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Some results for a supercritical Schrödinger-Poisson type system with (p, q)-Laplacian

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Abstract: In this work, we focus our attention on the existence of nontrivial solutions to the following supercritical Schrödinger-Poisson type system with (p, q)-Laplacian:

$$\begin{cases} -\Delta_p u - \Delta_q u + \phi|u|^{q-2}u = f(x, u) + \mu|u|^{s-2}u & \text{in } \Omega, \\ -\Delta\phi = |u|^q & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\mu > 0, N > 1$, and $-\Delta_\wp \varphi = div(|\nabla\varphi|^{\wp-2}\nabla\varphi)$, with $\wp \in \{p, q\}$, is the homogeneous $\wp$-Laplacian. $1 < p < q < \frac{q^*}{2}$, $q^* := \frac{Nq}{N-q} < s$, and q^* is the critical exponent to q. The proof is accomplished by the Moser iterative method, the mountain pass theorem, and the truncation technique. Furthermore, the (p, q)-Laplacian and the supercritical term appear simultaneously, which is the main innovation and difficulty of this paper.

Keywords: Schrödinger-Poisson type system; truncation technique; mountain pass theorem; variation methods; Moser iterative method
Mathematics Subject Classification: 35J20, 35R03, 35J60, 35J10

1. Introduction

This paper deals with the following Schrödinger-Poisson type system with supercritical growth:

$$\begin{cases} -\Delta_p u - \Delta_q u + \phi|u|^{q-2}u = f(x, u) + \mu|u|^{s-2}u & \text{in } \Omega, \\ -\Delta\phi = |u|^q & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{P}$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < p < q < \frac{q^*}{2}$, $q^* := \frac{Nq}{N-q} < s$, q^* is the critical exponent, $\mu > 0$, and $N > 1$. In addition, the nonlinearity $f \in \mathbf{C}(\bar{\Omega} \times \mathbb{R}^+, \mathbb{R})$ meets the following assumptions:

(f_1) There exist $\hbar \in (p, q^*)$ and $C > 0$ such that

$$|f(x,\xi)| \leq C|\xi|^{\hbar-1}, \quad \forall\, \xi \in \mathbb{R}^+.$$

(f_2) There exists $\iota \in (2q, q^*)$ such that

$$0 < \iota F(x,\xi) \leq \xi f(x,\xi), \quad \forall\, (x,\xi) \in \bar{\Omega} \times \mathbb{R}^+,$$

where $F(x,\xi) = \int_0^\xi f(x,t)dt$.

The Schrödinger-Poisson system has a strong physical background; for instance, it has been widely applied in fields like semiconductor theory [16, 20] and quantum mechanics models [8, 17]. An increasing number of scholars developed an interest in the system after Benci and Fortunato's groundbreaking work [9]. To be more precise, Du et al. [12] investigated the following quasilinear Schrödinger-Poisson system with p-Laplacian operator:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda\phi|u|^{p-2}u = |u|^{q-2}u & \text{in } \mathbb{R}^3 \\ -\Delta\phi = |u|^p & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0, \Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), 1 < p < 3, p < q < p^* := \frac{3p}{3-p}$. Combining the mountain pass theorem together with some scaling transformation and ingenious methods, they demonstrated the system has nontrivial solutions. In [10], Cassani et al. made use of the variational approximation method to obtain the existence of the solution of the auxiliary Choquard equation for a class of uniformly approximated variational logarithm kernels in fractional Sobolev space. For additional information on this system, readers who are interested might consult [1, 5, 21] and their references.

Recently, the subcritical, critical, and supercritical studies of Schrödinger-Poisson systems have attracted much attention in the field of mathematics. More precisely, Li et al. [18] illustrated that in subcritical and critical circumstances, the existence of infinitely many solutions to fractional Kirchhoff-Schrödinger-Poisson systems is obtained by using the variational method. Liu et al. [19] took into account a critical nonlocal Schrödinger-Poisson system on the Heisenberg group. They utilized the Clark critical point theorem, the mountain pass theorem, the Krasnoselskii genus theorem, and the Ekeland variational principle to give the existence and multiplicity of solutions to this problem. Gu et al. discussed fractional-order Schrödinger-Poisson systems with critical or supercritical nonlinearities in [15]. The existence results of ground state solutions and variable-sign solutions were proved by employing Moser iterative techniques and truncation methods. Readers can also refer to [4, 11, 24, 25] for more relevant content. On the other hand, we discover that investigations of Schrödinger-Poisson systems with (p, q)-Laplacian are scarce. Indeed, Song et al. [22] dealt with the Schrödinger-Poisson system as follows:

$$\begin{cases} -\Delta_p u - \Delta_q u + \left(|u|^{p-2} + |u|^{q-2}\right)u - \phi|u|^{q-2}u = h(x,u) + \lambda g(x) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^q & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$, h is a Carathéodory function, $\frac{3}{4} < p < q < 3$, and $\Delta_\varsigma = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $\varsigma = \{p, q\}$, is the ς-Laplacian. The authors admitted nontrivial solutions by applying fixed point theory. For the subcritical case, Du et al. [13] obtained the existence of solutions for the (p,q)-Schrödinger-Poisson system. Arora et al. [6] obtained the multiplicity results for double phase problems of Kirchhoff type with right-hand sides that include a parametric singular term and a nonlinear term of subcritical growth. However, as far as we know, there are no works in the literature that study the supercritical Schrödinger Poisson type system with (p,q) Laplacian. Hence, inspired by the studies mentioned above and Gao and Tan [14], in the present paper, we consider the existence of nontrivial solutions to problem $(\mathscr{P})$.

Next, we state our main results.

Theorem 1.1. *Assume that hypotheses (f_1) and (f_2) are satisfied. Then, there exists $\mu_* > 0$ such that problem $(\mathscr{P})$ has a nontrivial solution u for $\mu \in (0,\mu_*)$.*

Remark 1.1. *Here we need to point out that problem $(\mathscr{P})$ is driven by several nonstandard differential operators with unbalanced growth, whose associated energy is a double-phase variational functional, which generates an interesting double-phase associated energy. Furthermore, the interaction between the two operators needs to be analyzed in detail. From a mathematical point of view, this problem has great appeal because two features are present in it: the critical nonlinearity and (p,q)-Laplacian. Moreover, our results are new, even in the $p = q$ case.*

The paper is organized as follows. In Section 2, we review some significant properties about $D(\Omega)$, and a truncation argument is introduced. In Section 3, we present that the truncated problem has a nontrivial solution. Finally, we give our main result.

2. Preliminaries

In this section, we will work on some crucial embedding results and properties of $D(\Omega)$, which will be used in the rest of the paper.

To this end, we first show the functional spaces listed as follows:

- ♣ $D^{1,s}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{D^{1,s}} = \left(\int_\Omega |\nabla u|^s dx\right)^{\frac{1}{s}}.$$

- ♣ For all $r \in (p, q^*)$, $L^r(\Omega)$ denotes the Lebesgue space with the norm

$$\|u\|_r = \left(\int_\Omega |u|^r dx\right)^{\frac{1}{r}}.$$

- ♣ $D(\Omega) = D^{1,p}(\Omega) \cap D^{1,q}(\Omega)$, for all $u \in D(\Omega)$, is the natural space for the solutions of problem $(\mathscr{P})$, endowed with the natural norm

$$\|u\| = \|u\|_{D^{1,p}} + \|u\|_{D^{1,q}}.$$

It is known that the space $D(\Omega)$, $D^{1,\wp}(\Omega)$, and $L^r(\Omega)$ are reflexive and uniformly convex Banach spaces.

Lemma 2.1. $D(\Omega)$ *is the reflexive Banach space.*

Proof. Our proof consists of two steps.
Claim 1. $D(\Omega)$ is complete with respect to the norm

$$\|u\| = \|u\|_{D^{1,p}} + \|u\|_{D^{1,q}}.$$

In fact, let $\{u_n\}_n$ be a Cauchy sequence in $D(\Omega)$. Therefore, for any $\varepsilon > 0$, there exists $\mu_\varepsilon > 0$ such that if $n, m \geq \mu_\varepsilon$, we have

$$\|u_n - u_m\| = \|u_n - u_m\|_{D^{1,p}} + \|u_n - u_m\|_{D^{1,q}} < \varepsilon. \tag{2.1}$$

Applying the completeness of $L^p(\Omega)$, then there exists $u \in L^p(\Omega)$ such that $u_n \to u$ as $n \to \infty$. Consequently, there exists a subsequence $\{u_{n_k}\}$ in $D(\Omega)$ such that $u_n \to u$ a.e. in Ω as $k \to \infty$ (please see Brézis [7], Theorem 4.9). Therefore, with the aid of Fatou's lemma in (2.1), with $\varepsilon = 1$, we obtain

$$\begin{aligned}
\|u\| &= \|u\|_{D^{1,p}} + \|u\|_{D^{1,q}} \\
&\leq \liminf_{k\to\infty} \Big[\|u_{n_k}\|_{D^{1,p}} + \|u_{n_k}\|_{D^{1,q}}\Big] \\
&= \liminf_{k\to\infty} \Big[\|u_{n_k} - u_{\mu_1}\|_{D^{1,p}} + \|u_{\mu_1}\|_{D^{1,p}} + \|u_{n_k} - u_{\mu_1}\|_{D^{1,q}} + \|u_{\mu_1}\|_{D^{1,q}}\Big] \\
&= \liminf_{k\to\infty} \Big[\|u_{n_k} - u_{\mu_1}\|_{D^{1,p}} + \|u_{n_k} - u_{\mu_1}\|_{D^{1,q}} + \|u_{\mu_1}\|_{D^{1,p}} + \|u_{\mu_1}\|_{D^{1,q}}\Big] \\
&\leq 1 + \|u_{\mu_1}\| < \infty.
\end{aligned}$$

Hence, $u \in D(\Omega)$. Let $n \geq \mu_\varepsilon$. Using (2.1) and Fatou's lemma, we have

$$\|u_n - u\| \leq \liminf_{k\to\infty} \|u_n - u_{nk}\| < \varepsilon,$$

i.e., $u_n \to u$ in $D(\Omega)$ as $n \to \infty$.

Claim 2. We prove that $(D, \|\cdot\|)$ is uniformly convex. To this end, we fix $\varepsilon \in (0, 2)$ and $u, v \in D(\Omega)$, with

$$\|u\|_{D^{1,p}} = \|v\|_{D^{1,p}} = 2^{-\frac{1}{p}}, \quad \|u\|_{D^{1,q}} = \|v\|_{D^{1,q}} = 2^{-\frac{1}{q}}$$

and $\|u - v\|_{D^{1,p}} \geq \varepsilon$, $\|u - v\|_{D^{1,q}} \geq \varepsilon$.
Case $q > p \geq 2$. We have the following inequality (see Adams and Fournier [2]):

$$\Big|\frac{a+b}{2}\Big|^q + \Big|\frac{a-b}{2}\Big|^q \leq \frac{1}{2}(|a|^q + |b|^q) \;\; \forall a, b \in \mathbb{R}.$$

Now, we prove that

$$\begin{aligned}
\|\frac{u+v}{2}\|^q_{D^{1,q}} + \|\frac{u-v}{2}\|^q_{D^{1,q}} &= \int_\Omega \Big|\frac{\nabla u + \nabla v}{2}\Big|^q dx + \int_\Omega \Big|\frac{\nabla u - \nabla v}{2}\Big|^q dx \\
&= \int_\Omega \Big(\Big|\frac{\nabla u + \nabla v}{2}\Big|^q + \int_\Omega \Big|\frac{\nabla u - \nabla v}{2}\Big|^q\Big) dx \\
&\leq \frac{1}{2}\int_\Omega (|\nabla u|^p + \nabla v|^q) dx \\
&= \frac{1}{2}\Big(\int_\Omega |\nabla u|^q dx + \int_\Omega |\nabla v|^q dx\Big) \\
&= \frac{1}{2}(\|u\|^q_{D^{1,q}} + \|v\|^q_{D^{1,q}}) = \frac{1}{2}(\frac{1}{2} + \frac{1}{2}) = \frac{1}{2} < 1.
\end{aligned} \tag{2.2}$$

By (2.2), we get

$$\|\frac{u+v}{2}\|^q_{D^{1,q}} < 1 - \|\frac{u-v}{2}\|^q_{D^{1,q}} = 1 - (\frac{\varepsilon}{2})^q.$$

We take $\delta = \delta(\varepsilon)$ such that
$$1 - (\frac{\varepsilon}{2})^q = (\frac{1-\delta}{2})^q.$$
So, we have
$$\|\frac{u+v}{2}\|_{D^{1,q}}^q < (\frac{1-\delta}{2})^q.$$
Consequently,
$$\|\frac{u+v}{2}\|_{D^{1,q}} < \frac{1-\delta}{2} < 1-\delta.$$
Therefore, we get that $D^{1,q}(\Omega)$ is uniformly convex. Similarly, we also obtain that $D^{1,p}(\Omega)$ is uniformly convex. Moreover, we get that
$$\|\frac{u+v}{2}\| = \|\frac{u+v}{2}\|_{D^{1,p}} + \|\frac{u+v}{2}\|_{D^{1,q}} < \frac{1-\delta}{2} + \frac{1-\delta}{2} = 1-\delta.$$
Thus, $D(\Omega)$ is uniformly convex.
Case $1 < p < q < 2$. Obviously, the following formula holds:
$$\begin{aligned}
\|u\|_{D^{1,q}}^{q'} &= \left(\int_\Omega |\nabla u|^q dx\right)^{\frac{q'}{q}} \\
&= \left(\int_\Omega |\nabla u|^q dx\right)^{\frac{q}{q-1}\cdot\frac{1}{q}} \\
&= \left(\int_\Omega \left(|\nabla u|^{q'}\right)^{q-1} dx\right)^{\frac{1}{q-1}},
\end{aligned}$$
where $q' = \frac{q}{q-1}$. Following from the reverse Minkowski inequality (see [2], Theorem 2.13), we have
$$\begin{aligned}
&\left\|\frac{u+v}{2}\right\|_{D^{1,q}}^{q'} + \left\|\frac{u-v}{2}\right\|_{D^{1,q}}^{q'} \\
&= \left(\int_\Omega \left(\left|\frac{\nabla u + \nabla v}{2}\right|^{q'}\right)^{q-1} dx\right)^{\frac{1}{q-1}} + \left(\int_\Omega \left(\left|\frac{\nabla u - \nabla v}{2}\right|^{q'}\right)^{q-1} dx\right)^{\frac{1}{q-1}} \\
&= \left\|\left|\frac{\nabla u + \nabla v}{2}\right|^{q'}\right\|_{q-1} + \left\|\left|\frac{\nabla u - \nabla v}{2}\right|^{q'}\right\|_{q-1} \\
&\le \left\|\left|\frac{\nabla u + \nabla v}{2}\right|^{q'} + \left|\frac{\nabla u - \nabla v}{2}\right|^{q'}\right\|_{q-1} \\
&= \left[\int_\Omega \left(\left|\frac{\nabla u + \nabla v}{2}\right|^{q'} + \left|\frac{\nabla u - \nabla v}{2}\right|^{q'}\right)^{q-1} dx\right]^{\frac{1}{q-1}}.
\end{aligned} \tag{2.3}$$
By the inequality (see [2])
$$\left|\frac{a+b}{2}\right|^{q'} + \left|\frac{a-b}{2}\right|^{q'} \le \left(\frac{1}{2}\left(|a|^q + |b|^q\right)\right)^{\frac{1}{q-1}} \quad \forall a, b \in \mathbb{R},$$

we infer in addition to (2.3) that

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_{D^{1,q}}^{q'}+\left\|\frac{u-v}{2}\right\|_{D^{1,q}}^{q'} &\leq\left(\int_\Omega\left(\frac{1}{2}\left(|\nabla u|^q+|\nabla v|^q\right)\right)^{\frac{1}{q-1}\cdot(q-1)}dx\right)^{\frac{1}{q-1}}\\ &=\left(\frac{1}{2}\right)^{\frac{1}{q-1}}\int_\Omega\left(|\nabla u|^q+|\nabla v|^q\right)dx\\ &=\left(\frac{1}{2}\right)^{\frac{1}{q-1}}\left(\|\nabla u\|_{D^{1,q}}^q+\|\nabla v\|_{D^{1,q}}^q\right)\\ &=\left(\frac{1}{2}\right)^{\frac{1}{q-1}}<1.\end{aligned}\tag{2.4}$$

From (2.4), we have

$$\left\|\frac{u+v}{2}\right\|_{D^{1,q}}^{q'}<1-\left\|\frac{u-v}{2}\right\|_{D^{1,q}}^{q'}=1-\left(\frac{\varepsilon}{2}\right)^{q'}.$$

We take $\delta=\delta(\varepsilon)$ such that $1-\left(\frac{\varepsilon}{2}\right)^{q'}=(\frac{1-\delta}{2})^{q'}$. So, we have

$$\|\frac{u+v}{2}\|_{D^{1,q}}<\frac{1-\delta}{2}<1-\delta.$$

Therefore, we get that $D^{1,q}(\Omega)$ is uniformly convex. Similarly, we also obtain that $D^{1,p}(\Omega)$ is uniformly convex. Moreover, we get that

$$\|\frac{u+v}{2}\|=\|\frac{u+v}{2}\|_{D^{1,p}}+\|\frac{u+v}{2}\|_{D^{1,q}}<\frac{1-\delta}{2}+\frac{1-\delta}{2}=1-\delta.$$

Thus, $D(\Omega)$ is uniformly convex. By Theorem 1.21 in Adams and Fournier [2], we obtain that $D(\Omega)$ is a reflexive Banach space. Hence, the proof of Lemma 2.1 is finished. □

Proposition 2.1. *Let* $1<p<q<\frac{q^*}{2}$ *and* $q^*=\frac{Nq}{N-q}$ *hold. Then,*

- ♦ *for all* $r\in[p,q^*]$, *the embedding* $D(\Omega)\hookrightarrow L^r(\Omega)$ *is continuous;*
- ♦ *for all* $r\in[p,q^*)$, *the embedding* $D(\Omega)\hookrightarrow L^r(\Omega)$ *is compact.*

When we study problem $(\mathscr{P})$ by variational methods, the main difficulty lies in the fact that the presence of a supercritical term makes $\mathcal{J}_{\mu,H}$ unable to satisfy the *PS* condition. To solve such a difficulty, we introduce a truncation function. Let $H>0$, and define the following continuous function $m_{\mu,H}:\bar{\Omega}\times\mathbb{R}\to\mathbb{R}$:

$$m_{\mu,H}(x,\xi)=\begin{cases}0 & \text{if } \xi\leq 0,\\ f(x,\xi)+\mu\xi^{s-1} & \text{if } 0<\xi\leq H,\\ f(x,\xi)+\mu H^{s-\hbar}\xi^{\hbar-1} & \text{if } \xi>H,\end{cases}$$

where $p<\hbar<q^*, p\geq 1$, and $s>q^*$. We can easily check that the following properties are satisfied by employing (f_1) and (f_2):

(m_1) $\left|m_{\mu,H}(x,\xi)\right|\leq\left(C+\mu H^{s-\hbar}\right)|\xi|^{\hbar-1}$, where $C>0$ and $\forall\xi\in\mathbb{R}, x\in\bar{\Omega}$.

(m_2) $0 < \iota M_{\mu,H}(x,\xi) \leq \xi m_{\mu,H}(x,\xi)$, $\forall \xi > 0, x \in \bar{\Omega}$, where $\iota \in (2q, q^*)$ and

$$M_{\mu,H}(x,\xi) = \int_0^\xi m_{\mu,H}(x,t)dt.$$

In the next moment, we are committed to a truncation problem as follows:

$$\begin{cases} -\Delta_p u - \Delta_q u + \phi |u|^{q-2}u = m_{\mu,H}(x,\xi) & \text{in } \Omega, \\ -\Delta\phi = |u|^q & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathscr{Q}$}$$

Problem $(\mathscr{Q})$ is variational, and the associated energy functional $\Im_{\mu,H} : D(\Omega) \to \mathbb{R}$ is given by for any $u \in D(\Omega)$:

$$\Im_{\mu,H}(u,\phi) = \frac{1}{p}\|u\|_{D^{1,p}} + \frac{1}{q}\|u\|_{D^{1,q}} + \frac{1}{2q}\int_\Omega |\nabla\phi|^2 dx - \int_\Omega M_{\mu,H}(x,u)dx.$$

For any $u \in D(\Omega)$, by the Lax-Milgram Theorem, we can find a unique $\phi_u \in D^{1,2}(\Omega)$ satisfying

$$-\Delta\phi_u = |u|^q,$$

which yields that

$$0 \leq \int_\Omega |\nabla\phi|^2 dx = \int_\Omega \phi_u |u|^q.$$

Hence, we are able to define the one-variable functional $\mathscr{J}_{\mu,H} : D(\Omega) \to \mathbb{R}$ by

$$\mathscr{J}_{\mu,H}(u) = \Im_{\mu,H}(u,\phi) = \frac{1}{p}\|u\|^p_{D^{1,p}} + \frac{1}{q}\|u\|^q_{D^{1,q}} + \frac{1}{2q}\int_\Omega \phi_u |u|^q dx - \int_\Omega M_{\mu,H}(x,u)dx.$$

The derivative of this can be represented as

$$\left\langle \mathscr{J}'_{\mu,H}(u), v \right\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v + \int_\Omega \phi_u |u|^{q-2} u v dx - \int_\Omega m_{\mu,H}(x,u) v dx. \tag{2.5}$$

Thanks to Du et al. [12], we have the following vital proposition.

Proposition 2.2. *For any $u \in D(\Omega)$, the following results hold:*

(i) $\phi_u \geq 0$ *and* $\phi_{tu} = t^q \phi_u$ *for any* $t > 0$.
(ii) There exists $C > 0$ *such that* $\|\phi_u\|_{D^{1,2}} \leq C\|u\|^q$ *and* $\int_\Omega |\nabla\phi_u|^2 dx = \int_\Omega \phi_u |u|^q \leq C\|u\|^{2q}$.
(iii) If $u_n \rightharpoonup u$ *in* $D(\Omega)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\Omega)$, *and*

$$\int_\Omega \phi_{u_n} |u_n|^{q-2} u_n \varphi dx \to \int_\Omega \phi_u |u|^{q-2} u \varphi dx, \quad \forall \varphi \in D(\Omega).$$

Remark 2.1. *Note that if u is a nontrivial solution of problem $(\mathscr{Q})$ with $\|u\|_\infty \leq H$, then u is also a nontrivial solution of problem $(\mathscr{P})$.*

3. Proving the main result

To prove our desired conclusion, let us start with some information on the geometry of functional $\mathscr{J}_{\mu,H}$.

Lemma 3.1. *The functional $\mathscr{J}_{\mu,H}$ satisfies the following conditions:*

(i) There exist $\vartheta, \rho > 0$ such that

$$\mathscr{J}_{\mu,H}(u) \geq \vartheta, \quad if \ \ \|u\| = \rho.$$

(ii) Let $\omega \in D(\Omega)$ with $\omega \neq 0$, and we have

$$\limsup_{t\to\infty} \mathscr{J}_{\mu,H}(t\omega) = -\infty.$$

Proof. Notice that, as a consequence of $p < q < \frac{q^*}{2}$, Sobolev embeddings, $(m_1) - (m_2)$, and taking $\|u\| < 1$, we have

$$\begin{aligned}
\mathscr{J}_{\mu,H}(u) =& \frac{1}{p}\|u\|^p_{D^{1,p}} + \frac{1}{q}\|u\|^q_{D^{1,q}} + \frac{1}{2q}\int_\Omega \phi_u|u|^q dx - \int_\Omega M_{\mu,H}(x,u)dx \\
\geq& \frac{1}{q}2^{1-p}\|u\|^p - \frac{C+\mu H^{s-\hbar}}{\iota}\|u\|^{\hbar}_{\hbar} \\
\geq& \frac{1}{q}2^{1-p}\|u\|^p - C_{\mu,H}\|u\|^{\hbar},
\end{aligned}$$

where $C_{\mu,H} > 0$. It follows from $p < \hbar < q^*$ that item (i) follows. Now, let us prove the second term. It can be deduced from (f_2) that there exist positive constants C_1 and C_2 such that for all $\xi > 0$

$$F(x,\xi) > C_1\xi^\iota - C_2. \tag{3.1}$$

Let $\omega \in D(\Omega)$ with $\omega \neq 0$ and $t > 0$, and by means of $p < q < q^*$ and (3.1), we have

$$\begin{aligned}
\mathscr{J}_{\mu,H}(t\omega) =& \frac{t^p}{p}\|\omega\|^p_{D^{1,p}} + \frac{t^q}{q}\|\omega\|^q_{D^{1,q}} + \frac{t^{2q}}{2q}\int_\Omega \phi_\omega|\omega|^q dx - \int_\Omega M_{\mu,H}(x,t\omega)dx \\
\leq& \frac{1}{p}\|\omega\|^q(t^p + t^q) + \frac{t^{2q}}{2q}\int_\Omega \phi_\omega|\omega|^q dx - \int_\Omega F(x,t\omega)dx \\
\leq& \frac{1}{p}\|\omega\|^q(t^p + t^q) + \frac{t^{2q}}{2q}\int_\Omega \phi_\omega|\omega|^q dx - C_1 t^\iota \int_\Omega |\omega|^\iota dx + C_2|\Omega|.
\end{aligned}$$

In light of $\iota \in (2q, q^*)$, one has that there exists $\bar{\nu} = t\omega \in D(\Omega)$ (with t sufficiently large) such that

$$\limsup_{t\to\infty} \mathscr{J}_{\mu,H}(\bar{\nu}) = -\infty.$$

This finishes the proof of Lemma 3.1. $\square$

By Lemma 3.1, according to Willem [23], we know that there is a PS sequence $\{u_n\} \subset D(\Omega)$ at level $c_{\mu,H}$, where

$$c_{\mu,H} = \inf_{\eta\in\Gamma_\mu} \max_{t\in[0,1]} \mathscr{J}_\mu(\eta(t))$$

and

$$\Gamma_\mu := \{\eta \in C([0,1], D(\Omega)) : \eta(0) = 0, \eta(1) = \bar{v}\}.$$

In the following, we show that $\{u_n\}$ is bounded in $D(\Omega)$.

Lemma 3.2. *Assume $(f_1)-(f_2)$ hold. If $\{u_n\} \subset D(\Omega)$ is a $(PS)_c$ sequence, then $\{u_n\}$ is bounded in $D(\Omega)$.*

Proof. Combining $p < q < \frac{q^*}{2}$ and (m_2), it is easy to see that

$$\begin{aligned}
c + o_n(1) + o_n(1)\|u_n\| =& \mathscr{J}_{\mu,H}(u_n) - \frac{1}{\iota}\left\langle \mathscr{J}'_{\mu,H}(u_n), u_n\right\rangle\\
\geq& (\frac{1}{p} - \frac{1}{\iota})\|u_n\|^p_{D^{1,p}} + (\frac{1}{q} - \frac{1}{\iota})\|u_n\|^q_{D^{1,q}} + (\frac{1}{2q} - \frac{1}{\iota})\int_\Omega \phi_{u_n}|u_n|^q dx\\
&+ \int_\Omega \Big[\frac{1}{\iota} m_{\mu,H}(x, u_n)u_n - M_{\mu,H}(x, u_n)\Big]dx\\
\geq& (\frac{1}{p} - \frac{1}{\iota})\|u_n\|^p_{D^{1,p}} + (\frac{1}{q} - \frac{1}{\iota})\|u_n\|^q_{D^{1,q}} + (\frac{1}{2q} - \frac{1}{\iota})\int_\Omega \phi_{u_n}|u_n|^q dx\\
\geq& (\frac{1}{q} - \frac{1}{\iota})2^{1-p}\|u_n\|^p.
\end{aligned}$$

From $\iota \in (2q, q^*)$, one has that $\{u_n\}$ is bounded in $D(\Omega)$. □

In what follows, we are going to verify that the functional $\mathscr{J}_{\mu,H}$ satisfies the PS condition.

Lemma 3.3. *The functional $\mathscr{J}_{\mu,H}$ satisfies the $(PS)_c$ condition.*

Proof. Let $\{u_n\}$ be a PS sequence for $\mathscr{J}_{\mu,H}$ at level c. Note that Lemma 3.2 shows that the sequence $\{u_n\}$ is bounded in $D(\Omega)$. Thus, utilizing the reflexivity of $D(\Omega)$ and Proposition 2.1, we can get a subsequence still denoted by $\{u_n\}$ and $u \in D(\Omega)$ such that

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } D(\Omega),\\
&u_n \to u \text{ in } L^r(\Omega) \quad for\ any\ r \in [p, q^*),\\
&u_n \to u \text{ a.e. in } \Omega.
\end{aligned} \tag{3.2}$$

Making use of the same ideas as those found in Alves and Figueiredo [3], it is easy to see that u is a critical point of $\mathscr{J}_{\mu,H}$. Thus, owing to (m_1), one can easily show that

$$\begin{aligned}
\Big|\int_\Omega m_{\mu,H}(x, u_n)(u_n - u)dx\Big| \leq& \int_\Omega |m_{\mu,H}(x, u_n)||u_n - u|dx\\
\leq& \int_\Omega (C + \mu H^{s-\hbar})|u|^{\hbar-1}|u_n - u|dx\\
\leq& (C + \mu H^{s-\hbar})\|u\|^{\hbar-1}_{\hbar}\|u_n - u\|_{\hbar}.
\end{aligned}$$

We can use the Brezis-Lieb lemma and (3.2) to obtain that

$$\lim_{n\to\infty}\int_\Omega |u_n-u|^{\hbar}\ \mathrm{d}x=\lim_{n\to\infty}\int_\Omega \left(|u_n|^{\hbar}-|u|^{\hbar}\right)\mathrm{d}x=0.$$

Then, this implies immediately that

$$\lim_{n\to\infty}\int_\Omega m_{\mu,H}(x,u_n)(u_n-u)dx=0. \tag{3.3}$$

Furthermore, in view of Proposition 2.2, one has

$$\lim_{n\to\infty}\int_\Omega \phi_{u_n}|u_n|^{q-2}u_n(u_n-u)dx=0. \tag{3.4}$$

Next, we will present that, up to a subsequence, $u_n \to u$ in $D(\Omega)$.

In fact, with the help of the Brezis-Lieb lemma, $\mathcal{J}'_{\mu,H}(u)=0$, $\mathcal{J}'_{\mu,H}(u_n)(u_n-u)=o_n(1)$, (3.3), and (3.4), we admit

$$\begin{aligned}0=\lim_{n\to\infty}\int_\Omega m_{\mu,H}(x,u_n)(u_n-u)dx &\geq \lim_{n\to\infty}\Big[\int_\Omega(|\nabla u_n|^p-|\nabla u|^p)dx+\int_\Omega(|\nabla u_n|^q-|\nabla u|^q)dx\Big]\\ &=\lim_{n\to\infty}\Big[(\|u_n\|^p_{D^{1,p}}-\|u\|^p_{D^{1,p}})+(\|u_n\|^q_{D^{1,q}}-\|u_n\|^q_{D^{1,q}})\Big]\\ &=\|u_n-u\|^p_{D^{1,p}}+\|u_n-u\|^q_{D^{1,q}}+o_n(1)\\ &\geq 2^{1-p}\|u_n-u\|^p+o_n(1)\geq 0.\end{aligned}$$

This completes the proof of Lemma 3.3. □

In the following, the Moser iteration approach will be employed to demonstrate the following lemma, which displays an estimate of the problem $(\mathcal{Q})$ in L^∞. For simplicity, we denote $u_{\mu,H}$ by u, where $u_{\mu,H}$ is a nontrivial solution of problem $(\mathcal{Q})$.

Lemma 3.4. *There exist two constants $E_1, E_2 > 0$ independent of μ and H such that*

$$\|u\|_\infty \leq E_1(1+\mu H^{s-\hbar})^{E_2}.$$

Proof. For any $A > 0, \gamma > 1$, let

$$u_A(x):=\begin{cases}u(x), & u(x)\leq A,\\ A, & u(x)>A.\end{cases}$$

Moreover, take the following function:

$$Z(u)=Z_{A,\gamma}(u)=uu_A^{q(\gamma-1)}.$$

It is easy to see that Z is an increasing function, so we derive for each $a, b \in \mathbb{R}$

$$(a-b)(Z(a)-Z(b))\geq 0.$$

We define functions as follows:

$$\varpi(\xi)=\int_0^\xi (Z'(\tau))^{\frac{1}{q}}\,d\tau \quad \text{and} \quad \eta(\xi)=\frac{|\xi|^q}{q}.$$

From Jensen's inequality, we admit that for any $a > b$,

$$\begin{aligned}\eta'(a-b)(Z(a)-Z(b)) &= (a-b)^{q-1}(Z(a)-Z(b))\\ &= (a-b)^{q-1}\int_b^a Z'(\xi)d\xi\\ &= (a-b)^{q-1}\int_b^a (\varpi'(\xi))^q\, d\xi\\ &\geq \left(\int_b^a (\varpi'(\xi))\, d\xi\right)^q.\end{aligned}$$

Analogously, for each $a \leq b$, the above inequality also holds. This implies that for each $a, b \in \mathbb{R}$,

$$\eta'(a-b)(Z(a)-Z(b)) \geq |\varpi(a)-\varpi(b)|^q. \tag{3.5}$$

It can be acquired from (3.5) that

$$\begin{aligned}&|\varpi(u)(x)-\varpi(u)(y)|^q\\ \leq\ &|u(x)-u(y)|^{q-2}(u(x)-u(y))\left((uu_A^{q(\gamma-1)})(x)-(uu_A^{q(\gamma-1)})(y)\right).\end{aligned}\tag{3.6}$$

Consider $Z(u) = uu_A^{q(\gamma-1)}$ to be the test function, and then one has $\langle I'(u), Z(u)\rangle = 0$. In addition, putting together (3.6), $\varpi(u) \geq \frac{1}{\gamma}uu_A^{(\gamma-1)}$, and the Sobolev embedding $D^{1,q} \hookrightarrow L^{q^*}$, we infer that

$$\begin{aligned}\int_\Omega m_{\mu,H}(x,u)\left(uu_A^{q(\gamma-1)}\right)(x)\mathrm{d}x &\geq \|\varpi(|u(x)|)\|_{D^{1,q}}^q\\ &\geq C_3\|\varpi(|u(x)|)\|_{q^*}^q\\ &\geq C_3\frac{1}{\gamma^q}\||u|u_A^{\gamma-1}\|_{q^*}^q.\end{aligned}$$

The fact (m_1) gives that

$$\begin{aligned}C_3\frac{1}{\gamma^q}\||u|u_A^{\gamma-1}\|_{q^*}^q &\leq \int_\Omega m_{\mu,H}(x,u)\left(uu_L^{q(\gamma-1)}\right)(x)\mathrm{d}x\\ &\leq \left(C+\mu H^{s-\hbar}\right)\int_\Omega |u|^{\hbar}u_A^{q(\gamma-1)}\mathrm{d}x\\ &\leq \left(C+\mu H^{s-\hbar}\right)\int_\Omega |u|^{\hbar-q}\left(|u|u_A^{(\gamma-1)}\right)^q\,\mathrm{d}x.\end{aligned}$$

Consider $w_A = |u|u_A^{\gamma-1}$. Utilizing the Hölder inequality, we conclude

$$\|w_A\|_{q^*}^q \leq C\left(1+\mu H^{s-\hbar}\right)\gamma^q\left(\int_\Omega |u(x)|^{q^*}\,\mathrm{d}x\right)^{\frac{\hbar-q}{q^*}}\left(\int_\Omega |w_A(x)|^{\beta^*}\,\mathrm{d}x\right)^{\frac{q}{\beta^*}},$$

where

$$\beta^* := \frac{qq^*}{q^*-\hbar+q}.$$

Consequently, using Lemma 3.2 together with Proposition 2.1, one can easily know that

$$\|w_A\|_{q^*}^{q} \leq C\left(1+\mu H^{s-\mathscr{k}}\right)\gamma^q \|w_A\|_{\beta^*}^{q}. \tag{3.7}$$

Take $A \to +\infty$ in (3.7). Combining Fatou's lemma and $0 \leq u_A \leq |u|$, one gets

$$\|u\|_{\gamma q^*} \leq C(1+\mu H^{s-\mathscr{k}})^{\frac{1}{q\gamma}}\gamma^{\frac{1}{\gamma}}\|u\|_{\gamma\beta^*}. \tag{3.8}$$

Therefore,

$$|u|^{\gamma\beta^*} \in L^1(\Omega) \Rightarrow |u|^{\gamma q^*} \in L^1(\Omega).$$

Set $\gamma_0 := \gamma = \frac{q^*}{\beta^*} > 1$. We can employ $\gamma^2\beta^* = \gamma q^*$ and (3.8) to acquire that

$$\|u\|_{\gamma^2 q^*} \leq C(1+\mu H^{s-\mathscr{k}})^{\frac{1}{q\gamma^2}}\gamma^{\frac{2}{\gamma^2}}\|u\|_{\gamma^2\beta^*},$$

which shows that

$$\|u\|_{\gamma^2 q^*} \leq C(1+\mu H^{s-\mathscr{k}})^{\sum_{i=1}^{2}\frac{1}{q\gamma^i}}\gamma^{\sum_{i=1}^{2}\frac{i}{\gamma^i}}\|u\|_{\gamma\beta^*}.$$

Since $q^* = \gamma\beta^*$, repeating the arguments above for $\gamma^3, \gamma^4, \cdots$, we can know for any $d \in \mathbb{N}$

$$\|u\|_{\gamma^d q^*} \leq C(1+\mu H^{s-\mathscr{k}})^{\sum_{i=1}^{d}\frac{1}{q\gamma^i}}\gamma^{\sum_{i=1}^{d}\frac{i}{\gamma^i}}\|u\|_{q^*}. \tag{3.9}$$

Notice that $\sum_{i=1}^{d}\frac{1}{q\gamma^i}$ and $\sum_{i=1}^{d}\frac{i}{\gamma^i}$ are convergent series. Hence, by means of Lemma 3.2 and Proposition 2.1, taking $d \to +\infty$ in (3.9), one has that there are two constants $E_1, E_2 > 0$ independent of μ and H such that

$$\|u\|_\infty \leq E_1(1+\mu H^{s-\mathscr{k}})^{E_2}.$$

This completes the proof. □

Proof of Theorem 1.1. Due to Lemma 3.1, we infer that functional $\mathscr{J}_{\mu,H}$ admits mountain path structure. Therefore, putting Lemma 3.2 together with Lemma 3.3, one can get that problem $(\mathscr{Q})$ has a nontrivial solution. Lemmas 3.1-3.4 mean that, for any $\mu \in (0, \mu_*)$, there exists $\mu_* > 0$ such that $\|u\|_\infty \leq H$. Last but not least, according to Remark 2.1, we know that u is a nontrivial solution of problem $(\mathscr{P})$.

4. Conclusions

This paper studies a supercritical Schrödinger-Poisson type system with (p, q)-Laplacian in $\mathbb{R}^N$, and the existence of nontrivial solutions is discussed. First, we introduced a working space and a truncation argument and obtained the existence of solutions for the truncated problem. Then, by Moser iterative method, the solution of the problem is proved to be the solution of the original system. Finally, we obtain the existence of nontrivial solutions.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interests

The authors declare that they have no competing interests.

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