Y(t) \leq \frac{\delta_L a_1}{\eta_Y} = a_3.$$

Equation (2.5) implies that

$$\begin{aligned}
\dot{Z}(t) &= e^{-\alpha_5\omega_5}\delta_E E(t-\omega_5) - \eta_Z Z(t)\\
&\leq \delta_E a_2 - \eta_Z Z(t) \Longrightarrow \limsup_{t\to\infty} Z(t) \leq \frac{\delta_E a_2}{\eta_Z} = a_4.
\end{aligned}$$

Similarly from Eqs (2.6) and (2.7) we get

$$\dot{C}(t) \leq \theta_C a_3 - \eta_C C(t) \Longrightarrow \limsup_{t\to\infty} C(t) \leq \frac{\theta_C a_3}{\eta_C} = a_5,$$

$$\dot{B}(t) \leq \theta_B a_4 - \eta_B B(t) \Longrightarrow \limsup_{t\to\infty} B(t) \leq \frac{\theta_B a_4}{\eta_B} = a_6.$$

This completes the proof. □

Based on Proposition 1 we can show that

$$\Gamma = \left\{(H, L, Y, E, Z, C, B) \in \mathbb{C}^7_{\geq 0} : \|H\| \leq a_0, \|L\| \leq a_1, \|Y\| \leq a_3, \|E\| \leq a_2, \|Z\| \leq a_4, \|C\| \leq a_5, \|B\| \leq a_6\right\}$$

is positively invariant for system (2.1)–(2.7).

2.3. Equilibria

We calculate the model's equilibria and deduce the conditions of their existence. Any equilibrium point $\Xi = (H, L, Y, E, Z, C, B)$ satisfies

$$\begin{aligned}
0 &= \phi - \eta_H H - \frac{\gamma_{HC} HC}{1 + \psi_C C} - \frac{\gamma_{HB} HB}{1 + \psi_B B},\\
0 &= e^{-\alpha_1 \omega_1} \frac{\gamma_{HC} HC}{1 + \psi_C C} - (\eta_L + \delta_L) L,\\
0 &= e^{-\alpha_2 \omega_2} \delta_L L - \eta_Y Y,\\
0 &= e^{-\alpha_4 \omega_4} \frac{\gamma_{HB} HB}{1 + \psi_B B} - (\eta_E + \delta_E) E,\\
0 &= e^{-\alpha_5 \omega_5} \delta_E E - \eta_Z Z,\\
0 &= e^{-\alpha_3 \omega_3} \theta_C Y - \eta_C C,\\
0 &= e^{-\alpha_6 \omega_6} \theta_B Z - \eta_B B.
\end{aligned} \tag{2.9}$$

System (2.9) admits four equilibria.

(I) Infection-free equilibrium, $\Xi_0 = (H_0, 0, 0, 0, 0, 0, 0)$, where $H_0 = \phi/\eta_H$.

(II) Virus type C single-infection equilibrium $\Xi_1 = (H_1, L_1, Y_1, 0, 0, C_1, 0)$, where

$$\begin{aligned}
H_1 &= \frac{e^{\Sigma_{i=1}^{3} \alpha_i \omega_i} \eta_Y \eta_C (\eta_L + \delta_L) + \psi_C \phi \delta_L \theta_C}{\delta_L \theta_C (\eta_H \psi_C + \gamma_{HC})}, \quad L_1 = \frac{e^{\Sigma_{i=2}^{3} \alpha_i \omega_i} \eta_Y \eta_C \eta_H}{\delta_L \theta_C (\eta_H \psi_C + \gamma_{HC})} (\mathfrak{R}_1 - 1),\\
Y_1 &= \frac{e^{\alpha_3 \omega_3} \eta_C \eta_H}{\theta_C (\eta_H \psi_C + \gamma_{HC})} (\mathfrak{R}_1 - 1), \qquad C_1 = \frac{\eta_H}{(\eta_H \psi_C + \gamma_{HC})} (\mathfrak{R}_1 - 1),
\end{aligned}$$

and

$$\mathfrak{R}_1 = \frac{e^{-\Sigma_{i=1}^{3} \alpha_i \omega_i} H_0 \delta_L \theta_C \gamma_{HC}}{\eta_Y \eta_C (\eta_L + \delta_L)}, \tag{2.10}$$

which represents the basic reproduction number for virus type C single-infection. It follows that, Ξ_1 exists if $\mathfrak{R}_1 > 1$.

(III) Virus type B single-infection equilibrium $\Xi_2 = (H_2, 0, 0, E_2, Z_2, 0, B_2)$, where

$$\begin{aligned}
H_2 &= \frac{e^{\Sigma_{i=4}^{6} \alpha_i \omega_i} \eta_Z \eta_B (\eta_E + \delta_E) + \delta_E \psi_B \phi \theta_B}{\delta_E \theta_B (\eta_H \psi_B + \gamma_{HB})}, \quad E_2 = \frac{e^{\Sigma_{i=5}^{6} \alpha_i \omega_i} \eta_Z \eta_B \eta_H}{\delta_E \theta_B (\eta_H \psi_B + \gamma_{HB})} (\mathfrak{R}_2 - 1),\\
Z_2 &= \frac{e^{\alpha_6 \omega_6} \eta_B \eta_H}{\theta_B (\eta_H \psi_B + \gamma_{HB})} (\mathfrak{R}_2 - 1), \qquad B_2 = \frac{\eta_H}{(\eta_H \psi_B + \gamma_{HB})} (\mathfrak{R}_2 - 1),
\end{aligned}$$

and

$$\mathfrak{R}_2 = \frac{e^{-\Sigma_{i=4}^{6} \alpha_i \omega_i} H_0 \delta_E \theta_B \gamma_{HB}}{\eta_Z \eta_B (\eta_E + \delta_E)}. \tag{2.11}$$

which represents the basic reproduction number for virus type B single-infection. Therefore, Ξ_2 exists if $\mathfrak{R}_2 > 1$.

(IV) Coexistence equilibrium $\Xi_3 = (H_3, L_3, Y_3, E_3, Z_3, C_3, B_3)$, where

$$H_3 = \frac{\phi \psi_C \delta_L \theta_C \psi_B \delta_E \theta_B + \eta_Y \eta_C \psi_B \delta_E \theta_B (\eta_L + \delta_L) e^{\Sigma_{i=1}^{3} \alpha_i \omega_i} + \eta_Z \eta_B \psi_C \delta_L \theta_C (\eta_E + \delta_E) e^{\Sigma_{i=4}^{6} \alpha_i \omega_i}}{\delta_E \theta_B \delta_L \theta_C (\eta_H \psi_C \psi_B + \gamma_{HC} \psi_B + \gamma_{HB} \psi_C)},$$

$$L_3 = \frac{e^{\Sigma_{i=2}^{3}\alpha_i\omega_i}\eta_Y\eta_C(\eta_H\psi_B+\gamma_{HB})}{\delta_L\theta_C(\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C)}(\mathfrak{R}_3-1), \quad Y_3 = \frac{e^{\alpha_3\omega_3}\eta_C(\eta_H\psi_B+\gamma_{HB})}{\theta_C(\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C)}(\mathfrak{R}_3-1),$$
$$E_3 = \frac{e^{\Sigma_{i=5}^{6}\alpha_i\omega_i}\eta_Z\eta_B(\eta_H\psi_C+\gamma_{HC})}{\delta_E\theta_B(\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C)}(\mathfrak{R}_4-1), \quad Z_3 = \frac{e^{\alpha_6\omega_6}\eta_B(\eta_H\psi_C+\gamma_{HC})}{\theta_B(\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C)}(\mathfrak{R}_4-1),$$
$$C_3 = \frac{(\eta_H\psi_B+\gamma_{HB})}{(\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C)}(\mathfrak{R}_3-1), \qquad B_3 = \frac{(\eta_H\psi_C+\gamma_{HC})}{(\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C)}(\mathfrak{R}_4-1),$$

and

$$\mathfrak{R}_3 = \frac{\gamma_{HC}\delta_L\theta_C[\phi\psi_B\delta_E\theta_B+\eta_Z\eta_B(\eta_E+\delta_E)e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}]}{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y\eta_C\delta_E\theta_B(\eta_L+\delta_L)(\eta_H\psi_B+\gamma_{HB})},$$
$$\mathfrak{R}_4 = \frac{\gamma_{HB}\delta_E\theta_B[\phi\psi_C\delta_L\theta_C+\eta_Y\eta_C(\eta_L+\delta_L)e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}]}{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z\eta_B\delta_L\theta_C(\eta_E+\delta_E)(\eta_H\psi_C+\gamma_{HC})}.$$

Clearly, Ξ_3 exists when $\mathfrak{R}_3 > 1$ and $\mathfrak{R}_4 > 1$.

Remark 2. We note that we have calculated the basic reproduction numbers R_1 and R_2 from the existence's conditions of equilibria Ξ_1 and Ξ_2, respectively. It is worth noting that R_1 and R_2 can also be calculated using the next-generation matrix method of van den Driessche and Watmough [46] or by analyzing the local stability of the infection-free equilibrium.

2.4. Global stability analysis

In this part, we use the Lyapunov function construction approach described in [47] to demonstrate the global asymptotic stability of all equilibria. Let $\Lambda_j(H, L, Y, E, Z, C, B)$ be a Lyapunov function candidate and $\tilde{\Omega}_j$ be the largest invariant subset of

$$\Omega_j = \left\{(H, L, Y, E, Z, C, B) : \frac{d\Lambda_j}{dt} = 0\right\}, \quad j = 0, 1, 2, 4.$$

Define a function

$$F : (0, \infty) \longrightarrow [0, \infty)$$

as

$$F(\upsilon) = \upsilon - 1 - \ln \upsilon.$$

We denote

$$\begin{aligned}(H, L, Y, E, Z, C, B) &= (H(t), L(t), Y(t), E(t), Z(t), C(t), B(t)),\\ H_{\omega_1} = H(t-\omega_1), &\quad H_{\omega_4} = H(t-\omega_4), \quad L_{\omega_2} = L(t-\omega_2),\\ Y_{\omega_3} = Y(t-\omega_3), &\quad E_{\omega_5} = E(t-\omega_5), \quad Z_{\omega_6} = Z(t-\omega_6),\\ C_{\omega_1} = C(t-\omega_1), &\quad B_{\omega_4} = B(t-\omega_4).\end{aligned}$$

Theorem 1. Consider (2.1)–(2.7) and suppose that $\mathfrak{R}_1 \leq 1$ and $\mathfrak{R}_2 \leq 1$, then Ξ_0 is globally asymptotically stable (GAS).

Proof. Define

$$\begin{aligned}\Lambda_0 &= H_0 F\left(\frac{H}{H_0}\right) + e^{\alpha_1\omega_1}L + \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}(\eta_L+\delta_L)}{\delta_L}Y + e^{\alpha_4\omega_4}E + \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}(\eta_E+\delta_E)}{\delta_E}Z \\ &+ \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\theta_C\delta_L}C + \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\theta_B\delta_E}B + \gamma_{HC}\int_{t-\omega_1}^{t}\frac{H(u)C(u)}{1+\psi_C C(u)}du \\ &+ e^{\alpha_1\omega_1}(\eta_L+\delta_L)\int_{t-\omega_2}^{t}L(u)du + \gamma_{HB}\int_{t-\omega_4}^{t}\frac{H(u)B(u)}{1+\psi_B B(u)}du + e^{\alpha_4\omega_4}(\eta_E+\delta_E)\int_{t-\omega_5}^{t}E(u)du \\ &+ \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}\int_{t-\omega_3}^{t}Y(u)du + \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}\int_{t-\omega_6}^{t}Z(u)du.\end{aligned}$$

It is seen that, $\Lambda_0 > 0$ for all $H, L, Y, E, Z, C, B > 0$, and $\Lambda_0(H_0, 0, 0, 0, 0, 0, 0) = 0$. We calculate $\frac{d\Lambda_0}{dt}$ along the solutions of system (2.1)–(2.7) as:

$$\begin{aligned}\frac{d\Lambda_0}{dt} &= \left(1-\frac{H_0}{H}\right)\dot{H} + e^{\alpha_1\omega_1}\dot{L} + \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}(\eta_L+\delta_L)}{\delta_L}\dot{Y} + e^{\alpha_4\omega_4}\dot{E} + \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}(\eta_E+\delta_E)}{\delta_E}\dot{Z} \\ &+ \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\theta_C\delta_L}\dot{C} + \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\theta_B\delta_E}\dot{B} + \frac{\gamma_{HC}HC}{1+\psi_C C} \\ &- \frac{\gamma_{HC}H_{\omega_1}C_{\omega_1}}{1+\psi_C C_{\omega_1}} + e^{\alpha_1\omega_1}(\eta_L+\delta_L)[L-L_{\omega_2}] + \frac{\gamma_{HB}HB}{1+\psi_B B} - \frac{\gamma_{HB}H_{\omega_4}B_{\omega_4}}{1+\psi_B B_{\omega_4}} + e^{\alpha_4\omega_4}(\eta_E+\delta_E)[E-E_{\omega_5}] \\ &+ \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}[Y-Y_{\omega_3}] + \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}[Z-Z_{\omega_6}].\end{aligned}$$

From Eqs (2.1)–(2.7), we obtain

$$\begin{aligned}\frac{d\Lambda_0}{dt} &= \left(1-\frac{H_0}{H}\right)\left[\phi - \eta_H H - \frac{\gamma_{HC}HC}{1+\psi_C C} - \frac{\gamma_{HB}HB}{1+\psi_B B}\right] + \frac{\gamma_{HC}H_{\omega_1}C_{\omega_1}}{1+\psi_C C_{\omega_1}} - e^{\alpha_1\omega_1}(\eta_L+\delta_L)L \\ &+ \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}(\eta_L+\delta_L)}{\delta_L}[e^{-\alpha_2\omega_2}\delta_L L_{\omega_2} - \eta_Y Y] + \frac{\gamma_{HB}H_{\omega_4}B_{\omega_4}}{1+\psi_B B_{\omega_4}} - e^{\alpha_4\omega_4}(\eta_E+\delta_E)E \\ &+ \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}(\eta_E+\delta_E)}{\delta_E}[e^{-\alpha_5\omega_5}\delta_E E_{\omega_5} - \eta_Z Z] + \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\theta_C\delta_L}[e^{-\alpha_3\omega_3}\theta_C Y_{\omega_3} - \eta_C C] \\ &+ \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\theta_B\delta_E}[e^{-\alpha_6\omega_6}\theta_B Z_{\omega_6} - \eta_B B] + \frac{\gamma_{HC}HC}{1+\psi_C C} - \frac{\gamma_{HC}H_{\omega_1}C_{\omega_1}}{1+\psi_C C_{\omega_1}} \\ &+ e^{\alpha_1\omega_1}(\eta_L+\delta_L)L - e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_{\omega_2} + \frac{\gamma_{HB}HB}{1+\psi_B B} - \frac{\gamma_{HB}H_{\omega_4}B_{\omega_4}}{1+\psi_B B_{\omega_4}} + e^{\alpha_4\omega_4}(\eta_E+\delta_E)E \\ &- e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_{\omega_5} + \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y - \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y_{\omega_3} \\ &+ \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z - \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z_{\omega_6}.\end{aligned}$$

Then, we collect terms as:

$$\frac{d\Lambda_0}{dt} = \left(1 - \frac{H_0}{H}\right)(\phi - \eta_H H) + \gamma_{HC} H_0 C - \frac{\gamma_{HC}\psi_C H_0 C^2}{1+\psi_C C} + \gamma_{HB} H_0 B - \frac{\gamma_{HB}\psi_B H_0 B^2}{1+\psi_B B} - \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y\eta_C(\eta_L+\delta_L)}{\theta_C\delta_L}C - \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z\eta_B(\eta_E+\delta_E)}{\theta_B\delta_E}B.$$

Using the equilibrium condition $\phi = \eta_H H_0$, we get

$$\frac{d\Lambda_0}{dt} = -\eta_H\frac{(H-H_0)^2}{H} - \frac{\gamma_{HC}\psi_C H_0 C^2}{1+\psi_C C} - \frac{\gamma_{HB}\psi_B H_0 B^2}{1+\psi_B B} + \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y\eta_C(\eta_L+\delta_L)}{\theta_C\delta_L}(\mathfrak{R}_1 - 1)C + \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z\eta_B(\eta_E+\delta_E)}{\theta_B\delta_E}(\mathfrak{R}_2 - 1)B.$$

Since $\mathfrak{R}_1 \leq 1$ and $\mathfrak{R}_2 \leq 1$, then $\frac{d\Lambda_0}{dt} \leq 0$ for all $H, C, B > 0$. In addition $\frac{d\Lambda_0}{dt} = 0$ when $H = H_0$ and $C = B = 0$. The solutions of system (2.1)–(2.7) tend to $\tilde{\Omega}_0$ [40] where $C = B = 0$. Thus, $\dot{C} = \dot{B} = 0$ and from Eqs (2.6) and (2.7) we have

$$0 = \dot{C} = e^{-\alpha_3\omega_3}\theta_C Y_{\omega_3} \Longrightarrow Y(t) = 0, \text{ for any } t,$$
$$0 = \dot{B} = e^{-\alpha_6\omega_6}\theta_B Z_{\omega_6} \Longrightarrow Z(t) = 0, \text{ for any } t.$$

Then from Eqs (2.3) and (2.5) we get

$$0 = \dot{Y} = e^{-\alpha_2\omega_2}\delta_L L_{\omega_2} \Longrightarrow L(t) = 0, \text{ for any } t,$$
$$0 = \dot{Z} = e^{-\alpha_5\omega_5}\delta_E E_{\omega_5} \Longrightarrow E(t) = 0, \text{ for any } t.$$

Therefore, $\tilde{\Omega}_0 = \{\Xi_0\}$ and applying LaSalle's invariance principle (LIP) [48], we obtain that Ξ_0 is GAS. □

Theorem 2. Consider (2.1)–(2.7) and suppose that $\mathfrak{R}_1 > 1$ and $\mathfrak{R}_4 \leq 1$, then Ξ_1 is GAS.

Proof. Define Λ_1 as:

$$\Lambda_1 = H_1 F\left(\frac{H}{H_1}\right) + e^{\alpha_1\omega_1}L_1 F\left(\frac{L}{L_1}\right) + \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}(\eta_L+\delta_L)}{\delta_L}Y_1 F\left(\frac{Y}{Y_1}\right) + e^{\alpha_4\omega_4}E + \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}(\eta_E+\delta_E)}{\delta_E}Z + \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\theta_C\delta_L}C_1 F\left(\frac{C}{C_1}\right) + \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\theta_B\delta_E}B + \frac{\gamma_{HC}H_1C_1}{1+\psi_C C_1}\int_{t-\omega_1}^{t} F\left(\frac{H(u)C(u)(1+\psi_C C_1)}{H_1C_1(1+\psi_C C(u))}\right)du + e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_1\int_{t-\omega_2}^{t} F\left(\frac{L(u)}{L_1}\right)du + \gamma_{HB}\int_{t-\omega_4}^{t}\frac{H(u)B(u)}{1+\psi_B B(u)}du + e^{\alpha_4\omega_4}(\eta_E+\delta_E)\int_{t-\omega_5}^{t}E(u)du + \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y_1\int_{t-\omega_3}^{t}F\left(\frac{Y(u)}{Y_1}\right)du + \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}\int_{t-\omega_6}^{t}Z(u)du.$$

We calculate $\frac{d\Lambda_1}{dt}$ as:

$$\begin{aligned}\frac{d\Lambda_1}{dt} &= \left(1-\frac{H_1}{H}\right)\dot{H} + e^{\alpha_1\omega_1}\left(1-\frac{L_1}{L}\right)\dot{L} + \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}(\eta_L+\delta_L)}{\delta_L}\left(1-\frac{Y_1}{Y}\right)\dot{Y} + e^{\alpha_4\omega_4}\dot{E}\\
&+ \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}(\eta_E+\delta_E)}{\delta_E}\dot{Z} + \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\theta_C\delta_L}\left(1-\frac{C_1}{C}\right)\dot{C} + \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\theta_B\delta_E}\dot{B}\\
&+ \frac{\gamma_{HC}H_1C_1}{1+\psi_CC_1}\left[\frac{HC(1+\psi_CC_1)}{H_1C_1(1+\psi_CC)} - \ln\left(\frac{HC(1+\psi_CC_1)}{H_1C_1(1+\psi_CC)}\right) - \frac{H_{\omega_1}C_{\omega_1}(1+\psi_CC_1)}{H_1C_1(1+\psi_CC_{\omega_1})}\right.\\
&\left.+ \ln\left(\frac{H_{\omega_1}C_{\omega_1}(1+\psi_CC_1)}{H_1C_1(1+\psi_CC_{\omega_1})}\right)\right] + e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_1\left[\frac{L}{L_1} - \ln\left(\frac{L}{L_1}\right) - \frac{L_{\omega_2}}{L_1} + \ln\left(\frac{L_{\omega_2}}{L_1}\right)\right]\\
&+ \gamma_{HB}\left[\frac{HB}{1+\psi_BB} - \frac{H_{\omega_4}B_{\omega_4}}{1+\psi_BB_{\omega_4}}\right] + e^{\alpha_4\omega_4}(\eta_E+\delta_E)\left[E - E_{\omega_5}\right]\\
&+ \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y_1\left[\frac{Y}{Y_1} - \ln\left(\frac{Y}{Y_1}\right) - \frac{Y_{\omega_3}}{Y_1} + \ln\left(\frac{Y_{\omega_3}}{Y_1}\right)\right] + \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}\left[Z - Z_{\omega_6}\right].\end{aligned}$$

So, we get

$$\begin{aligned}\frac{d\Lambda_1}{dt} &= \left(1-\frac{H_1}{H}\right)\left[\phi - \eta_HH - \frac{\gamma_{HC}HC}{1+\psi_CC} - \frac{\gamma_{HB}HB}{1+\psi_BB}\right]\\
&+ e^{\alpha_1\omega_1}\left(1-\frac{L_1}{L}\right)\left[e^{-\alpha_1\omega_1}\frac{\gamma_{HC}H_{\omega_1}C_{\omega_1}}{1+\psi_CC_{\omega_1}} - (\eta_L+\delta_L)L\right]\\
&+ \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}(\eta_L+\delta_L)}{\delta_L}\left(1-\frac{Y_1}{Y}\right)\left[e^{-\alpha_2\omega_2}\delta_LL_{\omega_2} - \eta_YY\right]\\
&+ e^{\alpha_4\omega_4}\left[e^{-\alpha_4\omega_4}\frac{\gamma_{HB}H_{\omega_4}B_{\omega_4}}{1+\psi_BB_{\omega_4}} - (\eta_E+\delta_E)E\right] + \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}(\eta_E+\delta_E)}{\delta_E}[e^{-\alpha_5\omega_5}\delta_EE_{\omega_5} - \eta_ZZ]\\
&+ \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\theta_C\delta_L}\left(1-\frac{C_1}{C}\right)[e^{-\alpha_3\omega_3}\theta_CY_{\omega_3} - \eta_CC]\\
&+ \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\theta_B\delta_E}[e^{-\alpha_6\omega_6}\theta_BZ_{\omega_6} - \eta_BB]\\
&+ \frac{\gamma_{HC}HC}{1+\psi_CC} - \frac{\gamma_{HC}H_{\omega_1}C_{\omega_1}}{1+\psi_CC_{\omega_1}} + \frac{\gamma_{HC}H_1C_1}{1+\psi_CC_1}\ln\left(\frac{H_{\omega_1}C_{\omega_1}(1+\psi_CC)}{HC(1+\psi_CC_{\omega_1})}\right)\\
&+ e^{\alpha_1\omega_1}(\eta_L+\delta_L)L - e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_{\omega_2} + e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_1\ln\left(\frac{L_{\omega_2}}{L}\right)\\
&+ \frac{\gamma_{HB}HB}{1+\psi_BB} - \frac{\gamma_{HB}H_{\omega_4}B_{\omega_4}}{1+\psi_BB_{\omega_4}} + e^{\alpha_4\omega_4}(\eta_E+\delta_E)E - e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_{\omega_5}\\
&+ \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y - \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y_{\omega_3} + \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y_1\ln\left(\frac{Y_{\omega_3}}{Y}\right)\\
&+ \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z - \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z_{\omega_6}. \qquad (2.12)\end{aligned}$$

Simplifying Eq (2.12), and using the equilibrium conditions for Ξ_1:

$$\phi = \eta_H H_1 + \frac{\gamma_{HC} H_1 C_1}{1+\psi_C C_1}, \quad \frac{\gamma_{HC} H_1 C_1}{1+\psi_C C_1} = e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_1,$$
$$\delta_L L_1 = e^{\alpha_2\omega_2}\eta_Y Y_1, \qquad \theta_C Y_1 = e^{\alpha_3\omega_3}\eta_C C_1,$$

we obtain

$$\begin{aligned}\frac{d\Lambda_1}{dt} &= -\frac{\eta_H(H-H_1)^2}{H} + e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_1\left[4-\frac{H_1}{H}-\frac{L_1 H_{\omega_1} C_{\omega_1}(1+\psi_C C_1)}{L H_1 C_1(1+\psi_C C_{\omega_1})}-\frac{Y_1 L_{\omega_2}}{Y L_1}\right.\\ &\left.-\frac{C_1 Y_{\omega_3}}{C Y_1}+\ln\left(\frac{H_{\omega_1}C_{\omega_1}(1+\psi_C C)}{HC(1+\psi_C C_{\omega_1})}\right)+\ln\left(\frac{L_{\omega_2}}{L}\right)+\ln\left(\frac{Y_{\omega_3}}{Y}\right)\right]\\ &+ e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_1\left(\frac{(1+\psi_C C_1)C}{(1+\psi_C C)C_1}-\frac{C}{C_1}\right)\\ &+\frac{e^{\sum_{i=4}^{6}\alpha_i\omega_i}\eta_Z\eta_B(\eta_E+\delta_E)}{\theta_B\delta_E}\left[\frac{H_1\gamma_{HB}\theta_B\delta_E}{e^{\sum_{i=4}^{6}\alpha_i\omega_i}\eta_B\eta_Z(\eta_E+\delta_E)}-1\right]B-\frac{\gamma_{HB}\psi_B H_1 B^2}{1+\psi_B B}.\end{aligned}$$

Using the following equalities:

$$\begin{aligned}\ln\left(\frac{H_{\omega_1}C_{\omega_1}(1+\psi_C C)}{HC(1+\psi_C C_{\omega_1})}\right) &= \ln\left(\frac{L_i H_{\omega_1}C_{\omega_1}(1+\psi_C C_i)}{L H_i C_i(1+\psi_C C_{\omega_1})}\right)+\ln\left(\frac{1+\psi_C C}{1+\psi_C C_i}\right)+\ln\left(\frac{H_i}{H}\right)+\ln\left(\frac{C_i L}{C L_i}\right),\\ \ln\left(\frac{L_{\omega_2}}{L}\right) &= \ln\left(\frac{Y_i L_{\omega_2}}{Y L_i}\right)+\ln\left(\frac{Y L_i}{Y_i L}\right),\\ \ln\left(\frac{Y_{\omega_3}}{Y}\right) &= \ln\left(\frac{C_i Y_{\omega_3}}{C Y_i}\right)+\ln\left(\frac{C Y_i}{C_i Y}\right),\end{aligned} \tag{2.13}$$

where $i = 1, 3$, we get

$$\begin{aligned}\frac{d\Lambda_1}{dt} &= -\frac{\eta_H(H-H_1)^2}{H} - e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_1\left[F\left(\frac{H_1}{H}\right)+F\left(\frac{H_{\omega_1}C_{\omega_1}L_1(1+\psi_C C_1)}{H_1 C_1 L(1+\psi_C C_{\omega_1})}\right)\right.\\ &\left.+F\left(\frac{Y_1 L_{\omega_2}}{Y L_1}\right)+F\left(\frac{C_1 Y_{\omega_3}}{C Y_1}\right)+F\left(\frac{1+\psi_C C}{1+\psi_C C_1}\right)\right]-\frac{e^{\alpha_1\omega_1}\psi_C(\eta_L+\delta_L)(C-C_1)^2}{(1+\psi_C C)(1+\psi_C C_1)C_1}L_1\\ &+\frac{e^{\sum_{i=4}^{6}\alpha_i\omega_i}\eta_Z\eta_B(\eta_E+\delta_E)}{\theta_B\delta_E}(\mathfrak{R}_4-1)B-\frac{\gamma_{HB}\psi_B H_1 B^2}{1+\psi_B B}.\end{aligned}$$

Since $\mathfrak{R}_4 \le 1$ then, $\frac{d\Lambda_1}{dt} \le 0$ for all $H, L, Y, E, Z, C, B > 0$. Moreover, $\frac{d\Lambda_1}{dt} = 0$ when $H = H_1$, $L = L_1$, $Y = Y_1$, $C = C_1$, and $B = 0$. The solutions of system (2.1)–(2.7) tend to $\tilde{\Omega}_1$ which includes elements with $B = 0$ which gives $\dot{B} = 0$. From Eq (2.7) we get

$$0 = \dot{B} = e^{-\alpha_6\omega_6}\theta_B Z_{\omega_6} \Longrightarrow Z(t) = 0, \quad \text{for any } t.$$

Then from Eq (2.5) we have

$$0 = \dot{Z} = e^{-\alpha_5\omega_5}\delta_E E_{\omega_5} \Longrightarrow E(t) = 0, \quad \text{for any } t.$$

Hence, $\tilde{\Omega}_1 = \{\Xi_1\}$ and Ξ_1 is GAS using LIP. □

Theorem 3. Consider (2.1)–(2.7) and suppose that $\mathfrak{R}_2 > 1$ and $\mathfrak{R}_3 \le 1$, then Ξ_2 is GAS.

Proof. Consider

$$\begin{aligned}\Lambda_2 &= H_2F\left(\frac{H}{H_2}\right) + e^{\alpha_1\omega_1}L + \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}(\eta_L+\delta_L)}{\delta_L}Y + e^{\alpha_4\omega_4}E_2F\left(\frac{E}{E_2}\right) + \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}(\eta_E+\delta_E)}{\delta_E}Z_2F\left(\frac{Z}{Z_2}\right)\\ &+ \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\theta_C\delta_L}C + \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\theta_B\delta_E}B_2F\left(\frac{B}{B_2}\right) + \gamma_{HC}\int_{t-\omega_1}^{t}\frac{H(u)C(u)}{1+\psi_C C(u)}du\\ &+ e^{\alpha_1\omega_1}(\eta_L+\delta_L)\int_{t-\omega_2}^{t}L(u)du + \frac{\gamma_{HB}H_2B_2}{1+\psi_BB_2}\int_{t-\omega_4}^{t}F\left(\frac{H(u)B(u)(1+\psi_BB_2)}{H_2B_2(1+\psi_BB(u))}\right)du\\ &+ e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_2\int_{t-\omega_5}^{t}F\left(\frac{E(u)}{E_2}\right)du + \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}\int_{t-\omega_3}^{t}Y(u)du\\ &+ \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z_2\int_{t-\omega_6}^{t}F\left(\frac{Z(u)}{Z_2}\right)du.\end{aligned}$$

We calculate $\frac{d\Lambda_2}{dt}$ as:

$$\begin{aligned}\frac{d\Lambda_2}{dt} &= \left(1-\frac{H_2}{H}\right)\left[\phi-\eta_HH-\frac{\gamma_{HC}HC}{1+\psi_CC}-\frac{\gamma_{HB}HB}{1+\psi_BB}\right] + e^{\alpha_1\omega_1}\left[e^{-\alpha_1\omega_1}\frac{\gamma_{HC}H_{\omega_1}C_{\omega_1}}{1+\psi_CC_{\omega_1}}-(\eta_L+\delta_L)L\right]\\ &+ \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}(\eta_L+\delta_L)}{\delta_L}\left[e^{-\alpha_2\omega_2}\delta_LL_{\omega_2}-\eta_YY\right] + e^{\alpha_4\omega_4}\left(1-\frac{E_2}{E}\right)\left[e^{-\alpha_4\omega_4}\frac{\gamma_{HB}H_{\omega_4}B_{\omega_4}}{1+\psi_BB_{\omega_4}}-(\eta_E+\delta_E)E\right]\\ &+ \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}(\eta_E+\delta_E)}{\delta_E}\left(1-\frac{Z_2}{Z}\right)[e^{-\alpha_5\omega_5}\delta_EE_{\omega_5}-\eta_ZZ]\\ &+ \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\theta_C\delta_L}[e^{-\alpha_3\omega_3}\theta_CY_{\omega_3}-\eta_CC]\\ &+ \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\theta_B\delta_E}\left(1-\frac{B_2}{B}\right)[e^{-\alpha_6\omega_6}\theta_BZ_{\omega_6}-\eta_BB] + \frac{\gamma_{HC}HC}{1+\psi_CC}\\ &- \frac{\gamma_{HC}H_{\omega_1}C_{\omega_1}}{1+\psi_CC_{\omega_1}} + e^{\alpha_1\omega_1}(\eta_L+\delta_L)L - e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_{\omega_2} + \frac{\gamma_{HB}HB}{1+\psi_BB}\\ &- \frac{\gamma_{HB}H_{\omega_4}B_{\omega_4}}{1+\psi_BB_{\omega_4}} + \frac{\gamma_{HB}H_2B_2}{1+\psi_BB_2}\ln\left(\frac{H_{\omega_4}B_{\omega_4}(1+\psi_BB)}{HB(1+\psi_BB_{\omega_4})}\right) + e^{\alpha_4\omega_4}(\eta_E+\delta_E)E\\ &- e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_{\omega_5} + e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_2\ln\left(\frac{E_{\omega_5}}{E}\right) + \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y\\ &- \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y_{\omega_3} + \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z - \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z_{\omega_6}\\ &+ \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z_2\ln\left(\frac{Z_{\omega_6}}{Z}\right).\end{aligned} \tag{2.14}$$

Then simplifying Eq (2.14) and using the equilibrium conditions for Ξ_2:

$$\phi = \eta_H H_2 + \frac{\gamma_{HB} H_2 B_2}{1+\psi_B B_2}, \quad \frac{\gamma_{HB} H_2 B_2}{1+\psi_B B_2} = e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_2,$$
$$\delta_E E_2 = e^{\alpha_5\omega_5}\eta_Z Z_2, \qquad \theta_B Z_2 = e^{\alpha_6\omega_6}\eta_B B_2,$$

and using the following equalities:

$$\ln\left(\frac{H_{\omega_4}B_{\omega_4}(1+\psi_B B)}{HB(1+\psi_B B_{\omega_4})}\right) = \ln\left(\frac{E_i H_{\omega_4} B_{\omega_4}(1+\psi_B B_i)}{EH_i B_i(1+\psi_B B_{\omega_4})}\right) + \ln\left(\frac{1+\psi_B B}{1+\psi_B B_i}\right) + \ln\left(\frac{H_i}{H}\right) + \ln\left(\frac{B_i E}{BE_i}\right),$$
$$\ln\left(\frac{E_{\omega_5}}{E}\right) = \ln\left(\frac{Z_i E_{\omega_5}}{ZE_i}\right) + \ln\left(\frac{ZE_i}{Z_i E}\right),$$
$$\ln\left(\frac{Z_{\omega_6}}{Z}\right) = \ln\left(\frac{B_i Z_{\omega_6}}{BZ_i}\right) + \ln\left(\frac{BZ_i}{B_i Z}\right), \tag{2.15}$$

where $i = 2, 3$, we get:

$$\frac{d\Lambda_2}{dt} = -\frac{\eta_H(H-H_2)^2}{H} - e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_2\left[F\left(\frac{H_{\omega_4}B_{\omega_4}E_2(1+\psi_B B_2)}{H_2B_2E(1+\psi_B B_{\omega_4})}\right) + F\left(\frac{H_2}{H}\right)\right.$$
$$\left. + F\left(\frac{Z_2E_{\omega_5}}{ZE_2}\right) + F\left(\frac{B_2Z_{\omega_6}}{BZ_2}\right) + F\left(\frac{1+\psi_B B}{1+\psi_B B_2}\right)\right] - \frac{e^{\alpha_4\omega_4}\psi_B(\eta_E+\delta_E)(B-B_2)^2}{(1+\psi_B B)(1+\psi_B B_2)B_2}E_2$$
$$+ \frac{e^{\sum_{i=1}^{3}\alpha_i\omega_i}\eta_C\eta_Y(\eta_L+\delta_L)}{\theta_C\delta_L}(\Re_3 - 1)C - \frac{\gamma_{HC}\psi_C H_2 C^2}{1+\psi_C C}.$$

Since $\Re_3 \leq 1$ then, $\frac{d\Lambda_2}{dt} \leq 0$ for all $H, L, Y, E, Z, C, B > 0$. Further, $\frac{d\Lambda_2}{dt} = 0$ when $H = H_2$, $E = E_2$, $Z = Z_2$, $B = B_2$ and $C = 0$. The solutions of system (2.1)–(2.7) tend to $\tilde{\Omega}_2$ which contains elements with $C = 0$, which gives $\dot{C} = 0$. Equation (2.6) implies

$$0 = \dot{C} = e^{-\alpha_3\omega_3}\theta_C Y_{\omega_3} \Longrightarrow Y(t) = 0, \quad \text{for any } t.$$

Then, Eq (2.3) becomes

$$0 = \dot{Y} = e^{-\alpha_2\omega_2}\delta_L L_{\omega_2} \Longrightarrow L(t) = 0, \quad \text{for any } t.$$

Therefore, $\tilde{\Omega}_2 = \{\Xi_2\}$. Applying LIP, we get Ξ_2 is GAS. □

Theorem 4. Consider (2.1)–(2.7) and suppose that $\Re_3 > 1$ and $\Re_4 > 1$, then Ξ_3 is GAS.

Proof. Define

$$\Lambda_3 = H_3 F\left(\frac{H}{H_3}\right) + e^{\alpha_1\omega_1}L_3 F\left(\frac{L}{L_3}\right) + \frac{e^{\sum_{i=1}^{2}\alpha_i\omega_i}(\eta_L+\delta_L)}{\delta_L}Y_3 F\left(\frac{Y}{Y_3}\right) + e^{\alpha_4\omega_4}E_3 F\left(\frac{E}{E_3}\right)$$
$$+ \frac{e^{\sum_{i=4}^{5}\alpha_i\omega_i}(\eta_E+\delta_E)}{\delta_E}Z_3 F\left(\frac{Z}{Z_3}\right) + \frac{e^{\sum_{i=1}^{3}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\theta_C\delta_L}C_3 F\left(\frac{C}{C_3}\right)$$
$$+ \frac{e^{\sum_{i=4}^{6}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\theta_B\delta_E}B_3 F\left(\frac{B}{B_3}\right) + \frac{\gamma_{HC}H_3C_3}{1+\psi_C C_3}\int_{t-\omega_1}^{t} F\left(\frac{H(u)C(u)(1+\psi_C C_3)}{H_3C_3(1+\psi_C C(u))}\right)du$$

$$+ e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_3\int_{t-\omega_2}^{t}F\left(\frac{L(u)}{L_3}\right)du + \frac{\gamma_{HB}H_3B_3}{1+\psi_BB_3}\int_{t-\omega_4}^{t}F\left(\frac{H(u)B(u)(1+\psi_BB_3)}{H_3B_3(1+\psi_BB(u))}\right)du$$

$$+ e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_3\int_{t-\omega_5}^{t}F\left(\frac{E(u)}{E_3}\right)du + \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y_3\int_{t-\omega_3}^{t}F\left(\frac{Y(u)}{Y_3}\right)du$$

$$+ \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z_3\int_{t-\omega_6}^{t}F\left(\frac{Z(u)}{Z_3}\right)du.$$

We calculate $\frac{d\Lambda_3}{dt}$ as:

$$\begin{aligned}
\frac{d\Lambda_3}{dt} &= \left(1-\frac{H_3}{H}\right)\left[\phi-\eta_HH-\frac{\gamma_{HC}HC}{1+\psi_CC}-\frac{\gamma_{HB}HB}{1+\psi_BB}\right]\\
&+ e^{\alpha_1\omega_1}\left(1-\frac{L_3}{L}\right)\left[e^{-\alpha_1\omega_1}\frac{\gamma_{HC}H_{\omega_1}C_{\omega_1}}{1+\psi_CC_{\omega_1}}-(\eta_L+\delta_L)L\right]\\
&+ \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}(\eta_L+\delta_L)}{\delta_L}\left(1-\frac{Y_3}{Y}\right)\left[e^{-\alpha_2\omega_2}\delta_LL_{\omega_2}-\eta_YY\right]\\
&+ e^{\alpha_4\omega_4}\left(1-\frac{E_3}{E}\right)\left[e^{-\alpha_4\omega_4}\frac{\gamma_{HB}H_{\omega_4}B_{\omega_4}}{1+\psi_BB_{\omega_4}}-(\eta_E+\delta_E)E\right]\\
&+ \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}(\eta_E+\delta_E)}{\delta_E}\left(1-\frac{Z_3}{Z}\right)\left[e^{-\alpha_5\omega_5}\delta_EE_{\omega_5}-\eta_ZZ\right]\\
&+ \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\theta_C\delta_L}\left(1-\frac{C_3}{C}\right)[e^{-\alpha_3\omega_3}\theta_CY_{\omega_3}-\eta_CC]\\
&+ \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\theta_B\delta_E}\left(1-\frac{B_3}{B}\right)[e^{-\alpha_6\omega_6}\theta_BZ_{\omega_6}-\eta_BB]\\
&+ \frac{\gamma_{HC}HC}{1+\psi_CC}-\frac{\gamma_{HC}H_{\omega_1}C_{\omega_1}}{1+\psi_CC_{\omega_1}}+\frac{\gamma_{HC}H_3C_3}{1+\psi_CC_3}\ln\left(\frac{H_{\omega_1}C_{\omega_1}(1+\psi_CC)}{HC(1+\psi_CC_{\omega_1})}\right)\\
&+ e^{\alpha_1\omega_1}(\eta_L+\delta_L)L-e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_{\omega_2}+e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_3\ln\left(\frac{L_{\omega_2}}{L}\right)\\
&+ \frac{\gamma_{HB}HB}{1+\psi_BB}-\frac{\gamma_{HB}H_{\omega_4}B_{\omega_4}}{1+\psi_BB_{\omega_4}}+\frac{\gamma_{HB}H_3B_3}{1+\psi_BB_3}\ln\left(\frac{H_{\omega_4}B_{\omega_4}(1+\psi_BB)}{HB(1+\psi_BB_{\omega_4})}\right)\\
&+ e^{\alpha_4\omega_4}(\eta_E+\delta_E)E-e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_{\omega_5}+e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_3\ln\left(\frac{E_{\omega_5}}{E}\right)\\
&+ \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y-\frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y_{\omega_3}\\
&+ \frac{e^{\Sigma_{i=1}^{2}\alpha_i\omega_i}\eta_Y(\eta_L+\delta_L)}{\delta_L}Y_3\ln\left(\frac{Y_{\omega_3}}{Y}\right)+\frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z\\
&- \frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z_{\omega_6}+\frac{e^{\Sigma_{i=4}^{5}\alpha_i\omega_i}\eta_Z(\eta_E+\delta_E)}{\delta_E}Z_3\ln\left(\frac{Z_{\omega_6}}{Z}\right). \qquad (2.16)
\end{aligned}$$

Then collecting terms of Eq (2.16) and using the equilibrium conditions for Ξ_3:

$$\phi = \eta_H H_3 + \frac{\gamma_{HC} H_3 C_3}{1+\psi_C C_3} + \frac{\gamma_{HB} H_3 B_3}{1+\psi_B B_3}, \quad \frac{\gamma_{HC} H_3 C_3}{1+\psi_C C_3} = e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_3,$$
$$\delta_L L_3 = e^{\alpha_2\omega_2}\eta_Y Y_3, \quad \frac{\gamma_{HB} H_3 B_3}{1+\psi_B B_3} = e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_3, \quad \delta_E E_3 = e^{\alpha_5\omega_5}\eta_Z Z_3,$$
$$\theta_C Y_3 = e^{\alpha_3\omega_3}\eta_C C_3, \quad \theta_B Z_3 = e^{\alpha_6\omega_6}\eta_B B_3,$$

and equalities (2.13) and (2.15), we get:

$$\begin{aligned}\frac{d\Lambda_3}{dt} = &-\frac{\eta_H(H-H_3)^2}{H} - e^{\alpha_1\omega_1}(\eta_L+\delta_L)L_3\left[F\left(\frac{H_3}{H}\right)+F\left(\frac{H_{\omega_1}C_{\omega_1}L_3(1+\psi_C C_3)}{H_3C_3L(1+\psi_C C_{\omega_1})}\right)\right.\\ &\left.+F\left(\frac{Y_3L_{\omega_2}}{YL_3}\right)+F\left(\frac{C_3Y_{\omega_3}}{CY_3}\right)+F\left(\frac{1+\psi_C C}{1+\psi_C C_3}\right)\right]\\ &- e^{\alpha_4\omega_4}(\eta_E+\delta_E)E_3\left[F\left(\frac{H_3}{H}\right)+F\left(\frac{H_{\omega_4}B_{\omega_4}E_3(1+\psi_B B_3)}{H_3B_3E(1+\psi_B B_{\omega_4})}\right)+F\left(\frac{Z_3E_{\omega_5}}{ZE_3}\right)\right.\\ &\left.+F\left(\frac{B_3Z_{\omega_6}}{BZ_3}\right)+F\left(\frac{1+\psi_B B}{1+\psi_B B_3}\right)\right] - \frac{e^{\alpha_1\omega_1}\psi_C(\eta_L+\delta_L)(C-C_3)^2}{(1+\psi_C C)(1+\psi_C C_3)C_3}L_3\\ &- \frac{e^{\alpha_4\omega_4}\psi_B(\eta_E+\delta_E)(B-B_3)^2}{(1+\psi_B B)(1+\psi_B B_3)B_3}E_3.\end{aligned}$$

So, we get $\frac{d\Lambda_3}{dt} \leq 0$ for all $H, L, Y, E, Z, C, B > 0$. Further, $\frac{d\Lambda_3}{dt} = 0$ when $H = H_3$, $L = L_3$, $Y = Y_3$, $E = E_3$, $Z = Z_3$, $C = C_3$ and $Z = Z_3$. Therefore, $\tilde{\Omega}_3 = \{\Xi_3\}$. Applying LIP, we find that Ξ_3 is GAS. □

We have compiled the existence and global stability conditions for each equilibrium point in Table 1 based on the aforementioned results.

Table 1. Existence and global stability conditions of the equilibria of system (2.1)–(2.7).

Equilibrium point	Existence conditions	Global stability conditions
$\Xi_0 = (H_0, 0, 0, 0, 0, 0, 0)$	None	$\Re_1 \leq 1$ and $\Re_2 \leq 1$
$\Xi_1 = (H_1, L_1, Y_1, 0, 0, C_1, 0)$	$\Re_1 > 1$	$\Re_1 > 1$ and $\Re_4 \leq 1$
$\Xi_2 = (H_2, 0, 0, E_2, Z_2, 0, B_2)$	$\Re_2 > 1$	$\Re_2 > 1$ and $\Re_3 \leq 1$
$\Xi_3 = (H_3, L_3, Y_3, E_3, Z_3, C_3, B_3)$	$\Re_3 > 1$ and $\Re_4 > 1$	$\Re_3 > 1$ and $\Re_4 > 1$

3. Model with distributed-time delays

In the preceding section, we assumed that:

(i) The formation time of each latently infected cell is fixed;

(ii) The activation time of each latently infected cell is fixed;

(iii) The maturation time of each newly released viron is fixed. It is evident from a mathematical and biological perspective that the distributed delay (where the time delay is taken as a random variable drawn from the probability distribution function) is more appropriate in real-world scenarios than the discrete delay. There have been several studies on virus infection models with distributed-time delays (see [27, 31]).

3.1. Model formulation

In this section, we extend the two-virus model presented in the previous section by including six distributed-time delays as:

$$\dot{H}(t) = \phi - \eta_H H(t) - \frac{\gamma_{HC} H(t)C(t)}{1+\psi_C C(t)} - \frac{\gamma_{HB} H(t)B(t)}{1+\psi_B B(t)}, \tag{3.1}$$

$$\dot{L}(t) = \gamma_{HC} \int_0^{\varkappa_1} P_1(\omega) e^{-\alpha_1 \omega} \frac{H(t-\omega)C(t-\omega)}{1+\psi_C C(t-\omega)} d\omega - (\eta_L + \delta_L) L(t), \tag{3.2}$$

$$\dot{Y}(t) = \delta_L \int_0^{\varkappa_2} P_2(\omega) e^{-\alpha_2 \omega} L(t-\omega) d\omega - \eta_Y Y(t), \tag{3.3}$$

$$\dot{E}(t) = \gamma_{HB} \int_0^{\varkappa_4} P_4(\omega) e^{-\alpha_4 \omega} \frac{H(t-\omega)B(t-\omega)}{1+\psi_B B(t-\omega)} d\omega - (\eta_E + \delta_E) E(t), \tag{3.4}$$

$$\dot{Z}(t) = \delta_E \int_0^{\varkappa_5} P_5(\omega) e^{-\alpha_5 \omega} E(t-\omega) d\omega - \eta_Z Z(t), \tag{3.5}$$

$$\dot{C}(t) = \theta_C \int_0^{\varkappa_3} P_3(\omega) e^{-\alpha_3 \omega} Y(t-\omega) d\omega - \eta_C C(t), \tag{3.6}$$

$$\dot{B}(t) = \theta_B \int_0^{\varkappa_6} P_6(\omega) e^{-\alpha_6 \omega} Z(t-\omega) d\omega - \eta_B B(t). \tag{3.7}$$

Here ω is random taken from probability distributed function $P_i(\omega)$ during time interval $[0, \varkappa_i]$, where $\varkappa_i$ is the limit superior of the delay period, $i = 1, 2, \ldots, 6$. We have the assumptions:

(i) The probability of uninfected cells touched by viral types C and B at time $t-\omega$ surviving ω time units and becoming latent infected cells at time t are represented by factors $P_i(\omega)e^{-\alpha_i \omega}$, where $i = 1$ and 4, respectively.

(ii) The probability that latent cells infected with viruses type C and B at time $t-\omega$ would survive ω time units and become active are shown by factors $P_i(\omega)e^{-\alpha_i \omega}$, where $i = 2$ and 5, respectively.

(iii) The probability that immature viruses type C and B at time $t-\omega$ survive ω time units to become mature viruses at time t are shown by factors $P_i(\omega)e^{-\alpha_i \omega}$, where $i = 3$ and 6, respectively.

Function $P_i(\omega)$, $i = 1, 2, \ldots, 6$ satisfy $P_i(\omega) > 0$ and

$$\int_0^{\varkappa_i} P_i(\omega) d\omega = 1, \quad \int_0^{\varkappa_i} P_i(\omega) e^{-n\omega} d\omega < \infty,$$

where $n > 0$. Let us denote that

$$\mathcal{P}_i = \int_0^{\varkappa_i} P_i(\omega) e^{-\alpha_i \omega} d\omega, \;\; i = 1, 2, \ldots, 6.$$

This implies that $0 < \mathcal{P}_i \leq 1$. The initial conditions for system (3.1)–(3.7) are the same as given by Eq (2.8), where $\omega^* = \max\{\varkappa_1, \varkappa_2, \ldots, \varkappa_6\}$.

3.2. Preliminaries

Proposition 2. The solutions of system (3.1)–(3.7) with initial (2.8) are nonnegative and ultimately bounded.

Proof. We have that

$$\dot{H}\mid_{H=0}= \phi > 0.$$

Hence, $H(t) > 0$ for all $t \geq 0$. Moreover, for all $t \in [0, \omega^*]$, we have:

$$L(t) = \ell_2(0)e^{-(\eta_L+\delta_L)t} + \gamma_{HC}\int_0^t e^{-(\eta_L+\delta_L)(t-r)}\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}\frac{H(r-\omega)C(r-\omega)}{1+\psi_C C(r-\omega)}d\omega dr,$$

$$Y(t) = \ell_3(0)e^{-\eta_Y t} + \delta_L\int_0^t e^{-\eta_Y(t-r)}\int_0^{\varkappa_2} P_2(\omega)e^{-\alpha_2\omega}L(r-\omega)d\omega dr,$$

$$E(t) = \ell_4(0)e^{-(\eta_E+\delta_E)t} + \gamma_{HB}\int_0^t e^{-(\eta_E+\delta_E)(t-r)}\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}\frac{H(r-\omega)B(r-\omega)}{1+\psi_B B(r-\omega)}d\omega dr,$$

$$Z(t) = \ell_5(0)e^{-\eta_Z t} + \delta_E\int_0^t e^{-\eta_Z(t-r)}\int_0^{\varkappa_5} P_5(\omega)e^{-\alpha_5\omega}E(r-\omega)d\omega dr,$$

$$C(t) = \ell_6(0)e^{-\eta_C t} + \theta_C\int_0^t e^{-\eta_C(t-r)}\int_0^{\varkappa_3} P_3(\omega)e^{-\alpha_3\omega}Y(r-\omega)d\omega dr,$$

$$B(t) = \ell_7(0)e^{-\eta_B t} + \theta_B\int_0^t e^{-\eta_B(t-r)}\int_0^{\varkappa_6} P_6(\omega)e^{-\alpha_6\omega}Z(r-\omega)d\omega dr.$$

Hence, $L(t), Y(t), E(t), Z(t), C(t), B(t) \geq 0$ for all $t \in [0, \omega^*]$. Through recursive argumentation, we get $L(t), Y(t), E(t), Z(t), C(t), B(t)$ for all $t \geq 0$. Therefore, H, L, Y, E, Z, C and B are nonnegative.

The nonnegativity of the system's solution implies that:

$$\dot{H}(t) \leq \phi - \eta_H H(t) \implies \limsup_{t\to\infty} H(t) = \frac{\phi}{\eta_H} = a_0.$$

Let us define

$$\Phi_1(t) = \int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}H(t-\omega)d\omega + L(t).$$

Then

$$\begin{aligned}
\dot{\Phi}_1(t) &= \int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}\dot{H}(t-\omega)d\omega + \dot{L}(t)\\
&= \int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}\left[\phi - \eta_H H(t-\omega) - \frac{\gamma_{HC}H(t-\omega)C(t-\omega)}{1+\psi_C C(t-\omega)} - \frac{\gamma_{HB}H(t-\omega)B(t-\omega)}{1+\psi_B B(t-\omega)}\right]d\omega\\
&+ \gamma_{HC}\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}\frac{H(t-\omega)C(t-\omega)}{1+\psi_C C(t-\omega)}d\omega - (\eta_L+\delta_L)L(t)\\
&= \phi\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}d\omega - \eta_H\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}H(t-\omega)d\omega\\
&- \gamma_{HB}\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}\frac{H(t-\omega)B(t-\omega)}{1+\psi_B B(t-\omega)}d\omega - (\eta_L+\delta_L)L(t)
\end{aligned}$$

$$\leq \phi - \eta_H \int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}H(t-\omega)d\omega - (\eta_L + \delta_L)L(t)$$
$$\leq \phi - \sigma_1\left[\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}H(t-\omega)d\omega - L(t)\right] = \phi - \sigma_1\Phi(t).$$

It follows that

$$\limsup_{t\to\infty} \Phi_1(t) \leq \frac{\phi}{\sigma_1} = a_1 \Longrightarrow \limsup_{t\to\infty} L(t) \leq a_1.$$

Let

$$\Phi_2(t) = \int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}H(t-\omega)d\omega + E(t),$$

then,

$$\begin{aligned}
\dot{\Phi}_2(t) &= \int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}\dot{H}(t-\omega)d\omega + \dot{E}(t)\\
&= \int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}\left[\phi - \eta_H H(t-\omega) - \frac{\gamma_{HC}H(t-\omega)C(t-\omega)}{1+\psi_C C(t-\omega)} - \frac{\gamma_{HB}H(t-\omega)B(t-\omega)}{1+\psi_B B(t-\omega)}\right]d\omega\\
&+ \gamma_{HB}\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}\frac{H(t-\omega)B(t-\omega)}{1+\psi_B B(t-\omega)}d\omega - (\eta_E + \delta_E)E(t)\\
&= \phi\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}d\omega - \eta_H\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}H(t-\omega)d\omega\\
&- \gamma_{HC}\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}\frac{H(t-\omega)C(t-\omega)}{1+\psi_C C(t-\omega)} - (\eta_E + \delta_E)E(t)\\
&\leq \phi - \eta_H\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}H(t-\omega)d\omega - (\eta_E + \delta_E)E(t)\\
&\leq \phi - \sigma_2\left[\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}H(t-\omega)d\omega + E(t)\right] = \phi - \sigma_2\Phi(t).
\end{aligned}$$

It follows that

$$\limsup_{t\to\infty} \Phi_2(t) \leq \frac{\phi}{\sigma_2} = a_2 \Longrightarrow \limsup_{t\to\infty} E(t) \leq a_2.$$

From Eq (3.3) we get

$$\begin{aligned}
\dot{Y}(t) &= \delta_L\int_0^{\varkappa_2} P_2(\omega)e^{-\alpha_2\omega}L(t-\omega)d\omega - \eta_Y Y(t)\\
&\leq \delta_L a_1 - \eta_Y Y(t) \Longrightarrow \limsup_{t\to\infty} Y(t) \leq \frac{\delta_L a_1}{\eta_Y} = a_3.
\end{aligned}$$

Equation (3.5) implies that

$$\begin{aligned}
\dot{Z}(t) &= \delta_E\int_0^{\varkappa_5} P_5(\omega)e^{-\alpha_5\omega}E(t-\omega)d\omega - \eta_Z Z(t)\\
&\leq \delta_E a_2 - \eta_Z Z(t) \Longrightarrow \limsup_{t\to\infty} Z(t) \leq \frac{\delta_E a_2}{\eta_Z} = a_4.
\end{aligned}$$

Similarly from Eqs (3.6) and (3.7) we get

$$\dot{C}(t) \leq \theta_C a_3 - \eta_C C(t) \Longrightarrow \limsup_{t\to\infty} C(t) \leq \frac{\theta_C a_3}{\eta_C} = a_5,$$
$$\dot{B}(t) \leq \theta_B a_4 - \eta_B B(t) \Longrightarrow \limsup_{t\to\infty} B(t) \leq \frac{\theta_B a_4}{\eta_B} = a_6.$$

Based on Proposition 2 we can show that Γ is positively invariant for system (3.1)–(3.7). □

3.3. Equilibria

We calculate the model's equilibria deduce when they exist. Any equilibria point $\Xi = (H, L, Y, E, Z, C, B)$ satisfies:

$$\begin{aligned}
0 &= \phi - \eta_H H - \frac{\gamma_{HC} HC}{1+\psi_C C} - \frac{\gamma_{HB} HB}{1+\psi_B B},\\
0 &= \mathcal{P}_1 \frac{\gamma_{HC} HC}{1+\psi_C C} - (\eta_L + \delta_L)L,\\
0 &= \mathcal{P}_2 \delta_L L - \eta_Y Y,\\
0 &= \mathcal{P}_4 \frac{\gamma_{HB} HB}{1+\psi_B B} - (\eta_E + \delta_E)E,\\
0 &= \mathcal{P}_5 \delta_E E - \eta_Z Z,\\
0 &= \mathcal{P}_3 \theta_C Y - \eta_C C,\\
0 &= \mathcal{P}_6 \theta_B Z - \eta_B B.
\end{aligned} \tag{3.8}$$

System (3.8) admits four equilibria.

(I) Infection-free equilibrium, $\Xi_0 = (H_0, 0, 0, 0, 0, 0, 0)$, where $H_0 = \phi/\eta_H$.

(II) Virus type C single-infection equilibrium $\Xi_1 = (H_1, L_1, Y_1, 0, 0, C_1, 0)$, where

$$H_1 = \frac{\eta_Y \eta_C(\eta_L + \delta_L) + \psi_C \phi \delta_L \theta_C \Pi_{i=1}^3 \mathcal{P}_i}{\delta_L \theta_C(\eta_H \psi_C + \gamma_{HC})\Pi_{i=1}^3 \mathcal{P}_i}, \quad L_1 = \frac{\eta_Y \eta_C \eta_H}{\delta_L \theta_C(\eta_H \psi_C + \gamma_{HC})\Pi_{i=2}^3 \mathcal{P}_i}(\mathcal{R}_1 - 1),$$
$$Y_1 = \frac{\eta_C \eta_H}{\mathcal{P}_3 \theta_C(\eta_H \psi_C + \gamma_{HC})}(\mathcal{R}_1 - 1), \qquad C_1 = \frac{\eta_H}{(\eta_H \psi_C + \gamma_{HC})}(\mathcal{R}_1 - 1),$$

where, $\mathcal{R}_1$ is given by

$$\mathcal{R}_1 = \frac{H_0 \delta_L \theta_C \gamma_{HC} \Pi_{i=1}^3 \mathcal{P}_i}{\eta_Y \eta_C(\eta_L + \delta_L)}.$$

It follows that, Ξ_1 exists if $\mathcal{R}_1 > 1$.

(III) Virus type B single-infection equilibrium $\Xi_2 = (H_2, 0, 0, E_2, Z_2, 0, B_2)$, where

$$H_2 = \frac{\eta_Z \eta_B(\eta_E + \delta_E) + \delta_E \psi_B \phi \theta_B \Pi_{i=4}^6 \mathcal{P}_i}{\delta_E \theta_B(\eta_H \psi_B + \gamma_{HB})\Pi_{i=4}^6 \mathcal{P}_i}, \quad E_2 = \frac{\eta_Z \eta_B \eta_H}{\delta_E \theta_B(\eta_H \psi_B + \gamma_{HB})\Pi_{i=5}^6 \mathcal{P}_i}(\mathcal{R}_2 - 1),$$
$$Z_2 = \frac{\eta_B \eta_H}{\mathcal{P}_6 \theta_B(\eta_H \psi_B + \gamma_{HB})}(\mathcal{R}_2 - 1), \qquad B_2 = \frac{\eta_H}{(\eta_H \psi_B + \gamma_{HB})}(\mathcal{R}_2 - 1),$$

where

$$\mathcal{R}_2 = \frac{H_0\delta_E\theta_B\gamma_{HB}\Pi_{i=4}^{6}\mathcal{P}_i}{\eta_Z\eta_B(\eta_E+\delta_E)}.$$

Therefore, Ξ_2 exists if $\mathcal{R}_2 > 1$.

(IV) Coexistence equilibrium $\Xi_3 = (H_3, L_3, Y_3, E_3, Z_3, C_3, B_3)$, where

$$\begin{aligned}
H_3 &= \frac{\phi\psi_C\delta_L\theta_C\psi_B\delta_E\theta_B\Pi_{i=1}^{6}\mathcal{P}_i + \eta_Y\eta_C\psi_B\delta_E\theta_B(\eta_L+\delta_L)\Pi_{i=4}^{6}\mathcal{P}_i + \eta_Z\eta_B\psi_C\delta_L\theta_C(\eta_E+\delta_E)\Pi_{i=1}^{3}\mathcal{P}_i}{\delta_E\theta_B\delta_L\theta_C(\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C)\Pi_{i=1}^{6}\mathcal{P}_i},\\
L_3 &= \frac{\eta_Y\eta_C(\eta_H\psi_B+\gamma_{HB})}{\delta_L\theta_C(\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C)\Pi_{i=2}^{3}\mathcal{P}_i}(\mathcal{R}_3-1),\\
Y_3 &= \frac{\eta_C(\eta_H\psi_B+\gamma_{HB})}{\mathcal{P}_3\theta_C(\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C)}(\mathcal{R}_3-1),\\
E_3 &= \frac{\eta_Z\eta_B(\eta_H\psi_C+\gamma_{HC})}{\delta_E\theta_B(\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C)\Pi_{i=5}^{6}\mathcal{P}_i}(\mathcal{R}_4-1),\\
Z_3 &= \frac{\eta_B(\eta_H\psi_C+\gamma_{HC})}{\mathcal{P}_6\theta_B(\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C)}(\mathcal{R}_4-1),\\
C_3 &= \frac{\eta_H\psi_B+\gamma_{HB}}{\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C}(\mathcal{R}_3-1),\\
B_3 &= \frac{\eta_H\psi_C+\gamma_{HC}}{\eta_H\psi_C\psi_B+\gamma_{HC}\psi_B+\gamma_{HB}\psi_C}(\mathcal{R}_4-1),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_3 &= \frac{\gamma_{HC}\delta_L\theta_C[\phi\psi_B\delta_E\theta_B\Pi_{i=4}^{6}\mathcal{P}_i + \eta_Z\eta_B(\eta_E+\delta_E)]\Pi_{i=1}^{3}\mathcal{P}_i}{\eta_Y\eta_C\delta_E\theta_B(\eta_L+\delta_L)(\eta_H\psi_B+\gamma_{HB})\Pi_{i=4}^{6}\mathcal{P}_i},\\
\mathcal{R}_4 &= \frac{\gamma_{HB}\delta_E\theta_B[\phi\psi_C\delta_L\theta_C\Pi_{i=1}^{3}\mathcal{P}_i + \eta_Y\eta_C(\eta_L+\delta_L)]\Pi_{i=4}^{6}\mathcal{P}_i}{\eta_Z\eta_B\delta_L\theta_C(\eta_E+\delta_E)(\eta_H\psi_C+\gamma_{HC})\Pi_{i=1}^{3}\mathcal{P}_i}.
\end{aligned}$$

Clearly, Ξ_3 exists when $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$.

3.4. Global stability analysis

Let $\Theta_j(H, L, Y, E, Z, C, B)$ be a Lyapunov function candidate and $\tilde{\Upsilon}_j$ be the largest invariant subset of

$$\Upsilon_j = \left\{(H, L, Y, E, Z, C, B) : \frac{d\Theta_j}{dt} = 0\right\}, \quad j = 0, 1, 2, 4.$$

We denote

$$(H, L, Y, E, Z, C, B) = (H(t), L(t), Y(t), E(t), Z(t), C(t), B(t))$$

and

$$(H_\omega, L_\omega, Y_\omega, E_\omega, Z_\omega, C_\omega, B_\omega) = (H(t-\omega), L(t-\omega), Y(t-\omega), E(t-\omega), Z(t-\omega), C(t-\omega), B(t-\omega)).$$

Theorem 5. For system (3.1)–(3.7) suppose that $\mathcal{R}_1 \leq 1$ and $\mathcal{R}_2 \leq 1$, then Ξ_0 is GAS.

Proof. Define

$$\begin{aligned}\Theta_0 &= H_0 F\left(\frac{H}{H_0}\right) + \frac{1}{\mathcal{P}_1}L + \frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}Y + \frac{1}{\mathcal{P}_4}E + \frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}Z \\ &+ \frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\theta_C\delta_L}C + \frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\theta_B\delta_E}B + \frac{1}{\mathcal{P}_1}\gamma_{HC}\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}\int_{t-\omega}^{t}\frac{H(u)C(u)}{1+\psi_C C(u)}dud\omega \\ &+ \frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2}\int_0^{\varkappa_2} P_2(\omega)e^{-\alpha_2\omega}\int_{t-\omega}^{t}L(u)dud\omega + \frac{1}{\mathcal{P}_4}\gamma_{HB}\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}\int_{t-\omega}^{t}\frac{H(u)B(u)}{1+\psi_B B(u)}dud\omega \\ &+ \frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5}\int_0^{\varkappa_5} P_5(\omega)e^{-\alpha_5\omega}\int_{t-\omega}^{t}E(u)dud\omega + \frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\delta_L}\int_0^{\varkappa_3} P_3(\omega)e^{-\alpha_3\omega}\int_{t-\omega}^{t}Y(u)dud\omega \\ &+ \frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\delta_E}\int_0^{\varkappa_6} P_6(\omega)e^{-\alpha_6\omega}\int_{t-\omega}^{t}Z(u)dud\omega.\end{aligned}$$

It is seen that, $\Theta_0 > 0$ for all $H, L, Y, E, Z, C, B > 0$, and $\Theta_0(H_0, 0, 0, 0, 0, 0, 0) = 0$. Calculate $\frac{d\Theta_0}{dt}$ as:

$$\begin{aligned}\frac{d\Theta_0}{dt} &= \left(1-\frac{H_0}{H}\right)\left[\phi - \eta_H H - \frac{\gamma_{HC}HC}{1+\psi_C C} - \frac{\gamma_{HB}HB}{1+\psi_B B}\right] \\ &+ \frac{1}{\mathcal{P}_1}\left[\gamma_{HC}\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}\frac{H_\omega C_\omega}{1+\psi_C C_\omega}d\omega - (\eta_L+\delta_L)L\right] \\ &+ \frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}\left[\delta_L\int_0^{\varkappa_2} P_2(\omega)e^{-\alpha_2\omega}L_\omega d\omega - \eta_Y Y\right] \\ &+ \frac{1}{\mathcal{P}_4}\left[\gamma_{HB}\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_3\omega}\frac{H_\omega B_\omega}{1+\psi_B B_\omega}d\omega - (\eta_E+\delta_E)E\right] \\ &+ \frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}\left[\delta_E\int_0^{\varkappa_5} P_5(\omega)e^{-\alpha_5\omega}E_\omega d\omega - \eta_Z Z\right] \\ &+ \frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\theta_C\delta_L}\left[\theta_C\int_0^{\varkappa_3} P_3(\omega)e^{-\alpha_3\omega}Y_\omega d\omega - \eta_C C\right] \\ &+ \frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\theta_B\delta_E}\left[\theta_B\int_0^{\varkappa_6} P_6(\omega)e^{-\alpha_6\omega}Z_\omega d\omega - \eta_B B\right] + \frac{\gamma_{HC}HC}{1+\psi_C C} \\ &- \frac{1}{\mathcal{P}_1}\gamma_{HC}\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}\frac{H_\omega C_\omega}{1+\psi_C C_\omega}d\omega + \frac{(\eta_L+\delta_L)}{\mathcal{P}_1}L \\ &- \frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2}\int_0^{\varkappa_2} P_2(\omega)e^{-\alpha_2\omega}L_\omega d\omega + \frac{\gamma_{HB}HB}{1+\psi_B B} \\ &- \frac{1}{\mathcal{P}_4}\gamma_{HB}\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}\frac{H_\omega B_\omega}{1+\psi_B B_\omega}d\omega + \frac{(\eta_E+\delta_E)}{\mathcal{P}_4}E \\ &- \frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5}\int_0^{\varkappa_5} P_5(\omega)e^{-\alpha_5\omega}E_\omega d\omega + \frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}Y \\ &- \frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\delta_L}\int_0^{\varkappa_3} P_3(\omega)e^{-\alpha_3\omega}Y_\omega d\omega + \frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}Z - \frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\delta_E}\int_0^{\varkappa_6} P_6(\omega)e^{-\alpha_6\omega}Z_\omega d\omega.\end{aligned}$$

Collecting terms and using the equilibrium condition $\phi = \eta_H H_0$, we get:

$$\frac{d\Theta_0}{dt} = -\eta_H \frac{(H-H_0)^2}{H} - \frac{\gamma_{HC}\psi_C H_0 C^2}{1+\psi_C C} - \frac{\gamma_{HB}\psi_B H_0 B^2}{1+\psi_B B} + \frac{\eta_Y \eta_C(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\theta_C\delta_L}(\mathcal{R}_1-1)C + \frac{\eta_Z\eta_B(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\theta_B\delta_E}(\mathcal{R}_2-1)B.$$

Since $\mathcal{R}_1 \leq 1$ and $\mathcal{R}_2 \leq 1$, then $\frac{d\Theta_0}{dt} \leq 0$ for all $H, C, B > 0$. In addition $\frac{d\Theta_0}{dt} = 0$ when $H = H_0$ and $C = B = 0$. The solutions of system (3.1)–(3.7) tend to $\tilde{\Upsilon}_0$ [40] which includes elements with $C = B = 0$. Thus, $\dot{C} = \dot{B} = 0$ and from Eqs (3.6) and (3.7) we have:

$$0 = \dot{C} = \theta_C \int_0^{\varkappa_3} P_3(\omega)e^{-\alpha_3\omega}Y_\omega d\omega \Longrightarrow Y(t) = 0, \text{ for any } t,$$
$$0 = \dot{B} = \theta_B \int_0^{\varkappa_6} P_6(\omega)e^{-\alpha_6\omega}Z_\omega d\omega \Longrightarrow Z(t) = 0, \text{ for any } t.$$

Then from Eqs (3.3) and (3.5) we get

$$0 = \dot{Y} = \delta_L \int_0^{\varkappa_2} P_2(\omega)e^{-\alpha_2\omega}L_\omega d\omega \Longrightarrow L(t) = 0, \text{ for any } t,$$
$$0 = \dot{Z} = \delta_E \int_0^{\varkappa_5} P_5(\omega)e^{-\alpha_5\omega}E_\omega d\omega \Longrightarrow E(t) = 0, \text{ for any } t.$$

Therefore, $\tilde{\Upsilon}_0 = \{\Xi_0\}$ and applying LIP, we obtain that Ξ_0 is GAS. □

Theorem 6. For system (3.1)–(3.7) suppose that $\mathcal{R}_1 > 1$ and $\mathcal{R}_4 \leq 1$, then Ξ_1 is GAS.

Proof. Let us formulate a Lyapunov function Θ_1 as:

$$\begin{aligned}\Theta_1 = {} & H_1 F\left(\frac{H}{H_1}\right) + \frac{1}{\mathcal{P}_1}L_1 F\left(\frac{L}{L_1}\right) + \frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}Y_1 F\left(\frac{Y}{Y_1}\right) + \frac{1}{\mathcal{P}_4}E \\ & + \frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}Z + \frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\theta_C\delta_L}C_1 F\left(\frac{C}{C_1}\right) + \frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\theta_B\delta_E}B \\ & + \frac{1}{\mathcal{P}_1}\frac{\gamma_{HC}H_1C_1}{1+\psi_C C_1}\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}\int_{t-\omega}^{t} F\left(\frac{H(u)C(u)(1+\psi_C C_1)}{H_1C_1(1+\psi_C C(u))}\right)dud\omega \\ & + \frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2}L_1\int_0^{\varkappa_2} P_2(\omega)e^{-\alpha_2\omega}\int_{t-\omega}^{t} F\left(\frac{L(u)}{L_1}\right)dud\omega \\ & + \frac{1}{\mathcal{P}_4}\gamma_{HB}\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}\int_{t-\omega}^{t}\frac{H(u)B(u)}{1+\psi_B B(u)}dud\omega + \frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5}\int_0^{\varkappa_5} P_5(\omega)e^{-\alpha_5\omega}\int_{t-\omega}^{t}E(u)dud\omega \\ & + \frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\delta_L}Y_1\int_0^{\varkappa_3} P_3(\omega)e^{-\alpha_3\omega}\int_{t-\omega}^{t} + \frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\delta_E}\int_0^{\varkappa_6} P_6(\omega)e^{-\alpha_6\omega}\int_{t-\omega}^{t}Z(u)dud\omega.\end{aligned}$$

We calculate $\frac{d\Theta_1}{dt}$ as:

$$\begin{aligned}
\frac{d\Theta_1}{dt} &= \left(1-\frac{H_1}{H}\right)\left[\phi-\eta_H H-\frac{\gamma_{HC}HC}{1+\psi_C C}-\frac{\gamma_{HB}HB}{1+\psi_B B}\right]\\
&+\frac{1}{\mathcal{P}_1}\left(1-\frac{L_1}{L}\right)\left[\gamma_{HC}\int_0^{\varkappa_1}P_1(\omega)e^{-\alpha_1\omega}\frac{H_\omega C_\omega}{1+\psi_C C_\omega}d\omega-(\eta_L+\delta_L)L\right]\\
&+\frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}\left(1-\frac{Y_1}{Y}\right)\left[\delta_L\int_0^{\varkappa_2}P_2(\omega)e^{-\alpha_2\omega}L_\omega d\omega-\eta_Y Y\right]\\
&+\frac{1}{\mathcal{P}_4}\left[\gamma_{HB}\int_0^{\varkappa_4}P_4(\omega)e^{-\alpha_4\omega}\frac{H_\omega B_\omega}{1+\psi_B B_\omega}d\omega-(\eta_E+\delta_E)E\right]\\
&+\frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}[\delta_E\int_0^{\varkappa_5}P_5(\omega)e^{-\alpha_5\omega}E_\omega d\omega-\eta_Z Z]\\
&+\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\theta_C\delta_L}\left(1-\frac{C_1}{C}\right)\left[\theta_C\int_0^{\varkappa_3}P_3(\omega)e^{-\alpha_3\omega}Y_\omega d\omega-\eta_C C\right]\\
&+\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\theta_B\delta_E}\left[\theta_B\int_0^{\varkappa_6}P_6(\omega)e^{-\alpha_6\omega}Z_\omega d\omega-\eta_B B\right]\\
&+\frac{\gamma_{HC}HC}{1+\psi_C C}-\frac{1}{\mathcal{P}_1}\gamma_{HC}\int_0^{\varkappa_1}P_1(\omega)e^{-\alpha_1\omega}\frac{H_\omega C_\omega}{1+\psi_C C_\omega}d\omega\\
&+\frac{1}{\mathcal{P}_1}\frac{\gamma_{HC}H_1C_1}{1+\psi_C C_1}\int_0^{\varkappa_1}P_1(\omega)e^{-\alpha_1\omega}\ln\left(\frac{H_\omega C_\omega(1+\psi_C C)}{HC(1+\psi_C C_\omega)}\right)d\omega\\
&+\frac{(\eta_L+\delta_L)}{\mathcal{P}_1}L-\frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2}\int_0^{\varkappa_2}P_2(\omega)e^{-\alpha_2\omega}L_\omega d\omega\\
&+\frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2}L_1\int_0^{\varkappa_2}P_2(\omega)e^{-\alpha_2\omega}\ln\left(\frac{L_\omega}{L}\right)d\omega+\frac{\gamma_{HB}HB}{1+\psi_B B}\\
&-\frac{1}{\mathcal{P}_4}\gamma_{HB}\int_0^{\varkappa_4}P_4(\omega)e^{-\alpha_4\omega}\frac{H_\omega B_\omega}{1+\psi_B B_\omega}d\omega+\frac{(\eta_E+\delta_E)}{\mathcal{P}_4}E\\
&-\frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5}\int_0^{\varkappa_5}P_5(\omega)e^{-\alpha_5\omega}E_\omega d\omega+\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}Y\\
&-\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\delta_L}\int_0^{\varkappa_3}P_3(\omega)e^{-\alpha_3\omega}Y_\omega d\omega+\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\delta_L}Y_1\int_0^{\varkappa_3}P_3(\omega)e^{-\alpha_3\omega}\ln\left(\frac{Y_\omega}{Y}\right)d\omega\\
&+\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}Z-\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\delta_E}\int_0^{\varkappa_6}P_6(\omega)e^{-\alpha_6\omega}Z_\omega d\omega. \qquad (3.9)
\end{aligned}$$

Simplifying Eq (3.9) and using the equilibrium conditions for Ξ_1:

$$\begin{aligned}
\phi &= \eta_H H_1+\frac{\gamma_{HC}H_1C_1}{1+\psi_C C_1},\quad \frac{\gamma_{HC}H_1C_1}{1+\psi_C C_1}=\frac{1}{\mathcal{P}_1}(\eta_L+\delta_L)L_1,\\
\mathcal{P}_2\delta_L L_1 &= \eta_Y Y_1,\quad \mathcal{P}_3\theta_C Y_1=\eta_C C_1,
\end{aligned}$$

and the following equalities:

$$\ln\left(\frac{H_\omega C_\omega(1+\psi_C C)}{HC(1+\psi_C C_\omega)}\right)=\ln\left(\frac{L_i H_\omega C_\omega(1+\psi_C C_i)}{LH_i C_i(1+\psi_C C_\omega)}\right)+\ln\left(\frac{1+\psi_C C}{1+\psi_C C_i}\right)+\ln\left(\frac{H_i}{H}\right)+\ln\left(\frac{C_i L}{CL_i}\right),$$

$$\ln\left(\frac{L_\omega}{L}\right) = \ln\left(\frac{Y_i L_\omega}{Y L_i}\right) + \ln\left(\frac{Y L_i}{Y_i L}\right),$$
$$\ln\left(\frac{Y_\omega}{Y}\right) = \ln\left(\frac{C_i Y_\omega}{C Y_i}\right) + \ln\left(\frac{C Y_i}{C_i Y}\right), \tag{3.10}$$

where $i = 1, 3$, we get:

$$\begin{aligned}\frac{d\Theta_1}{dt} &= -\frac{\eta_H(H-H_1)^2}{H} - \frac{\gamma_{HB}\psi_B H_1 B^2}{1+\psi_B B} - \frac{1}{\mathcal{P}_1}(\eta_L+\delta_L)L_1\left[F\left(\frac{H_1}{H}\right) + F\left(\frac{1+\psi_C C}{1+\psi_C C_1}\right)\right] \\ &- \frac{\psi_C(\eta_L+\delta_L)(C-C_1)^2}{\mathcal{P}_1(1+\psi_C C)(1+\psi_C C_1)C_1}L_1 - \frac{1}{\mathcal{P}_1^2}(\eta_L+\delta_L)L_1\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}F\left(\frac{H_\omega C_\omega L_1(1+\psi_C C_1)}{H_1 C_1 L(1+\psi_C C_\omega)}\right)d\omega \\ &- \frac{1}{\mathcal{P}_1\mathcal{P}_2}(\eta_L+\delta_L)L_1\int_0^{\varkappa_2} P_2(\omega)e^{-\alpha_2\omega}F\left(\frac{Y_1 L_\omega}{Y L_1}\right)d\omega \\ &- \frac{1}{\mathcal{P}_1\mathcal{P}_3}(\eta_L+\delta_L)L_1\int_0^{\varkappa_3} P_3(\omega)e^{-\alpha_3\omega}F\left(\frac{C_1 Y_\omega}{C Y_1}\right)d\omega + \frac{\eta_Z\eta_B(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\theta_B\delta_E}(\mathcal{R}_4-1)B.\end{aligned}$$

Since $\mathcal{R}_4 \leq 1$ then, $\frac{d\Theta_1}{dt} \leq 0$ for all $H, L, Y, E, Z, C, B > 0$. Moreover, $\frac{d\Theta_1}{dt} = 0$ when $H = H_1$, $L = L_1$, $Y = Y_1$, $C = C_1$, and $B = 0$. The solutions of system (3.1)–(3.7) tend to $\tilde{\Upsilon}_1$ which includes elements with $B = 0$ which gives $\dot{B} = 0$. From Eq (3.7) we get

$$0 = \dot{B} = \theta_B\int_0^{\varkappa_6} P_6(\omega)e^{-\alpha_6\omega}Z_\omega d\omega \Longrightarrow Z(t) = 0, \quad \text{for any } t.$$

Then from Eq (3.5) we have

$$0 = \dot{Z} = \delta_E\int_0^{\varkappa_5} P_5(\omega)e^{-\alpha_5\omega}E_\omega d\omega \Longrightarrow E(t) = 0, \quad \text{for any } t.$$

Hence, $\tilde{\Upsilon}_1 = \{\Xi_1\}$ and Ξ_1 is GAS using LIP. □

Theorem 7. For system (3.1)–(3.7) suppose that $\mathcal{R}_2 > 1$ and $\mathcal{R}_3 \leq 1$, then Ξ_2 is GAS.

Proof. Consider

$$\begin{aligned}\Theta_2 &= H_2 F\left(\frac{H}{H_2}\right) + \frac{1}{\mathcal{P}_1}L + \frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}Y + \frac{1}{\mathcal{P}_4}E_2 F\left(\frac{E}{E_2}\right) + \frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}Z_2 F\left(\frac{Z}{Z_2}\right) \\ &+ \frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\theta_C\delta_L}C + \frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\theta_B\delta_E}B_2 F\left(\frac{B}{B_2}\right) \\ &+ \frac{1}{\mathcal{P}_1}\gamma_{HC}\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}\int_{t-\omega}^{t}\frac{H(u)C(u)}{1+\psi_C C(u)}dud\omega + \frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2}\int_0^{\varkappa_2} P_2(\omega)e^{-\alpha_2\omega}\int_{t-\omega}^{t}L(u)dud\omega \\ &+ \frac{1}{\mathcal{P}_4}\gamma_{HB}\frac{H_2B_2}{1+\psi_B B_2}\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}\int_{t-\omega}^{t}F\left(\frac{H(u)B(u)(1+\psi_B B_2)}{H_2B_2(1+\psi_B B(u))}\right)dud\omega \\ &+ \frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5}E_2\int_0^{\varkappa_5} P_5(\omega)e^{-\alpha_5\omega}\int_{t-\omega}^{t}F\left(\frac{E(u)}{E_2}\right)dud\omega\end{aligned}$$

$$+\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\delta_L}\int_0^{\varkappa_3}P_3(\omega)e^{-\alpha_3\omega}\int_{t-\omega}^{t}Y(u)dud\omega+\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\delta_E}Z_2\int_0^{\varkappa_6}P_6(\omega)e^{-\alpha_6\omega}\int_{t-\omega}^{t}F\left(\frac{Z(u)}{Z_2}\right)dud\omega.$$

We calculate $\frac{d\Theta_2}{dt}$ as:

$$\begin{aligned}
\frac{d\Theta_2}{dt}&=\left(1-\frac{H_2}{H}\right)\left[\phi-\eta_H H-\frac{\gamma_{HC}HC}{1+\psi_C C}-\frac{\gamma_{HB}HB}{1+\psi_B B}\right]\\
&+\frac{1}{\mathcal{P}_1}\left[\gamma_{HC}\int_0^{\varkappa_1}P_1(\omega)e^{-\alpha_1\omega}\frac{H_\omega C_\omega}{1+\psi_C C_\omega}d\omega-(\eta_L+\delta_L)L\right]\\
&+\frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}\left[\delta_L\int_0^{\varkappa_2}P_2(\omega)e^{-\alpha_2\omega}L_\omega d\omega-\eta_Y Y\right]\\
&+\frac{1}{\mathcal{P}_4}\left(1-\frac{E_2}{E}\right)\left[\gamma_{HB}\int_0^{\varkappa_4}P_4(\omega)e^{-\alpha_4\omega}\frac{H_\omega B_\omega}{1+\psi_B B_\omega}d\omega-(\eta_E+\delta_E)E\right]\\
&+\frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}\left(1-\frac{Z_2}{Z}\right)\left[\delta_E\int_0^{\varkappa_5}P_5(\omega)e^{-\alpha_5\omega}E_\omega d\omega-\eta_Z Z\right]\\
&+\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\theta_C\delta_L}\left[\theta_C\int_0^{\varkappa_3}P_3(\omega)e^{-\alpha_3\omega}Y_\omega d\omega-\eta_C C\right]\\
&+\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\theta_B\delta_E}\left(1-\frac{B_2}{B}\right)\left[\theta_B\int_0^{\varkappa_6}P_6(\omega)e^{-\alpha_6\omega}Z_\omega d\omega-\eta_B B\right]\\
&+\frac{\gamma_{HC}HC}{1+\psi_C C}-\frac{1}{\mathcal{P}_1}\gamma_{HC}\int_0^{\varkappa_1}P_1(\omega)e^{-\alpha_1\omega}\frac{H_\omega C_\omega}{1+\psi_C C_\omega}d\omega+\frac{(\eta_L+\delta_L)}{\mathcal{P}_1}L\\
&-\frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2}\int_0^{\varkappa_2}P_2(\omega)e^{-\alpha_2\omega}L_\omega d\omega+\frac{\gamma_{HB}HB}{1+\psi_B B}-\frac{1}{\mathcal{P}_4}\gamma_{HB}\int_0^{\varkappa_4}P_4(\omega)e^{-\alpha_4\omega}\frac{H_\omega B_\omega}{1+\psi_B B_\omega}d\omega\\
&+\frac{1}{\mathcal{P}_4}\frac{\gamma_{HB}H_2B_2}{1+\psi_B B_2}\int_0^{\varkappa_4}P_4(\omega)e^{-\alpha_4\omega}\ln\left(\frac{H_\omega B_\omega(1+\psi_B B)}{HB(1+\psi_B B_\omega)}\right)d\omega\\
&+\frac{(\eta_E+\delta_E)}{\mathcal{P}_4}E-\frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5}\int_0^{\varkappa_5}P_5(\omega)e^{-\alpha_5\omega}E_\omega d\omega\\
&+\frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5}E_2\int_0^{\varkappa_5}P_5(\omega)e^{-\alpha_5\omega}\ln\left(\frac{E_\omega}{E}\right)d\omega+\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}Y\\
&-\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\delta_L}\int_0^{\varkappa_3}P_3(\omega)e^{-\alpha_3\omega}Y_\omega d\omega+\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}Z\\
&-\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\delta_E}\int_0^{\varkappa_6}P_6(\omega)e^{-\alpha_6\omega}Z_\omega d\omega+\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\delta_E}Z_2\int_0^{\varkappa_6}P_6(\omega)e^{-\alpha_6\omega}\ln\left(\frac{Z_\omega}{Z}\right)d\omega. \qquad (3.11)
\end{aligned}$$

Then simplifying Eq (3.11) and using the equilibrium conditions for Ξ_2:

$$\phi=\eta_H H_2+\frac{\gamma_{HB}H_2B_2}{1+\psi_B B_2},\quad \frac{\gamma_{HB}H_2B_2}{1+\psi_B B_2}=\frac{1}{\mathcal{P}_4}(\eta_E+\delta_E)E_2,$$
$$\mathcal{P}_5\delta_E E_2=\eta_Z Z_2,\quad \mathcal{P}_6\theta_B Z_2=\eta_B B_2,$$

and the following equalities:

$$\ln\left(\frac{H_\omega B_\omega(1+\psi_B B)}{HB(1+\psi_B B_\omega)}\right)=\ln\left(\frac{E_i H_\omega B_\omega(1+\psi_B B_i)}{EH_i B_i(1+\psi_B B_\omega)}\right)+\ln\left(\frac{1+\psi_B B}{1+\psi_B B_i}\right)+\ln\left(\frac{H_i}{H}\right)+\ln\left(\frac{B_i E}{BE_i}\right),$$

$$\ln\left(\frac{E_\omega}{E}\right) = \ln\left(\frac{Z_i E_\omega}{Z E_i}\right) + \ln\left(\frac{Z E_i}{Z_i E}\right),$$
$$\ln\left(\frac{Z_\omega}{Z}\right) = \ln\left(\frac{B_i Z_\omega}{B Z_i}\right) + \ln\left(\frac{B Z_i}{B_i Z}\right), \tag{3.12}$$

where $i = 2, 3$, we get:

$$\begin{aligned}\frac{d\Theta_2}{dt} &= -\frac{\eta_H(H-H_2)^2}{H} - \frac{\gamma_{HC}\psi_C H_2 C^2}{1+\psi_C C} - \frac{1}{\mathcal{P}_4}(\eta_E+\delta_E)E_2\left[F\left(\frac{H_2}{H}\right) + F\left(\frac{1+\psi_B B}{1+\psi_B B_2}\right)\right] \\ &- \frac{\psi_B(\eta_E+\delta_E)(B-B_2)^2}{\mathcal{P}_4(1+\psi_B B)(1+\psi_B B_2)B_2}E_2 - \frac{1}{\mathcal{P}_4^2}(\eta_E+\delta_E)E_2\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}F\left(\frac{H_\omega B_\omega E_2(1+\psi_B B_2)}{H_2 B_2 E(1+\psi_B B_\omega)}\right)d\omega \\ &- \frac{1}{\mathcal{P}_4\mathcal{P}_5}(\eta_E+\delta_E)E_2\int_0^{\varkappa_5} P_5(\omega)e^{-\alpha_5\omega}F\left(\frac{Z_2 E_\omega}{Z E_2}\right)d\omega \\ &- \frac{1}{\mathcal{P}_4\mathcal{P}_6}(\eta_E+\delta_E)E_2\int_0^{\varkappa_6} P_6(\omega)e^{-\alpha_6\omega}F\left(\frac{B_2 Z_\omega}{B Z_2}\right)d\omega + \frac{\eta_C\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\theta_C\delta_L}(\mathcal{R}_3-1)C.\end{aligned}$$

Since $\mathcal{R}_3 \leq 1$ then, $\frac{d\Theta_2}{dt} \leq 0$ for all $H, L, Y, E, Z, C, B > 0$. Further, $\frac{d\Theta_2}{dt} = 0$ when $H = H_2$, $E = E_2$, $Z = Z_2$, $B = B_2$ and $C = 0$. The solutions of system (3.1)–(3.7) tend to $\tilde{\Upsilon}_2$ which contains elements with $C = 0$, which gives $\dot{C} = 0$. Equation (3.6) implies

$$0 = \dot{C} = \theta_C\int_0^{\varkappa_3} P_3(\omega)e^{-\alpha_3\omega}Y_\omega d\omega \Longrightarrow Y(t) = 0, \quad \text{for any } t.$$

Then, Eq (3.3) becomes

$$0 = \dot{Y} = \delta_L\int_0^{\varkappa_2} P_2(\omega)e^{-\alpha_2\omega}L_\omega d\omega \Longrightarrow L(t) = 0, \quad \text{for any } t.$$

Therefore, $\tilde{\Upsilon}_2 = \{\Xi_2\}$. Applying LIP, we get Ξ_2 is GAS. □

Theorem 8. For system (3.1)–(3.7) assume that $\mathcal{R}_3 > 1$ and $\mathcal{R}_4 > 1$, then Ξ_3 is GAS.

Proof. Define a function Θ_3 as:

$$\begin{aligned}\Theta_3 &= H_3 F\left(\frac{H}{H_3}\right) + \frac{1}{\mathcal{P}_1}L_3 F\left(\frac{L}{L_3}\right) + \frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}Y_3 F\left(\frac{Y}{Y_3}\right) + \frac{1}{\mathcal{P}_4}E_3 F\left(\frac{E}{E_3}\right) \\ &+ \frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}Z_3 F\left(\frac{Z}{Z_3}\right) + \frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\theta_C\delta_L}C_3 F\left(\frac{C}{C_3}\right) + \frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\theta_B\delta_E}B_3 F\left(\frac{B}{B_3}\right) \\ &+ \frac{1}{\mathcal{P}_1}\frac{\gamma_{HC}H_3C_3}{1+\psi_C C_3}\int_0^{\varkappa_1} P_1(\omega)e^{-\alpha_1\omega}\int_{t-\omega}^{t} F\left(\frac{H(u)C(u)(1+\psi_C C_3)}{H_3C_3(1+\psi_C C(u))}\right)dud\omega \\ &+ \frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2}L_3\int_0^{\varkappa_2} P_2(\omega)e^{-\alpha_2\omega}\int_{t-\omega}^{t} F\left(\frac{L(u)}{L_3}\right)dud\omega \\ &+ \frac{1}{\mathcal{P}_4}\frac{\gamma_{HB}H_3B_3}{1+\psi_B B_3}\int_0^{\varkappa_4} P_4(\omega)e^{-\alpha_4\omega}\int_{t-\omega}^{t} F\left(\frac{H(u)B(u)(1+\psi_B B_3)}{H_3B_3(1+\psi_B B(u))}\right)dud\omega\end{aligned}$$

$$+\frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5}E_3\int_0^{\varkappa_5}P_5(\omega)e^{-\alpha_5\omega}\int_{t-\omega}^{t}F\left(\frac{E(u)}{E_3}\right)dud\omega$$

$$+\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\delta_L}Y_3\int_0^{\varkappa_3}P_3(\omega)e^{-\alpha_3\omega}\int_{t-\omega}^{t}F\left(\frac{Y(u)}{Y_3}\right)dud\omega$$

$$+\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\delta_E}Z_3\int_0^{\varkappa_6}P_6(\omega)e^{-\alpha_6\omega}\int_{t-\omega}^{t}F\left(\frac{Z(u)}{Z_3}\right)dud\omega.$$

We calculate $\frac{d\Theta_3}{dt}$ as:

$$\begin{aligned}
\frac{d\Theta_3}{dt} &= \left(1-\frac{H_3}{H}\right)\left[\phi-\eta_H H-\frac{\gamma_{HC}HC}{1+\psi_C C}-\frac{\gamma_{HB}HB}{1+\psi_B B}\right]\\
&+\frac{1}{\mathcal{P}_1}\left(1-\frac{L_3}{L}\right)\left[\gamma_{HC}\int_0^{\varkappa_1}P_1(\omega)e^{-\alpha_1\omega}\frac{H_\omega C_\omega}{1+\psi_C C_\omega}d\omega-(\eta_L+\delta_L)L\right]\\
&+\frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}\left(1-\frac{Y_3}{Y}\right)\left[\delta_L\int_0^{\varkappa_2}P_2(\omega)e^{-\alpha_2\omega}L_\omega d\omega-\eta_Y Y\right]\\
&+\frac{1}{\mathcal{P}_4}\left(1-\frac{E_3}{E}\right)\left[\gamma_{HB}\int_0^{\varkappa_4}P_4(\omega)e^{-\alpha_4\omega}\frac{H_\omega B_\omega}{1+\psi_B B_\omega}d\omega-(\eta_E+\delta_E)E\right]\\
&+\frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}\left(1-\frac{Z_3}{Z}\right)\left[\delta_E\int_0^{\varkappa_5}P_5(\omega)e^{-\alpha_5\omega}E_\omega d\omega-\eta_Z Z\right]\\
&+\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\theta_C\delta_L}\left(1-\frac{C_3}{C}\right)\left[\theta_C\int_0^{\varkappa_3}P_3(\omega)e^{-\alpha_3\omega}Y_\omega d\omega-\eta_C C\right]\\
&+\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\theta_B\delta_E}\left(1-\frac{B_3}{B}\right)\left[\theta_B\int_0^{\varkappa_6}P_6(\omega)e^{-\alpha_6\omega}Z_\omega d\omega-\eta_B B\right]+\frac{\gamma_{HC}HC}{1+\psi_C C}\\
&-\frac{1}{\mathcal{P}_1}\gamma_{HC}\int_0^{\varkappa_1}P_1(\omega)e^{-\alpha_1\omega}\frac{H_\omega C_\omega}{1+\psi_C C_\omega}d\omega+\frac{1}{\mathcal{P}_1}\frac{\gamma_{HC}H_3C_3}{1+\psi_C C_3}\int_0^{\varkappa_1}P_1(\omega)e^{-\alpha_1\omega}\ln\left(\frac{H_\omega C_\omega(1+\psi_C C)}{HC(1+\psi_C C_\omega)}\right)d\omega\\
&+\frac{(\eta_L+\delta_L)}{\mathcal{P}_1}L-\frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2}\int_0^{\varkappa_2}P_2(\omega)e^{-\alpha_2\omega}L_\omega d\omega+\frac{(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2}L_3\int_0^{\varkappa_2}P_2(\omega)e^{-\alpha_2\omega}\ln\left(\frac{L_\omega}{L}\right)d\omega\\
&+\frac{\gamma_{HB}HB}{1+\psi_B B}-\frac{1}{\mathcal{P}_4}\gamma_{HB}\int_0^{\varkappa_4}P_4(\omega)e^{-\alpha_4\omega}\frac{H_\omega B_\omega}{1+\psi_B B_\omega}d\omega\\
&+\frac{1}{\mathcal{P}_4}\frac{\gamma_{HB}H_3B_3}{1+\psi_B B_3}\int_0^{\varkappa_4}P_4(\omega)e^{-\alpha_4\omega}\ln\left(\frac{H_\omega B_\omega(1+\psi_B B)}{HB(1+\psi_B B_\omega)}\right)d\omega+\frac{(\eta_E+\delta_E)}{\mathcal{P}_4}E\\
&-\frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5}\int_0^{\varkappa_5}P_5(\omega)e^{-\alpha_5\omega}E_\omega d\omega+\frac{(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5}E_3\int_0^{\varkappa_5}P_5(\omega)e^{-\alpha_5\omega}\ln\left(\frac{E_\omega}{E}\right)d\omega+\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\delta_L}Y\\
&-\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\delta_L}\int_0^{\varkappa_3}P_3(\omega)e^{-\alpha_3\omega}Y_\omega d\omega+\frac{\eta_Y(\eta_L+\delta_L)}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\delta_L}Y_3\int_0^{\varkappa_3}P_3(\omega)e^{-\alpha_3\omega}\ln\left(\frac{Y_\omega}{Y}\right)d\omega+\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\delta_E}Z\\
&-\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\delta_E}\int_0^{\varkappa_6}P_6(\omega)e^{-\alpha_6\omega}Z_\omega d\omega+\frac{\eta_Z(\eta_E+\delta_E)}{\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6\delta_E}Z_3\int_0^{\varkappa_6}P_6(\omega)e^{-\alpha_6\omega}\ln\left(\frac{Z_\omega}{Z}\right)d\omega. \qquad (3.13)
\end{aligned}$$

Then collecting terms of Eq (3.13) and using the equilibrium conditions for Ξ_3:

$$\phi = \eta_H H_3 + \frac{\gamma_{HC} H_3 C_3}{1+\psi_C C_3} + \frac{\gamma_{HB} H_3 B_3}{1+\psi_B B_3}, \quad \frac{\gamma_{HC} H_3 C_3}{1+\psi_C C_3} = \frac{1}{\mathcal{P}_1}(\eta_L + \delta_L) L_3,$$
$$\mathcal{P}_2\, \delta_L L_3 = \eta_Y Y_3, \quad \frac{\gamma_{HB} H_3 B_3}{1+\psi_B B_3} = \frac{1}{\mathcal{P}_4}(\eta_E + \delta_E) E_3, \quad \mathcal{P}_5 \delta_E E_3 = \eta_Z Z_3,$$
$$\mathcal{P}_3 \theta_C Y_3 = \eta_C C_3, \quad \mathcal{P}_6 \theta_B Z_3 = \eta_B B_3,$$

and equalities (3.10) and (3.12), we get:

$$\begin{aligned}
\frac{d\Theta_3}{dt} = &-\frac{\eta_H (H-H_3)^2}{H} - \frac{1}{\mathcal{P}_1}(\eta_L+\delta_L) L_3 \left[F\left(\frac{H_3}{H}\right) + F\left(\frac{1+\psi_C C}{1+\psi_C C_3}\right)\right] - \frac{\psi_C (\eta_L+\delta_L)(C-C_3)^2}{\mathcal{P}_1 (1+\psi_C C)(1+\psi_C C_3) C_3} L_3 \\
&- \frac{1}{\mathcal{P}_4}(\eta_E+\delta_E) E_3 \left[F\left(\frac{H_3}{H}\right) + F\left(\frac{1+\psi_B B}{1+\psi_B B_3}\right)\right] - \frac{\psi_B (\eta_E+\delta_E)(B-B_3)^2}{\mathcal{P}_4 (1+\psi_B B)(1+\psi_B B_3) B_3} E_3 \\
&- \frac{(\eta_L+\delta_L)}{\mathcal{P}_1^2} L_3 \int_0^{\varkappa_1} P_1(\omega) e^{-\alpha_1 \omega} F\left(\frac{H_\omega C_\omega L_3 (1+\psi_C C_3)}{H_3 C_3 L (1+\psi_C C_\omega)}\right) d\omega \\
&- \frac{(\eta_L+\delta_L)}{\mathcal{P}_1 \mathcal{P}_2} L_3 \int_0^{\varkappa_2} P_2(\omega) e^{-\alpha_2 \omega} F\left(\frac{Y_3 L_\omega}{Y L_3}\right) d\omega \\
&- \frac{(\eta_E+\delta_E)}{\mathcal{P}_4^2} E_3 \int_0^{\varkappa_4} P_4(\omega) e^{-\alpha_4 \omega} F\left(\frac{H_\omega B_\omega E_3 (1+\psi_B B_3)}{H_3 B_3 E (1+\psi_B B_\omega)}\right) d\omega \\
&- \frac{(\eta_E+\delta_E)}{\mathcal{P}_4 \mathcal{P}_5} E_3 \int_0^{\varkappa_5} P_5(\omega) e^{-\alpha_5 \omega} F\left(\frac{Z_3 E_\omega}{Z E_3}\right) d\omega - \frac{(\eta_L+\delta_L)}{\mathcal{P}_1 \mathcal{P}_3} L_3 \int_0^{\varkappa_3} P_3(\omega) e^{-\alpha_3 \omega} F\left(\frac{C_3 Y_\omega}{C Y_3}\right) d\omega \\
&- \frac{(\eta_E+\delta_E)}{\mathcal{P}_4 \mathcal{P}_6} E_3 \int_0^{\varkappa_6} P_6(\omega) e^{-\alpha_6 \omega} F\left(\frac{B_3 Z_\omega}{B Z_3}\right) d\omega.
\end{aligned}$$

So, we get $\frac{d\Theta_3}{dt} \leq 0$ for all $H, L, Y, E, Z, C, B > 0$. Further, $\frac{d\Theta_3}{dt} = 0$ when $H = H_3$, $L = L_3$, $Y = Y_3$, $E = E_3$, $Z = Z_3$, $C = C_3$ and $Z = Z_3$. Therefore, $\tilde{\Upsilon}_3 = \{\Xi_3\}$. Applying LIP, we find that Ξ_3 is GAS. □

4. Comparison results

We compare our model (2.1)–(2.7) with the following model, where saturation is ignored:

$$\dot{H} = \phi - \eta_H H - \gamma_{HC} HC - \gamma_{HB} HB, \tag{4.1}$$
$$\dot{L} = e^{-\alpha_1 \omega_1} \gamma_{HC} H_{\omega_1} C_{\omega_1} - (\eta_L + \delta_L) L, \tag{4.2}$$
$$\dot{Y} = e^{-\alpha_2 \omega_2} \delta_L L_{\omega_2} - \eta_Y Y, \tag{4.3}$$
$$\dot{E} = e^{-\alpha_4 \omega_4} \gamma_{HB} H_{\omega_4} B_{\omega_4} - (\eta_E + \delta_E) E, \tag{4.4}$$
$$\dot{Z} = e^{-\alpha_5 \omega_5} \delta_E E_{\omega_5} - \eta_Z Z, \tag{4.5}$$
$$\dot{C} = e^{-\alpha_3 \omega_3} \theta_C Y_{\omega_3} - \eta_C C, \tag{4.6}$$
$$\dot{B} = e^{-\alpha_6 \omega_6} \theta_B Z_{\omega_6} - \eta_B B. \tag{4.7}$$

Model (4.1)–(4.7) has only three equilibria:

(I) Infection-free equilibrium, $\tilde{\Xi}_0 = (\tilde{H}_0, 0, 0, 0, 0, 0, 0)$, where $\tilde{H}_0 = \phi/\eta_H$.

(II) Virus type C single-infection equilibrium $\tilde{\Xi}_1 = (\tilde{H}_1, \tilde{L}_1, \tilde{Y}_1, 0, 0, \tilde{C}_1, 0)$, where

$$\tilde{H}_1 = \frac{e^{\Sigma_{i=1}^{3}\alpha_i\omega_i}\eta_Y\eta_C(\eta_L+\delta_L)}{\delta_L\theta_C\gamma_{HC}}, \quad \tilde{L}_1 = \frac{e^{\Sigma_{i=2}^{3}\alpha_i\omega_i}\eta_Y\eta_C\eta_H}{\delta_L\theta_C\gamma_{HC}}(\Re_1-1),$$
$$\tilde{Y}_1 = \frac{e^{\alpha_3\omega_3}\eta_C\eta_H}{\theta_C\gamma_{HC}}(\Re_1-1), \quad \tilde{C}_1 = \frac{\eta_H}{\gamma_{HC}}(\Re_1-1).$$

(III) Virus type B single-infection equilibrium $\tilde{\Xi}_2 = (\tilde{H}_2, 0, 0, \tilde{E}_2, \tilde{Z}_2, 0, \tilde{B}_2)$, where

$$\tilde{H}_2 = \frac{e^{\Sigma_{i=4}^{6}\alpha_i\omega_i}\eta_Z\eta_B(\eta_E+\delta_E)}{\delta_E\theta_B\gamma_{HB}}, \quad \tilde{E}_2 = \frac{e^{\Sigma_{i=5}^{6}\alpha_i\omega_i}\eta_Z\eta_B\eta_H}{\delta_E\theta_B\gamma_{HB}}(\Re_2-1),$$
$$\tilde{Z}_2 = \frac{e^{\alpha_6\omega_6}\eta_B\eta_H}{\theta_B\gamma_{HB}}(\Re_2-1), \quad \tilde{B}_2 = \frac{\eta_H}{\gamma_{HB}}(\Re_2-1).$$

We note that the basic reproduction numbers $\Re_1$ and $\Re_2$ do not depend on the saturation parameters ψ_B and ψ_C. It should be noted that the co-existence equilibrium is absent from this model and a number of other models that have been published in the literature (see, for example, [7,23]). Consequently, one of the elements that can lead to the coexistence of the two rival viruses is saturation.

Remark 1. When $\psi_B = 0$ and $\psi_C = 0$, then $\Re_3 = \frac{\Re_1}{\Re_2}$ and $\Re_4 = \frac{\Re_2}{\Re 1}$.

Corollary 1. Consider (4.1)–(4.7) then:

(i) If $\Re_1 \leq 1$ and $\Re_2 \leq 1$, then $\tilde{\Xi}_0$ is GAS;

(ii) If $\Re_1 > 1$ and $\Re_1 \geq \Re_2$, then $\tilde{\Xi}_1$ is GAS;

(iii) If $\Re_2 > 1$ and $\Re_2 \geq \Re_1$, then $\tilde{\Xi}_2$ is GAS.

Next we show the range of the saturation parameters ψ_B and ψ_C that ensure the coexistence equilibrium will appear.

4.1. Coexistence conditions of the two competing viruses in terms of the saturation parameters ψ_B and ψ_C

We have shown that the coexistence equilibrium Ξ_3 appears when $\Re_3 > 1$ and $\Re_4 > 1$. When all other parameters are fixed, $\Re_3$ and $\Re_4$ are functions of ψ_B and ψ_C, respectively. In addition, we have

$$\frac{\partial\Re_3}{\partial\psi_B} = \frac{e^{-\Sigma_{i=1}^{3}\alpha_i\omega_i}\gamma_{HC}\delta_L\eta_B\eta_H\eta_Z\theta_C(\eta_E+\delta_E)}{e^{-\Sigma_{i=4}^{6}\alpha_i\omega_i}\delta_E\eta_C\eta_Y\theta_B(\eta_L+\delta_L)(\eta_H\psi_B+\gamma_{HB})^2}(\Re_2-1),$$
$$\frac{\partial\Re_4}{\partial\psi_C} = \frac{e^{-\Sigma_{i=4}^{6}\alpha_i\omega_i}\gamma_{HB}\delta_E\eta_C\eta_H\eta_Y\theta_B(\eta_L+\delta_L)}{e^{-\Sigma_{i=1}^{3}\alpha_i\omega_i}\delta_L\eta_B\eta_Z\theta_C(\eta_E+\delta_E)(\eta_H\psi_C+\gamma_{HC})^2}(\Re_1-1).$$

Hence, if $\Re_1 > 1$ and $\Re_2 > 1$, then $\Re_3$ and $\Re_4$ are increasing functions of ψ_B and ψ_C, respectively.

Let us compute $\psi_B^{\min}$ and $\psi_C^{\min}$ such that

$$\Re_3 > 1, \text{ for all } \psi_B > \psi_B^{\min},$$
$$\Re_4 > 1, \text{ for all } \psi_C > \psi_C^{\min},$$

$$\psi_B^{\min} = \max\left\{0, \frac{\eta_B\eta_Z(\eta_E+\delta_E)}{e^{-\Sigma_{i=4}^{6}\alpha_i\omega_i}\delta_E\theta_B\phi}\left(\frac{\Re_2-\Re_1}{\Re_1-1}\right)\right\},$$
$$\psi_C^{\min} = \max\left\{0, \frac{\eta_C\eta_Y(\eta_L+\delta_L)}{e^{-\Sigma_{i=1}^{3}\alpha_i\omega_i}\delta_L\theta_C\phi}\left(\frac{\Re_1-\Re_2}{\Re_2-1}\right)\right\}.$$

Then the coexistence conditions are

$$\Re_1 > 1, \quad \Re_2 > 1, \quad \psi_B > \psi_B^{\min} \quad \text{and} \quad \psi_C > \psi_C^{\min}. \tag{4.8}$$

We have two cases:
(i) If $\Re_2 > \Re_1 > 1$, then

$$\psi_B^{\min} = \frac{\eta_B\eta_Z(\eta_E+\delta_E)}{e^{-\Sigma_{i=4}^{6}\alpha_i\omega_i}\delta_E\theta_B\phi}\left(\frac{\Re_2-\Re_1}{\Re_1-1}\right),$$
$$\psi_C^{\min} = 0.$$

(ii) If $\Re_1 > \Re_2 > 1$, then

$$\psi_B^{\min} = 0,$$
$$\psi_C^{\min} = \frac{\eta_C\eta_Y(\eta_L+\delta_L)}{e^{-\Sigma_{i=1}^{3}\alpha_i\omega_i}\delta_L\theta_C\phi}\left(\frac{\Re_1-\Re_2}{\Re_2-1}\right).$$

We see that model (4.1)–(4.7) does not include the scenario where two types of viruses coexist. It was noted that several patients in the studies reported in [14] had co-infections with two different strains of HIV. As such, ignoring saturation might not provide an adequate description of the coinfection dynamics between the two types of viruses (or strains). This lends credence to the notion of including saturation in the coinfection model of two types of viruses, in which the coexistence of two types of viruses is seen. Chronic viral coinfections can also result from other reasons, including immune response [16] and superinfection [49].

4.2. Models under the influence of antiviral treatment

To demonstrate why it is crucial to incorporate latently infected cells and time delays in our proposed model, we examine the model (2.1)–(2.7) in the presence of reverse transcriptase inhibitors (RTIs), a class of medications that may effectively block the free viral infection of target cells. Let $\epsilon_C \in [0,1]$ and $\epsilon_B \in [0,1]$ be the efficacies of RTIs for the viruses types C and B, respectively [22]. Under the effect of RTIs, model (2.1)–(2.7) becomes:

$$\dot{H} = \phi - \eta_H H - \frac{\gamma_{HC}(1-\epsilon_C)HC}{1+\psi_C C} - \frac{\gamma_{HB}(1-\epsilon_B)HB}{1+\psi_B B}, \tag{4.9}$$
$$\dot{L} = e^{-\alpha_1\omega_1}\frac{\gamma_{HC}(1-\epsilon_C)H_{\omega_1}C_{\omega_1}}{1+\psi_C C_{\omega_1}} - (\eta_L+\delta_L)L, \tag{4.10}$$
$$\dot{Y} = e^{-\alpha_2\omega_2}\delta_L L_{\omega_2} - \eta_Y Y, \tag{4.11}$$
$$\dot{E} = e^{-\alpha_4\omega_4}\frac{\gamma_{HB}(1-\epsilon_B)H_{\omega_4}B_{\omega_4}}{1+\psi_B B_{\omega_4}} - (\eta_E+\delta_E)E, \tag{4.12}$$

$$\dot{Z} = e^{-\alpha_5\omega_5}\delta_E E_{\omega_5} - \eta_Z Z, \tag{4.13}$$
$$\dot{C} = e^{-\alpha_3\omega_3}\theta_C Y_{\omega_3} - \eta_C C, \tag{4.14}$$
$$\dot{B} = e^{-\alpha_6\omega_6}\theta_B Z_{\omega_6} - \eta_B B. \tag{4.15}$$

The basic reproduction numbers of system (4.9)–(4.15) are:

$$\mathfrak{R}_1^{\epsilon_C} = (1-\epsilon_C)\frac{e^{-\Sigma_{i=1}^{3}\alpha_i\omega_i}H_0\delta_L\theta_C\gamma_{HC}}{\eta_Y\eta_C(\eta_L+\delta_L)} = (1-\epsilon_C)\mathfrak{R}_1,$$
$$\mathfrak{R}_2^{\epsilon_B} = (1-\epsilon_B)\frac{e^{-\Sigma_{i=4}^{6}\alpha_i\omega_i}H_0\delta_E\theta_B\gamma_{HB}}{\eta_Z\eta_B(\eta_E+\delta_E)} = (1-\epsilon_B)\mathfrak{R}_2.$$

We now compute the drug efficacies ϵ_C and ϵ_B, which make $\mathfrak{R}_1^{\epsilon_C} \leq 1$ and $\mathfrak{R}_1^{\epsilon_B} \leq 1$, and then stabilize system (4.9)–(4.15) about the infection-free equilibrium Ξ_0

$$\mathfrak{R}_1^{\epsilon_C} \leq 1 \Longleftrightarrow 1 \geq \epsilon_C \geq \epsilon_C^{\min} = \max\left\{\frac{\mathfrak{R}_1 - 1}{\mathfrak{R}_1}, 0\right\}, \tag{4.16}$$
$$\mathfrak{R}_1^{\epsilon_B} \leq 1 \Longleftrightarrow 1 \geq \epsilon_B \geq \epsilon_B^{\min} = \max\left\{\frac{\mathfrak{R}_2 - 1}{\mathfrak{R}_2}, 0\right\}. \tag{4.17}$$

If the latently infected cells in model (4.9)–(4.15) are disregarded, we get

$$\dot{H} = \phi - \eta_H H - \frac{\gamma_{HC}(1-\epsilon_C)HC}{1+\psi_C C} - \frac{\gamma_{HB}(1-\epsilon_B)HB}{1+\psi_B B}, \tag{4.18}$$
$$\dot{Y} = e^{-\alpha_1\omega_1}\frac{\gamma_{HC}(1-\epsilon_C)H_{\omega_1}C_{\omega_1}}{1+\psi_C C_{\omega_1}} - \eta_Y Y, \tag{4.19}$$
$$\dot{Z} = e^{-\alpha_4\omega_4}\frac{\gamma_{HB}(1-\epsilon_B)H_{\omega_4}B_{\omega_4}}{1+\psi_B B_{\omega_4}} - \eta_Z Z, \tag{4.20}$$
$$\dot{C} = e^{-\alpha_3\omega_3}\theta_C Y_{\omega_3} - \eta_C C, \tag{4.21}$$
$$\dot{B} = e^{-\alpha_6\omega_6}\theta_B Z_{\omega_6} - \eta_B B, \tag{4.22}$$

and the basic reproduction numbers of model (4.18)–(4.22) are given by

$$\bar{\mathfrak{R}}_1^{\epsilon_C} = (1-\epsilon_C)\frac{e^{-(\alpha_1\omega_1+\alpha_3\omega_3)}H_0\theta_C\gamma_{HC}}{\eta_Y\eta_C} = (1-\epsilon_C)\bar{\mathfrak{R}}_1,$$
$$\bar{\mathfrak{R}}_2^{\epsilon_B} = (1-\epsilon_B)\frac{e^{-(\alpha_4\omega_4+\alpha_6\omega_6)}H_0\theta_B\gamma_{HB}}{\eta_Z\eta_B} = (1-\epsilon_B)\bar{\mathfrak{R}}_2.$$

We determine the drug efficacies ϵ_C and ϵ_B, which make $\bar{\mathfrak{R}}_1^{\epsilon_C} \leq 1$ and $\bar{\mathfrak{R}}_1^{\epsilon_B} \leq 1$ and stabilizes the infection-free equilibrium of system (4.18)–(4.22) as:

$$\bar{\mathfrak{R}}_1^{\epsilon_C} \leq 1 \Longleftrightarrow 1 \geq \epsilon_C \geq \bar{\epsilon}_C^{\min} = \max\left\{\frac{\bar{\mathfrak{R}}_1 - 1}{\bar{\mathfrak{R}}_1}, 0\right\}, \tag{4.23}$$
$$\bar{\mathfrak{R}}_1^{\epsilon_B} \leq 1 \Longleftrightarrow 1 \geq \epsilon_B \geq \bar{\epsilon}_B^{\min} = \max\left\{\frac{\bar{\mathfrak{R}}_2 - 1}{\bar{\mathfrak{R}}_2}, 0\right\}. \tag{4.24}$$

Since $0 < e^{-\alpha_2\omega_2} \leq 1$ and $0 < e^{-\alpha_5\omega_5} \leq 1$, then

$$\mathfrak{R}_1 = \frac{e^{-\Sigma_{i=1}^{3}\alpha_i\omega_i}H_0\delta_L\theta_C\gamma_{HC}}{\eta_Y\eta_C(\eta_L+\delta_L)} \leq \frac{e^{-(\alpha_1\omega_1+\alpha_3\omega_3)}H_0\delta_L\theta_C\gamma_{HC}}{\eta_Y\eta_C(\eta_L+\delta_L)} < \frac{e^{-(\alpha_1\omega_1+\alpha_3\omega_3)}H_0\theta_C\gamma_{HC}}{\eta_Y\eta_C} = \bar{\mathfrak{R}}_1,$$
$$\mathfrak{R}_2 = \frac{e^{-\Sigma_{i=4}^{6}\alpha_i\omega_i}H_0\delta_E\theta_B\gamma_{HB}}{\eta_Z\eta_B(\eta_E+\delta_E)} \leq \frac{e^{-(\alpha_4\omega_4+\alpha_6\omega_6)}H_0\delta_E\theta_B\gamma_{HB}}{\eta_Z\eta_B(\eta_E+\delta_E)} < \frac{e^{-(\alpha_4\omega_4+\alpha_6\omega_6)}H_0\theta_B\gamma_{HB}}{\eta_Z\eta_B} = \bar{\mathfrak{R}}_2.$$

Hence, $\mathfrak{R}_1^{\epsilon_C} < \bar{\mathfrak{R}}_1^{\epsilon_C}$ and $\mathfrak{R}_2^{\epsilon_B} < \bar{\mathfrak{R}}_2^{\epsilon_B}$ and thus eliminating the latently infected cells from the two-virus co-dynamics model would lead to an overestimation of the basic reproduction numbers. By comparing Eqs (4.16), (4.17) and (4.23), (4.24) we get that $\epsilon_C^{\min} < \bar{\epsilon}_C^{\min}$ and $\epsilon_B^{\min} < \bar{\epsilon}_B^{\min}$. Thus, in order to keep the system at the infection-free equilibrium and remove the two types of viruses from the body, fewer treatment efficacies will be needed when utilizing a model with latently infected cells.

Similar to the discussion, one can find that the presence of time delays reduces the basic reproduction numbers. Then, when using a model with time delays, fewer treatment efficacies will be needed to keep the system at infection-free equilibrium and remove the two types of viruses from the body.

5. Numerical simulations

We perform numerical simulation for the model with discrete-time delays (2.1)–(2.7) in this section. We use the values of the parameters given in Table 2. Using MATLAB's dde23 solver, the system of DDEs is numerically solved.

Table 2. The values of parameters of system (2.1)–(2.7).

Parameter	Value	Source	Parameter	Value	Source	Parameter	Value	Source
ϕ	10	[29,44,50]	θ_C	1	[30]	α_1	0.2	[51]
η_H	0.01	[50,52]	θ_B	1.2	Assumed	α_2	0.3	[51]
γ_{HC}	Varied	-	η_C	2.4	[29,44]	α_3	0.4	[51]
γ_{HB}	Varied	-	η_B	2.4	[29,44]	α_4	0.5	[51]
ψ_C	Varied	-	η_L	0.02	[44,53]	α_5	0.6	[51]
ψ_B	Varied	-	η_E	0.01	[50]	α_6	0.9	[51]
η_Y	0.5	[28,30]	δ_L	0.2	[53]			
η_Z	0.2	[54]	δ_E	0.003	[55]			

5.1. Stability of the equilibria

In this subsection, we fix the delay parameters as: $\omega_1 = 1$, $\omega_2 = 0.8$, $\omega_3 = 0.6$, $\omega_4 = 0.4$, $\omega_5 = 0.2$, and $\omega_6 = 0.1$. Moreover, we solve system (2.1)–(2.7) with the following initial conditions:

I-1: $(H(u), L(u), Y(u), E(u), Z(u), C(u), B(u)) = (800, 5, 1.5, 100, 1, 1, 0.6)$,

I-2: $(H(u), L(u), Y(u), E(u), Z(u), C(u), B(u)) = (600, 10, 3, 150, 2, 1.5, 1)$,

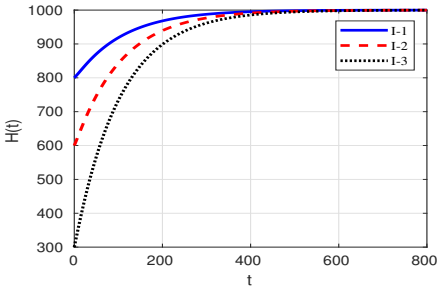
I-3: $(H(u), L(u), Y(u), E(u), Z(u), C(u), B(u)) = (300, 15, 4.5, 200, 3, 2, 1.4)$,

where $u \in [-1, 0]$.

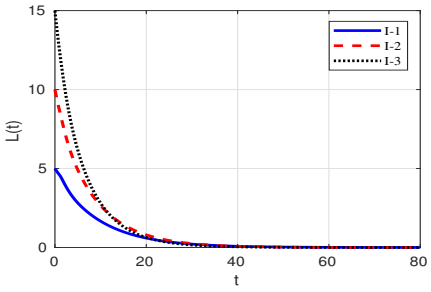
Selecting the values of γ_{HC}, γ_{HB},ψ_C and ψ_B leads to the following plans:

Plan 1. $\gamma_{HC} = 0.001$, $\gamma_{HB} = 0.0003$ and $\psi_C = \psi_B = 0.01$. These values yield $\mathfrak{R}_1 = 0.38 < 1$ and $\mathfrak{R}_2 = 0.11 < 1$. Figure 1 displays that the solutions initiating with I-1, I-2 and I-3 converge the

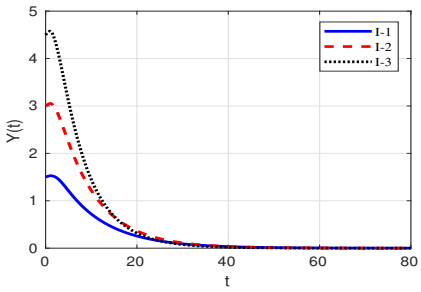
equilibrium $\Xi_0 = (1000, 0, 0, 0, 0, 0, 0)$. This illustrates the global asymptotic stability of Ξ_0 proven Theorem 1. Both virus types C and B will finally be eradicated in this case.



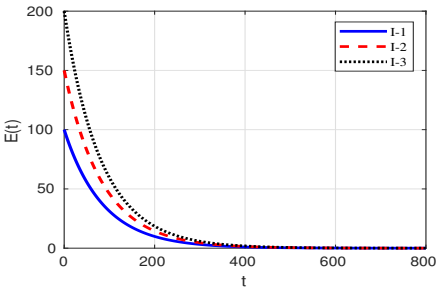
(a) Uninfected cells



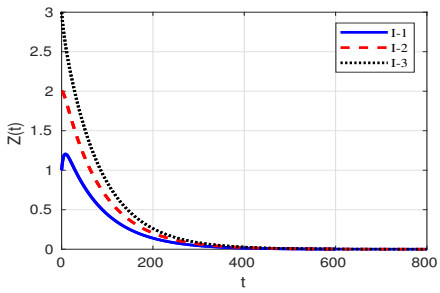
(b) Latently infected cells by virus type C



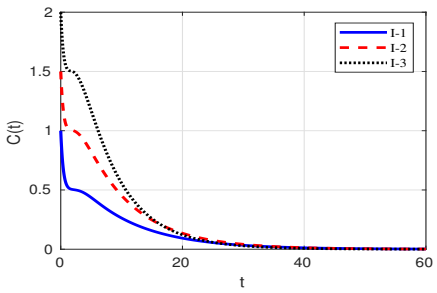
(c) Actively infected cells by virus type C



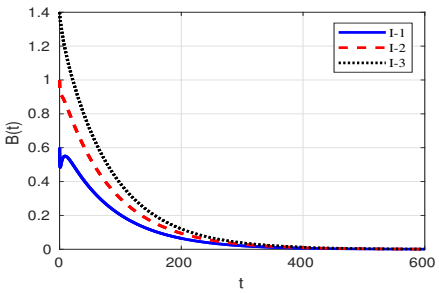
(d) Latently infected cells by virus type B



(e) Actively infected cells by virus type B



(f) Free viruses type C

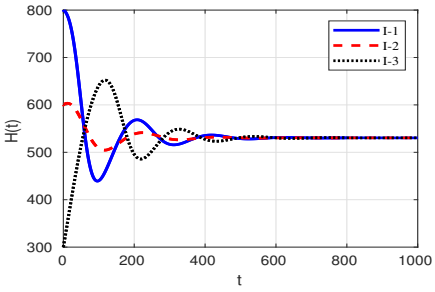


(g) Free viruses type B

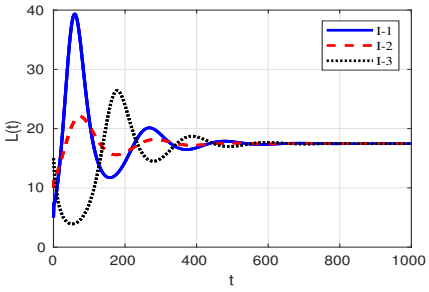
Figure 1. Numerical solutions of system (2.1)–(2.7) when $\Re_1 \leq 1$ and $\Re_2 \leq 1$ (Plan 1).

Plan 2. $\gamma_{HC} = 0.005$, $\gamma_{HB} = 0.0005$ and $\psi_C = \psi_B = 0.01$. For such choice, we have $\Re_1 = 1.92 > 1$ and $\Re_4 = 0.10 < 1$. It is evident from Figure 2 that for the three selected initial values, the system's

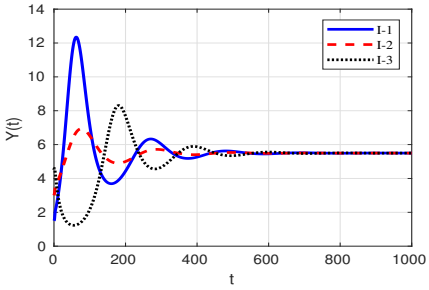
solutions converge to the equilibrium $\Xi_1 = (530.5, 17.47, 5.5, 0, 0, 1.8, 0)$. This shows that Ξ_1 is GAS, according to Theorem 2. This situation represents the infection of virus type C only.



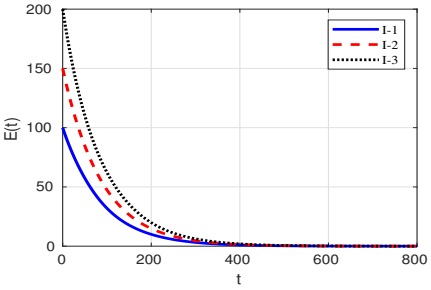
(a) Uninfected cells



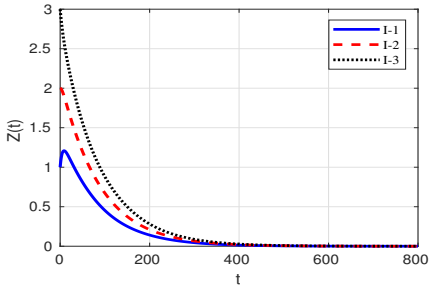
(b) Latently infected cells by virus type C



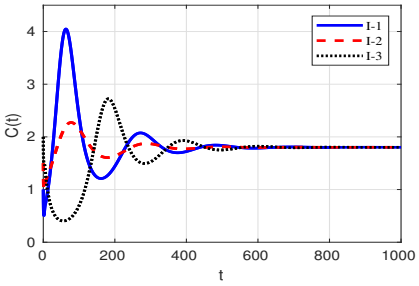
(c) Actively infected cells by virus type C



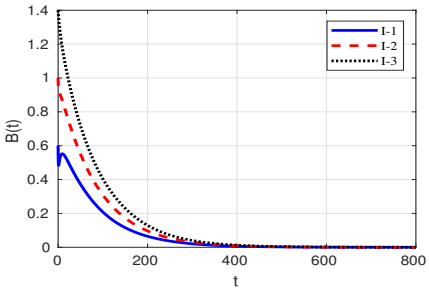
(d) Latently infected cells by virus type B



(e) Actively infected cells by virus type B



(f) Free viruses type C

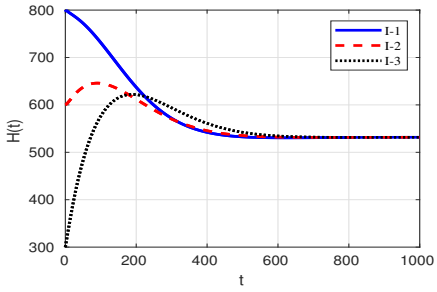


(g) Free viruses type B

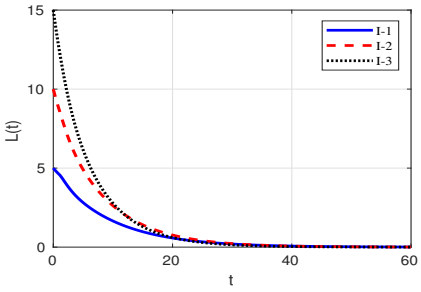
Figure 2. Numerical solutions of system (2.1)–(2.7) when $\mathfrak{R}_1 > 1$ and $\mathfrak{R}_4 \leq 1$ (Plan 2).

Plan 3. $\gamma_{HC} = 0.001, \gamma_{HB} = 0.005$ and $\psi_C = \psi_B = 0.01$. These parameters provide that $\mathfrak{R}_2 = 1.91 > 1$ and $\mathfrak{R}_3 = 0.20 < 1$. In Figure 3, we display that the equilibrium

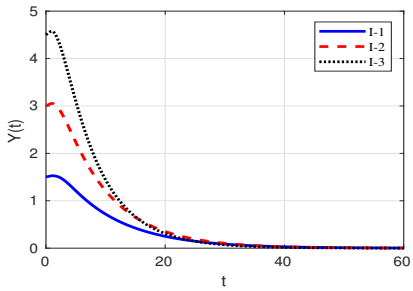
$\Xi_2 = (531.73, 0, 0, 294.91, 3.92, 0, 1.79)$ is reached for all initials I-1, I-2 and I-3. This shows that Ξ_2 is GAS, according to Theorem 3. This situation represents the infection of virus type B only.



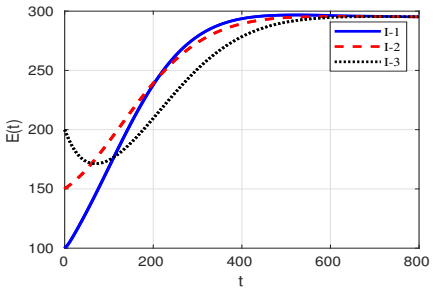
(a) Uninfected cells



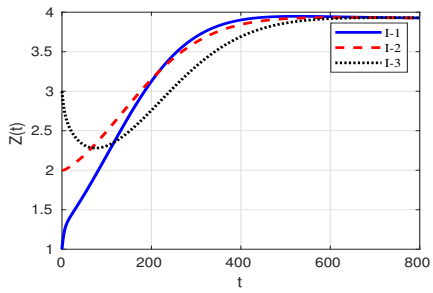
(b) Latently infected cells by virus type C



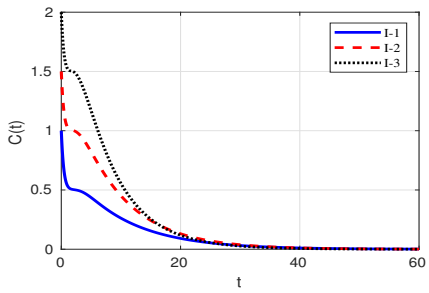
(c) Actively infected cells by virus type C



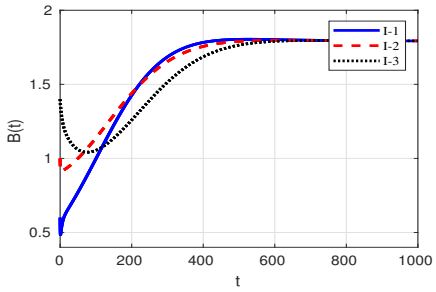
(d) Latently infected cells by virus type B



(e) Actively infected cells by virus type B



(f) Free viruses type C

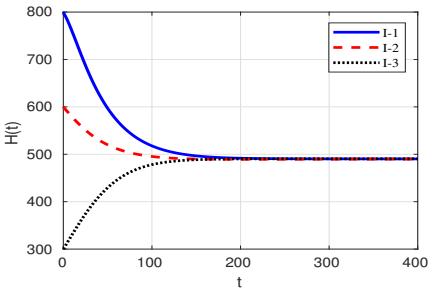


(g) Free viruses type B

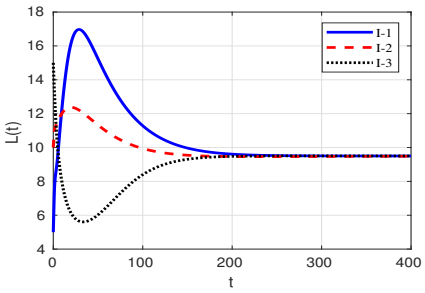
Figure 3. Numerical solutions of system (2.1)–(2.7) when $\mathfrak{R}_2 > 1$ and $\mathfrak{R}_3 \leq 1$ (Plan 3).

Plan 4. $\gamma_{HC} = \gamma_{HB} = 0.01$ and $\psi_C = \psi_B = 0.9$. For such values, we get that $\mathfrak{R}_3 = 2.35 > 1$ and $\mathfrak{R}_4 = 2.34 > 1$. From Figure 4, we can see that the equilibrium

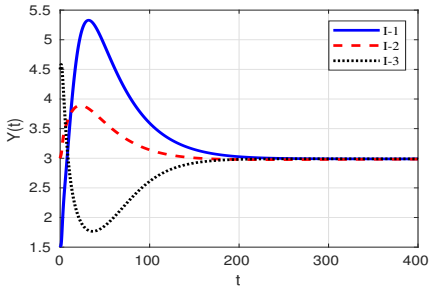
$\Xi_3 = (490.25, 9.5, 2.99, 160.3, 2.13, 0.98, 0.97)$ is reached for the three selected initials. This illustrates the global asymptotic stability of Ξ_3 proven Theorem 1. This case show the coexistence of the two type of viruses.



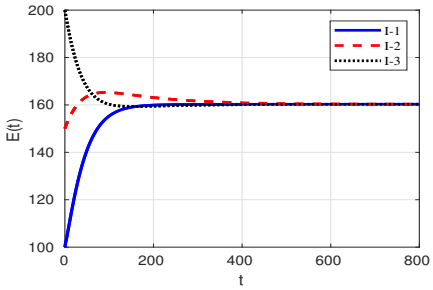
(a) Uninfected cells



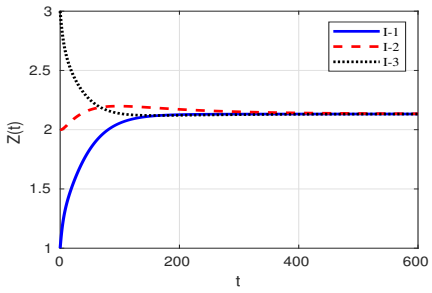
(b) Latently infected cells by virus type C



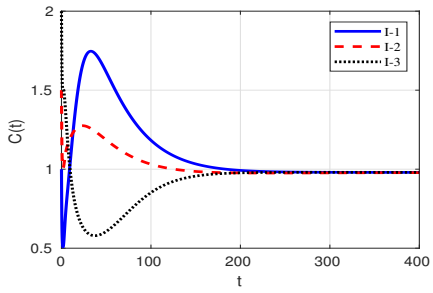
(c) Actively infected cells by virus type C



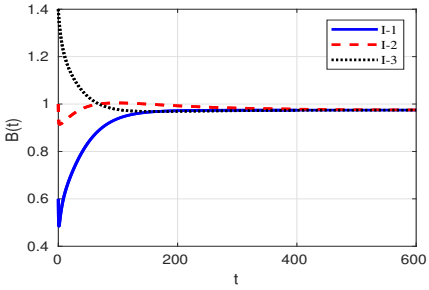
(d) Latently infected cells by virus type B



(e) Actively infected cells by virus type B



(f) Free viruses type C



(g) Free viruses type B

Figure 4. Numerical solutions of system (2.1)–(2.7) when $\mathfrak{R}_3 > 1$ and $\mathfrak{R}_4 > 1$ (Plan 4).

5.2. Impact of saturation on viral co-dynamics

In this part we show the effect of the saturation parameters ψ_B and ψ_C on viral co-dynamics. Here, we fix the delay parameters as

$$\omega_1 = 1, \quad \omega_2 = 0.8, \quad \omega_3 = 0.6, \quad \omega_4 = 0.4, \quad \omega_5 = 0.2 \quad \text{and} \quad \omega_6 = 0.1.$$

We have two situations to identify the values of ψ_B and ψ_C that lead to the coexistence of the two types of viruses:

Situation (I). $\mathfrak{R}_2 > \mathfrak{R}_1 > 1$. In this case we choose

$$\gamma_{HC} = 0.008 \quad \text{and} \quad \gamma_{HB} = 0.009.$$

Then we get

$$\mathfrak{R}_1 = 3.07041 > 1, \quad \mathfrak{R}_2 = 3.44588 > 1, \quad \psi_B^{\min} = 0.0473656 \quad \text{and} \quad \psi_C^{\min} = 0.$$

Therefore, Ξ_3 exists when

$$\psi_B > 0.0473656 \quad \text{and} \quad \psi_C > 0.$$

Situation (II). $\mathfrak{R}_1 > \mathfrak{R}_2 > 1$. We take

$$\gamma_{HC} = 0.009 \quad \text{and} \quad \gamma_{HB} = 0.008.$$

Then we get

$$\mathfrak{R}_1 = 3.45421, \quad \mathfrak{R}_2 = 3.063, \quad \psi_B^{\min} = 0.0 \quad \text{and} \quad \psi_C^{\min} = 0.0494083.$$

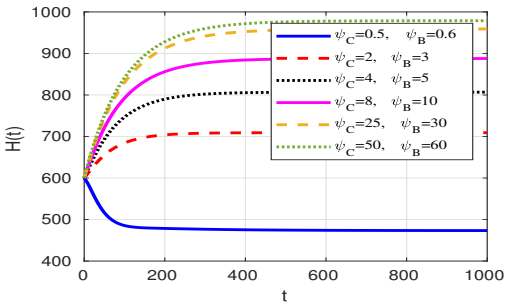
It follows that, Ξ_3 exists when

$$\psi_B > 0 \quad \text{and} \quad \psi_C > 0.0494083.$$

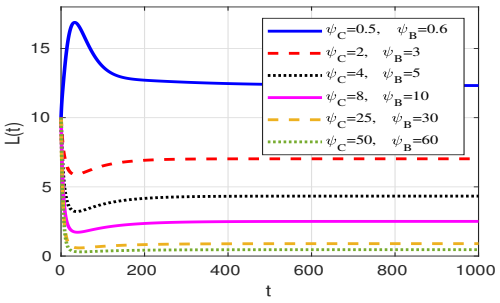
Now we show the effect of the saturation parameters ψ_B and ψ_C on the solutions of model (2.1)–(2.7) in case of Situation (II). We consider the initial condition

I-4: $(H(u), L(u), Y(u), E(u), Z(u), C(u), B(u)) = (600, 10, 3, 80, 1, 1, 0.5), \quad u \in [-\omega^*, 0]$.

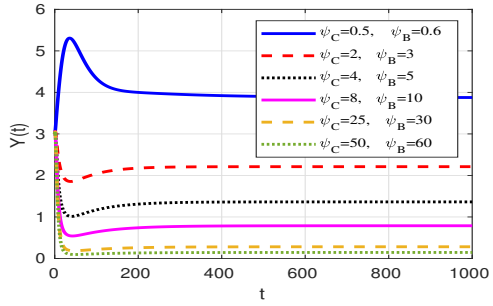
Figure 5 shows the effect of saturation on the viral co-dynamics. We note that a rise in ψ_C and ψ_B results in a drop in the incidence rate between uninfected cells and the two types of viruses. This decrease is followed by an increase in the concentration of uninfected cells and a decrease in the concentrations of infected cells and free viruses. The infection-free equilibrium Ξ_0 will not be maintained by raising ψ_B and ψ_C because the basic reproduction numbers $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are independent of the saturation parameters.



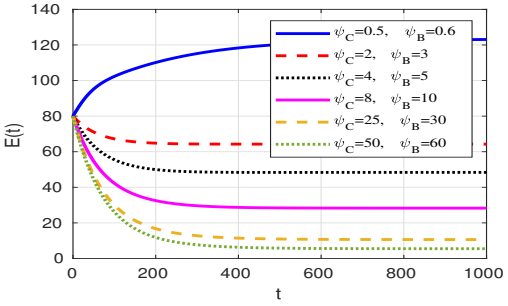
(a) Uninfected cells



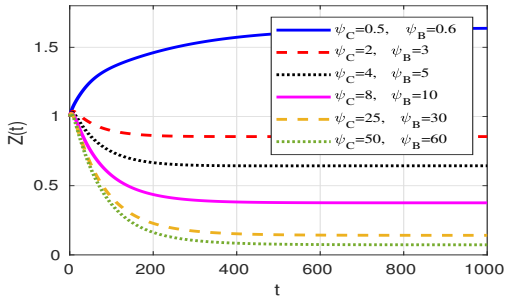
(b) Latently infected cells by virus type C



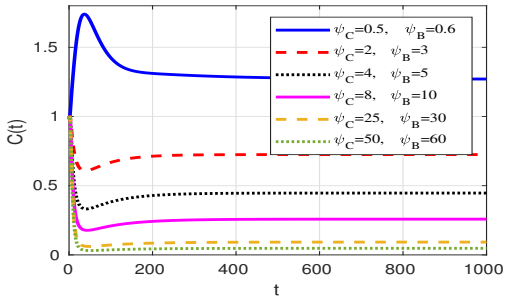
(c) Actively infected cells by virus type C



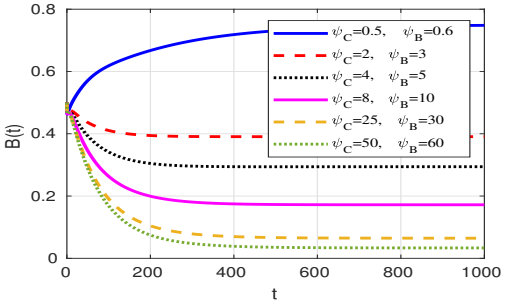
(d) Latently infected cells by virus type B



(e) Actively infected cells by virus type B



(f) Free viruses type C



(g) Free viruses type B

Figure 5. Numerical solutions of system (2.1)–(2.7) with different saturation parameters ψ_C, ψ_B.

5.3. Effect of the delay parameters on viral co-dynamics

In this subsection, we analyze the impact of time delay parameters ω_i, $i = 1, 2, \ldots, 6$ on the co-dynamics of the two types of viruses. We fix the parameters that $\gamma_{HC} = 0.009$, $\gamma_{HB} = 0.08$, and $\psi_C = \psi_B = 0.9$. We consider the scenarios given in Table 3.

Table 3. Different scenarios for the delay parameters.

Scenario	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
S1	1	0.9	0.8	0.7	0.6	0.5
S2	1.5	1.4	1.3	1.2	1	0.9
S3	2	1.8	1.7	1.6	1.5	1.4
S4	2.1329	2.1329	2.1329	1.916	1.916	1.916
S5	7	6	5	4	3	2
S6	9	8	7	6	5	4

We solve the system (2.1)–(2.7) under the following initial condition:

I-5: $(H(u), L(u), Y(u), E(u), Z(u), C(u), B(u)) = (600, 10, 3, 150, 2, 1.5, 1), \quad u \in [-\omega^*, 0]$.

The numerical results are shown in Figure 6. It is observed that time delays might lead to a notable rise in the quantity of uninfected cells and a decrease in the quantity of remaining compartments. We note that, $\mathfrak{R}_1$ and $\mathfrak{R}_2$ given by Eqs (2.10) and (2.11) depend on ω_i, $i = 1, 2, \ldots, 6$ when all other parameters are fixed. We observe from Table 4 that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ decrease if ω_i increases; hence, the stability of Ξ_0 will be changed.

Now, we need to calculate the critical values of the delay parameters ω_i, $i = 1, 2, \ldots, 6$ that make the system stable around the equilibrium point Ξ_0. Let $\omega_C = \omega_1 = \omega_2 = \omega_3$ and $\omega_B = \omega_4 = \omega_5 = \omega_6$, and we write $\mathfrak{R}_1(\omega_C)$ and $\mathfrak{R}_2(\omega_B)$ as:

$$\mathfrak{R}_1(\omega_C) = \frac{e^{-(\alpha_1+\alpha_2+\alpha_3)\omega_C} H_0 \delta_L \theta_C \gamma_{HC}}{\eta_Y \eta_C (\eta_L + \delta_L)}, \qquad \mathfrak{R}_2(\omega_B) = \frac{e^{-(\alpha_4+\alpha_5+\alpha_6)\omega_B} H_0 \delta_E \theta_B \gamma_{HB}}{\eta_Z \eta_B (\eta_E + \delta_E)}.$$

Clearly, when all other parameters are fixed, $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are decreasing functions of ω_C and ω_B, respectively. Let us calculate $\omega_C^{\min}$ and $\omega_B^{\min}$ such that $\mathfrak{R}_1(\omega_C^{\min}) = 1$ and $\mathfrak{R}_2(\omega_B^{\min}) = 1$ as:

$$\omega_C^{\min} = \max\left\{0, \frac{1}{\alpha_1+\alpha_2+\alpha_3} \ln\left(\frac{H_0 \delta_L \theta_C \gamma_{HC}}{\eta_Y \eta_C (\eta_L + \delta_L)}\right)\right\},$$
$$\omega_B^{\min} = \max\left\{0, \frac{1}{\alpha_4+\alpha_5+\alpha_6} \ln\left(\frac{H_0 \delta_E \theta_B \gamma_{HB}}{\eta_Z \eta_B (\eta_E + \delta_E)}\right)\right\}.$$

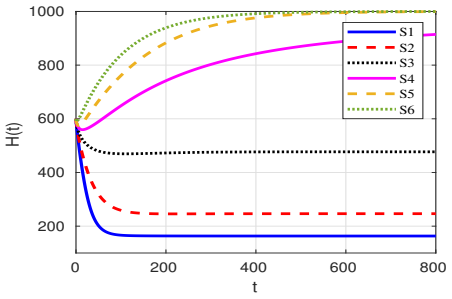
Consequently,

$$\mathfrak{R}_1(\omega_C) \leq 1, \text{ for all } \omega_C \geq \omega_C^{\min},$$
$$\mathfrak{R}_2(\omega_B) \leq 1, \text{ for all } \omega_B \geq \omega_B^{\min}.$$

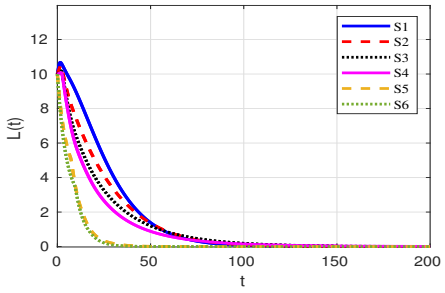
Therefore, Ξ_0 is GAS when $\omega_C \geq \omega_C^{\min}$ and $\omega_B \geq \omega_B^{\min}$. Using the values of the parameters in Table 2, we get $\omega_C = 2.1329$ and $\omega_B = 1.9160$. It follows that:

(i) If $\omega_C \geq 2.1329$ and $\omega_B \geq 1.9160$, then $\mathfrak{R}_1(\omega_C) \leq 1$ and $\mathfrak{R}_2(\omega_B) \leq 1$, and then Ξ_0 is GAS.

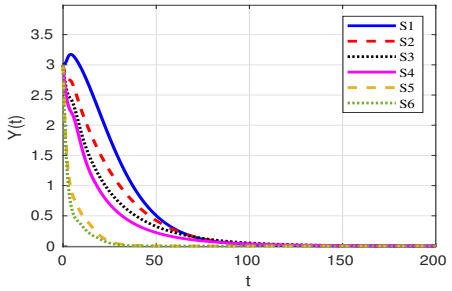
(ii) If $\omega_C < 2.1329$ and/or $\omega_B < 1.9160$, then $\mathfrak{R}_1(\omega_C) > 1$ and/or $\mathfrak{R}_2(\omega_B) > 1$. In this case, Ξ_0 will lose its stability. We see that the effects of antiviral medications and time delays can be comparable. This can help scientists develop novel therapies that lengthen time delays in cases of coinfection between the C and B viruses.



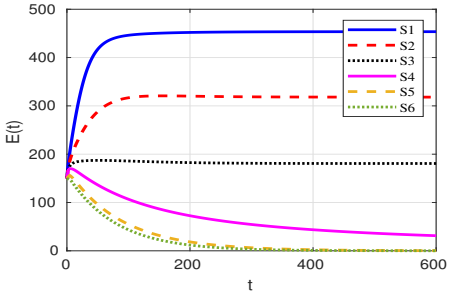
(a) Uninfected cells



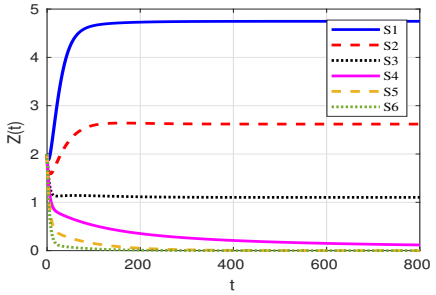
(b) Latently infected cells by virus type C



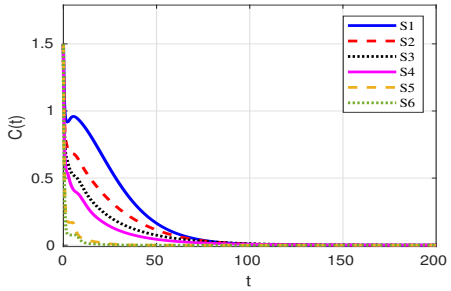
(c) Actively infected cells by virus type C



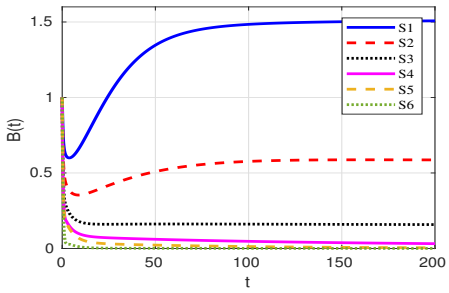
(d) Latently infected cells by virus type B



(e) Actively infected cells by virus type B



(f) Free viruses type C



(g) Free viruses type B

Figure 6. The impact of the delay on the co-infection dynamics.

Table 4. The values of parameters of system (2.1)–(2.7).

Delay parameters	$\mathfrak{R}_1$	$\mathfrak{R}_2$
$\omega_1 = 1,\ \omega_2 = 0.9, \omega_3 = 0.8,\ \omega_4 = 0.7,\ \ \omega_5 = 0.6,\ \ \omega_6 = 0.5$	3.09	14.47
$\omega_1 = 1.5,\ \omega_2 = 1.4, \omega_3 = 1.3,\ \omega_4 = 1.2,\ \ \omega_5 = 1,\ \ \omega_6 = 0.9$	1.97	6.18
$\omega_1 = 2,\ \omega_2 = 1.8,\ \omega_3 = 1.7,\ \omega_4 = 1.6,\ \omega_5 = 1.5,\ \omega_6 = 1.4$	1.35	2.39
$\omega_1 = \omega_2 = \omega_3 = 2.1329,\ \ \omega_4 = \ \omega_5 = \omega_6 = 1.916$	1	1
$\omega_1 = 7,\ \omega_2 = 6,\ \omega_3 = 5,\ \omega_4 = 4,\ \omega_5 = 3,\ \ \omega_6 = 2$	0.038	0.17
$\omega_1 = 9,\ \omega_2 = 8,\ \omega_3 = 7,\ \omega_4 = 6,\ \omega_5 = 5,\ \ \omega_6 = 4$	0.006	0.003

6. Conclusions and future perspectives

In this work, we examined a mathematical model of the population co-dynamics of two types of viruses (or virus variants) infecting the same target cells. The infection rate is given by the saturated incidence. The model included the latently infected cells. Three kinds of discrete (or distributed) time delays were incorporated into the model:

(i) The formation delay of latently infected cells;

(ii) The activation delay of latently infected cells;

(iii) The maturation delay of newly released virions.

First, we demonstrated nonnegativity and boundedness, which are the key characteristics of the solutions. Second, we proved that the model admits four equilibria. We derived four threshold parameters, which decide whether the model's equilibria exist and are globally asymptotically stable. We demonstrated the global asymptotic stability for every equilibrium point using the Lyapunov approach. We used a numerical method to solve the model, and then we displayed the findings graphically. We found a correlation between the theoretical and numerical results. Our findings are contrasted with models that do not account for saturation incidence, latently infected cells, or time delays. We have the following observations:

- When saturation is not present, only one type of virus with the highest reproduction number can survive in equilibrium. The rivalry between two virus kinds for shared resources leads to the survival of just one viral type with the highest reproduction number. In our proposed model with saturated incidence, two types of viruses can coexist in equilibrium. We can think of this situation as follows: Two viral kinds can coexist because saturation incidence lowers infection rates, which also lessens rivalry between the two virus types. We established conditions under which these types of viruses can coexist. The coexistence conditions are formulated in terms of saturation constants.

- The presence of latently infected cells and/or time delays reduces the basic reproduction numbers. Then, when using a model with latently infected cells and/or time delays, fewer treatment efficacies will be needed to keep the system at infection-free equilibrium and remove the two types of viruses from the body. This may help scientists develop novel therapies that lengthen time delays.

- Our results indicate that saturated, latently infected cells and time delay are essential elements of the two-virus model that cannot be ignored.

Our study's main flaw is that we were unable to use actual data to estimate the model's parameter values. The following are the causes:

(i) Genuine data on two-virus infections is lacking;

(ii) Comparing our findings to a limited number of genuine studies may not be particularly reliable;

(iii) It can be challenging to collect real data from patients who have two virus infections.

Our proposed model can be extended by considereing different incidene rate forms, such as: Beddington-DeAngelis incidence $\frac{\gamma_{HV}HV}{1+\psi_V V+\psi_H H}$ [56]; Crowley-Martin incidence $\frac{\gamma_{HV}HV}{(1+\psi_V V)(1+\psi_H H)}$ [57]; Hattaf-Yousf incidence $\frac{\gamma_{HV}HV}{\psi_0+\psi_V V+\psi_H H+\psi_{VH}HV}$, [58]; general incidence $\xi(H,V)$ [59], where V is the concentration of the viruses, $\psi_0, \psi_V, \psi_H, \psi_{VH} \geq 0$ and ξ is a general function. Investigating the memory effect on the dynamics of our model using fractional differential equations (FDEs) sounds like a fascinating direction [60]. FDEs can capture non-local and memory-dependent effects, which are often crucial in viological systems [61], and epidemical systems [62, 63]. Additionally, we would like to contrast the results with real data from people who have the infection.

We observe that active-particle methods have recently been used to model epidemics through a detailed description of the immune competition at the cellular scale. Specifically, the competition between the primary virus and variants has been considered, see [64, 65]. This approach has not yet considered comorbidities. We believe that our approach can contribute to include different types of viruses within the model proposed in [64, 65].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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