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Research article

Analysis on existence of system of coupled multifractional nonlinear hybrid differential equations with coupled boundary conditions

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Abstract: This article dealt with a class of coupled hybrid fractional differential system. It consisted of a mixed type of Caputo and Hilfer fractional derivatives with respect to two different kernel functions, ψ_1 and ψ_2, respectively, in addition to coupled boundary conditions. The existence of the solution of the system was investigated using the Dhage fixed point theorem. Finally, an illustration was presented to validate our findings.

Keywords: ψ-Hilfer and ψ-Caputo fractional derivatives; hybrid fractional differential equations; fixed point theory; existence
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1. Introduction

Fractional calculus is the study of fractional order integral and derivative operators over real or complex domains. The importance of utilizing fractional derivatives in differential equations arises from their capacity to model complicated events when compared to traditional integer-order derivatives. The nonlocal property allows us to describe long-range correlations in various physical systems [1–5]. Fractional differential equations (FDEs) are employed to mathematically represent circumstances and processes observed in a variety of engineering and scientific fields. These disciplines include electrodynamics of complex mediums, aerodynamics, signal and image processing, blood flow phenomena, economics, biophysics, control theory, and more [6–10]. Due to their effectiveness in simulating intricate real-world processes, FDEs have attracted the interest of many researchers.

The study of coupled systems, including FDEs, is significant, as such systems appear in a variety of practical problems, such as the Lorentz system [11], the fractional Duffing system [12], etc. A number of theoretical investigations and research results on coupled systems of FDEs are found in [13, 14]. Certain physical problems are also nonlinear in nature. Perturbing such problems enables a smooth study of their characteristics. Systems perturbed in a quadratic manner are referred to as hybrid differential equations (HDEs). Dhage and Lakshmikanthan [15] first investigated the existence and uniqueness of the solution to the ordinary first-order HDEs with perturbations of the first and second kinds. Recent works on HDEs can be found in [16–18].

Significant advancements have been made in addressing the qualitative analysis and numerical computation of solutions to boundary value problems associated with nonlinear FDEs. Among several fractional derivatives, the generalized (ψ-) fractional derivatives are effectively used for investigating FDEs. Boundary value problems involving generalized fractional derivatives were studied by several authors; see, for example, [19–22] and references therein.

Recently in [23], we studied the coupled system of the ψ-Hilfer nonlinear implicit fractional multipoint boundary value problem of the form

$$\begin{cases} {}^H D_{a^+}^{\alpha_1,\beta;\psi} x(t) = f(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)), \ t \in \mathbb{J} = [a,b], \\ {}^H D_{a^+}^{\alpha_2,\beta;\psi} y(t) = g(t, x(t), {}^H D_{a^+}^{u,v;\psi} x(t)), \ t \in \mathbb{J} = [a,b], \\ x(a) = 0, \ x(b) = \sum_{i=1}^{m} \varphi_i {}^H D_{a^+}^{\delta_i,\beta;\psi} y(\eta_i) + \sum_{j=1}^{n} \sigma_j y(\omega_j), \\ y(a) = 0, \ y(b) = \sum_{r=1}^{p} \lambda_r {}^H D_{a^+}^{\theta_r,\beta;\psi} x(\zeta_r) + \sum_{s=1}^{q} \mu_s x(\xi_s), \end{cases} \tag{1.1}$$

where ${}^H D_{a^+}^{\alpha_1,\beta;\psi}$, ${}^H D_{a^+}^{\alpha_2,\beta;\psi}$, ${}^H D_{a^+}^{u,v;\psi}$, ${}^H D_{a^+}^{\delta_i,\beta;\psi}$ and ${}^H D_{a^+}^{\theta_r,\beta;\psi}$ are the ψ-Hilfer fractional derivatives of order α_1, α_2, u, δ_i, and θ_r, respectively, with $1 < \delta_i, \theta_r < u < \alpha_1, \alpha_2 < 2$, and type $0 \le \beta, v \le 1$, $\varphi_i, \sigma_j, \lambda_r, \mu_s, \in \mathbb{R}^+$, $\eta_i, \omega_j, \zeta_r, \xi_s \in \mathbb{J}$, $f, g : \mathbb{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

We now extend and develop our investigation to analyze the coupled system of multifractional HDEs with coupled boundary conditions of the form

$$\begin{cases} {}^C D_{a^+}^{\delta_1:\psi_1} \left[{}^H D_{a^+}^{\alpha_1,\beta_1:\psi_2} \left(\dfrac{\varphi(\varepsilon)}{g_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon))} \right) + \lambda_1 \varphi(\varepsilon) \right] = f_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon)), \\ {}^C D_{a^+}^{\delta_2:\psi_1} \left[{}^H D_{a^+}^{\alpha_2,\beta_2:\psi_2} \left(\dfrac{\rho(\varepsilon)}{g_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon))} \right) + \lambda_2 \rho(\varepsilon) \right] = f_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)), \ \varepsilon \in [a,b] = \mathcal{J}, \\ \varphi(a) = 0, \qquad \varphi(b) = \zeta_1 \rho(\sigma_1), \\ \rho(a) = 0, \qquad \rho(b) = \zeta_2 \varphi(\sigma_2), \qquad \sigma_1, \sigma_2 \in \mathcal{J}, \quad \zeta_1, \zeta_2 \in \mathcal{R}^+, \end{cases} \tag{1.2}$$

where ${}^C D^{\delta_1:\psi_1}$ and ${}^C D^{\delta_2:\psi_1}$ are the ψ_1-Caputo fractional derivatives of order δ_1 and δ_2, respectively, $0 < \delta_1, \delta_2 < 1$, ${}^H D_{a^+}^{\alpha_1,\beta_1:\psi_2}$, and ${}^H D_{a^+}^{\alpha_2,\beta_2:\psi_2}$ are the ψ_2-Hilfer fractional derivatives of order α_1 and α_2, type β_1 and β_2, respectively, $0 < \alpha_1, \alpha_2 < 1$, $0 \le \beta_1, \beta_2 \le 1$, $\lambda_1, \lambda_2 \in \mathcal{R}$, and the functions $f_1, f_2 : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ and $g_1, g_2 : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R} \backslash \{0\}$ are all continuous.

We emphasize that the present study is novel, more general, and contributes significantly to the existing literature on the topic. The proposed problem includes:

- fractional derivatives of different types, namely, ψ-Caputo and ψ-Hilfer fractional derivatives;
- fractional derivatives with respect to different kernel functions ψ_1 and ψ_2;
- a hybrid system of fractional differential equations with boundary conditions.

The combination of ψ-Caputo and ψ-Hilfer derivatives in a coupled system provides a more flexible and comprehensive framework and enhances the modeling capability by capturing different types of fractional behavior simultaneously. The ψ-Caputo fractional derivative is suitable for modeling systems with memory effects, where past states strongly influence the present behavior. The ψ-Hilfer fractional derivative provides a more generalized fractional derivative that includes both memory and anticipation effects. The parameters β_1 and β_2 in (1.2) allow for tuning the balance between the past and future contributions. The ψ-Caputo derivative is sensitive to initial conditions, while the ψ-Hilfer derivative is known for its behavior near singularities. Combining both allows for a more nuanced approach to systems with complex dynamics.

Different kernels in the derivatives allow for the modeling of different decay and growth behaviors, enabling a more accurate representation of the physical processes.

Multifractional derivatives provide a natural framework for capturing nonuniform memory effects in coupled systems. The dynamics of coupled systems with multifractional derivatives are significant for developing more effective control and optimization algorithms for complex interconnected systems.

Understanding the qualitative aspects of the system of FDEs helps in establishing the mathematical framework necessary for analyzing and solving these equations. Thus, our objective is to investigate the existence of a solution to a coupled system of multifractional HDEs (1.2).

The article is organized as follows: Section 2 presents the core definitions, lemmas, and theorems for the study. In Section 3, we derive a solution to (1.2). Section 4 establishes the existence of a solution to (1.2). Section 5 includes an example that illustrates our findings, along with graphical representations of the results.

2. Preliminaries

In this section, we present various lemmas, theorems, definitions, and notations that are significant for our study.

Let $C(\mathcal{J},\mathcal{R})$ and $C^m(\mathcal{J},\mathcal{R})$ be the spaces of all continuous and m-times continuously differentiable functions, respectively.

The weighted space of a function h is given by [24]

$$C^m_{\kappa,\gamma}(\mathcal{J}) = \{h : (a,b] \to \mathcal{R};\ h^{[m-1]} \in C(\mathcal{J}),\ h^{[m]} \in C_{\kappa,\gamma}(\mathcal{J})\},$$

where

$$C_{\kappa,\gamma}(\mathcal{J}) = \{h : (a,b] \to \mathcal{R};\ (\psi(\varepsilon) - \psi(a))^{\gamma} h(\varepsilon) \in C(\mathcal{J})\},\ 0 \le \gamma < 1.$$

Also, $\psi \in C(\mathcal{J},\mathcal{R})$ is an increasing function such that $\psi'(\varepsilon) > 0$ for all $t \in \mathcal{J}$.

Definition 2.1. *[24] Let $(a,b) \in \mathcal{R}$ and $\kappa > 0$. The ψ-Riemann-Liouville fractional integral of a function h with respect to ψ is defined by*

$$I^{\kappa;\psi}_{a^+} h(\varepsilon) = \frac{1}{\Gamma(\kappa)} \int_a^{\varepsilon} \psi'(s)(\psi(\varepsilon) - \psi(s))^{\kappa-1} h(s)ds, \quad \varepsilon > a > 0,$$

where $\Gamma(.)$ is the Gamma function.

Lemma 2.1. *[24] Let $\alpha_1, \delta_1 > 0$ and $l > 0$ be constants, then* *(i)* $I_{a^+}^{\alpha_1;\psi} I_{a^+}^{\delta_1;\psi} h(\varepsilon) = I_{a^+}^{\alpha_1+\delta_1;\psi} h(\varepsilon),\ \varepsilon > a,$

(ii) $I_{a^+}^{\delta_1;\psi} (\psi(\varepsilon) - \psi(a))^{l-1}(\varepsilon) = \frac{\Gamma(l)}{\Gamma(l+\delta_1)} (\psi(\varepsilon) - \psi(a))^{l+\delta_1-1}.$

Definition 2.2. *[25] Let $\delta_1 > 0$ and $m - 1 < \delta_1 < m$. The Caputo fractional derivative of a function $h \in C^m(\mathcal{J}, \mathcal{R})$ with respect to ψ is defined by*

$$\begin{aligned}{}^C D_{a^+}^{\delta_1;\psi} h(\varepsilon) =& I_{a^+}^{m-\delta_1;\psi} \Big(\frac{1}{\psi'(\varepsilon)} \frac{d}{d\varepsilon}\Big)^m h(\varepsilon) \\ =& \frac{1}{\Gamma(m - \delta_1)} \int_a^\varepsilon \psi'(s)(\psi(\varepsilon) - \psi(s))^{m-\delta_1-1} h_\psi^{[m]}(s) ds,\end{aligned}$$

where $m = [\delta_1] + 1$, and $[\delta_1]$ is the integer part of $\delta_1 \in \mathcal{R}$.

Definition 2.3. *[26] Let $\alpha_1 > 0$ and $m - 1 < \alpha_1 < m$. The ψ-Hilfer fractional derivative of a function $h \in C^m(\mathcal{J}, \mathcal{R})$ of order α_1 and type $0 \le \beta_1 \le 1$ is defined by*

$${}^H D_{a^+}^{\alpha_1,\beta_1;\psi} h(\varepsilon) = I_{a^+}^{\beta_1(m-\alpha_1);\psi} \Big(\frac{1}{\psi'(\varepsilon)} \frac{d}{d\varepsilon}\Big)^m I_{a^+}^{(1-\beta_1)(m-\alpha_1);\psi} h(\varepsilon),$$

where $m = [\alpha_1] + 1$, $[\alpha_1]$ is the integer part of $\alpha_1 \in \mathcal{R}$, and $\gamma = \alpha_1 + \beta_2(m - \alpha_1)$.

Lemma 2.2. *[25] If $h \in C^m(\mathcal{J}, \mathcal{R})$, $m - 1 < \delta_1 < m$, then*

$$I_{a^+}^{\delta_1;\psi}\, {}^C D_{a^+}^{\delta_1;\psi} h(\varepsilon) = h(\varepsilon) - \sum_{k=0}^{m} \frac{h^{[k]}(a)}{\Gamma(k + 1)} (\psi(\varepsilon) - \psi(a))^k,$$

for all $\varepsilon \in [a, b]$, where $h_\psi^{[m]} h(\varepsilon) = (\frac{1}{\psi'(\varepsilon)} \frac{d}{d\varepsilon})^k h(\varepsilon)$.

Lemma 2.3. *[26] If $h \in C^m(\mathcal{J}, \mathcal{R})$, $m - 1 < \alpha_1 < m$, $0 \le \beta_1 \le 1$, and $\gamma = \alpha_1 + \beta_1(m - \alpha_1)$, then*

$$I_{a^+}^{\alpha_1;\psi}\, {}^H D_{a^+}^{\alpha_1,\beta_1;\psi} h(\varepsilon) = h(\varepsilon) - \sum_{k=1}^{m} \frac{(\psi(\varepsilon) - \psi(a))^{\gamma-k}}{\Gamma(\gamma - k + 1)} h_\psi^{[m-k]} I_{a^+}^{(1-\beta_1)(m-\alpha_1);\psi} h(a),$$

for all $\varepsilon \in [a, b]$, where $h_\psi^{[m]} h(\varepsilon) = (\frac{1}{\psi'(\varepsilon)} \frac{d}{d\varepsilon})^m h(\varepsilon)$.

Let $\mathcal{S} = C(\mathcal{J}, \mathcal{R})$. Clearly, $\mathcal{S}$ is a Banach space endowed with the norm $\|\varphi\| = \sup_{\varepsilon\in\mathcal{J}} |\varphi|$, and also a Banach algebra under the multiplication defined by $\varphi\rho(\varepsilon) = \varphi(\varepsilon)\rho(\varepsilon),\ \varphi, \rho \in \mathcal{S},\ \varepsilon \in \mathcal{J}$.

Consequently, the product space $\mathcal{E} = \mathcal{S} \times \mathcal{S}$ is a Banach space with the norm $\|(\varphi, \rho)\| = \|\varphi\| + \|\rho\|$. $\mathcal{E}$ is also a Banach algebra [27] under the multiplication $((\varphi, \rho).(\bar\varphi, \bar\rho))(\varepsilon) = (\varphi, \rho)(\varepsilon).(\bar\varphi, \bar\rho)(\varepsilon) = (\varphi(\varepsilon)\bar\varphi(\varepsilon), \rho(\varepsilon)\bar\rho(\varepsilon)),\ (\varphi, \rho),\ (\bar\varphi, \bar\rho) \in \mathcal{E},\ \varepsilon \in \mathcal{J}$.

Theorem 2.1. *[28] Let $\mathcal{S}$ be a convex, bounded, and closed set contained in the Banach algebra $\mathcal{E}$, and operators $\mathcal{A} : \mathcal{E} \longrightarrow \mathcal{E}$ and $\mathcal{B} : \mathcal{S} \longrightarrow \mathcal{E}$ be such that*

(i) *$\mathcal{A}$ is a Lipschitz map with Lipschitz constant $\mathcal{L}$;*
(ii) *$\mathcal{B}$ is completely continuous;*
(iii) $\varphi = \mathcal{A}(\varphi)\mathcal{B}(\rho) \Leftrightarrow \varphi \in \mathcal{S}\ \forall\ \rho \in \mathcal{S}$;
(iv) $\mathcal{L}\mathcal{M} < 1$, *where* $\mathcal{M} = \|\mathcal{B}(\mathcal{S})\|$.

Thus, the operator equation $\varphi = \mathcal{A}(\varphi)\mathcal{B}(\rho)$ has a solution in $\mathcal{S}$.

3. An auxiliary result

The solution of the boundary value problem (1.2) is derived in this section.

Lemma 3.1. *Let* $0 < \delta_1, \delta_2, \alpha_1, \alpha_2 < 1$, $0 \leq \beta_1, \beta_2 \leq 1$, $\gamma = \alpha_1 + \beta_1(1-\alpha_1)$, $\bar{\gamma} = \alpha_2 + \beta_2(1-\alpha_2)$, $a \geq 0$, *and* $\Omega = \Omega_1\Omega_4 - \Omega_2\Omega_3 \neq 0$*, then for* $f_1, f_2 : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ *and* $g_1, g_2 : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}\backslash\{0\}$*, the solution of the system (1.2) is given by*

$$\varphi(\varepsilon) = \begin{cases} g_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon))\Big\{ I_{a^+}^{\alpha_1:\psi_2} I_{a^+}^{\delta_1:\psi_1} f_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon)) - \lambda_1 I_{a^+}^{\alpha_1:\psi_2}\varphi(\varepsilon) + \dfrac{(\psi_2(\varepsilon)-\psi_2(a))^{\alpha_1}}{\Omega\cdot\Gamma(\alpha_1+1)} \\ \times \Big(\Omega_4\Big[\zeta_1 g_2(\sigma_1, \rho(\sigma_1), \varphi(\sigma_1)) I_{a^+}^{\alpha_2:\psi_2} I_{a^+}^{\delta_2:\psi_1} f_2(\sigma_1, \rho(\sigma_1), \varphi(\sigma_1)) - g_1(b, \varphi(b), \rho(b)) \\ \times I_{a^+}^{\alpha_1:\psi_2} I_{a^+}^{\delta_1:\psi_1} f_1(b, \varphi(b), \rho(b)) - \zeta_1\lambda_2 g_2(\sigma_1, \rho(\sigma_1), \varphi(\sigma_1)) I_{a^+}^{\alpha_2:\psi_2}\rho(\sigma_1) + \lambda_1 g_1(b, \varphi(b), \rho(b)) \\ \times I_{a^+}^{\alpha_1:\psi_2}\varphi(b)\Big] - \Omega_2\Big[\zeta_2 g_1(\sigma_2, \varphi(\sigma_2), \rho(\sigma_2)) I_{a^+}^{\alpha_1:\psi_2} I_{a^+}^{\delta_1:\psi_1} f_1(\sigma_2, \varphi(\sigma_2), \rho(\sigma_2)) \\ - g_2(b, \rho(b), \varphi(b)) I_{a^+}^{\alpha_2:\psi_2} I_{a^+}^{\delta_2:\psi_1} f_2(b, \rho(b), \varphi(b)) - \zeta_2\lambda_1 g_1(\sigma_2, \varphi(\sigma_2), \rho(\sigma_2)) I_{a^+}^{\alpha_1:\psi_2}\varphi(\sigma_2) \\ + \lambda_2 g_2(b, \varphi(b), \rho(b)) I_{a^+}^{\alpha_2:\psi_2}\rho(b)\Big]\Big)\Big\}, \end{cases} \tag{3.1}$$

$$\rho(\varepsilon) = \begin{cases} g_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon))\Big\{ I_{a^+}^{\alpha_2:\psi_2} I_{a^+}^{\delta_2:\psi_1} f_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)) - \lambda_2 I_{a^+}^{\alpha_2:\psi_2}\rho(\varepsilon) + \dfrac{(\psi_2(\varepsilon)-\psi_2(a))^{\alpha_2}}{\Omega\cdot\Gamma(\alpha_2+1)} \\ \times \Big(\Omega_3\Big[\zeta_1 g_2(\sigma_1, \rho(\sigma_1), \varphi(\sigma_1)) I_{a^+}^{\alpha_2:\psi_2} I_{a^+}^{\delta_2:\psi_1} f_2(\sigma_1, \rho(\sigma_1), \varphi(\sigma_1)) - g_1(b, \varphi(b), \rho(b)) \\ \times I_{a^+}^{\alpha_1:\psi_2} I_{a^+}^{\delta_1:\psi_1} f_1(b, \varphi(b), \rho(b)) - \zeta_1\lambda_2 g_2(\sigma_1, \rho(\sigma_1), \varphi(\sigma_1)) I_{a^+}^{\alpha_2:\psi_2}\rho(\sigma_1) + \lambda_1 g_1(b, \varphi(b), \rho(b)) \\ \times I_{a^+}^{\alpha_1:\psi_2}\varphi(b)\Big] - \Omega_1\Big[\zeta_2 g_1(\sigma_2, \varphi(\sigma_2), \rho(\sigma_2)) I_{a^+}^{\alpha_1:\psi_2} I_{a^+}^{\delta_1:\psi_1} f_1(\sigma_2, \varphi(\sigma_2), \rho(\sigma_2)) \\ - g_2(b, \rho(b), \varphi(b)) I_{a^+}^{\alpha_2:\psi_2} I_{a^+}^{\delta_2:\psi_1} f_2(b, \rho(b), \varphi(b)) - \zeta_2\lambda_1 g_1(\sigma_2, \varphi(\sigma_2), \rho(\sigma_2)) I_{a^+}^{\alpha_1:\psi_2}\varphi(\sigma_2) \\ + \lambda_2 g_2(b, \varphi(b), \rho(b)) I_{a^+}^{\alpha_2:\psi_2}\rho(b)\Big]\Big)\Big\}, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} \Omega_1 &= g_1(b, \varphi(b), \rho(b))\frac{(\psi_2(b)-\psi_2(a))^{\alpha_1}}{\Gamma(\alpha_1+1)}, \\ \Omega_2 &= \zeta_1 g_2(\sigma_1, \rho(\sigma_1), \varphi(\sigma_1))\frac{(\psi_2(\sigma_1)-\psi_2(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}, \\ \Omega_3 &= \zeta_2 g_1(\sigma_2, \varphi(\sigma_2), \rho(\sigma_2))\frac{(\psi_2(\sigma_2)-\psi_2(a))^{\alpha_1}}{\Gamma(\alpha_1+1)}, \\ \Omega_4 &= g_2(b, \rho(b), \varphi(b))\frac{(\psi_2(b)-\psi_2(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}. \end{aligned}$$

Proof. Using Lemma 2.2 and applying $I_{a^+}^{\delta_1:\psi_1}$ and $I_{a^+}^{\delta_2:\psi_1}$ on both sides of the HDEs in (1.2), we obtain

$$\begin{aligned} {}^H D_{a^+}^{\alpha_1,\beta_1:\psi_2}\Big(\frac{\varphi(\varepsilon)}{g_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon))}\Big) &= I_{a^+}^{\delta_1:\psi_1} f_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon)) - \lambda_1\varphi(\varepsilon) + c_1, \\ {}^H D_{a^+}^{\alpha_2,\beta_2:\psi_2}\Big(\frac{\rho(\varepsilon)}{g_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon))}\Big) &= I_{a^+}^{\delta_2:\psi_1} f_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)) - \lambda_2\rho(\varepsilon) + d_1. \end{aligned}$$

Now, using Lemma 2.3 and applying $I_{a^+}^{\alpha_1:\psi_2}$ and $I_{a^+}^{\alpha_2:\psi_2}$, we have

$$\begin{aligned}\frac{\varphi(\varepsilon)}{g_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon))} =& I_{a^+}^{\alpha_1:\psi_2} I_{a^+}^{\delta_1:\psi_1} f_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon)) - \lambda_1 I_{a^+}^{\alpha_2:\psi_2}\varphi(\varepsilon) + c_1 \frac{(\psi_2(\varepsilon)-\psi_2(a))^{\alpha_1}}{\Gamma(\alpha_1+1)} \\ &+ c_2 \frac{(\psi_2(\varepsilon)-\psi_2(a))^{\gamma-1}}{\Gamma(\gamma)},\end{aligned}$$

$$\begin{aligned}\frac{\rho(\varepsilon)}{g_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))} =& I_{a^+}^{\alpha_2:\psi_2} I_{a^+}^{\delta_2:\psi_1} f_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)) - \lambda_2 I_{a^+}^{\alpha_2:\psi_2}\rho(\varepsilon) + d_1 \frac{(\psi_2(\varepsilon)-\psi_2(a))^{\alpha_2}}{\Gamma(\alpha_2+1)} \\ &+ d_2 \frac{(\psi_2(\varepsilon)-\psi_2(a))^{\bar{\gamma}-1}}{\Gamma(\bar{\gamma})}.\end{aligned}$$

From $\varphi(a) = \rho(a) = 0$, we get $c_2 = d_2 = 0$, then the above equations reduce to

$$\frac{\varphi(\varepsilon)}{g_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon))} = I_{a^+}^{\alpha_1:\psi_2} I_{a^+}^{\delta_1:\psi_1} f_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon)) - \lambda_1 I_{a^+}^{\alpha_2:\psi_2}\varphi(\varepsilon) + c_1 \frac{(\psi_2(\varepsilon)-\psi_2(a))^{\alpha_1}}{\Gamma(\alpha_1+1)}, \tag{3.3}$$

$$\frac{\rho(\varepsilon)}{g_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))} = I_{a^+}^{\alpha_2:\psi_2} I_{a^+}^{\delta_2:\psi_1} f_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)) - \lambda_2 I_{a^+}^{\alpha_2:\psi_2}\rho(\varepsilon) + d_1 \frac{(\psi_2(\varepsilon)-\psi_2(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}. \tag{3.4}$$

Using $\varphi(b) = \zeta_1\rho(\sigma_1),\ \rho(b) = \zeta_2\varphi(\sigma_2)$ and solving the equations, we obtain

$$\begin{aligned}c_1 =& \frac{1}{\Omega}\Big(\Omega_4\Big[\zeta_1 g_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1)) I_{a^+}^{\alpha_2:\psi_2} I_{a^+}^{\delta_2:\psi_1} f_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1)) - g_1(b,\varphi(b),\rho(b)) \\ &\times I_{a^+}^{\alpha_1:\psi_2} I_{a^+}^{\delta_1:\psi_1} f_1(b,\varphi(b),\rho(b)) - \zeta_1\lambda_2 g_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1)) I_{a^+}^{\alpha_2:\psi_2}\rho(\sigma_1) + \lambda_1 g_1(b,\varphi(b),\rho(b)) \\ &\times I_{a^+}^{\alpha_1:\psi_2}\varphi(b)\Big] - \Omega_2\Big[\zeta_2 g_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2)) I_{a^+}^{\alpha_1:\psi_2} I_{a^+}^{\delta_1:\psi_1} f_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2)) \\ &- g_2(b,\rho(b),\varphi(b)) I_{a^+}^{\alpha_2:\psi_2} I_{a^+}^{\delta_2:\psi_1} f_2(b,\rho(b),\varphi(b)) - \zeta_2\lambda_1 g_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2)) I_{a^+}^{\alpha_1:\psi_2}\varphi(\sigma_2) \\ &+ \lambda_2 g_2(b,\varphi(b),\rho(b)) I_{a^+}^{\alpha_2:\psi_2}\rho(b)\Big]\Big),\end{aligned}$$

and

$$\begin{aligned}d_1 =& \frac{1}{\Omega}\Big(\Omega_3\Big[\zeta_1 g_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1)) I_{a^+}^{\alpha_2:\psi_2} I_{a^+}^{\delta_2:\psi_1} f_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1)) - g_1(b,\varphi(b),\rho(b)) \\ &\times I_{a^+}^{\alpha_1:\psi_2} I_{a^+}^{\delta_1:\psi_1} f_1(b,\varphi(b),\rho(b)) - \zeta_1\lambda_2 g_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1)) I_{a^+}^{\alpha_2:\psi_2}\rho(\sigma_1) + \lambda_1 g_1(b,\varphi(b),\rho(b)) \\ &\times I_{a^+}^{\alpha_1:\psi_2}\varphi(b)\Big] - \Omega_1\Big[\zeta_2 g_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2)) I_{a^+}^{\alpha_1:\psi_2} I_{a^+}^{\delta_1:\psi_1} f_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2)) \\ &- g_2(b,\rho(b),\varphi(b)) I_{a^+}^{\alpha_2:\psi_2} I_{a^+}^{\delta_2:\psi_1} f_2(b,\rho(b),\varphi(b)) - \zeta_2\lambda_1 g_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2)) I_{a^+}^{\alpha_1:\psi_2}\varphi(\sigma_2) \\ &+ \lambda_2 g_2(b,\varphi(b),\rho(b)) I_{a^+}^{\alpha_2:\psi_2}\rho(b)\Big]\Big).\end{aligned}$$

By substituting c_1 and d_1 in (3.3) and (3.4), we obtain (3.1) and (3.2), respectively. Conversely, we can verify that (3.1) and (3.2) satisfy (1.2) by direct computation. □

For simplicity in computation, we introduce the following notations:

$$\mathcal{X}^{k_1,k_2}_{\psi_1,\psi_2}(b) = I^{k_2:\psi_2}_{a^+} I^{k_1:\psi_1}_{a^+}(1)(b), \qquad \Upsilon^{k}_{\psi}(b) = I^{k:\psi}_{a^+}(1)(b),$$

$$\mathcal{H}_i(\varepsilon) = K_{f_i}\mathcal{X}^{\delta_i,\alpha_i}_{\psi_1,\psi_2}(\varepsilon) + \lambda_i\Upsilon^{\alpha_i}_{\psi_2}(\varepsilon),\ i = 1,2, \qquad \mathcal{J}_i(\varepsilon) = \mathcal{M}_{f_i} + \lambda_i\Upsilon^{\alpha_i}_{\psi_2}(\varepsilon)\ i = 1,2,$$

$$Q_1 = \frac{\Upsilon^{\alpha_1}_{\psi_2}(b)}{\Omega}[\Omega_2\zeta_2\mathcal{X}^{\delta_1,\alpha_1}_{\psi_1,\psi_2}(\sigma_2) + \Omega_4\mathcal{X}^{\delta_1,\alpha_1}_{\psi_1,\psi_2}(b)], \qquad Q_2 = \frac{\Upsilon^{\alpha_1}_{\psi_2}(b)}{\Omega}[\Omega_4\zeta_1\mathcal{X}^{\delta_2,\alpha_2}_{\psi_1,\psi_2}(\sigma_1) + \Omega_2\mathcal{X}^{\delta_2,\alpha_2}_{\psi_1,\psi_2}(b)],$$

$$Q_3 = \frac{\Upsilon^{\alpha_1}_{\psi_2}(b)}{\Omega}\Big[\Omega_2\zeta_2\lambda_1\Upsilon^{\alpha_1}_{\psi_2}(\sigma_2) + \Omega_4\lambda_1\Upsilon^{\alpha_1}_{\psi_2}(b)\Big], \qquad Q_4 = \frac{\Upsilon^{\alpha_1}_{\psi_2}(b)}{\Omega}\Big[\Omega_4\zeta_1\lambda_2\Upsilon^{\alpha_2}_{\psi_2}(\sigma_1) + \Omega_2\lambda_2\Upsilon^{\alpha_2}_{\psi_2}(b)\Big],$$

$$Q_5 = \frac{\Upsilon^{\alpha_2}_{\psi_2}(b)}{\Omega}[\Omega_3\zeta_1\mathcal{X}^{\delta_2,\alpha_2}_{\psi_1,\psi_2}(\sigma_1) + \Omega_1\mathcal{X}^{\delta_2,\alpha_2}_{\psi_1,\psi_2}(b)], \qquad Q_6 = \frac{\Upsilon^{\alpha_2}_{\psi_2}(b)}{\Omega}[\Omega_1\zeta_2\mathcal{X}^{\delta_1,\alpha_1}_{\psi_1,\psi_2}(\sigma_2) + \Omega_3\mathcal{X}^{\delta_1,\alpha_1}_{\psi_1,\psi_2}(b)],$$

$$Q_7 = \frac{\Upsilon^{\alpha_2}_{\psi_2}(b)}{\Omega}\Big[\Omega_1\zeta_2\lambda_1\Upsilon^{\alpha_1}_{\psi_2}(\sigma_2) + \Omega_3\lambda_1\Upsilon^{\alpha_1}_{\psi_2}(b)\Big], \qquad Q_8 = \frac{\Upsilon^{\alpha_2}_{\psi_2}(b)}{\Omega}\Big[\Omega_3\zeta_1\lambda_2\Upsilon^{\alpha_2}_{\psi_2}(\sigma_1) + \Omega_1\lambda_2\Upsilon^{\alpha_2}_{\psi_2}(b)\Big].$$

4. Existence of a solution

The existence of the solution to (1.2) is established in this section.

We transform our system into a fixed point problem.

Define an operator $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ by $\mathcal{T}(\varphi,\rho)(\varepsilon) = (\mathcal{T}_1(\varphi,\rho)(\varepsilon), \mathcal{T}_2(\varphi,\rho)(\varepsilon)))$, where

$$\mathcal{T}_1(\varphi,\rho)(\varepsilon) = \begin{cases} g_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon))\Big\{I^{\alpha_1:\psi_2}_{a^+}I^{\delta_1:\psi_1}_{a^+}f_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon)) - \lambda_1 I^{\alpha_1:\psi_2}_{a^+}\varphi(\varepsilon) + \dfrac{(\psi_2(\varepsilon)-\psi_2(a))^{\alpha_1}}{\Omega\cdot\Gamma(\alpha_1+1)} \\ \times\Big(\Omega_4\Big[\zeta_1 g_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1))I^{\alpha_2:\psi_2}_{a^+}I^{\delta_2:\psi_1}_{a^+}f_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1)) - g_1(b,\varphi(b),\rho(b)) \\ \times I^{\alpha_1:\psi_2}_{a^+}I^{\delta_1:\psi_1}_{a^+}f_1(b,\varphi(b),\rho(b)) - \zeta_1\lambda_2 g_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1))I^{\alpha_2:\psi_2}_{a^+}\rho(\sigma_1) \\ + \lambda_1 g_1(b,\varphi(b),\rho(b))I^{\alpha_1:\psi_2}_{a^+}\varphi(b)\Big] - \Omega_2\Big[\zeta_2 g_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2)) \\ \times I^{\alpha_1:\psi_2}_{a^+}I^{\delta_1:\psi_1}_{a^+}f_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2)) - g_2(b,\rho(b),\varphi(b))I^{\alpha_2:\psi_2}_{a^+}I^{\delta_2:\psi_1}_{a^+}f_2(b,\rho(b),\varphi(b)) \\ - \zeta_2\lambda_1 g_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2))I^{\alpha_1:\psi_2}_{a^+}\varphi(\sigma_2) + \lambda_2 g_2(b,\varphi(b),\rho(b))I^{\alpha_2:\psi_2}_{a^+}\rho(b)\Big]\Big)\Big\}, \end{cases}$$

$$\mathcal{T}_2(\varphi,\rho)(\varepsilon) = \begin{cases} g_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))\Big\{I^{\alpha_2:\psi_2}_{a^+}I^{\delta_2:\psi_1}_{a^+}f_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)) - \lambda_2 I^{\alpha_2:\psi_2}_{a^+}\rho(\varepsilon) + \dfrac{(\psi_2(\varepsilon)-\psi_2(a))^{\alpha_2}}{\Omega\cdot\Gamma(\alpha_2+1)} \\ \times\Big(\Omega_3\Big[\zeta_1 g_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1))I^{\alpha_2:\psi_2}_{a^+}I^{\delta_2:\psi_1}_{a^+}f_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1)) - g_1(b,\varphi(b),\rho(b)) \\ \times I^{\alpha_1:\psi_2}_{a^+}I^{\delta_1:\psi_1}_{a^+}f_1(b,\varphi(b),\rho(b)) - \zeta_1\lambda_2 g_2(\sigma_1,\rho(\sigma_1),\varphi(\sigma_1))I^{\alpha_2:\psi_2}_{a^+}\rho(\sigma_1) \\ + \lambda_1 g_1(b,\varphi(b),\rho(b))I^{\alpha_1:\psi_2}_{a^+}\varphi(b)\Big] - \Omega_1\Big[\zeta_2 g_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2)) \\ \times I^{\alpha_1:\psi_2}_{a^+}I^{\delta_1:\psi_1}_{a^+}f_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2)) - g_2(b,\rho(b),\varphi(b))I^{\alpha_2:\psi_2}_{a^+}I^{\delta_2:\psi_1}_{a^+}f_2(b,\rho(b),\varphi(b)) \\ - \zeta_2\lambda_1 g_1(\sigma_2,\varphi(\sigma_2),\rho(\sigma_2))I^{\alpha_1:\psi_2}_{a^+}\varphi(\sigma_2) + \lambda_2 g_2(b,\varphi(b),\rho(b))I^{\alpha_2:\psi_2}_{a^+}\rho(b)\Big]\Big)\Big\}. \end{cases}$$

The coupled system (1.2) has a solution if the operator equation $\mathcal{T}(\varphi,\rho)(\varepsilon) = (\varphi,\rho)(\varepsilon)$ has a fixed point.

We consider the following hypotheses:

($\mathbf{H_1}$) The functions $f_i : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ and $g_i : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}\backslash\{0\}$, $i = 1, 2$ are continuous.
($\mathbf{H_2}$) There exist constants $\mathcal{L}_{f_1}$, $\mathcal{L}_{f_2}$, $\mathcal{L}_{g_1}$, and $\mathcal{L}_{g_2}$ such that

$$\begin{aligned}
|f_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon)) - f_1(\varepsilon, \bar{\varphi}(\varepsilon), \bar{\rho}(\varepsilon))| &\leq \mathcal{L}_{f_1}(|\varphi - \bar{\varphi}| + |\rho - \bar{\rho}|),\\
|f_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)) - f_2(\varepsilon, \bar{\rho}(\varepsilon), \bar{\varphi}(\varepsilon))| &\leq \mathcal{L}_{f_2}(|\rho - \bar{\rho}| + |\varphi - \bar{\varphi}|),\\
|g_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon)) - g_1(\varepsilon, \bar{\varphi}(\varepsilon), \bar{\rho}(\varepsilon))| &\leq \mathcal{L}_{g_1}(|\varphi - \bar{\varphi}| + |\rho - \bar{\rho}|),\\
|g_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)) - g_2(\varepsilon, \bar{\rho}(\varepsilon), \bar{\varphi}(\varepsilon))| &\leq \mathcal{L}_{g_2}(|\rho - \bar{\rho}| + |\varphi - \bar{\varphi}|),
\end{aligned}$$

for all $(\varepsilon, \varphi, \rho)$, $(\varepsilon, \bar{\varphi}, \bar{\rho}) \in \mathcal{J} \times \mathcal{R} \times \mathcal{R}$.
($\mathbf{H_3}$) There exist functions p_1, $p_2 \in \mathcal{S}$ and continuous nondecreasing functions q_1, q_2, r_1, r_2 : $[0, \infty) \to [0, \infty)$ such that

$$\begin{aligned}
|f_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon))| &\leq p_1(\varepsilon)\, q_1(|\varphi|)\, r_1(|\rho|),\\
|f_2(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon))| &\leq p_2(\varepsilon)\, q_2(|\varphi|)\, r_2(|\rho|),
\end{aligned}$$

for all $(\varepsilon, \varphi, \rho) \in \mathcal{J} \times \mathcal{R} \times \mathcal{R}$.
($\mathbf{H_4}$) There exists a number $\varpi > 0$ such that

$$\varpi \geq \frac{\mathcal{M}_1 \mathcal{Q}_{\varpi_1} + \mathcal{M}_2 \mathcal{Q}_{\varpi_2}}{1 - (\mathcal{L}_{g_1} \mathcal{Q}_{\varpi_1} + \mathcal{L}_{g_2} \mathcal{Q}_{\varpi_2})} \quad \text{and} \quad \Delta = (\mathcal{L}_{g_1} + \mathcal{L}_{g_2}) \mathcal{Q}_{\varpi} < 1,$$

where $\mathcal{M}_1 = \sup_{\varepsilon \in \mathcal{J}} |g_1(\varepsilon, 0, 0)|$, $\mathcal{M}_2 = \sup_{\varepsilon \in \mathcal{J}} |g_2(\varepsilon, 0, 0)|$, $\mathcal{Q}_{\varpi} = \mathcal{Q}_{\varpi_1} + \mathcal{Q}_{\varpi_2}$,

$$\begin{aligned}
\mathcal{Q}_{\varpi_1} =& (\mathcal{X}^{\delta_1, \alpha_1}_{\psi_1, \psi_2}(\varepsilon) + \mathcal{Q}_1 \mathcal{M}_1) \|p_1\| q_1(\varpi) r_1(\varpi) + \mathcal{Q}_2 \mathcal{M}_2 \|p_2\| q_2(\varpi) r_2(\varpi)\\
&+ \varpi(\lambda_1 \Upsilon^{\alpha_1}_{\psi_2}(\varepsilon) + \mathcal{Q}_1 \mathcal{L}_{g_1} \|p_1\| q_1(\varpi) r_1(\varpi) + \mathcal{Q}_2 \mathcal{L}_{g_2} \|p_2\| q_2(\varpi) r_2(\varpi)\\
&+ \mathcal{Q}_4 \mathcal{M}_2 + \mathcal{Q}_3 \mathcal{M}_1) + \varpi^2 (\mathcal{Q}_4 \mathcal{L}_{g_2} + \mathcal{Q}_3 \mathcal{L}_{g_1}),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}_{\varpi_2} =& (\mathcal{X}^{\delta_2, \alpha_2}_{\psi_1, \psi_2}(\varepsilon) + \mathcal{Q}_5 \mathcal{M}_2) \|p_2\| q_2(\varpi) r_2(\varpi) + \mathcal{Q}_6 \mathcal{M}_1 \|p_1\| q_1(\varpi) r_1(\varpi)\\
&+ \varpi(\lambda_2 \Upsilon^{\alpha_2}_{\psi_2}(\varepsilon) + \mathcal{Q}_5 \mathcal{L}_{g_2} \|p_2\| q_2(\varpi) r_2(\varpi) + \mathcal{Q}_6 \mathcal{L}_{g_1} \|p_1\| q_1(\varpi) r_1(\varpi)\\
&+ \mathcal{Q}_7 \mathcal{M}_1 + \mathcal{Q}_8 \mathcal{M}_2) + \varpi^2 (\mathcal{Q}_7 \mathcal{L}_{g_1} + \mathcal{Q}_8 \mathcal{L}_{g_2}).
\end{aligned}$$

Theorem 4.1. *If* ($\mathbf{H_1}$)–($\mathbf{H_4}$) *hold, then the coupled system (1.2) has a coupled solution on* $\mathcal{J}$.

Proof. Let us define a subset $\mathcal{K}$ of the Banach space $\mathcal{E}$ by $\mathcal{K} = \{(\varphi, \rho) \in \mathcal{E} : \|(\varphi, \rho)\| \leq \varpi\}$.
Clearly, $\mathcal{K}$ is a closed, bounded, and convex subset of $\mathcal{E}$.
Define operators $\mathcal{F}^{\varphi\rho}_1$, $\mathcal{F}^{\rho\varphi}_2$, $\mathcal{G}^{\varphi\rho}_1$, $\mathcal{G}^{\rho\varphi}_2 : \mathcal{J} \to \mathcal{R}$ by

$$\mathcal{F}^{\varphi\rho}_1 = I^{\alpha_1 : \psi_2}_{a^+} I^{\delta_1 : \psi_1}_{a^+} f_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon)), \qquad \mathcal{F}^{\rho\varphi}_2 = I^{\alpha_2 : \psi_2}_{a^+} I^{\delta_2 : \psi_1}_{a^+} f_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)),$$

$$\mathcal{G}^{\varphi\rho}_1 = g_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon)), \qquad \mathcal{G}^{\rho\varphi}_2 = g_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)).$$

For $(\varphi,\rho),\ (\bar{\varphi},\bar{\rho})\in\mathcal{E}$, we obtain

$$\begin{aligned}
&|\mathcal{F}_1^{\varphi\rho}-\mathcal{F}_1^{\bar{\varphi}\bar{\rho}}|\le \mathcal{X}_{\psi_1,\psi_2}^{\delta_1,\alpha_1}(\varepsilon)\mathcal{L}_{f_1}(|\varphi-\bar{\varphi}|+|\rho-\bar{\rho}|),\\
&|\mathcal{F}_2^{\rho\varphi}-\mathcal{F}_2^{\bar{\rho}\bar{\varphi}}|\le \mathcal{X}_{\psi_1,\psi_2}^{\delta_2,\alpha_2}(\varepsilon)\mathcal{L}_{f_2}(|\rho-\bar{\rho}|+|\varphi-\bar{\varphi}|),\\
&|\mathcal{G}_1^{\varphi\rho}-\mathcal{G}_1^{\bar{\varphi}\bar{\rho}}|\le \mathcal{L}_{g_1}(|\varphi-\bar{\varphi}|+|\rho-\bar{\rho}|),\qquad |\mathcal{G}_2^{\rho\varphi}-\mathcal{G}_2^{\bar{\rho}\bar{\varphi}}|\le \mathcal{L}_{g_2}(|\rho-\bar{\rho}|+|\varphi-\bar{\varphi}|),\\
&|\mathcal{F}_1^{\varphi\rho}|\le \mathcal{X}_{\psi_1,\psi_2}^{\delta_1,\alpha_1}(\varepsilon)\|p_1\|q_1(\varpi)r_1(\varpi),\qquad |\mathcal{F}_2^{\rho\varphi}|\le \mathcal{X}_{\psi_1,\psi_2}^{\delta_2,\alpha_2}(\varepsilon)\|p_2\|q_2(\varpi)r_2(\varpi),\\
&|\mathcal{G}_1^{\varphi\rho}|\le \mathcal{L}_{g_1}(|\varphi|+|\rho|)+\mathcal{M}_1,\qquad |\mathcal{G}_2^{\rho\varphi}|\le \mathcal{L}_{g_2}(|\rho|+|\varphi|)+\mathcal{M}_2.
\end{aligned}$$

Now, let us define operators $\mathcal{A}=(\mathcal{A}_1,\mathcal{A}_2):\mathcal{E}\to\mathcal{E}$ and $\mathcal{B}=(\mathcal{B}_1,\mathcal{B}_2):\mathcal{K}\to\mathcal{E}$ by

$$\mathcal{A}_1(\varphi,\rho)=\mathcal{G}_1^{\varphi\rho}(\varepsilon),\qquad \mathcal{A}_2(\varphi,\rho)=\mathcal{G}_2^{\rho\varphi}(\varepsilon),$$

$$\mathcal{B}_1(\varphi,\rho)=\begin{cases}\mathcal{F}_1^{\varphi\rho}(\varepsilon)-\lambda_1 I_{a^+}^{\alpha_1:\psi_2}\varphi(\varepsilon)+\dfrac{(\psi_2(\varepsilon)-\psi_2(a))^{\alpha_1}}{\Omega\cdot\Gamma(\alpha_1+1)}\Big(\Omega_4\Big[\zeta_1\mathcal{G}_2^{\rho\varphi}(\sigma_1)\mathcal{F}_2^{\rho\varphi}(\sigma_1)-\mathcal{G}_1^{\varphi\rho}(b)\mathcal{F}_1^{\varphi\rho}(b)\\ -\zeta_1\lambda_2\mathcal{G}_2^{\rho\varphi}(\sigma_1)I_{a^+}^{\alpha_2:\psi_2}\rho(\sigma_1)+\lambda_1\mathcal{G}_1^{\varphi\rho}(b)I_{a^+}^{\alpha_1:\psi_2}\varphi(b)\Big]-\Omega_2\Big[\zeta_2\mathcal{G}_1^{\varphi\rho}(\sigma_2)\mathcal{F}_1^{\varphi\rho}(\sigma_2)\\ -\mathcal{G}_2^{\rho\varphi}(b)\mathcal{F}_2^{\rho\varphi}(b)-\zeta_2\lambda_1\mathcal{G}_1^{\varphi\rho}(\sigma_2)I_{a^+}^{\alpha_1:\psi_2}\varphi(\sigma_2)+\lambda_2\mathcal{G}_2^{\rho\varphi}(b)I_{a^+}^{\alpha_2:\psi_2}\rho(b)\Big]\Big),\end{cases}$$

and

$$\mathcal{B}_2(\varphi,\rho)=\begin{cases}\mathcal{F}_2^{\rho\varphi}(\varepsilon)-\lambda_2 I_{a^+}^{\alpha_2:\psi_2}\rho(\varepsilon)+\dfrac{(\psi_2(\varepsilon)-\psi_2(a))^{\alpha_2}}{\Omega\cdot\Gamma(\alpha_2+1)}\Big(\Omega_3\Big[\zeta_1\mathcal{G}_2^{\rho\varphi}(\sigma_1)\mathcal{F}_2^{\rho\varphi}(\sigma_1)-\mathcal{G}_1^{\varphi\rho}(b)\mathcal{F}_1^{\varphi\rho}(b)\\ -\zeta_1\lambda_2\mathcal{G}_2^{\rho\varphi}(\sigma_1)I_{a^+}^{\alpha_2:\psi_2}\rho(\sigma_1)+\lambda_1\mathcal{G}_1^{\varphi\rho}(b)I_{a^+}^{\alpha_1:\psi_2}\varphi(b)\Big]-\Omega_1\Big[\zeta_2\mathcal{G}_1^{\varphi\rho}(\sigma_2)\mathcal{F}_1^{\varphi\rho}(\sigma_2)\\ -\mathcal{G}_2^{\rho\varphi}(b)\mathcal{F}_2^{\rho\varphi}(b)-\zeta_2\lambda_1\mathcal{G}_1^{\varphi\rho}(\sigma_2)I_{a^+}^{\alpha_1:\psi_2}\varphi(\sigma_2)+\lambda_2\mathcal{G}_2^{\rho\varphi}(b)I_{a^+}^{\alpha_2:\psi_2}\rho(b)\Big]\Big).\end{cases}$$

Thus, we observe that $\mathcal{T}(\varphi,\rho)(\varepsilon)=\mathcal{A}(\varphi,\rho)\cdot\mathcal{B}(\varphi,\rho)$.
Now, we prove that $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions of Theorem 2.1.
Step 1: To show that $\mathcal{A}$ is a Lipschitzian on $\mathcal{E}$.
Let $(\varphi,\rho),\ (\bar{\varphi},\bar{\rho})\in\mathcal{E}$, then we obtain

$$\|\mathcal{A}_1(\varphi,\rho)(\varepsilon)-\mathcal{A}_1(\bar{\varphi},\bar{\rho})\|\le \mathcal{L}_{g_1}(\|\varphi-\bar{\varphi}\|+\|\rho-\bar{\rho}\|),$$

and

$$\|\mathcal{A}_2(\varphi,\rho)(\varepsilon)-\mathcal{A}_2(\bar{\varphi},\bar{\rho})\|\le \mathcal{L}_{g_2}(\|\varphi-\bar{\varphi}\|+\|\rho-\bar{\rho}\|).$$

This implies that

$$\|\mathcal{A}(\varphi,\rho)(\varepsilon)-\mathcal{A}(\bar{\varphi},\bar{\rho})\|\le (\mathcal{L}_{g_1}+\mathcal{L}_{g_2})(\|\varphi-\bar{\varphi}\|+\|\rho-\bar{\rho}\|).$$

It means that $\mathcal{A}$ is a Lipschitzian with a Lipschitz constant $\mathcal{L}_{\mathcal{A}}=\mathcal{L}_{g_1}+\mathcal{L}_{g_2}$.
Step 2: To prove $\mathcal{B}$ is completely continuous from $\mathcal{K}$ to $\mathcal{E}$.
Based on the continuity of functions $f_1,\ f_2,\ g_1$, and g_2, the operator $\mathcal{B}$ is continuous.
Now, we show that $\mathcal{B}(\mathcal{K})$ is uniformly bounded in $\mathcal{K}$.

For any $(\varphi,\rho)\in\mathcal{E}$, we obtain

$$
\begin{aligned}
&|\mathcal{B}_1(\varphi,\rho)(\varepsilon)|\\
&\le |\mathcal{F}_1^{\varphi\rho}(\varepsilon)| - |\lambda_1|\, I_{a^+}^{\alpha_1:\psi_2}|\varphi(\varepsilon)| + \frac{(\psi_2(\varepsilon)-\psi_2(a))^{\alpha_1}}{\Omega\cdot\Gamma(\alpha_1+1)}\Big(|\Omega_4|\Big[\zeta_1\, |\mathcal{G}_2^{\rho\varphi}(\sigma_1)|\\
&\quad\times|\mathcal{F}_2^{\rho\varphi}(\sigma_1)| - |\mathcal{G}_1^{\varphi\rho}(b)|\,|\mathcal{F}_1^{\varphi\rho}(b)| - \zeta_1\,|\lambda_2|\,|\mathcal{G}_2^{\rho\varphi}(\sigma_1)|\, I_{a^+}^{\alpha_2:\psi_2}|\rho(\sigma_1)|\\
&\quad+|\lambda_1|\,|\mathcal{G}_1^{\varphi\rho}(b)|\, I_{a^+}^{\alpha_1:\psi_2}|\varphi(b)|\Big] - |\Omega_2|\Big[\zeta_2\,|\mathcal{G}_1^{\varphi\rho}(\sigma_2)|\,|\mathcal{F}_1^{\varphi\rho}(\sigma_2)| - |\mathcal{G}_2^{\rho\varphi}(b)|\\
&\quad\times|\mathcal{F}_2^{\rho\varphi}(b)| - \zeta_2\,|\lambda_1|\,|\mathcal{G}_1^{\varphi\rho}(\sigma_2)|\, I_{a^+}^{\alpha_1:\psi_2}|\varphi(\sigma_2)| + |\lambda_2|\,|\mathcal{G}_2^{\rho\varphi}(b)|\, I_{a^+}^{\alpha_2:\psi_2}|\rho(b)|\Big]\Big)\\
&\le \mathcal{X}_{\psi_1,\psi_2}^{\delta_1,\alpha_1}(\varepsilon)\|p_1\|q_1(\varpi)r_1(\varpi) + |\lambda_1|\Upsilon_{\psi_2}^{\alpha_1}(\varepsilon)\varpi + \frac{\Upsilon_{\psi_2}^{\alpha_1}(\varepsilon)}{\Omega}\Big[|\Omega_4|\Big(\zeta_1(\mathcal{L}_{g_2}\varpi+\mathcal{M}_2)\mathcal{X}_{\psi_1,\psi_2}^{\delta_2,\alpha_2}(\sigma_1)\\
&\quad\times\|p_2\|q_2(\varpi)r_2(\varpi) + (\mathcal{L}_{g_1}\varpi+\mathcal{M}_1)\mathcal{X}_{\psi_1,\psi_2}^{\delta_1,\alpha_1}(b)\|p_1\|q_1(\varpi)r_1(\varpi) + \zeta_1|\lambda_2|(\mathcal{L}_{g_2}\varpi+\mathcal{M}_2)\\
&\quad\times\Upsilon_{\psi_2}^{\alpha_2}(\sigma_1)\varpi + \lambda_1(\mathcal{L}_{g_1}\varpi+\mathcal{M}_1)\Upsilon_{\psi_2}^{\alpha_1}(b)\varpi\Big) + |\Omega_2|\Big(\zeta_2(\mathcal{L}_{g_1}\varpi+\mathcal{M}_1)\mathcal{X}_{\psi_1,\psi_2}^{\delta_1,\alpha_1}(\sigma_2)\\
&\quad\times\|p_1\|q_1(\varpi)r_1(\varpi) + (\mathcal{L}_{g_2}\varpi+\mathcal{M}_2)\mathcal{X}_{\psi_1,\psi_2}^{\delta_2,\alpha_2}(b)\|p_2\|q_2(\varpi)r_2(\varpi) + \zeta_2|\lambda_1|(\mathcal{L}_{g_1}\varpi+\mathcal{M}_1)\\
&\quad\times\Upsilon_{\psi_2}^{\alpha_1}(\sigma_2)\varpi + \lambda_2(\mathcal{L}_{g_2}\varpi+\mathcal{M}_2)\Upsilon_{\psi_2}^{\alpha_2}(b)\varpi\Big)\Big]\\
&\le \mathcal{X}_{\psi_1,\psi_2}^{\delta_1,\alpha_1}(\varepsilon)\|p_1\|q_1(\varpi)r_1(\varpi) + \frac{\Upsilon_{\psi_2}^{\alpha_1}(\varepsilon)}{\Omega}\Big[|\Omega_4|\zeta_1\mathcal{X}_{\psi_1,\psi_2}^{\delta_2,\alpha_2}(\sigma_1) + |\Omega_2|\mathcal{X}_{\psi_1,\psi_2}^{\delta_2,\alpha_2}(b)\Big]\|p_2\|q_2(\varpi)r_2(\varpi)\mathcal{M}_2\\
&\quad+\frac{\Upsilon_{\psi_2}^{\alpha_1}(b)}{\Omega}[|\Omega_2|\zeta_2\mathcal{X}_{\psi_1,\psi_2}^{\delta_1,\alpha_1}(\sigma_2) + |\Omega_4|\mathcal{X}_{\psi_1,\psi_2}^{\delta_1,\alpha_1}(b)]\|p_1\|q_1(\varpi)r_1(\varpi)\mathcal{M}_1 + \Big(|\lambda_1|\Upsilon_{\psi_2}^{\alpha_1}(b) + \frac{\Upsilon_{\psi_2}^{\alpha_1}(\varepsilon)}{\Omega}\\
&\quad\times\Big[|\Omega_4|\zeta_1\mathcal{X}_{\psi_1,\psi_2}^{\delta_2,\alpha_2}(\sigma_1) + |\Omega_2|\mathcal{X}_{\psi_1,\psi_2}^{\delta_2,\alpha_2}(b)\Big]\|p_2\|q_2(\varpi)r_2(\varpi)\mathcal{L}_{g_2} + \frac{\Upsilon_{\psi_2}^{\alpha_1}(b)}{\Omega}[|\Omega_2|\zeta_2\mathcal{X}_{\psi_1,\psi_2}^{\delta_1,\alpha_1}(\sigma_2)\\
&\quad+|\Omega_4|\mathcal{X}_{\psi_1,\psi_2}^{\delta_1,\alpha_1}(b)]\|p_1\|q_1(\varpi)r_1(\varpi)\mathcal{L}_{g_1} + \frac{\Upsilon_{\psi_2}^{\alpha_1}(b)}{\Omega}\Big[|\Omega_4|\zeta_1|\lambda_2|\Upsilon_{\psi_2}^{\alpha_2}(\sigma_1) + |\Omega_2|\,|\lambda_2|\Upsilon_{\psi_2}^{\alpha_2}(b)\Big]\mathcal{M}_2\\
&\quad+\frac{\Upsilon_{\psi_2}^{\alpha_1}(b)}{\Omega}\Big[|\Omega_2|\zeta_2|\lambda_1|\Upsilon_{\psi_2}^{\alpha_1}(\sigma_2) + |\Omega_4|\,|\lambda_1|\Upsilon_{\psi_2}^{\alpha_1}(b)\Big]\mathcal{M}_1\Big)\varpi + \Big(\frac{\Upsilon_{\psi_2}^{\alpha_1}(b)}{\Omega}\Big[|\Omega_4|\zeta_1|\lambda_2|\Upsilon_{\psi_2}^{\alpha_2}(\sigma_1)\\
&\quad+|\Omega_2|\,|\lambda_2|\Upsilon_{\psi_2}^{\alpha_2}(b)\Big]\mathcal{L}_{g_2} + \frac{\Upsilon_{\psi_2}^{\alpha_1}(b)}{\Omega}\Big[|\Omega_2|\zeta_2|\lambda_1|\Upsilon_{\psi_2}^{\alpha_1}(\sigma_2) + |\Omega_4|\,|\lambda_1|\Upsilon_{\psi_2}^{\alpha_1}(b)\Big]\mathcal{L}_{g_1}\Big)\varpi^2\\
&\le (\mathcal{X}_{\psi_1,\psi_2}^{\delta_1,\alpha_1}(\varepsilon) + \mathcal{Q}_1\mathcal{M}_1)\|p_1\|q_1(\varpi)r_1(\varpi) + \mathcal{Q}_2\mathcal{M}_2\|p_2\|q_2(\varpi)r_2(\varpi)\\
&\quad+\varpi(|\lambda_1|\Upsilon_{\psi_2}^{\alpha_1}(b) + \mathcal{Q}_1\|p_1\|q_1(\varpi)r_1(\varpi) + \mathcal{Q}_2\|p_2\|q_2(\varpi)r_2(\varpi) + \mathcal{Q}_4\mathcal{M}_3 + \mathcal{Q}_3\mathcal{M}_1)\\
&\quad+\varpi^2(\mathcal{Q}_4\mathcal{L}_{g_2} + \mathcal{Q}_3\mathcal{L}_{g_1}).
\end{aligned}
$$

Thus, $\|\mathcal{B}_1(\varphi,\rho)\|\le\mathcal{Q}_{\varpi_1}$. This implies $\mathcal{B}_1$ is uniformly bounded on $\mathcal{K}$.
Similarly, we obtain $\mathcal{B}_2$ is uniformly bounded on $\mathcal{K}$.
Consequently, $\mathcal{B}$ is uniformly bounded on $\mathcal{K}$.
Next, we show that $\mathcal{B}$ is equicontinuous.
Let $\varepsilon_1,\ \varepsilon_2\in\mathcal{J}$ with $\varepsilon_1\le\varepsilon_2$, then we obtain

$$
\begin{aligned}
&|\mathcal{B}_1(\varphi,\rho)(\varepsilon_2)-\mathcal{B}_1(\varphi,\rho)(\varepsilon_1)|\\
=&\Big|\mathcal{F}_1^{\varphi\rho}(\varepsilon_2)-\lambda_1\, I_{a^+}^{\alpha_1:\psi_2}\varphi(\varepsilon_2)-\mathcal{F}_1^{\varphi\rho}(\varepsilon_1)+\lambda_1\, I_{a^+}^{\alpha_1:\psi_2}\varphi(\varepsilon_1)\\
&+\frac{(\psi_2(\varepsilon_2)-\psi_2(a))^{\alpha_1}-(\psi_2(\varepsilon_1)-\psi_2(a))^{\alpha_1}}{\Omega\cdot\Gamma(\alpha_1+1)}\Big(\Omega_4\Big[\zeta_1\,\mathcal{G}_2^{\rho\varphi}(\sigma_1)\mathcal{F}_2^{\rho\varphi}(\sigma_1)\\
&-\mathcal{G}_1^{\varphi\rho}(b)\,\mathcal{F}_1^{\varphi\rho}(b)-\zeta_1\,\lambda_2\,\mathcal{G}_2^{\rho\varphi}(\sigma_1)\,I_{a^+}^{\alpha_2:\psi_2}\rho(\sigma_1)+\lambda_1\,\mathcal{G}_1^{\varphi\rho}(b)\,I_{a^+}^{\alpha_1:\psi_2}\varphi(b)\Big]\\
&-\Omega_2\Big[\zeta_2\,\mathcal{G}_1^{\varphi\rho}(\sigma_2)\,\mathcal{F}_1^{\varphi\rho}(\sigma_2)-\mathcal{G}_2^{\rho\varphi}(b)\mathcal{F}_2^{\rho\varphi}(b)-\zeta_2\,\lambda_1\,\mathcal{G}_1^{\varphi\rho}(\sigma_2)\,I_{a^+}^{\alpha_1:\psi_2}\varphi(\sigma_2)+\lambda_2\,\mathcal{G}_2^{\rho\varphi}(b)\,I_{a^+}^{\alpha_2:\psi_2}\rho(b)\Big]\Big)\Big|\\
=&\Big|\frac{1}{\Gamma(\delta_1+1)\cdot\Gamma(\alpha_1)}\int_0^{\varepsilon_2}\psi_2'(u)(\psi_1(u)-\psi_1(a))_1^{\delta}\,(\psi_2(\varepsilon_2)-\psi_1(u))^{\alpha_1-1}f_1(u,\varphi(u),\rho(u))\,du\\
&-\frac{1}{\Gamma(\delta_1+1)\cdot\Gamma(\alpha_1)}\int_0^{\varepsilon_1}\psi_2'(u)(\psi_1(u)-\psi_1(a))_1^{\delta}\,(\psi_2(\varepsilon_1)-\psi_1(u))^{\alpha_1-1}f_1(u,\varphi(u),\rho(u))\,du\\
&-\frac{\lambda_1}{\Gamma(\alpha_1)}\int_0^{\varepsilon_2}\psi_2'(u)(\psi_2(\varepsilon_2)-\psi_1(u))^{\alpha_1-1}\varphi(u)\,du+\frac{\lambda_1}{\Gamma(\alpha_1)}\int_0^{\varepsilon_1}\psi_2'(u)(\psi_2(\varepsilon_1)-\psi_1(u))^{\alpha_1-1}\varphi(u)\\
&+\frac{(\psi_2(\varepsilon_2)-\psi_2(a))^{\alpha_1}-(\psi_2(\varepsilon_1)-\psi_2(a))^{\alpha_1}}{\Omega\cdot\Gamma(\alpha_1+1)}\Big(\Omega_4\Big[\zeta_1\,\mathcal{G}_2^{\rho\varphi}(\sigma_1)\mathcal{F}_2^{\rho\varphi}(\sigma_1)\\
&-\mathcal{G}_1^{\varphi\rho}(b)\,\mathcal{F}_1^{\varphi\rho}(b)-\zeta_1\,\lambda_2\,\mathcal{G}_2^{\rho\varphi}(\sigma_1)\,I_{a^+}^{\alpha_2:\psi_2}\rho(\sigma_1)+\lambda_1\,\mathcal{G}_1^{\varphi\rho}(b)\,I_{a^+}^{\alpha_1:\psi_2}\varphi(b)\Big]\\
&-\Omega_2\Big[\zeta_2\,\mathcal{G}_1^{\varphi\rho}(\sigma_2)\,\mathcal{F}_1^{\varphi\rho}(\sigma_2)-\mathcal{G}_2^{\rho\varphi}(b)\mathcal{F}_2^{\rho\varphi}(b)-\zeta_2\,\lambda_1\,\mathcal{G}_1^{\varphi\rho}(\sigma_2)\,I_{a^+}^{\alpha_1:\psi_2}\varphi(\sigma_2)+\lambda_2\,\mathcal{G}_2^{\rho\varphi}(b)\,I_{a^+}^{\alpha_2:\psi_2}\rho(b)\Big]\Big)\Big|\\
\leq&\frac{\|p_1\|q_1(\varpi)r_1(\varpi)}{\Gamma(\delta_1+1)\cdot\Gamma(\alpha_1)}\Big|\int_0^{\varepsilon_1}\psi_2'(u)(\psi_1(u)-\psi_1(a))_1^{\delta}\Big[(\psi_2(\varepsilon_2)-\psi_1(u))^{\alpha_1-1}-(\psi_2(\varepsilon_1)-\psi_1(u))^{\alpha_1-1}\Big]\,du\\
&+\int_{\varepsilon_1}^{\varepsilon_2}\psi_2'(u)(\psi_1(u)-\psi_1(a))_1^{\delta}\,(\psi_2(\varepsilon_2)-\psi_1(u))^{\alpha_1-1}\,du\Big|+\frac{\lambda_1\cdot\varpi}{\Gamma(\alpha_1)}\Big|\int_0^{\varepsilon_1}\psi_2'(u)\Big[(\psi_2(\varepsilon_2)-\psi_1(u))^{\alpha_1-1}\\
&-(\psi_2(\varepsilon_2)-\psi_1(u))^{\alpha_1-1}\Big]\,du+\int_{\varepsilon_1}^{\varepsilon_2}\psi_2'(u)(\psi_2(\varepsilon_2)-\psi_1(u))^{\alpha_1-1}\,du\Big|\\
&+\frac{(\psi_2(\varepsilon_2)-\psi_2(a))^{\alpha_1}-(\psi_2(\varepsilon_1)-\psi_2(a))^{\alpha_1}}{\Omega\cdot\Gamma(\alpha_1+1)}\Big(\Omega_4\Big[\zeta_1\,\mathcal{G}_2^{\rho\varphi}(\sigma_1)\mathcal{F}_2^{\rho\varphi}(\sigma_1)\\
&-\mathcal{G}_1^{\varphi\rho}(b)\,\mathcal{F}_1^{\varphi\rho}(b)-\zeta_1\,\lambda_2\,\mathcal{G}_2^{\rho\varphi}(\sigma_1)\,I_{a^+}^{\alpha_2:\psi_2}\rho(\sigma_1)+\lambda_1\,\mathcal{G}_1^{\varphi\rho}(b)\,I_{a^+}^{\alpha_1:\psi_2}\varphi(b)\Big]\\
&-\Omega_2\Big[\zeta_2\,\mathcal{G}_1^{\varphi\rho}(\sigma_2)\,\mathcal{F}_1^{\varphi\rho}(\sigma_2)-\mathcal{G}_2^{\rho\varphi}(b)\mathcal{F}_2^{\rho\varphi}(b)-\zeta_2\,\lambda_1\,\mathcal{G}_1^{\varphi\rho}(\sigma_2)\,I_{a^+}^{\alpha_1:\psi_2}\varphi(\sigma_2)+\lambda_2\,\mathcal{G}_2^{\rho\varphi}(b)\,I_{a^+}^{\alpha_2:\psi_2}\rho(b)\Big]\Big)\Big|\\
&\longrightarrow 0 \text{ as } \varepsilon_2\longrightarrow\varepsilon_1.
\end{aligned}
$$

Similarly, we can prove that $|\mathcal{B}_2(\varphi,\rho)(\varepsilon_2)-\mathcal{B}_2(\varphi,\rho)(\varepsilon_1)|\longrightarrow 0$ as $\varepsilon_2\longrightarrow\varepsilon_1$.

Consequently, $|\mathcal{B}(\varphi,\rho)(\varepsilon_2)-\mathcal{B}(\varphi,\rho)(\varepsilon_1)| \longrightarrow 0$ as $\varepsilon_2 \longrightarrow \varepsilon_1$, which implies that $\mathcal{B}$ is equicontinuous. By the Arzela-Ascoli theorem [29], $\mathcal{B}$ is completely continuous.

Step 3: To prove that condition (iii) of Theorem 2.1 holds.

Let $(\varphi,\rho) \in \mathcal{E}$ such that

$$(\varphi,\rho) = \mathcal{A}(\varphi,\rho)\cdot\mathcal{B}(\varphi,\rho) = \mathcal{T}(\varphi,\rho),$$

then

$$\begin{aligned}|\varphi(\varepsilon)| &= |\mathcal{A}_1(\varphi,\rho)(\varepsilon)\ \mathcal{B}_1(\varphi,\rho)(\varepsilon)| \le |\mathcal{G}_1^{\varphi\rho}(\varepsilon)|Q_{\varpi_1} \le \Big[\mathcal{L}_{g_1}\|(\varphi,\rho)\| + \mathcal{M}_1\Big]Q_{\varpi_1},\\ |\rho(\varepsilon)| &= |\mathcal{A}_2(\varphi,\rho)(\varepsilon)\ \mathcal{B}_2(\varphi,\rho)(\varepsilon)| \le |\mathcal{G}_2^{\rho\varphi}(\varepsilon)|Q_{\varpi_2} \le \Big[\mathcal{L}_{g_2}\|(\varphi,\rho)\| + \mathcal{M}_2\Big]Q_{\varpi_2}.\end{aligned}$$

This implies

$$\begin{aligned}\|(\varphi,\rho)\| = \|\varphi\| + \|\rho\| &\le \Big[\mathcal{L}_{g_1}\|(\varphi,\rho)\| + \mathcal{M}_1\Big]Q_{\varpi_1} + \Big[\mathcal{L}_{g_2}\|(\varphi,\rho)\| + \mathcal{M}_2\Big]Q_{\varpi_2}\\ &\le (\mathcal{L}_{g_1}Q_{\varpi_1} + \mathcal{L}_{g_2}Q_{\varpi_2})\|(\varphi,\rho)\| + \mathcal{M}_1Q_{\varpi_1} + \mathcal{M}_2Q_{\varpi_2}\\ &\le \frac{\mathcal{M}_1Q_{\varpi_1} + \mathcal{M}_2Q_{\varpi_2}}{1-(\mathcal{L}_{g_1}Q_{\varpi_1} + \mathcal{L}_{g_2}Q_{\varpi_2})}.\end{aligned}$$

From $(\mathbf{H_2})$, we obtain $\|(\varphi,\rho)\| \le \varpi$.

Step 4: To prove that condition (iv) of Theorem 2.1 holds.

We have

$$\begin{aligned}\mathcal{M} = \|\mathcal{B}(\mathcal{K})\| &= \sup\{\|\mathcal{B}(\varphi,\rho)\|\}\\ &= \sup\{\|\mathcal{B}_1(\varphi,\rho)\| + \|\mathcal{B}_2(\varphi,\rho)\|\}\\ &\le Q_{\varpi_1} + Q_{\varpi_2} = Q_{\varpi}.\end{aligned}$$

From the above equation, we get $\mathcal{L}_{\mathcal{A}}\mathcal{M} \le (\mathcal{L}_{g_1} + \mathcal{L}_{g_2})Q_{\varpi} \le 1$.

Thus, all the conditions of the Theorem 2.1 are satisfied and the equation $(\varphi,\rho) = \mathcal{T}(\varphi,\rho)$ has a solution in $\mathcal{K}$. Consequently, the coupled system (1.2) has a coupled solution. □

5. Example

This section includes an illustration to demonstrate the credibility of our findings. We also interpret the numerical solution of the system. The system describes a dynamic and interconnected process where two quantities $\varphi(\varepsilon)$ and $\rho(\varepsilon)$ evolve over time, influenced by memory effects, nonlocal interactions, external forcings, and mutual dependence on each other. This type of system is relevant in electrical engineering and circuit analysis. The state variables may represent the voltage across a capacitor and the current through an inductor. The system is coupled because the voltage and current influence each other through the interaction terms, reflecting the interdependence of voltage and current in electrical circuits.

Example 5.1. Consider the coupled multifractional nonlinear hybrid differential equations with coupled boundary conditions:

$$\begin{cases} {}^{C}D_{0^+}^{0.7:\varepsilon^2}\left[{}^{H}D_{0^+}^{0.4,0.8:2\varepsilon}\left(\dfrac{\varphi(\varepsilon)}{g_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon))}\right)+\lambda_1\varphi(\varepsilon)\right]=f_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon)),\\ {}^{C}D_{0^+}^{0.8:\varepsilon^2}\left[{}^{H}D_{0^+}^{0.6,0.3:2\varepsilon}\left(\dfrac{\rho(\varepsilon)}{g_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))}\right)+\lambda_2\rho(\varepsilon)\right]=f_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)),\ \varepsilon\in[0,1],\\ \varphi(0)=0, \qquad \varphi(1)=\dfrac{2}{5}\,\rho\left(\dfrac{3}{4}\right),\\ \rho(0)=0, \qquad \rho(1)=\dfrac{1}{2}\,\varphi\left(\dfrac{5}{9}\right). \end{cases} \tag{5.1}$$

Here,

$$\delta_1=0.7,\ \alpha_1=0.4,\ \beta_1=0.8,\ \delta_2=0.8,\ \alpha_2=0.6,\ \beta_2=0.3,$$
$$a=0,\ b=1,\ \zeta_1=\frac{2}{5},\ \sigma_1=\frac{3}{4},\ \zeta_2=\frac{1}{2},\ \sigma_2=\frac{5}{9},\ \psi_1(\varepsilon)=\varepsilon^2,\ \psi_2(\varepsilon)=2\,\varepsilon.$$

(i) Consider the functions g_1, g_2, f_1, and f_2,

$$g_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon))=\frac{\sin|\varphi|}{3+\varepsilon^2}+\frac{|\rho|}{1+|\rho|}+\frac{2}{3},$$
$$g_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))=\frac{\cos|\rho|}{4+\varepsilon}+|\varphi|+\frac{4}{5},$$
$$f_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon))=\frac{1}{18+\varepsilon^2}\left[\frac{|\varphi|}{1+|\varphi|}+\frac{2\,|\rho|}{1+|\rho|}\right],$$
$$f_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))=\frac{2\varepsilon+3}{3\varepsilon^2+8}\left[\frac{|\rho|}{1+|\rho|}+|\varphi|\right],$$

and let $\lambda_1=0.02$, $\lambda_2=0.03$.

For $(\varphi,\rho),\ (\bar{\varphi},\bar{\rho})\in\mathcal{E}$, we obtain

$$|f_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon))-f_1(\varepsilon,\bar{\varphi}(\varepsilon),\bar{\rho}(\varepsilon))|\le\frac{1}{9}(|\varphi-\bar{\varphi}|+|\rho-\bar{\rho}|),$$
$$|f_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))-f_2(\varepsilon,\bar{\rho}(\varepsilon),\bar{\varphi}(\varepsilon))|\le\frac{5}{8}(|\rho-\bar{\rho}|+|\varphi-\bar{\varphi}|),$$
$$|g_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon))-g_1(\varepsilon,\bar{\varphi}(\varepsilon),\bar{\rho}(\varepsilon))|\le\frac{1}{3}(|\varphi-\bar{\varphi}|+|\rho-\bar{\rho}|),$$
$$|g_2(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))-g_2(\varepsilon,\bar{\rho}(\varepsilon),\bar{\varphi}(\varepsilon))|\le\frac{1}{4}(|\rho-\bar{\rho}|+|\varphi-\bar{\varphi}|),$$
$$|f_1(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon))|\le\frac{1}{18}\left(\frac{|\varphi|}{1+|\varphi|}\right)\left(\frac{2\,|\rho|}{1+|\rho|}\right),$$
$$|f_2(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon))|\le\frac{5}{8}\left(\frac{|\rho|}{2+|\rho|}\right)|\varphi|,$$

where

$$\|p_1\|=\frac{1}{18},\ \|p_2\|=\frac{5}{8},\ q_1(|\varphi|)=\left(\frac{|\varphi|}{1+|\varphi|}\right),\ q_2(|\varphi|)=\left(\frac{|\rho|}{2+|\rho|}\right),\ r_1(|\rho|)=\left(\frac{2\,|\rho|}{1+|\rho|}\right),\ r_2(|\rho|)=|\varphi|.$$

$(\mathbf{H_2})$ and $(\mathbf{H_3})$ are satisfied with

$$\mathcal{L}_{f_1} = \frac{1}{9},\ \mathcal{L}_{f_2} = \frac{5}{8},\ \mathcal{L}_{g_1} = \frac{1}{3},\ \mathcal{L}_{g_2} = \frac{1}{4}, \mathcal{M}_1 = \frac{2}{3},\ \mathcal{M}_2 = \frac{4}{5}.$$

We compute

$$\Omega \approx 5.9518 \neq 0,\ Q_{\varpi_1} \approx 0.1945,\ \text{and}\ Q_{\varpi_2} \approx 0.3702.$$

Thus, we have

$$\Delta = (\mathcal{L}_{g_1} + \mathcal{L}_{g_2})Q_{\varpi} \approx 0.3294 < 1.$$

As a result, the hypothesis of Theorem 4.1 is satisfied and (5.1) has at least one solution on $\mathcal{J}$.

The numerical results and graphical representation of Δ for various values of $t \in [0, 1]$ and orders $\alpha_1, \delta_1, \alpha_2, \delta_2$ are shown in Table 1 and Figure 1, respectively.

Table 1. Δ for different values of α_1, δ_1, α_2, and δ_2.

t	Δ				
	$\delta_1 = 0.5,\ \alpha_1 = 0.7,$ $\delta_2 = 0.5,\ \alpha_2 = 0.7$	$\delta_1 = 0.55,\ \alpha_1 = 0.75,$ $\delta_2 = 0.55,\ \alpha_2 = 0.75$	$\delta_1 = 0.6,\ \alpha_1 = 0.8,$ $\delta_2 = 0.6,\ \alpha_2 = 0.8$	$\delta_1 = 0.65,\ \alpha_1 = 0.85,$ $\delta_2 = 0.65,\ \alpha_2 = 0.85$	$\delta_1 = 0.7,\ \alpha_1 = 0.9,$ $\delta_2 = 0.7,\ \alpha_2 = 0.9$
0	0.1850	0.1798	0.1740	0.1679	0.1616
0.1	0.2252	0.2140	0.2031	0.1926	0.1824
0.2	0.2502	0.2373	0.2247	0.2124	0.2005
0.3	0.2717	0.2578	0.2441	0.2307	0.2177
0.4	0.2910	0.2765	0.2622	0.2481	0.2343
0.5	0.3089	0.2942	0.2794	0.2648	0.2505
0.6	0.3257	0.3110	0.2960	0.2810	0.2663
0.7	0.3418	0.3270	0.3120	0.2969	0.2819
0.8	0.3571	0.3425	0.3276	0.3124	0.2973
0.9	0.3719	0.3576	0.3427	0.3276	0.3124
1	0.3862	0.3722	0.3576	0.3426	0.3274

We observe that for an increase in time, Δ increases gradually, and for an increase in order, Δ decreases gradually and is clearly less than 1. The results are graphically presented in Figure 1.

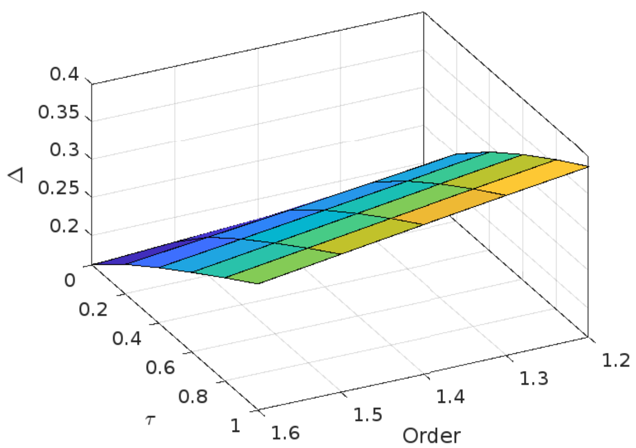

Figure 1. Representation of impact of fractional order α_1, δ_1, α_2, and δ_2 on Δ.

(ii) Consider the functions g_1, g_2, f_1, and f_2,

$$g_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon)) = \varepsilon, \ g_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)) = \varepsilon, \ f_1(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon)) = 1, \ f_2(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)) = 1.$$

Using Picard's approximation technique, under consideration of the above functions g_1, g_2, f_1, and f_2, we have approximated the solution of the system (5.1) in the time interval $[0, 1]$. The convergence is obtained in the fifth iteration. Figure 2 is a graphical representation of the approximate coupled solution to the system (5.1).

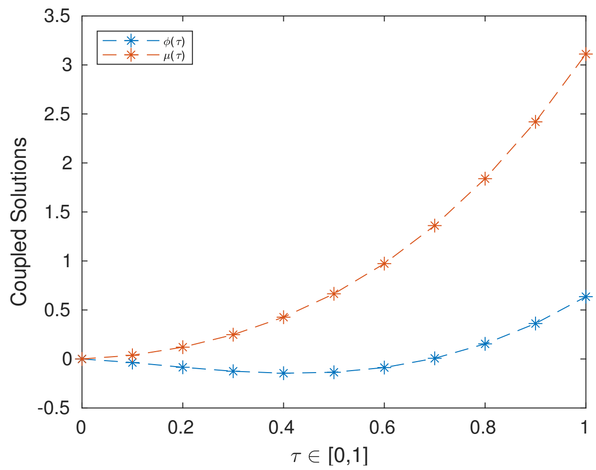

Figure 2. Solution of the system (5.1).

6. Conclusions

In this article, we considered the coupled system of multifractional HDEs with coupled boundary conditions. The system consists of a mixed type of fractional derivatives involving the ψ_1-Caputo and ψ_2-Hilfer fractional derivatives. The fractional derivatives with different kernels in a differential equation provided a flexible and powerful tool for capturing a wide range of memory and nonlocal effects in complex systems. The existence of the solution was established with the aid of the Dhage-fixed point theorem. We emphasized our findings by providing an example. Also, we had obtained the approximate solution of the system through a numerical approach, which was represented graphically.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no conflicts of interest disclosed by the authors.

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