AIMS Mathematics

AIMS Mathematics, 9(5): 13492–13507.
DOI: 10.3934/math.2024657
Received: 08 March 2024
Revised: 24 March 2024
Accepted: 01 April 2024
Published: 11 April 2024

https://www.aimspress.com/journal/Math

Research article

The study on the complex nature of a predator-prey model with fractional-order derivatives incorporating refuge and nonlinear prey harvesting

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Abstract: The main objective of our research was to explore and develop a fractional-order derivative within the predator-prey framework. The framework includes prey refuge and selective nonlinear harvesting, where the harvesting progressively approaches a threshold value as the density of the harvested population advances. For memory effect, a non-integer order derivative is better than an integer-order derivative. The solutions to the fractional framework were shown to be existence, uniqueness, non-negativity, and boundedness. Matignon's condition was used for analysing local stability, and a suitable Lyapunov function provided global stability. While discussing the Hopf bifurcation's existence condition, we explored derivative order and refuge as bifurcation parameters. We aimed at redefining the predator-prey framework to incorporate fractional order, refuge, and harvesting. This kind of nonlinear harvesting is more realistic and reasonable than the model with constant yield harvesting and constant effort harvesting. The Adams-Bashforth-Moulton PECE algorithm in MATLAB software was used to simulate the proposed outcomes, investigate the impact on various factors, and analyse harvesting’s effect on non-integer order predator-prey interactions.

Keywords: fractional order; refuge; harvesting; stability; Hopf bifurcation
Mathematics Subject Classification: 34A08, 37N25, 34C23

1. Introduction

The use of mathematics to model and comprehend biological processes has grown significantly in recent years. The prey-predator framework is an essential tool in mathematical biology that aids in interpreting the dynamics of population interactions in natural environments. In order to provide predictions, these mathematical models are built using data gathered from experiments and observations. The specific dynamics are crucial for comprehending the effects on the environment throughout the modelling phase. Mathematical systems, mostly representing biological events using differential equations, encapsulate their dynamics. Not only do they have extensive uses in mathematics but also fields as diverse as engineering, biology, physics, and meteorology. All of ecology's activities are essentially the scientific investigation and creative analysis of the complex interactions that exist between living things and their surroundings. It includes not only living things but also the communities they build and the inanimate objects in their surroundings, all of which are constantly interacting with one another. The relationship between predators and their prey has always had significance in both the disciplines of ecology and mathematical ecology due to its global relevance [1–3]. A key component of population dynamics is the relationship between these two entities. Numerous investigators have worked to enhance the fundamental predator-prey system by including various biological factors, such as the prey's fear effect [4], prey refuge [5], selective and mixed-form harvesting [6–8], and many more.

There has been a lot of mathematical model building around the effects of prey refuge in recent years [9–12]. Wang et al. [13] used both analytical and numerical techniques to investigate an impact on prey refuge as well as predator fear on anti-predator activity. Prey refuges generally manifest in one of two ways: either the number of refuges is directly related to the density of prey population, or a fixed maximal capacity is in place [14–19]. In our model, we focus on the first kind, where ε ($0 \le \varepsilon < 1$) is the fixed maximum availability of refuge for prey. Another important aspect of predator-prey dynamics is the harvesting of either group. Fisheries, forests, and wildlife management are common places to find biological resources being harvested. In order to ensure that renewable resources are preserved for future generations, bio-economic models have been created to aid in the scientific management of these assets. The concept of predator-prey interactions is discussed in [20], where the predator incorporates a prey refuge and the prey engages in harvesting. In addition to discussing the impact of harvesting, Mukhopadhyay and Bhattacharyya [21] examined a scenario in which two predators were vying for only one prey. The predator-prey model with various harvesting scenarios is examined from an economic perspective in [22]. Nonlinear harvesting allows our predator-prey system to choose to harvest a single species while protecting the other one. In reality, we limit population growth by harvesting prey or predators. This formulates a predator-prey framework based on the Holling type II dynamics, including the effects of nonlinear harvesting.

$$\begin{aligned} \frac{dx_1}{dt} &= rx_1\left(1-\frac{x_1}{k}\right)-\frac{b\,x_1\,x_2}{a+x_1}-\frac{h\,x_1}{h+x_1}, \\ \frac{dx_2}{dt} &= \frac{b_1 x_1\,x_2}{a+x_1}-cx_2. \end{aligned} \tag{1}$$

The above model is predicated on the following assumptions:

(i) Prey and predator population densities at a given instance of time t are denoted by the variables x_1 and x_2, respectively.

(ii) The inherent growth rate and environmental carrying capacity of prey are r and k, respectively.

(iii) The term $\frac{b x_1 x_2}{a + x_1}$ denotes Holling type II functional response that supports predator species progress, a is the half-saturation constant, $b(>0)$ is the predator's maximal consumption rate, $b_1(>0)$ is the biomass transformation amount $(0 < b_1 < b)$, and $c(>0)$ represents the predator's inherent level of mortality. Here, we can see that if $b_1 < c$, the right portion of the second equation of (1) never becomes positive. Therefore, we assume that $b_1 - c > 0$ for model (1).

(iv) The maximum prey harvesting rate is represented by the parameter h. With an increase in the harvesting population, the harvesting function $\frac{h x_1}{h + x_1}$ reaches gradually the saturation limit value h. From a biological point of view, we think this harvesting role makes more sense.

The suggested model (1) was built using a couple of first-order derivative nonlinear differential equations, in which the instantaneous population density determines how the population density changes over time. However, in actuality, a phenomenon known as the memory effect occurs, wherein the history of all past situations also influences the present state [23]. A fractional differential system may depict phenomena or systems with memory and genetic properties [24]. Drawing on the work of L'Hôpital and Leibniz, who provided a rigorous evaluation of derivatives of order 1/2, Liouville [25] developed the notion of fractional-order derivative. Riemann reinterpreted Liouville's formulation as the Riemann-Liouville fractional derivative operator [26], which he achieved by directly generalising the Cauchy formula. Euler's investigation of non-integer integration particularly inspired him to develop the Gamma function to be a generalisation of the factorial notion for non-integer numbers, which is used to explain the non-integer order derivative idea proposed by Liouville and Riemann [27]. When solving differential equations, Michele Caputo changed the Riemann-Liouville operator in 1967 to eliminate the need for beginning conditions. The modified operator's definition has become known as the Caputo fractional-order derivative operator. This derivative takes into account the memory effects and long-range dependencies present in many real-world systems, allowing for a more accurate representation of their behaviour. The Caputo fractional derivative can handle both smooth and non-smooth functions, making it applicable to a wide range of systems and phenomena. In addition, it possesses desirable mathematical properties, including linearity, compatibility with initial conditions, and the ability to preserve the order of differentiability. Many studies have been conducted on predator-prey frameworks involving non-integer derivatives of the Caputo type [28–35]. So, using the Caputo non-integer order derivative operator, we apply several modifications and analyses within the predator-prey framework (1) that include selective nonlinear harvesting and refuge for prey.

The following is the outline of the article. Section 2 outlines the process of developing the model and its fundamental attributes. One of the fundamental characteristics is the ability to confirm that the model's solutions are non-negative, exist, unique, and bounded. Section 3 presents the outcomes of the dynamical analysis. The findings include the presence and consistency of equilibrium points. The Hopf bifurcation is the one that is studied, and both local and global stability are examined. Verification of analytical findings is accomplished in Section 4 using numerical simulations and interpretations. In Section 5, we summarise our findings.

2. Formulation and fundamental characteristics of the model

The model is created using the Caputo non-integer order derivative operator to the left side of the framework (1), which incorporates selective nonlinear harvesting and refuge for the prey.

$$\begin{aligned} {}^{C}D_0^{\xi} x_1(t) &= r x_1\left(1-\frac{x_1}{k}\right) - \frac{b(1-\varepsilon)x_1 x_2}{a+(1-\varepsilon)x_1} - \frac{h x_1}{h+x_1}, \\ {}^{C}D_0^{\xi} x_2(t) &= \frac{b_1(1-\varepsilon)x_1 x_2}{a+(1-\varepsilon)x_1} - c x_2, \end{aligned} \tag{2}$$

with $\xi \in \mathbb{R}$, $0<\xi\le 1$, and ${}^{C}D_0^{\xi}$ indicates the ξ^{th} order of Caputo non-integer derivative operator given by $D_t^{\xi} f(t) = \frac{1}{\Gamma(1-\xi)}\int_0^t (t-\zeta)^{-\xi} f(\zeta)\, d\zeta$. All of system (2)'s parameters are defined in terms of system (1)'s assumptions.

Theorem 1. [36] Let $\frac{d^{\xi}u}{dt^{\xi}} = Au;\ u(0)=u_0$ be the linear and non-integer order differential equation, where $0<\xi\le 1$, $x\in\mathbb{R}^n$ and A is any arbitrary matrix of size n. In addition, there are:

(i) For $u=0$ to be asymptotically resilient, it is necessary and sufficient that every latent value λ_i of A fulfil $\left|\arg(\lambda_i)\right| > \frac{\xi\pi}{2}$,

(ii) for $u=0$ to be a stable solution, each latent value λ_i of A must satisfy $\left|\arg(\lambda_i)\right| \ge \frac{\xi\pi}{2}$, and all latent values that fulfil $\left|\arg(\lambda_i)\right| = \frac{\xi\pi}{2}$ must have the equal geometric and algebraic multiplicity.

Theorem 2. [37] Let $\frac{d^{\xi}u}{dt^{\xi}} = f(u);\ u(0)=u_0$, be the fractional order differential equation with $0<\xi\le 1$, $x\in\mathbb{R}^n$. The aforementioned system's equilibrium points may be found by solving $f(u)=0$. If the equilibrium points meets the criteria $\left|\arg(\lambda_i)\right| > \frac{\xi\pi}{2}$, then the system is locally asymptotically stable for each latent value λ_i of the community matrix $J=\frac{\partial f}{\partial u}$.

Lemma 1. [38] Assume that ξ is a member of $(0,1]$. Let us take a function $f(t)$ that is a member of $C[0,a]$ such that $D_t^{\xi} f(t) \in C[0,a]$. If $D_t^{\xi} f(t) \ge 0$, for every $t\in(0,a)$, then it is non-decreasing function for all $t\in[0,a]$. If $D_t^{\xi} f(t) \le 0$, then it is non-increasing function for all $t\in[0,a]$.

2.1. Existence, uniqueness, non-negativity, and boundedness

Theorem 3. The framework (2) has a unique solution to every non-negative starting condition.

Proof. Establishing this theorem requires finding a unique solution to the dynamical framework within the domain $[0,\infty)\times\chi$, where $\chi=\left\{(x_1, x_2)\in\mathbb{R}^2:\max\left\{|x_1|,|x_2|\right\}\le T\right\}$.

Let us take $V=(x_1, x_2), \bar{V}=(\bar{x}_1, \bar{x}_2)$ and now consider a mapping $H(V)=(H_1(V),H_2(V))$, where

$$\begin{aligned} H_1(V) &= rx_1\left(1-\frac{x_1}{k}\right)-\frac{b(1-\varepsilon)x_1 x_2}{a+(1-\varepsilon)x_1}-\frac{h x_1}{h+x_1}, \\ H_2(V) &= \frac{b_1(1-\varepsilon)x_1 x_2}{a+(1-\varepsilon)x_1}-c x_2. \end{aligned} \tag{3}$$

Then, for any $V, \bar{V}\in\chi$ we have

$$\left\|H(V)-H(\bar{V})\right\|=\left|H_1(V)-H_1(\bar{V})\right|+\left|H_2(V)-H_2(\bar{V})\right|,$$

$$\le\left(r+\frac{2rT}{k}+\frac{h^2}{(h+T)^2}+\frac{a(1-\varepsilon)T}{(a+(1-\varepsilon)T)^2}(b+b_1)\right)|x_1-\bar{x}_1|+\left(c+\frac{(1-\varepsilon)T}{a+(1-\varepsilon)T}(b+b_1)\right)|x_2-\bar{x}_2|,$$

$$\le L\left\|V-\bar{V}\right\|, \tag{4}$$

where $L=\max\left\{\left(r+\frac{2rT}{k}+\frac{h^2}{(h+T)^2}+\frac{a(1-\varepsilon)T}{(a+(1-\varepsilon)T)^2}(b+b_1)\right),\left(c+\frac{(1-\varepsilon)T}{a+(1-\varepsilon)T}(b+b_1)\right)\right\}$.

Consequently, the Lipschitz criteria is satisfied by the mapping $H(V)$. Therefore, it may be concluded that, given the starting condition $V_0=(x_{1_0}, x_{2_0})$, where $x_{1_0}\ge 0$ and $x_{2_0}\ge 0$, the existence and uniqueness theorem of [39] states that there is a single solution to the system of differential equations (2).

Theorem 4. There exist non-negative and uniformly bounded solutions to a pair of differential equations (2) that begin in the area $\mathbb{R}^2_+$.

Proof. We begin by determining if the solution, which originates in the domain $\mathbb{R}^2_+$, is non-negativity. Hence, it is necessary to establish that, for any $t\ge 0$, $x_1(t)\ge 0$, and $x_2(t)\ge 0$. Assume that not each $t\ge 0$ achieve the criteria of $x_1(t)\ge 0$. Then, there exist $t_1>0$ such that $x_1(t)>0$ for $t\in[0, t_1)$, $x_1(t_1)=0$ and $x_1({t^+}_1)<0$. Subsequently, we may deduce from the initial equation in (2) that $D_t^{\xi}x_1(t)|_{t=t_1}=0$. We find that $x_1(t)\ge 0$ for every $t\ge 0$ because Lemma 1 suggests that $x_1({t^+}_1)=0$, which goes against the assumption $x_1({t^+}_1)<0$. It is simple to establish that the remaining equation of (2) has a non-negative solution that begins within the domain of $\mathbb{R}^2_+$ using a similar procedure. Thus, for any $t\ge 0$, it provides $x_2(t)\ge 0$ in the same approach. Following that, we have to establish the uniform boundedness of the (2) solutions, which begin in the domain $\mathbb{R}^2_+$.

Considering $S(t)=x_1(t)+\frac{b}{b_1}x_2(t)$, we can use Eq (2) to get

$$D_t^{\xi} S(t) + cS(t) = D_t^{\xi} x_1(t) + \frac{b}{b_1} D_t^{\xi} x_2(t) + c\left(x_1(t) + \frac{b}{b_1} x_2(t) \right),$$

$$\leq -\frac{r}{k}\left(x_1 - \frac{k(r+c)}{2r} \right)^2 + \frac{k(r+c)^2}{4r},$$

$$D_t^{\xi} S(t) + cS(t) \leq \frac{k(r+c)^2}{4r}. \tag{5}$$

It is deduced from Lemma 3 of [40] that

$$S(t) \leq \left(S(0) - \frac{k(r+c)^2}{4r} \right) E_{\xi}\left(-(r+c)t^{\xi} \right) + \frac{k(r+c)^2}{4r},$$

where E_{ξ} represents the Mittag-Leffler function. By applying Lemma 5 and Corollary 6 from [41], we obtain $S(t) \leq \frac{k(r+c)^2}{4r}$, $t \to \infty$. Consequently, any solution to (2) that begins in $\mathbb{R}_+^2$ is confined to the region $\Omega = \left\{ (x_1, x_2) \in \mathbb{R}_+^2 : S \leq \frac{k(r+c)^2}{4r} + \delta,\ \delta > 0 \right\}$.

3. Dynamical analysis

3.1. Existence of equilibrium points

The steady state condition is solved at the equilibrium points of the framework (2), which are found by making $D_0^{\xi} x_1(t) = 0$ and $D_0^{\xi} x_2(t) = 0$. We can obtain the following equilibrium points:

(i) The extinction equilibrium $E_0 = (0,0)$,

(ii) the auxiliary equilibrium point $E_1 = (x_1, 0)$, where $rx_1^2 + r(h-k)x_1 + kh(1-r) = 0$, $x_1 = \frac{r(k-h) + \sqrt{r^2(h+k)^2 - 4rkh}}{2r}$ and x_1 is positive for $k > h$, $\frac{r(h+k)^2}{4kh} > 1$,

(iii) the interior equilibrium point $E_2 = (x_1^*, x_2^*)$, where $x_1^* = \frac{ac}{(b_1 - c)(1-\varepsilon)}$ which is positive for $\varepsilon < 1$ and $x_2^* = \frac{ab_1\left[kh(r-1)(b_1-c)^2(1-\varepsilon)^2 + rac(k-h)(b_1-c)(1-\varepsilon) - ra^2c^2 \right]}{kb(b_1-c)^2(1-\varepsilon)^2\left[h(b_1-c)(1-\varepsilon) + ac \right]}$ which is positive for $r > 1$ and $\frac{(k-h)(b_1-c)(1-\varepsilon)}{ac} > 1$.

3.2. Local stability

The local stability of the equilibrium points of the framework (2) is shown by the findings of the community matrix's latent values and the application of the stability criteria theorem in Petras [37].

Assuming that E^* is an equilibrium point of the framework (2). According to the Matignon stability criteria theorem, E^* is local asymptotically stable provided each of the latent values λ_i of a community matrix,

$$J\left(E^*\right)=\begin{pmatrix} r-\frac{2rx_1}{k}-\frac{b(1-\varepsilon)(a+x_1)x_2}{\left(a+(1-\varepsilon)x_1\right)^2}-\frac{h^2}{\left(h+x_1\right)^2} & -\frac{b(1-\varepsilon)x_1}{a+(1-\varepsilon)x_1} \\ \frac{b_1(1-\varepsilon)(a+x_1)x_2}{\left(a+(1-\varepsilon)x_1\right)^2} & \frac{b(1-\varepsilon)x_1}{a+(1-\varepsilon)x_1}-c \end{pmatrix}, \tag{6}$$

that satisfies $\left|\arg\left(\lambda_i\right)\right|>\frac{\xi\pi}{2}$.

Theorem 5. The origin point $E_0(0,0)$ is locally asymptotically stable if $r<1$ and saddle point if $r>1$.

Proof. The Jacobian matrix for $E_0=(0,0)$ is

$$J\left(E_0\right)=\begin{pmatrix} r-1 & 0 \\ 0 & -c \end{pmatrix}. \tag{7}$$

The latent values of $J\left(E_0\right)$ are obtained as $\lambda_1=r-1$ and $\lambda_2=-c<0$. When $r<1$, then the eigenvalue λ_1 is not positive. Consequently, we get $\left|\arg\left(\lambda_1\right)\right|=\left|\arg\left(\lambda_2\right)\right|=\pi$, indicating that the criterion $\left|\arg\left(\lambda_i\right)\right|>\frac{\xi\pi}{2}$, $i=1, 2$ has been achieved. Thus, E_0 is asymptotically stable. If $r>1$, then one latent value is positive and another is negative, with $\left|\arg\left(\lambda_1\right)\right|=0$ and $\left|\arg\left(\lambda_2\right)\right|=\pi$. Thus, we have the situation where one latent value meets the requirement $\left|\arg\left(\lambda_i\right)\right|>\frac{\xi\pi}{2}$, another meets the requirement $\left|\arg\left(\lambda_i\right)\right|<\frac{\xi\pi}{2}$. E_0 is the saddle point as a result.

Remark 1. According to the aforementioned study, the extinction equilibrium E_0 remains unstable $h=0$ (the system has no harvesting).

Theorem 6. The equilibrium point $E_1=\left(x_1^*,0\right)$ is locally asymptotically stable if $\sqrt{hk/r}-h<x_1^*<\frac{ac}{(1-\varepsilon)\left(b_1-c\right)}$.

Proof. The community matrix for E_1 is

$$J\left(E_1\right)=\begin{pmatrix} r-\frac{2rx_1}{k}-\frac{h^2}{\left(h+x_1\right)^2} & -\frac{b(1-\varepsilon)x_1}{a+(1-\varepsilon)x_1} \\ 0 & \frac{b_1(1-\varepsilon)x_1}{a+(1-\varepsilon)x_1}-c \end{pmatrix}. \tag{8}$$

The latent values of $J\left(E_1\right)$ are $\lambda_1=-\left[\frac{rx_1}{k}-\frac{hx_1}{\left(h+x_1\right)^2}\right]$, which is negative if $x_1>\left(\sqrt{hk/r}-h\right)$ and $\lambda_2=\frac{b_1\left(1-\varepsilon\right)x_1}{a+\left(1-\varepsilon\right)x_1}-c$, which is negative if $x_1<\frac{ac}{\left(1-\varepsilon\right)\left(b_1-c\right)}$. Consequently, we get $\left|\arg\left(\lambda_1\right)\right|=\left|\arg\left(\lambda_2\right)\right|=\pi$, indicating that the criterion $\left|\arg\left(\lambda_i\right)\right|>\frac{\xi\pi}{2}$, $i=1, 2$ has been achieved. Thus, E_1 is asymptotically stable if $\sqrt{hk/r}-h<x_1^*<\frac{ac}{\left(1-\varepsilon\right)\left(b_1-c\right)}$.

Theorem 7. The positive equilibrium point $E^*=\left(x_1^*,x_2^*\right)$ is locally asymptotically stable if any of the conditions listed below are fulfilled:

i) $\operatorname{trace}\left(J\left(E^*\right)\right)\le 0$,

ii) $\operatorname{trace}\left(J\left(E^*\right)\right)>0$, $\operatorname{trace}^2\left(J\left(E^*\right)\right)-4\det\left(J\left(E^*\right)\right)<0$ and $\operatorname{sqrt}\left(\left|\operatorname{trace}^2\left(J\left(E^*\right)\right)-4\det\left(J\left(E^*\right)\right)\right|\right)>\operatorname{trace}\left(J\left(E^*\right)\right)\tan\left(\frac{\xi\pi}{2}\right)$.

Proof. The community matrix for $E^*=\left(x_1^*,x_2^*\right)$ is given by

$$J\left(E^*\right)=\begin{pmatrix}-\frac{rx_1}{k}+\frac{b\left(1-\varepsilon\right)^2x_1x_2}{\left(a+\left(1-\varepsilon\right)x_1\right)^2}+\frac{hx_1}{\left(h+x_1\right)^2} & -\frac{b\left(1-\varepsilon\right)x_1}{a+\left(1-\varepsilon\right)x_1}\\ \frac{ab_1\left(1-\varepsilon\right)x_2}{\left(a+\left(1-\varepsilon\right)x_1\right)^2} & 0\end{pmatrix}. \tag{9}$$

Then, $\operatorname{trace}\left(J\left(E^*\right)\right)=-\frac{rx_1}{k}+\frac{b\left(1-\varepsilon\right)^2x_1x_2}{\left(a+\left(1-\varepsilon\right)x_1\right)^2}+\frac{hx_1}{\left(h+x_1\right)^2}$ and $\det\left(J\left(E^*\right)\right)=\frac{abb_1\left(1-\varepsilon\right)^2x_1x_2}{\left(a+\left(1-\varepsilon\right)x_1\right)^3}$

(1) If $\operatorname{trace}\left(J\left(E^*\right)\right)\le 0$, then we have to consider three cases:

Case i) For $\operatorname{trace}\left(J\left(E^*\right)\right)=0$, we will get $\left|\arg\left(\lambda_i\right)\right|=\frac{\pi}{2}$, $i=1, 2$ and E^* is locally asymptotically stable.

Case ii) For $\operatorname{trace}\left(J\left(E^*\right)\right)<0$ and $\operatorname{trace}^2\left(J\left(E^*\right)\right)-4\det\left(J\left(E^*\right)\right)\ge 0$, then both latent roots are negative and $\left|\arg\left(\lambda_i\right)\right|=\pi, i=1,2$. Thus, E^* is locally asymptotically stable.

Case iii) For $\operatorname{trace}\left(J\left(E^*\right)\right)<0$ and $\operatorname{trace}^2\left(J\left(E^*\right)\right)-4\det\left(J\left(E^*\right)\right)<0$, then both latent roots are complex conjugates with non-positive real part and $\left|\arg\left(\lambda_i\right)\right|>\frac{\xi\pi}{2}, i=1,2$. Therefore, E^* is locally asymptotically stable for any ξ.

(2) If $\operatorname{trace}\left(J\left(E^*\right)\right)>0$, $\operatorname{trace}^2\left(J\left(E^*\right)\right)-4\det\left(J\left(E^*\right)\right)<0$ and $\operatorname{sqrt}\left(\left|\operatorname{trace}^2\left(J\left(E^*\right)\right)-4\det\left(J\left(E^*\right)\right)\right|\right)>\operatorname{trace}\left(J\left(E^*\right)\right)\tan\left(\frac{\xi\pi}{2}\right)$, then we obtain

$\operatorname{Im} g(\lambda_1) = -\operatorname{Im} g(\lambda_2) = 4\det\left(J\left(E^*\right)\right) - \text{trace}^2\left(J\left(E^*\right)\right)$ and $\operatorname{Re}(\lambda_i) = \text{trace}\left(J\left(E^*\right)\right) > 0, i = 1,2$. From these we can conclude that $\left|\operatorname{Im} g(\lambda_i)\right| > \tan\left(\frac{\xi\pi}{2}\right)\left(\operatorname{Re}(\lambda_i)\right), i = 1,2$. Consequently, E^* is locally asymptotically stable if $\left|\arg(\lambda_i)\right| > \frac{\xi\pi}{2}, i = 1,2$.

3.3. Global stability

We next proceed to examine E^*'s global stability. To achieve this objective, Lemma 3.1 of [42] and the generalised LaSalle's invariance principle by Huo et al. [43] are employed.
Now, consider a function $V(x_1, x_2) = \left(x_1 - x_1^* - x_1^* \log\frac{x_1}{x_1^*}\right) + a_1\left(x_2 - x_2^* - x_2^* \log\frac{x_2}{x_2^*}\right)$, where $a_1 > 0$ is a constant. Lemma 3.1 of above reference and the ξ-order non-integer derivative of $V(x_1, x_2)$ along the solution of (2) give us

$$\frac{d^\xi V}{dt^\xi} \le \left(\frac{x_1 - x_1^*}{x_1}\right)\frac{d^\xi x_1}{dt^\xi} + a_1\left(\frac{x_2 - x_2^*}{x_2}\right)\frac{d^\xi x_2}{dt^\xi},$$

$$= \left(\frac{x_1 - x_1^*}{x_1}\right)\left(rx_1\left(1 - \frac{x_1}{k}\right) - \frac{b(1-\varepsilon)x_1 x_2}{a + (1-\varepsilon)x_1} - \frac{h x_1}{h + x_1}\right) + a_1\left(\frac{x_2 - x_2^*}{x_2}\right)\left(\frac{b_1(1-\varepsilon)x_1 x_2}{a + (1-\varepsilon)x_1} - c x_2\right),$$

$$= -\frac{r}{k}\left(x_1 - x_1^*\right)^2 - b(1-\varepsilon)\left(x_1 - x_1^*\right)\left(\frac{a\left(x_2 - x_2^*\right) + (1-\varepsilon)\left(x_1^* x_2 - x_1 x_2^*\right)}{\left(a + (1-\varepsilon)x_1^*\right)\left(a + (1-\varepsilon)x_1\right)}\right) +$$
$$a_1 b_1(1-\varepsilon)\left(x_2 - x_2^*\right)\left(\frac{a\left(x_1 - x_1^*\right)}{\left(a + (1-\varepsilon)x_1^*\right)\left(a + (1-\varepsilon)x_1\right)}\right),$$

$$= -\left[\frac{r}{k} - \frac{b(1-\varepsilon)^2 x_2^*}{\left(a + (1-\varepsilon)x_1^*\right)\left(a + (1-\varepsilon)x_1\right)}\right]\left(x_1 - x_1^*\right)^2 +$$
$$(1-\varepsilon)\left[\frac{a a_1 b_1 - \left(ab + b(1-\varepsilon)x_1^*\right)}{\left(a + (1-\varepsilon)x_1^*\right)\left(a + (1-\varepsilon)x_1\right)}\right]\left(x_1 - x_1^*\right)\left(x_2 - x_2^*\right),$$

the following can be derived by assuming $a a_1 b_1 = \left(ab + b(1-\varepsilon)x_1^*\right)$,

$$\frac{d^\xi V}{dt^\xi} = -\left[\frac{r}{k} - \frac{b(1-\varepsilon)^2 x_2^*}{\left(a + (1-\varepsilon)x_1^*\right)\left(a + (1-\varepsilon)x_1\right)}\right]\left(x_1 - x_1^*\right)^2, \frac{d^\xi V}{dt^\xi} \le 0.$$

Therefore, the positive equilibrium point E^* is globally asymptotically stable.

3.4. Existence of Hopf bifurcation

The transition from stable spiral to unstable behaviour at the equilibrium point causes Hopf bifurcation to occur when the bifurcation parameter reaches the critical value. It is obvious from Lemma 1 that the order of the derivative ξ determines the stability of a dynamical system. Therefore, ξ may be considered a bifurcation parameter, even if the non-integer order system does not possess the identical Hopf bifurcation condition as the integer-order framework.

Let us examine a system with fractional order

$$D_t^{\xi} x(t) = f\left(m, x\right), \tag{10}$$

where $\xi \in \left[0,1\right), x \in \Re^2$ and let E^* is an equilibrium point. Based on the analysis of a non-integer order Hopf bifurcation in [44,45], it can be concluded that, under certain circumstances, system (10) experiences a Hopf bifurcation given that the value m passes the threshold m^* around the equilibria E^*.

i) The characteristic equation has two complex conjugate eigenvalues

$$\lambda_{1,2}\left(m\right) = \nu\left(m\right) \pm i\pi\left(m\right),$$

ii) $\nu_{1,2}\left(\xi, m^*\right) = 0,$ where $\nu_{1,2} = \dfrac{\xi\pi}{2} - \left|\arg\left(\lambda_i\left(m\right)\right)\right|$,

iii) $\left.\dfrac{\partial \nu_{1,2}}{\partial m}\right|_{m=m^*} \neq 0$.

4. Numerical simulations

Theoretical results have been validated through numerical simulations employing the predictor-corrector technique on non-integer differential equations. In order to resolve fractional order nonlinear differential equations, numerical methods are employed. This research explores the impact of altering several factors on the non-integer order predator-prey dynamical framework (2).

Figure 1 illustrates the phase portraits of the framework (2) about E_2 for parameter values as $r = 2.5, k = 0.899, \beta = 0.945, \varepsilon = 0.058, \alpha = 0.559, h = 0.291,$ $d = 0.059,$ $\beta_1 = 0.895$ and three different values of $\xi = 0.98, 0.92, \text{and}\, 0.85$ in order to study the influence of the derivative order (ξ) on the predator-prey dynamics. Based on these numbers, we know that system (2) goes from unstable to stable and that at $\xi = 0.92$, it shows evidence of Hopf bifurcation. Hence, we deduce that the dynamics of the system under consideration are significantly affected by the non-integer order derivative.

The influence of prey refuge rate ε has been visually explored for $\xi = 1, 0.92$, with the values of $\varepsilon = 0.08, 0.6$, and 0.8, as well as the set of parameter values $r = 2.5, k = 0.899,$ $\beta = 0.945, \varepsilon = 0.058, h = 0.291, d = 0.059, \beta_1 = 0.895$ shown in Figures 2 and 3. The equilibrium point E_2, as seen in Figure 2, is not stable at $\xi = 1$ and $\varepsilon = 0.08$ but, it should be emphasised, becomes stable at $\xi = 0.92$ and $\varepsilon = 0.08$. This leads us to the conclusion that the existence of prey refuge affects the prey-predator system, and fractional order ξ contributes to species persistence by stabilising the system.

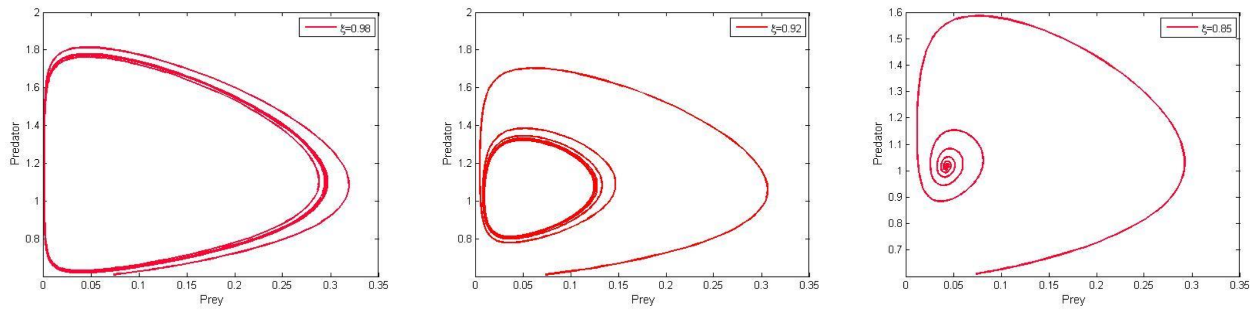

Figure 1. Time series diagram using various fractional order derivative (ξ) values.

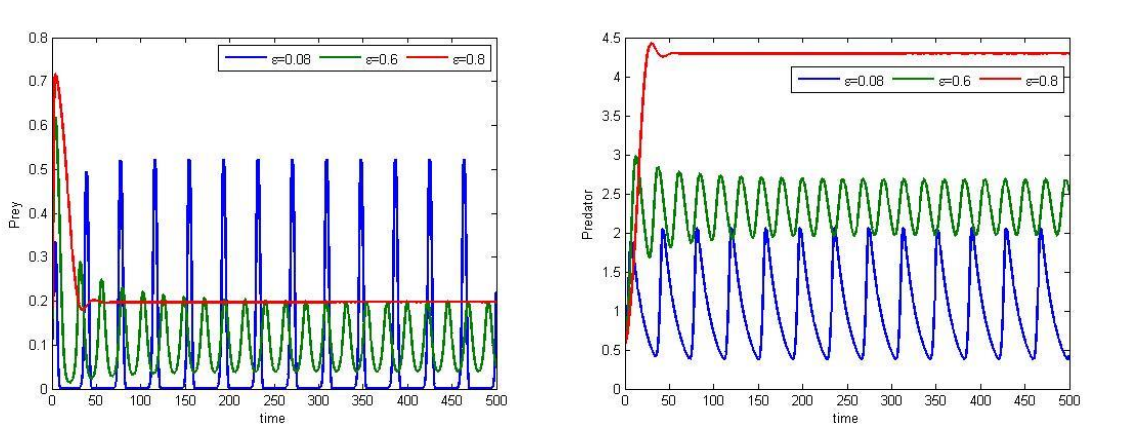

Figure 2. Time series diagram for various prey refuge values for $\xi = 1$.

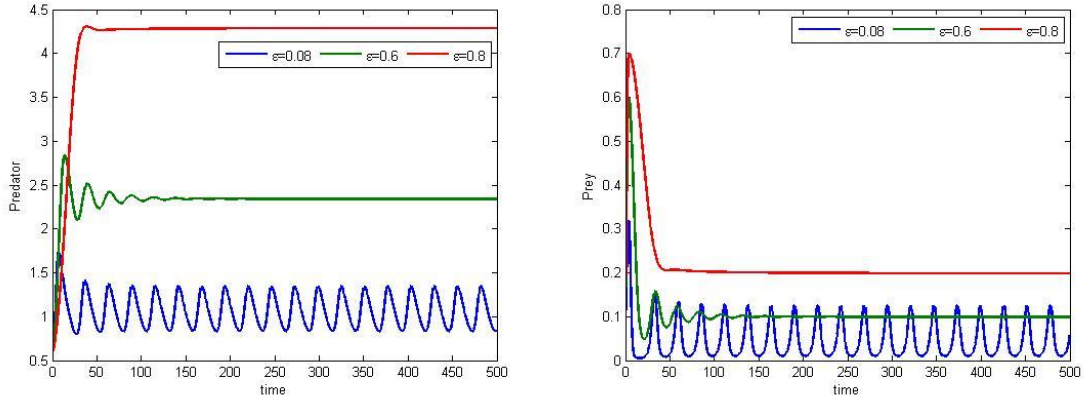

Figure 3. Time series plot for different values of prey refuge for $\xi = 0.92$.

Considering the identical selection of parameter values as illustrated in Figure 1, the influence of prey harvesting rate h has been visually explored for $\xi = 0.98$ and 1. Figures 4 and 5 illustrate that, when $\xi = 0.98$, the system (2) is stable at $h = 0.5$ then not stable for $h = 0.1$. In connection with the parameter h, bifurcation diagrams are provided in Figure 6.

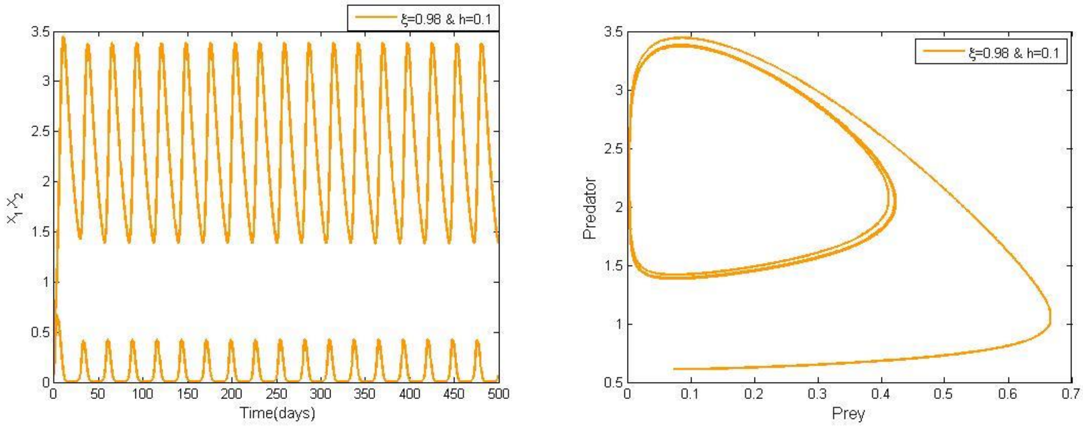

Figure 4. Time series and phase portrait plot for $\xi = 0.98, h = 0.1$.

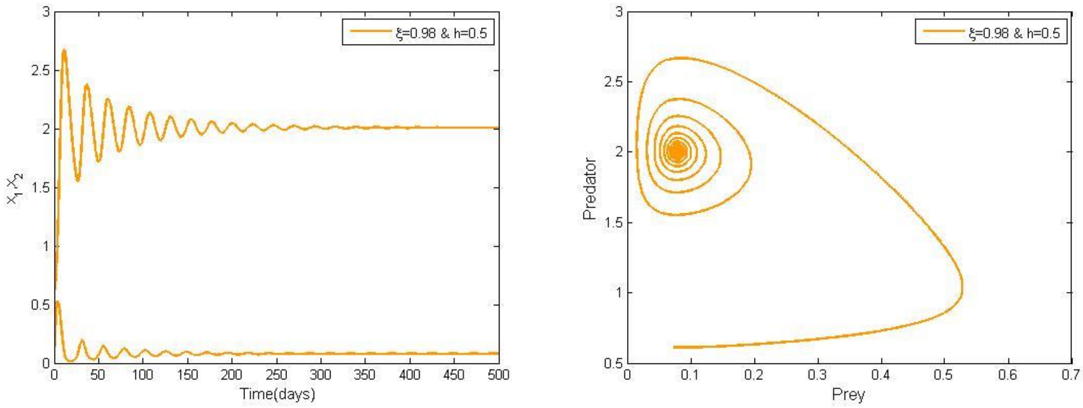

Figure 5. Time series and phase portrait plot for $\xi = 0.98, h = 0.5$.

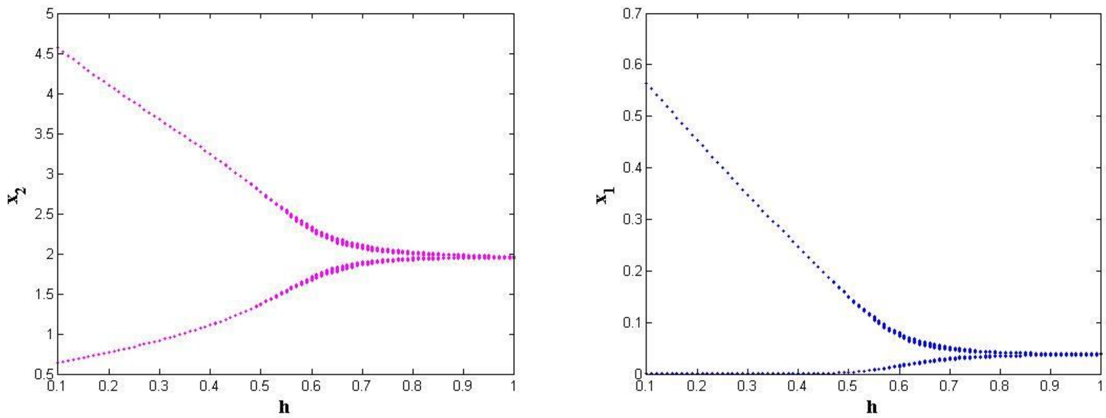

Figure 6. Bifurcation depictions for the system (2) with $\xi = 1$ and h being the bifurcation parameter.

5. Conclusions

In the present study, we explored a fractional-order Holling type II predator-prey framework that includes selective nonlinear harvesting and refuge in prey. As soon as the density of the harvested population reaches a certain limit, the harvesting grows smoothly to that value. The basic objective of this research is to understand how fractional order, refuge, and nonlinear harvesting affect the model's dynamic behaviour. The existence, uniqueness, non-negativity, and boundedness of the system's solution were defined first (2). Next, the local resilience of each feasible equilibrium point in the predator-prey framework was investigated using Matignon's criterion. Through the subsequent development of a suitable Lyapunov function, a global stability analysis of system (2) has been explored. Next, using ξ and h as parameters, the presence of Hopf bifurcation was investigated. Furthermore, the theoretical conclusions were numerically confirmed by employing the Adams-Bashforth PECE method (time-domain).

The system is stabilised by the prey refuge rate, according to numerical simulations. The framework exhibits stability in both integer-order as well as fractional orders when the refuge rates of prey populations are reduced. When the rate of prey refuge increases at a certain level of predation and harvesting, the system becomes unstable. Furthermore, it has been shown that nonlinear harvesting plays a vital part in maintaining system stability. The system shows stable behaviour at larger values of prey harvesting, whereas fractional order $\xi = 0.98$ shows stability at lower prey population harvesting. Therefore, given a constant value of the prey refuge amount, the system shows stable behaviour when there is a significant quantity of prey harvesting. Analysing the influence of spatial organisation on the overall behaviour of the system in the presence of a nonlinear harvesting term might provide valuable insights. Although this model can include spatial structure inpattern-forming Turing dynamics, it may be challenging to conduct comparable analyses and simulations for nonlinear harvesting techniques in this particular scenario. Therefore, more investigation into this matter is necessary. Also, the incorporation of prey-refuging strategy switching based on the prey-to-predator density ratio in the proposed model offers a more realistic ecological representation. These results underscore the importance of threshold-based refuge seeking for preserving species and maintaining ecosystem stability.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2024/01/78916).

Conflict of interest

The authors state that none of the work presented in this study may have been influenced by any known conflicting financial interests or personal relationships.

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