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Research article

Solution approximation of fractional boundary value problems and convergence analysis using AA-iterative scheme

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Abstract: Addressing the boundary value problems of fractional-order differential equations hold significant importance due to their applications in various fields. The aim of this paper was to approximate solutions for a class of boundary value problems involving Caputo fractional-order differential equations employing the AA-iterative scheme. Moreover, the stability and data dependence results of the iterative scheme were given for a certain class of mappings. Finally, a numerical experiment was illustrated to support the results presented herein. The results presented in this paper extend and unify some well-known comparable results in the existing literature.

Keywords: AA iteration; boundary value problems; fractional differential equation; stability; data dependence
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1. Introduction and preliminaries

The potential to simulate complex processes including memory effects, anomalous diffusion, and non-local interactions is one of the key benefits of the fractional calculus. The modeling of physical systems appearing in materials science, fluid mechanics, and signal processing makes fractional calculus extremely valuable. A family of boundary value problems of fractional-order differential equations involves conditions that are defined at the boundaries of the domain of their definitions. In recent years, these problems have attracted the attention of several mathematicians due to their applications to different fields of mathematics and beyond. These problems include non-local effects and memory features, and hence pose considerable analytical and numerical hurdles. Addressing these problems is critical not only for advancing our understanding of complex phenomena but also for implementation of fractional calculus in practical applications.

Boundary value problems of fractional order also play a crucial role in the development of numerical methods for approximating the solution of fractional differential equations. Evaluating, approximating, and characterizing the solution of these problems have become active areas of research, with applications in numerous branches of science and engineering. Moreover, it is anticipated that further research in this area will lead to significant discoveries and breakthroughs. For more details in this direction, we refer to [4, 15, 20].

Consider the fractional boundary value problem [24] as follows:

$$\begin{cases} {}^{c}D_{\varrho}^{\zeta} p(t) = \mathcal{G}(t, p(t)), \ for \ t \in \mathcal{J} = [\varrho, \theta], \ n-1 < \zeta < n \\ p^{(k)}(\varrho) = c_k, \ k = 0, 1, 2, \ldots, n-2; \ p^{(n-1)}(\theta) = c_\theta, \end{cases} \tag{1.1}$$

where, ${}^{c}D_{\varrho}^{\zeta}$ denotes the Caputo fractional derivative, $\mathcal{G} : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $c_0, c_1, c_2, \ldots, c_{n-2}, c_b$ are real constants and n is an integer.

A function $p \in C^{(n-1)}(\mathcal{J}, X)$ that satisfies (1.1) is called a solution of (1.1).

We assume that there exists a function $\mathcal{K} \in C(\mathbb{R}_+)$ such that

$$\|\mathcal{G}(t, u_1) - \mathcal{G}(t, v_1)\| \leq \mathcal{K}(t) \|u_1 - v_1\|. \tag{1.2}$$

Throughout this paper, we assume that the (1.2) satisfies.

The techniques for approximations to the solutions of fractional differential equations that cannot be solved analytically, are helpful in simulating and analyzing complicated systems. Moreover, the research work carried out in this direction has provided some useful tools and mathematical methods in the setup of fractional calculus in general and fractional differential equations in particular. These methods are now being applied in different fields of mathematical and engineering sciences.

Throughout this work, the set $\{0, 1, 2, ...\}$ is denoted by $\mathbb{Z}^+$. Let X be a normed space with norm $\|.\|$, C a non-empty closed convex subset of X and T a self mapping on C. The set $\{p^* \in C : p^* = Tp^*\}$ of all fixed points of T is denoted by $F(T)$.

There is a variety of fixed-point iteration schemes that approximate the solution of fixed-point equations, specially linear/nonlinear differential or integral equations. There are many factors that help to decide the preference of one iterative scheme over the others. One of the most important factors is to choose an iterative scheme that improves the rate of convergence of comparable existing schemes, that is, an iterative scheme that approximates the solution in a lower number of steps when compared with its counterparts.

One of the simplest iterative schemes [19], known as Picard iteration scheme, is defined as: Choose a point p_0 in C and obtain the successive approximations $\{p_n : n \in \mathbb{Z}^+\}$ of the solution of fixed-point equation involving a certain operator T by

$$p_{n+1} = Tp_n, \ n \in \mathbb{Z}^+.$$

The convergence of Picard iterative scheme depends not only on topological properties of the domain of an operator T but also on the nature of T itself. A well-known Banach contraction principle provides the necessary conditions for its convergence. However, it does not converge to the solution of fixed-point equation involving nonexpansive mapping.

We now recall some other known iterative schemes.

Let us choose an initial guess p_0 in C. Then, the sequence $\{p_n : n \in \mathbb{Z}^+\}$ defined by

$$p_{n+1} = (1 - k_n)p_n + k_n T p_n, \quad n \in \mathbb{Z}^+$$

is known as Mann iterative sequence [16], where the sequence $\{k_n\}$ of parameters satisfies certain conditions.

The sequence $\{p_n : n \in \mathbb{Z}^+\}$ defined by

$$\begin{cases} p_0 \in C \\ p_{n+1} = \quad (1 - k_n)p_n + k_n T q_n \\ q_n = \quad (1 - o_n)p_n + o_n T p_n, \; n \in \mathbb{Z}^+ \end{cases}$$

is known as Ishikawa iterative scheme [13], where $\{o_n\}$ and $\{k_n\}$ are some appropriate sequences in $(0, 1)$.

Noor [17] proposed a three-steps iterative scheme given by a sequence $\{p_n : n \in \mathbb{Z}^+\}$ as follows:

$$\begin{cases} p_0 \in C \\ p_{n+1} = \quad (1 - k_n)p_n + k_n T q_n \\ q_n = \quad (1 - o_n)p_n + o_n T r_n \\ r_n = \quad (1 - w_n)p_n + w_n T p_n, \quad \forall\, n \in \mathbb{Z}^+ \end{cases}$$

where $\{w_n\}, \{o_n\}, \{k_n\}$ in $(0, 1)$ satisfy certain conditions.

In 2007, Agarwal et al. [3] proposed an iterative scheme $\{p_n : n \in \mathbb{Z}^+\}$ known as S-iteration scheme, given by

$$\begin{cases} p_0 \in C \\ p_{n+1} = \quad (1 - k_n)T p_n + k_n T q_n \\ q_n = \quad (1 - o_n)p_n + o_n T p_n, \quad n \in \mathbb{Z}^+ \end{cases}$$

where $\{o_n\}, \{k_n\}$ are appropriate sequences in $(0, 1)$.

The convergence behavior of S-iterative scheme is the same as the Picard iterative scheme but faster than the Mann iterative scheme [3].

An iterative scheme $\{p_n : n \in \mathbb{Z}^+\}$ introduced by Abbas and Nazir in [2] has a faster rate of convergence than S- iteration. This three-steps iterative scheme is given as:

$$\begin{cases} p_0 \in C \\ p_{n+1} = \quad (1 - k_n)T q_n + k_n T r_n \\ q_n = \quad (1 - o_n)T p_n + o_n T r_n \\ r_n = \quad (1 - w_n)p_n + w_n T p_n, \quad \forall\, n \in \mathbb{Z}^+ \end{cases}$$

where $\{w_n\}, \{o_n\}$, and $\{k_n\}$ in $(0, 1)$ satisfy certain appropriate conditions.

Thakur et al. [22] defined a three-steps iterative scheme that has a better rate of convergence than the scheme in [2]. An iterative sequence $\{p_n : n \in \mathbb{Z}^+\}$ defined in [22] is given by

$$\begin{cases} p_0 \in C \\ p_{n+1} = & (1 - k_n)Tr_n + k_nTq_n \\ q_n = & (1 - o_n)r_n + o_nTr_n \\ r_n = & (1 - w_n)p_n + w_nTp_n, \quad \forall\, n \in \mathbb{Z}^+, \end{cases}$$

where the sequences $\{w_n\}, \{o_n\}$, and $\{k_n\}$ are given sequences in $(0, 1)$.

In 2018, Ullah et al. [14] defined M-iteration sequence $\{p_n : n \in \mathbb{Z}^+\}$ by

$$\begin{cases} p_0 \in C \\ p_{n+1} = & Tq_n \\ q_n = & Tr_n \\ r_n = & (1 - w_n)p_n + w_nTp_n, \quad \forall\, n \in \mathbb{Z}^+ \end{cases}$$

for approximating the fixed points of Suzuki's generalized nonexpansive mappings, where $\{w_n\} \subset (0, 1)$.

Let $\{k_n\}, \{o_n\}$, and $\{w_n\}$ be real sequences in $(0, 1)$ such that $k \leq k_n \leq 1$, $o \leq o_n \leq 1$ and $w \leq w_n \leq 1$ for all $n \in \mathbb{N}$ and for some $k, o, w > 0$. For a given $p_0 \in C$, the AA-iterative scheme $\{p_n : n \in \mathbb{Z}^+\}$ is defined as follows:

$$\text{AA-iteration process} : \begin{cases} p_{n+1} = Tq_n. \\ q_n = T((1 - k_n)Ts_n + k_nTr_n). \\ r_n = T((1 - o_n)s_n + o_nTs_n), \\ s_n = (1 - w_n)p_n + w_nTp_n, \ \forall n \in \mathbb{Z}^+. \end{cases} \tag{1.3}$$

It was shown in [1] that the AA-iteration scheme is faster than all the other iteration processes presented before.

Moreover, numerous research, such as [5, 6, 8], have featured the extensively used AA-iterative scheme, which keeps innovating computational methods in approximating the solution of fixed points and some other nonlinear problems.

Let us now recall some known definitions and results needed in this sequel.

Throughout this paper, we denote $\mathcal{J} = [\varrho, \theta]$ an interval in the set $\mathbb{R}$ of all real numbers. Consider the normed space of all $n - 1$ times continuously differentiable functions from $\mathcal{J}$ into X denoted by $C^{(n-1)}(\mathcal{J}, X) = B$ and is equipped with the norm given by

$$\|p\|_B = \sup\{\|p(t)\| : p \in B\}.$$

Definition 1.1. *[11] The Riemann-Liouville fractional integral of a function $\mathcal{G}$ of order $\zeta \in \mathbb{R}^+$ is defined by*

$${}^cI_\varrho^\zeta \mathcal{G}(t) = \frac{1}{\Gamma(\zeta)} \int_\varrho^t (t - s)^{\zeta - 1} \mathcal{G}(s) ds, \quad t > 0.$$

Definition 1.2. *[11] The Caputo fractional derivative of a function $\mathcal{G}$ of order $\zeta \in \mathbb{R}^+$ is defined by*

$$^{c}D_{\varrho}^{\zeta}\mathcal{G}(t) = \frac{1}{\Gamma(n-\zeta)}\int_{\varrho}^{t}(t-s)^{n-\zeta-1}\mathcal{G}^{(n)}(s)ds,$$

Where n is a positive integer and $n-1 < \zeta < n$, and the symbol Γ stands for the Gamma function given by

$$\Gamma(\zeta) = \int_{\varrho}^{\infty}\exp(-s)s^{\zeta-1}ds, \quad \Gamma(\zeta+1) = \zeta\Gamma(\zeta), \quad Re\{\zeta\} > 0.$$

Also, note that if $-n < Re\{\zeta\} \leq -n+1$, then

$$\Gamma(\zeta) = \frac{\Gamma(\zeta+n)}{\zeta(\zeta+1)(\zeta+2)\ldots(\zeta+n-1)}.$$

If $0 < \zeta < 1$, then the above Caputo fractional derivative of order $\zeta > 0$ becomes

$$^{c}D_{\varrho}^{\zeta}\mathcal{G}(t) = \frac{1}{\Gamma(1-\zeta)}\int_{\varrho}^{t}(t-s)^{-\zeta}\mathcal{G}'(s)ds.$$

Lemma 1.3. *[24] If $\zeta > 0$, then the differential equation*

$$^{c}D_{\varrho}^{\zeta}\mathcal{G}(t) = 0$$

has solutions

$$\mathcal{G}(t) = c_0 + c_1(t-\varrho) + c_2(t-\varrho)^2 + c_3(t-\varrho)^3 + \cdots + c_{n-1}(t-\varrho)^{n-1}, \quad c_i \in \mathbb{R},\ i = 0,1,2,\ldots,n-1,\ n = [\zeta]+1.$$

Lemma 1.4. *If $\zeta > 0$, then*

$$I_{\varrho}^{\zeta}\, {}^{c}D_{\varrho}^{\zeta}\mathcal{G}(t) = \mathcal{G}(t) + c_0 + c_1(t-\varrho) + c_2(t-\varrho)^2 + c_3(t-\varrho)^3 + \cdots + c_{n-1}(t-\varrho)^{n-1}.$$

Lemma 1.5. *[24] The relation*

$$^{c}D_{\varrho}^{\zeta}I_{\varrho}^{\zeta}\mathcal{G}(t) = \mathcal{G}(t), \quad I_{\varrho}^{\zeta}I_{\varrho}^{\beta}\mathcal{G}(t) = I_{\varrho}^{\zeta+\beta}\mathcal{G}(t)$$

is valid for

$$Re(\zeta) > 0, \quad Re(\beta) > 0, \mathcal{G}(t) \in L^1(\varrho, \theta).$$

As a consequence of Lemmas 1.3–1.5, the following results can be established:

Lemma 1.6. *[12, 24] A function p is a solution of the fractional boundary value problem defined in* (1.1) *if and only if $p(t)$ is a solution of the fractional integral equation*

$$\begin{aligned} p(t) = &\sum_{k=0}^{n-2}\frac{c_k}{k!}(t-\varrho)^k + \left(\frac{c_\theta}{(n-1)!} + \frac{\mathcal{G}(\varrho, p(\varrho))(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\ &- \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{G}(s,p(s))ds + \frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{G}(s,p(s))ds. \end{aligned}$$

Recently, Tidke and Patil [25] used the S-iteration to approximate the solution of a boundary value problem of fractional order.

This paper is organized as follows: In first place we highlighted the need to address boundary value problems related to fractional-order differential equations. We will then describe our technique, which uses the AA-iterative scheme to approximate solutions of the boundary value problems for Caputo fractional differential equations given in (1.1). We then discuss the stability and data dependence features of the iterative scheme used herein. Finally, we give a numerical experiment that validates our technique, extending and unifying certain well-known results from the current literature.

2. Solution approximation

We now present the following result.

Theorem 2.1. *Suppose that* (1.2) *holds, and if*

$$\Phi = \left[\frac{(\theta-\varrho)^{\zeta}\mathcal{K}(\varrho)}{(n-2)!\Gamma(\zeta-n+2)} + \frac{(\theta-\varrho)^{n-1}}{(n-1)!} I_{\varrho}^{\zeta-n+1}\mathcal{K}(\theta) + I_{\varrho}^{\zeta}\mathcal{K}(t)\right] < 1,$$

then the AA-iteration scheme (1.3) *converges to the solution of BVP* (1.1).

Proof. Let $p \in C^{(n-1)}(\mathcal{J}, X)$. Define the operator T on $C^{(n-1)}(\mathcal{J}, X)$ by

$$\begin{aligned}(Tp)(t) = & \sum_{k=0}^{n-2} \frac{c_k}{k!}(t-\varrho)^k + \left(\frac{c_\theta}{(n-1)!} + \frac{\mathcal{G}(\varrho, p(\varrho))(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\ & - \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{G}(s,p(s))ds + \frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{G}(s,p(s))ds.\end{aligned} \tag{2.1}$$

Let $\{p_n\}_{n=0}^{\infty}$ be an iterative process generated by the AA-iteration method (1.3) for the operator given by (2.1). We need to show that $p_n \to p$ as $n \to \infty$.

From (1.3) and (2.1), we have

$$\begin{aligned}\|p_{n+1}(t) - p(t)\| &= \|(Tq_n)(t) - (Tp)(t)\| \\ &= \Bigg\| \sum_{n=0}^{n-2} \frac{c_k}{k!}(t-\varrho)^k + \left(\frac{c_\theta}{(n-1)!} + \frac{\mathcal{G}(\varrho, q_n(\varrho))(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\ &\quad - \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{G}(s,q_n(s))ds + \frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{G}(s,q_n(s))ds \\ &\quad - \sum_{n=0}^{n-2} \frac{c_k}{k!}(t-\varrho)^k - \left(\frac{c_\theta}{(n-1)!} + \frac{\mathcal{G}(\varrho, p(\varrho))(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\ &\quad + \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{G}(s,p(s))ds - \frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{G}(s,p(s))ds \Bigg\| \\ &\le \left(\frac{\|\mathcal{G}(\varrho, q_n(\varrho)) - \mathcal{G}(\varrho, p(\varrho))\|(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\ &\quad + \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\|\mathcal{G}(s,q_n(s)) - \mathcal{G}(s,p(s))\|ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\left\|\mathcal{G}(s,q_n(s))-\mathcal{G}(s,p(s))\right\|ds\\
&\le\left(\frac{\mathcal{K}(\varrho)\left\|q_n(\varrho)-p(\varrho)\right\|(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1}\\
&+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{K}(s)\left\|q_n(s)-p(s)\right\|ds\\
&+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{K}(s)\left\|q_n(s)-p(s)\right\|ds.
\end{aligned}
$$

Taking norm on both sides of the above inequality, we have

$$
\begin{aligned}
\|p_{n+1}-p\|_B &\le\left(\frac{\mathcal{K}(\varrho)(t-\varrho)^{n-1}(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)\|q_n-p\|_B+\frac{(t-\varrho)^{n-1}\|q_n-p\|_B}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{K}(s)ds\\
&+\frac{\|q_n-p\|_B}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{K}(s)ds\\
&\le\left(\frac{\mathcal{K}(\varrho)(\theta-\varrho)^{n-1}(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)\|q_n-p\|_B+\frac{(\theta-\varrho)^{n-1}\|q_n-p\|_B}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{K}(s)ds\\
&+\frac{\|q_n-p\|_B}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{K}(s)ds\\
&=\left(\frac{\mathcal{K}(\varrho)(\theta-\varrho)^{n-1}(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)\|q_n-p\|_B+\left[\frac{(\theta-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{K}(s)ds\right.\\
&\left.+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{K}(s)ds\right]\|q_n-p\|_B\\
&\le\left[\frac{(\theta-\varrho)^{\zeta}\mathcal{K}(\varrho)}{(n-2)!\Gamma(\zeta-n+2)}+\frac{(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}\mathcal{K}(\theta)+I_{\varrho}^{\zeta}\mathcal{K}(t)\right]\|q_n-p\|_B\\
&=\Phi\|q_n-p\|_B,
\end{aligned}
$$

we get

$$
\|p_{n+1}-p\|_B=\|Tq_n-Tp\|_B\le\Phi\|q_n-p\|_B<\|q_n-p\|_B. \tag{2.2}
$$

Note that,

$$
\begin{aligned}
\|s_n-p\|_B &=\|(1-w_n)p_n+w_nTp_n-p\|_B\\
&=\|(1-w_n)p_n+w_nTp_n-w_np+w_np-p\|_B\\
&=\|(1-w_n)p_n-(1-w_n)p+w_nTp_n-w_np\|_B\\
&<(1-w_n)\|p_n-p\|_B+w_n\|p_n-p\|_B\\
&=\|p_n-p\|_B.
\end{aligned}
$$

From the AA-iterative process (1.3), we obtain that

$$\begin{aligned}\|r_n - p\|_B &= \|T((1-o_n)s_n + o_nTs_n) - p\|_B \\ &< \|(1-o_n)s_n + o_nTs_n - p\|_B \\ &\leq (1-o_n)\|s_n - p\|_B + o_n\|Ts_n - p\|_B \\ &< (1-o_n)\|s_n - p\|_B + o_n\|s_n - p\|_B \\ &= \|s_n - p\|_B \\ &\leq \|p_n - p\|_B \quad (\text{since } \|s_n - p\|_B \leq \|p_n - p\|_B).\end{aligned}$$

Also,

$$\begin{aligned}\|q_n - p\|_B &= \|T((1-k_n)Ts_n + k_nTr_n) - p\|_B \\ &< \|(1-k_n)Ts_n + k_nTr_n - p\|_B \\ &\leq \|(1-k_n)Ts_n + k_nTr_n - k_np + k_np - p\|_B \\ &\leq \|(1-k_n)Ts_n - (1-k_n)p + k_nTr_n - k_np\|_B \\ &\leq (1-k_n)\|Ts_n - p\|_B + k_n\|Tr_n - p\|_B \\ &\leq (1-k_n)\|s_n - p\|_B + k_n\|r_n - p\|_B \\ &\leq (1-k_n)\|p_n - p\|_B + k_n\|p_n - p\|_B \\ &= \|p_n - p\|_B .\end{aligned}$$

Now, by (2.2), we have

$$\|p_{n+1} - p\|_B \leq \Phi\|q_n - p\|_B \leq \Phi\|p_n - p\|_B .$$

By induction, we have

$$\|p_{n+1} - p\|_B \leq \Phi^{n+1}\|p_0 - p\|_B . \tag{2.3}$$

As $\Phi < 1$, we conclude that $\lim_{n\to\infty}\|p_{n+1} - p\|_B = 0$.

3. Convergence analysis

3.1. Stability analysis

Let us recall the following Definition.

Definition 3.1. *[18] Suppose that the iteration scheme $p_{n+1} = \phi(T, p_n)$ defined by some function ϕ and mapping T converges to a fixed point p of self mapping T on $C^{(n-1)}(\mathcal{J}, X)$ and $\{\varsigma_n\}$ is an approximate sequence of $\{p_n\}$ in a subset $C^{(n-1)}(\mathcal{J}, X)$ of a Banach space $C^{(n)}(\mathcal{J}, X)$. Then, the sequence p_n is said to be T-stable or stable with respect to T provided that* $\lim_{n\to\infty} z_n = 0$ *if and only if* $\lim_{n\to\infty}\varsigma_n = p$, *where $\{z_n\}$ is given by*

$$z_n = \|\varsigma_{n+1} - \phi(T, \varsigma_n)\|_B, \ \forall\, n \in \mathbb{Z}^+.$$

Lemma 3.2. *[9] Let $\{u_n\}$ and $\{z_n\}$ be sequences of positive real numbers satisfying the following inequality:*

$$u_{n+1} \leq (1-\upsilon_n)u_n + z_n,$$

where $\upsilon_n \in (0,1)$ for all $n \in \mathbb{Z}^+$ with $\Sigma_{n=0}^{\infty}\upsilon_n = \infty$. If $\lim_{n\to\infty}\frac{z_n}{\upsilon_n} = 0$, *then* $\lim_{n\to\infty} u_n = 0$.

Theorem 3.3. *If $C^{(n-1)}(\mathcal{J},X)$ is non-empty closed and convex subset of a Banach space $C^{(n)}(\mathcal{J},X)$ and T is defined as in Theorem 2.1, then the iterative scheme given in (1.3) is T-stable.*

Proof. Let $\{\varsigma_n\}$ be an approximate sequence of $\{p_n\}$ in $C^{(n-1)}(\mathcal{J},X)$. The sequence defined by iteration (1.3) is: $p_{n+1} = \phi(T,p_n)$ and $z_n = \|\varsigma_{n+1} - \phi(T,\varsigma_n)\|_B$, $n \in \mathbb{N}$.

We now show that $\lim_{n\to\infty} z_n = 0$ if and only if $\lim_{n\to\infty} \varsigma_n = p$.

Suppose that $\lim_{n\to\infty} z_n = 0$. It follows from (1.3) that

$$\begin{aligned}\|\varsigma_{n+1} - p\|_B &\le \|\varsigma_{n+1} - \phi(T,\varsigma_n)\|_B + \|\phi(T,\varsigma_n) - p\|_B\\ &= z_n + \|p_{n+1} - p\|_B.\end{aligned}$$

By Theorem 2.1, we have

$$\|\varsigma_{n+1} - p\|_B \le z_n + \left[\frac{(\theta-\varrho)^{\zeta}\mathcal{K}(\varrho)}{(n-2)!\Gamma(\zeta-n+2)} + \frac{(\theta-\varrho)^{n-1}}{(n-1)!} I_{\varrho}^{\zeta-n+1}\mathcal{K}(\theta) + I_{\varrho}^{\zeta}\mathcal{K}(t)\right]\|\varsigma_n - p\|_B.$$

Let

$$\zeta_n = \|\varsigma_n - p\|_B, \text{ and } \beta_n = \left[\frac{(\theta-\varrho)^{\zeta}\mathcal{K}(\varrho)}{(n-2)!\Gamma(\zeta-n+2)} + \frac{(\theta-\varrho)^{n-1}}{(n-1)!} I_{\varrho}^{\zeta-n+1}\mathcal{K}(\theta) + I_{\varrho}^{\zeta}\mathcal{K}(t)\right].$$

Then,

$$\zeta_n \le (1-\beta_n)\zeta_n + z_n.$$

As $\lim_{n\to\infty} z_n = 0$, $\frac{z_n}{\beta_n} \to 0$ as $n \to \infty$, Lemma 3.2 gives that $\lim_{n\to\infty} \zeta_n = 0$ and hence $\lim_{n\to\infty} \varsigma_n = p$.

Now, if $\lim_{n\to\infty} \varsigma_n = p$, then we have

$$\begin{aligned}z_n &= \|\varsigma_{n+1} - \phi(T,\varsigma_n)\|_B\\ &\le \|\varsigma_{n+1} - p\|_B + \|\phi(T,\varsigma_n) - p\|_B\\ &\le \|\varsigma_{n+1} - p\|_B + \left[\frac{(\theta-\varrho)^{\zeta}\mathcal{K}(\varrho)}{(n-2)!\Gamma(\zeta-n+2)} + \frac{(\theta-\varrho)^{n-1}}{(n-1)!} I_{\varrho}^{\zeta-n+1}\mathcal{K}(\theta) + I_{\varrho}^{\zeta}\mathcal{K}(t)\right]\|\varsigma_n - p\|_B,\end{aligned}$$

which implies that $\lim_{n\to\infty} z_n = 0$ and hence the iterative scheme (1.3) is T-stable.

3.2. Dependence on boundary data

Definition 3.4. *[10] Let $T_1, T_2 : C^{(n-1)}(\mathcal{J},X) \to C^{(n-1)}(\mathcal{J},X)$. Then, T_2 is said to be an approximate operator of T_1 if there exists $\varepsilon > 0$ such that*

$$\|T_1 p - T_2 p\|_B \le \varepsilon \ \forall\ p \in C^{(n-1)}(\mathcal{J},X).$$

Suppose p and $\bar{p}$ are solutions of (1.1) with boundary data

$$\begin{aligned}p^{(k)}(\varrho) &= c_k,\ k =, 1, 2, \ldots, n-2;\ p^{(n-1)}(\theta) = c_\theta,\\ \bar{p}^{(k)}(\varrho) &= d_k,\ k =, 1, 2, \ldots, n-2;\ \bar{p}^{(n-1)}(\theta) = \bar{c}_\theta,\end{aligned}$$

where, c_k, d_k $(k = 0, 1, 2, \ldots n-2)$, $c_\theta, \bar{c}_\theta$ are given elements in X.

Define an operator $\bar{T}$ as follows:

$$\left(\bar{T}\bar{p}\right)(t) = \sum_{k=0}^{n-2} \frac{d_k}{k!}(t-\varrho)^k + \left(\frac{\bar{c}_\theta}{(n-1)!} + \frac{\mathcal{G}(\varrho, \bar{p}(\varrho))(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1}$$
$$- \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)} \int_\varrho^\theta (\theta-s)^{\zeta-n}\mathcal{G}(s,\bar{p}(s))ds + \frac{1}{\Gamma(\zeta)}\int_\varrho^t (t-s)^{(\zeta-1)}\mathcal{G}(s,\bar{p}(s))ds.$$

To establish the continuous dependence of solutions of Eq (1.1) on the given boundary data, we prove the following result according to [23].

Theorem 3.5. *Let $\{p_n\}_{n=0}^\infty$ be an iterative sequence generated by the AA-iteration* (1.3) *associated with the operator T defined in* (2.1)*. Define an approximate sequence $\{\bar{p}_n\}_{n=0}^\infty$ generated by the AA-iterative scheme as follows*

$$\begin{cases} \bar{p}_{n+1} = \bar{T}\bar{q}_n. \\ \bar{q}_n = \bar{T}((1-k_n)\bar{T}\bar{s}_n + k_n\bar{T}\bar{r}_n). \\ \bar{r}_n = \bar{T}((1-o_n)\bar{s}_n + o_n\bar{T}\bar{s}_n), \\ \bar{s}_n = (1-w_n)\bar{p}_n + w_n\bar{T}\bar{p}_n, \ n \in \mathbb{N}, \end{cases}$$

with the real sequences $\{k_n\}_{n=0}^\infty$, $\{o_n\}_{n=0}^\infty$ and $\{w_n\}_{n=0}^\infty$ in $(0,1)$ *satisfying $k_n o_n \geq \frac{1}{3}$ for all $n \in \mathbb{N}$. If the sequence $\{\bar{p}_n\}_{n=0}^\infty$ converges to $\bar{p}$, then we have*

$$\|p - \bar{p}\|_B \leq \frac{7M}{1-\Phi},$$

where,

$$M = \sum_{j=0}^{n-2} \frac{\|c_k - d_k\|}{k!}(\theta-\varrho)^k + \frac{\|c_\theta - \bar{c}_\theta\|}{(n-1)!}(\theta-\varrho)^{n-1}.$$

Proof. Suppose the sequences $\{p_n\}_{n=0}^\infty$ and $\{\bar{p}_n\}_{n=0}^\infty$ with the real sequences $\{k_n\}_{n=0}^\infty$, $\{o_n\}_{n=0}^\infty$ and $\{w_n\}_{n=0}^\infty$ in $(0,1)$ satisfying $\frac{1}{2} \leq \zeta_n, \beta_n$ for all $n \in \mathbb{N}$.

Note that,

$$\|p_{n+1}(t) - \bar{p}_{n+1}(t)\| = \left\|(Tq_n)(t) - (\bar{T}\bar{q}_n)(t)\right\|$$
$$= \Bigg\| \sum_{n=0}^{n-2} \frac{c_k}{k!}(t-\varrho)^k + \left(\frac{c_\theta}{(n-1)!} + \frac{\mathcal{G}(\varrho, q_n(\varrho))(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1}$$
$$- \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)} \int_\varrho^\theta (\theta-s)^{\zeta-n}\mathcal{G}(s,q_n(s))ds + \frac{1}{\Gamma(\zeta)}\int_\varrho^t (t-s)^{(\zeta-1)}\mathcal{G}(s,q_n(s))ds$$
$$- \sum_{n=0}^{n-2} \frac{d_k}{k!}(t-\varrho)^k - \left(\frac{\bar{c}_\theta}{(n-1)!} + \frac{\mathcal{G}(\varrho, \bar{q}_n(\varrho))(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1}$$
$$+ \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)} \int_\varrho^\theta (\theta-s)^{\zeta-n}\mathcal{G}(s,\bar{q}_n(s))ds - \frac{1}{\Gamma(\zeta)}\int_\varrho^t (t-s)^{(\zeta-1)}\mathcal{G}(s,\bar{q}_n(s))ds \Bigg\|$$
$$\leq \sum_{j=0}^{n-2} \frac{\|c_k - d_k\|}{k!}(\theta-\varrho)^k + \frac{\|c_\theta - \bar{c}_\theta\|}{(n-1)!}(\theta-\varrho)^{n-1}$$

$$
\begin{aligned}
&+\left(\frac{\|\mathcal{G}(\varrho, q_n(\varrho)) - \mathcal{G}(\varrho, \bar{q}_n(\varrho))\| (\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\
&+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\|\mathcal{G}(s, q_n(s)) - \mathcal{G}(s, \bar{q}_n(s))\| ds \\
&+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\|\mathcal{G}(s, q_n(s)) - \mathcal{G}(s, \bar{q}_n(s))\| ds \\
&\le M+\left(\frac{\mathcal{K}(\varrho)\|q_n(\varrho)-\bar{q}_n(\varrho)\|(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\
&+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{K}(s)\|q_n(s)-\bar{q}_n(s)\| ds \\
&+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{K}(s)\|q_n(s)-\bar{q}_n(s)\| ds.
\end{aligned}
$$

Taking supremum norm on both sides of the above inequality, we have

$$
\begin{aligned}
\|p_{n+1}-\bar{p}_{n+1}\|_B &\le M+\left(\frac{\mathcal{K}(\varrho)(t-\varrho)^{n-1}(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)\|q_n-\bar{q}_n\|_B+\frac{(t-\varrho)^{n-1}\|q_n-\bar{q}_n\|_B}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{K}(s)ds \\
&+\frac{\|q_n-\bar{q}_n\|_B}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{K}(s)ds \\
&\le M+\left(\frac{\mathcal{K}(\varrho)(\theta-\varrho)^{n-1}(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)\|q_n-\bar{q}_n\|_B+\frac{(\theta-\varrho)^{n-1}\|q_n-\bar{q}_n\|_B}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{K}(s)ds \\
&+\frac{\|q_n-\bar{q}_n\|_B}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{K}(s)ds \\
&= M+\left(\frac{\mathcal{K}(\varrho)(\theta-\varrho)^{n-1}(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)\|q_n-\bar{q}_n\|_B+\left[\frac{(\theta-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{K}(s)ds\right. \\
&\left.+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{K}(s)ds\right]\|q_n-\bar{q}_n\|_B \\
&\le M+\left[\frac{(\theta-\varrho)^{\zeta}\mathcal{K}(\varrho)}{(n-2)!\Gamma(\zeta-n+2)}+\frac{(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}\mathcal{K}(\theta)+I_{\varrho}^{\zeta}\mathcal{K}(t)\right]\|q_n-\bar{q}_n\|_B \\
&\le M+\left[\frac{(\theta-\varrho)^{\zeta}\mathcal{K}(\varrho)}{(n-2)!\Gamma(\zeta-n+2)}+\frac{(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}\mathcal{K}(\theta)+I_{\varrho}^{\zeta}\mathcal{K}(t)\right]\|q_n-\bar{q}_n\|_B \\
&= M+\Phi\|q_n-\bar{q}_n\|_B.
\end{aligned}
$$

Now, if $u_n = (1-k_n)s_n + k_n T r_n$, then we get

$$
\begin{aligned}
\|q_n-\bar{q}_n\|_B &= \left\|Tu_n-\bar{T}\bar{u}_n\right\|_B \\
&\le M+\Phi\|u_n-\bar{u}_n\|_B,
\end{aligned}
$$

and

$$
\begin{aligned}
\|u_n-\bar{u}_n\|_B &= \left\|(1-k_n)s_n+k_nTr_n-(1-k_n)\bar{s}_n-k_n\bar{T}\bar{r}_n\right\|_B \\
&\le (1-k_n)\left\|Ts_n-\bar{T}\bar{s}_n\right\|_B+k_n\left\|Tr_n-\bar{T}\bar{r}_n\right\|_B.
\end{aligned}
$$

For $v_n = (1 - o_n)s_n + o_n T s_n$, we get

$$
\begin{aligned}
\left\|Tr_n - \bar{T}\bar{r}_n\right\|_B &\leq M + \Phi \left\|r_n - \bar{r}_n\right\|_B \\
&= M + \Phi \left\|Tv_n - \bar{T}\bar{v}_n\right\|_B \\
&= M + \Phi\Big[M + \Phi \left\|v_n - \bar{v}_n\right\|_B\Big] \\
&= M + \Phi\Big[M + \Phi[\|(1 - o_n)s_n + o_n T s_n - (1 - o_n)\bar{s}_n - o_n \bar{T}\bar{s}_n\|_B]\Big] \\
&= M + \Phi\Big[M + \Phi[(1 - o_n)\left\|s_n - \bar{s}_n\right\|_B + o_n \left\|Ts_n - \bar{T}\bar{s}_n\right\|_B]\Big] \\
&\leq M + \Phi\Big[M + \Phi[(1 - o_n)\left\|s_n - \bar{s}_n\right\|_B + o_n (M + \Phi \left\|s_n - \bar{s}_n\right\|_B)]\Big] \\
&= M + \Phi\Big[M + \Phi[o_n M + (1 - o_n(1 - \Phi))\left\|s_n - \bar{s}_n\right\|_B]\Big].
\end{aligned}
$$

Also,

$$
\begin{aligned}
\left\|u_n - \bar{u}_n\right\|_B &\leq (1 - k_n)\left\|Ts_n - \bar{T}\bar{s}_n\right\|_B + k_n \left\|Tr_n - \bar{T}\bar{r}_n\right\|_B \\
&\leq (1 - k_n)\Big[M + \Phi \left\|s_n - \bar{s}_n\right\|_B\Big] + k_n \left\|Tr_n - \bar{T}\bar{r}_n\right\|_B \\
&\leq (1 - k_n)\Big[M + \Phi\Big(w_n M + (1 - w_n(1 - \Phi))\left\|p_n - \bar{p}_n\right\|_B\Big)\Big] \\
&+ k_n\Big[M + \Phi\Big(M + \Phi[o_n M + (1 - o_n(1 - \Phi))\left\|s_n - \bar{s}_n\right\|_B]\Big)\Big] \\
&\leq (1 - k_n)\Big[M + \Phi\Big(w_n M + (1 - w_n(1 - \Phi))\left\|p_n - \bar{p}_n\right\|_B\Big)\Big] \\
&+ k_n\Big[M + \Phi\Big(M + \Phi\Big[o_n M + (1 - o_n(1 - \Phi)) \\
&\times \Big(w_n M + (1 - w_n(1 - \Phi))\left\|p_n - \bar{p}_n\right\|_B\Big)\Big]\Big)\Big].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\left\|p_{n+1} - \bar{p}_{n+1}\right\|_B &\leq M + \Phi\Big((1 - k_n)\Big[M + \Phi\Big(w_n M + (1 - w_n(1 - \Phi))\left\|p_n - \bar{p}_n\right\|_B\Big)\Big] \\
&+ k_n\Big[M + \Phi\Big(M + \Phi\Big[o_n M + (1 - o_n(1 - \Phi)) \times \Big(w_n M + (1 - w_n(1 - \Phi))\left\|p_n - \bar{p}_n\right\|_B\Big)\Big]\Big)\Big]\Big) \\
&\leq M + M\Phi + \Phi^2(1 - k_n)M + \Phi^3(1 - k_n)\Big(w_n M + (1 - w_n(1 - \Phi))\Big)\left\|p_n - \bar{p}_n\right\|_B + k_n M \\
&+ k_n \Phi M + k_n \Phi^2 M o_n + \Phi^2 k_n(1 - o_n(1 - \Phi))\Big(w_n M + (1 - w_n(1 - \Phi))\Big)\left\|p_n - \bar{p}_n\right\|_B.
\end{aligned}
$$

As $\Phi < 1$, we obtain that

$$\begin{aligned}\|p_{n+1} - \bar{p}_{n+1}\|_B &\leq 3M - k_n M + (1 - k_n)\Big(w_n M + (1 - w_n(1 - \Phi))\Big)\|p_n - \bar{p}_n\|_B + k_n M \\ &\quad + k_n M + k_n M o_n + k_n(1 - o_n(1 - \Phi))\Big(w_n M + (1 - w_n(1 - \Phi))\Big)\|p_n - \bar{p}_n\|_B \\ &\leq \{1 - k_n o_n(1 - \Phi)\}(1 - w_n(1 - \Phi))\|p_n - \bar{p}_n\|_B + (3 + k_n + k_n o_n)M \\ &\leq \{1 - k_n o_n(1 - \Phi)\}(1 - w_n(1 - \Phi))\|p_n - \bar{p}_n\|_B + (3k_n o_n + 3k_n o_n + k_n o_n)M \\ &\leq \{1 - k_n o_n(1 - \Phi)\}(1 - w_n(1 - \Phi))\|p_n - \bar{p}_n\|_B + \frac{7k_n o_n(1 - \Phi)M}{(1 - \Phi)}.\end{aligned}$$

Note that $w_n M < 1 \implies (1 - w_n(1 - \Phi)) < 1$ and we get that

$$\|p_{n+1} - \bar{p}_{n+1}\|_B \leq \{1 - k_n o_n(1 - \Phi)\}\|p_n - \bar{p}_n\|_B + \frac{7k_n o_n(1 - \Phi)M}{(1 - \Phi)}.$$

Setting $\mu_n = k_n o_n(1 - \Phi)$, results in

$$\|p_{n+1} - \bar{p}_{n+1}\|_B \leq (1 - \mu_n)\|p_n - \bar{p}_n\|_B + \mu_n \frac{7M}{(1 - \Phi)}.$$

Let us denote $\|p_n - \bar{p}_n\|_B$ by ζ_n and $\frac{7M}{(1-\Phi)}$ by w_n. Obviously, $\mu_n \in (0, 1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \mu_n = \infty$ and $w_n \geq 0$. Thus, assumptions of the Lemma 3.2 are satisfied, and hence we have

$$\begin{aligned}&0 \leq \limsup_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} w_n \\ &\implies 0 \leq \limsup_{n\to\infty} \|p_n - \bar{p}_n\|_B \leq \limsup_{n\to\infty} \frac{7M}{(1 - \Phi)}, \\ &\implies 0 \leq \limsup_{n\to\infty} \|p_n - \bar{p}_n\|_B \leq \frac{7M}{(1 - \Phi)}.\end{aligned}$$

Since $\{p_n\}_{n=0}^{\infty}$ converges to p, and $\{\bar{p}_n\}_{n=0}^{\infty}$ converges to $\bar{p}$, we have

$$\|p - \bar{p}\|_B \leq \frac{7M}{(1 - \Phi)}.$$

3.3. Error bounds of solutions of two boundary value problems

Let us recall a fractional boundary value problem as follows,

$$\begin{cases} {}^c D_{\varrho}^{\zeta} p(t) = \mathcal{G}(t, p(t)), \text{ for } t \in \mathcal{J} = [\varrho, \theta], \ n - 1 < \zeta < n \\ p^{(k)}(\varrho) = c_k, \ k = 0, 1, 2, \ldots, n - 2; \ p^{(n-1)}(\theta) = c_\theta. \end{cases} \tag{3.1}$$

We now consider another fractional boundary value problem given as:

$$\begin{cases} {}^c D_{\varrho}^{\zeta} \bar{p}(t) = \bar{\mathcal{G}}(t, \bar{p}(t)), \text{ for } t \in \mathcal{J} = [\varrho, \theta], \ n - 1 < \zeta < n \\ \bar{p}^{(k)}(\varrho) = c_k, \ k =, 1, 2, \ldots, n - 2; \ \bar{p}^{(n-1)}(\theta) = \bar{c}_\theta, \end{cases} \tag{3.2}$$

where, $\bar{\mathcal{G}} : \mathcal{J} \times X \to X$ is a continuous function.

Define the operator $\bar{T}$ corresponding to 3.2 as follows:

$$\begin{aligned}\left(\bar{T}\bar{p}\right)(t) &= \sum_{k=0}^{n-2}\frac{d_k}{k!}(t-\varrho)^k + \left(\frac{\bar{c}_\theta}{(n-1)!} + \frac{\bar{\mathcal{G}}(\varrho,\bar{p}(\varrho))(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\ &- \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_\varrho^\theta(\theta-s)^{\zeta-n}\bar{\mathcal{G}}(s,\bar{p}(s))ds + \frac{1}{\Gamma(\zeta)}\int_\varrho^t(t-s)^{(\zeta-1)}\bar{\mathcal{G}}(s,\bar{p}(s))ds.\end{aligned} \tag{3.3}$$

Suppose that,

(i) The conditions of Theorem 2.1 hold and p and $\bar{p}$ are solutions of (3.1) and (3.2), respectively.
(ii) There exists positive constant ϵ such that

$$\left\|\mathcal{G}(t,u_1)-\bar{\mathcal{G}}(t,u_1)\right\| \leq \epsilon \;\forall\; t\in\mathcal{J}.$$

Then, the following result gives an upper bound of the error between the solutions of two fractional boundary value problems provided that the error between the $\mathcal{G}$ and $\bar{\mathcal{G}}$ in (3.1) and (3.2) is given.

Theorem 3.6. *Consider the sequences $\{p_n\}_{n=1}^\infty$ and $\{\bar{p}_n\}_{n=1}^\infty$ defined with the operators T in* (2.1) *and $\bar{T}$ in* (3.3)*, respectively, such that* (*i*) − (*ii*) *hold, where the real sequences $\{k_n\}_{n=0}^\infty$, $\{o_n\}_{n=0}^\infty$ and $\{w_n\}_{n=0}^\infty$ are in* (0, 1) *satisfying $\frac{1}{3}\leq k_n, o_n$ for all $n\in\mathbb{N}$. If the sequence $\{\bar{p}_n\}_{n=1}^\infty$ converges to $\bar{p}$, then*

$$\|p-\bar{p}\|_B \leq \frac{5\left[M+\epsilon(\theta-\varrho)^\zeta\left(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\right)\right]}{1-\Phi}. \tag{3.4}$$

Proof. Note that,

$$\begin{aligned}\|p_{n+1}(t)-\bar{p}_{n+1}(t)\| &= \left\|(Tq_n)(t)-(\bar{T}\bar{q}_n)(t)\right\| \\ &= \Bigg\|\sum_{n=0}^{n-2}\frac{c_k}{k!}(t-\varrho)^k + \left(\frac{c_\theta}{(n-1)!}+\frac{\mathcal{G}(\varrho,q_n(\varrho))(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\ &-\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_\varrho^\theta(\theta-s)^{\zeta-n}\mathcal{G}(s,q_n(s))ds + \frac{1}{\Gamma(\zeta)}\int_\varrho^t(t-s)^{(\zeta-1)}\mathcal{G}(s,q_n(s))ds \\ &-\sum_{n=0}^{n-2}\frac{d_k}{k!}(t-\varrho)^k - \left(\frac{\bar{c}_\theta}{(n-1)!}+\frac{\bar{\mathcal{G}}(\varrho,\bar{q}_n(\varrho))(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\ &+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_\varrho^\theta(\theta-s)^{\zeta-n}\bar{\mathcal{G}}(s,\bar{q}_n(s))ds - \frac{1}{\Gamma(\zeta)}\int_\varrho^t(t-s)^{(\zeta-1)}\bar{\mathcal{G}}(s,\bar{q}_n(s))ds\Bigg\| \\ &\leq \sum_{j=0}^{n-2}\frac{\|c_k-d_k\|}{k!}(\theta-\varrho)^k + \frac{\|c_\theta-\bar{c}_\theta\|}{(n-1)!}(\theta-\varrho)^{n-1} \\ &+\left(\frac{\left\|\mathcal{G}(\varrho,q_n(\varrho))-\bar{\mathcal{G}}(\varrho,\bar{q}_n(\varrho))\right\|(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\ &+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_\varrho^\theta(\theta-s)^{\zeta-n}\left\|\mathcal{G}(s,q_n(s))-\bar{\mathcal{G}}(s,\bar{q}_n(s))\right\|ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\left\|\mathcal{G}(s,q_n(s))-\bar{\mathcal{G}}(s,\bar{q}_n(s))\right\|ds\\
&\leq\sum_{j=0}^{n-2}\frac{\|c_k-d_k\|}{k!}(\theta-\varrho)^k+\frac{\|c_\theta-\bar{c}_\theta\|}{(n-1)!}(\theta-\varrho)^{n-1}\\
&+\left(\frac{\left\|\mathcal{G}(\varrho,\bar{q}_n(\varrho))-\bar{\mathcal{G}}(\varrho,\bar{q}_n(\varrho))\right\|(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1}\\
&+\left(\frac{\|\mathcal{G}(\varrho,q_n(\varrho))-\mathcal{G}(\varrho,\bar{q}_n(\varrho))\|(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1}\\
&+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\left\|\mathcal{G}(s,\bar{q}_n(s))-\bar{\mathcal{G}}(s,\bar{q}_n(s))\right\|ds\\
&+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\|\mathcal{G}(s,q_n(s))-\mathcal{G}(s,\bar{q}_n(s))\|ds\\
&+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\left\|\mathcal{G}(s,\bar{q}_n(s))-\bar{\mathcal{G}}(s,\bar{q}_n(s))\right\|ds\\
&+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\|\mathcal{G}(s,q_n(s))-\mathcal{G}(s,\bar{q}_n(s))\|ds\\
&\leq M+\left(\frac{\epsilon(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1}\\
&+\left(\frac{\mathcal{K}(\varrho)\|q_n(\varrho)-\bar{q}_n(\varrho)\|(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1}\\
&+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\epsilon ds\\
&+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{K}(s)\|q_n(s)-\bar{q}_n(s)\|ds\\
&+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\epsilon ds\\
&+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{K}(s)\|q_n(s)-\bar{q}_n(s)\|ds.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\|p_{n+1}(t)-\bar{p}_{n+1}(t)\| \leq\; & M+\frac{\epsilon(\theta-\varrho)^{\zeta-n+1}(\theta-\varrho)^{n-1}}{(n-2)!\Gamma(\zeta-n+2)}+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\epsilon ds \\
& +\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\epsilon ds+\left(\frac{\mathcal{K}(\varrho)\,\|q_n(\varrho)-\bar{q}_n(\varrho)\|\,(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\
& +\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{K}(s)\,\|q_n(s)-\bar{q}_n(s)\|\,ds \\
& +\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{K}(s)\,\|q_n(s)-\bar{q}_n(s)\|\,ds \\
\leq\; & M+\frac{\epsilon(\theta-\varrho)^{\zeta-n+1}(\theta-\varrho)^{n-1}}{(n-2)!\Gamma(\zeta-n+2)}+\frac{\epsilon(\theta-\varrho)^{n-1}(\theta-\varrho)^{\zeta-n+1}}{(n-1)!\Gamma(\zeta-n+2)}+\frac{\epsilon(\theta-\varrho)^{\zeta}}{\Gamma(\zeta+1)} \\
& +\left(\frac{\mathcal{K}(\varrho)\,\|q_n(\varrho)-\bar{q}_n(\varrho)\|\,(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\
& +\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{K}(s)\,\|q_n(s)-\bar{q}_n(s)\|\,ds \\
& +\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{K}(s)\,\|q_n(s)-\bar{q}_n(s)\|\,ds.
\end{aligned}$$

Taking supremum norm on both sides of the above inequality and simplifying, we have

$$\|p_{n+1}-\bar{p}_{n+1}\|_B \leq M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big)+\Phi\,\|q_n-\bar{q}_n\|_B\,.$$

Following arguments similar to those given in the proof of Theorem 3.5, we get

$$\|q_n-\bar{q}_n\|_B=\left\|Tu_n-\bar{T}\bar{u}_n\right\|_B \leq M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big)+\Phi\,\|u_n-\bar{u}_n\|_B\,.$$

And

$$\|u_n-\bar{u}_n\|_B \leq (1-k_n)\left\|Ts_n-\bar{T}\bar{s}_n\right\|_B+k_n\left\|Tr_n-\bar{T}\bar{r}_n\right\|_B\,.$$

Also,

$$\begin{aligned}
\left\|Tr_n-\bar{T}\bar{r}_n\right\|_B \leq\; & M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big)+\Phi\,\|r_n-\bar{r}_n\|_B \\
=\; & M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big)+\Phi\left\|Tv_n-\bar{T}\bar{v}_n\right\|_B \\
=\; & M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big) \\
& +\Phi\Big[M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big)+\Phi\,\|v_n-\bar{v}_n\|_B\Big] \\
=\; & M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big) \\
& +\Phi\Big[M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big) \\
& +\Phi\Big[o_n\Big(M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big) \\
& +(1+o_n(1-\Phi))\,\|s_n-\bar{s}_n\|_B\Big)\Big]\Big].
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\|u_n - \bar{u}_n\|_B &\leq (1-k_n)\left\|Ts_n - \bar{T}\bar{s}_n\right\|_B + k_n\left\|Tr_n - \bar{T}\bar{r}_n\right\|_B \\
&= (1-k_n)\Big(M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ \Phi\Big(\Big(w_nM + (1-w_n(1-\Phi))\Big)\|p_n - \bar{p}_n\|_B\Big)\Big) \\
&+ k_n\Big(M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ \Phi\Big[M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ \Phi\Big[o_n\Big(M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ (1+o_n(1-\Phi))\|s_n - \bar{s}_n\|_B\Big)\Big]\Big]\Big).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\|p_{n+1} - \bar{p}_{n+1}\|_B &\leq M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) + \Phi\|q_n - \bar{q}_n\|_B \\
&\leq M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ \Phi\Big(M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ \Phi\|u_n - \bar{u}_n\|_B\Big) \\
&\leq M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ \Phi\Big(M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ \Phi\Big[(1-k_n)\Big(M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ \Phi\Big(\Big(w_nM + (1-w_n(1-\Phi))\Big)\|p_n - \bar{p}_n\|_B\Big)\Big) \\
&+ k_n\Big(M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ \Phi\Big[M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ \Phi\Big[o_n\Big(M + \epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)} + \frac{1}{(n-1)!\Gamma(\zeta-n+2)} + \frac{1}{\Gamma(\zeta+1)}\Big) \\
&+ (1+o_n(1-\Phi))\|s_n - \bar{s}_n\|_B\Big)\Big]\Big]\Big)\Big]\Big) \\
&\leq [1 - k_n o_n(1-\Phi)]\Big(w_nM + (1-w_n(1-\Phi))\Big)\|p_n - \bar{p}_n\|_B + 3 + k_n
\end{aligned}
$$

$$+\, o_n\Big(M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big)\Big).$$

Now, by taking $\mu_n = k_n o_n(1-\Phi)$, we have

$$\|p_{n+1}-\bar{p}_{n+1}\|_B \leq (1-\mu_n)\,\|p_n-\bar{p}_n\|_B + \mu_n \frac{5\Big[M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big)\Big]}{1-\Phi}.$$

Using the Lemma 3.2, we arrive at

$$\|p-\bar{p}\|_B \leq \frac{5\Big[M+\epsilon(\theta-\varrho)^{\zeta}\Big(\frac{1}{(n-2)!\Gamma(\zeta-n+2)}+\frac{1}{(n-1)!\Gamma(\zeta-n+2)}+\frac{1}{\Gamma(\zeta+1)}\Big)\Big]}{1-\Phi}. \tag{3.5}$$

The inequality (3.5) shows the relationship between solutions of the BVP (3.1) and (3.2), in the sense that if $\epsilon \to 0$, that is, $\mathcal{G}$ and $\bar{\mathcal{G}}$ are sufficiently close to each other, then not only will the solutions of the two BVPs be close to each other, but will also depend continuously on the functions involved therein and the boundary data.

3.4. Dependence on parameters

Consider the fractional boundary value problems

$$\begin{cases} {}^{c}D^{\zeta}_{\varrho}p(t) = \mathcal{G}(t,p(t),\mu_1),\ for\ t\in\mathcal{J}=[\varrho,\theta],\ n-1<\zeta<n \\ p^{(k)}(\varrho)=c_k,\ k=,1,2,\dots,n-2;\ p^{(n-1)}(\theta)=c_\theta, \end{cases}$$

and

$$\begin{cases} {}^{c}D^{\zeta}_{\varrho}\bar{p}(t) = \mathcal{G}(t,\bar{p}(t),\mu_2),\ for\ t\in\mathcal{J}=[\varrho,\theta],\ n-1<\zeta<n \\ \bar{p}^{(k)}(\varrho)=d_k,\ k=,1,2,\dots,n-2;\ \bar{p}^{(n-1)}(\theta)=\bar{c}_\theta. \end{cases}$$

Let p and $\bar{p}\in C^{(n-1)}(\mathcal{J},X)$ as given in the previous Theorem. Define the operators T and $\bar{T}$ as follows:

$$\begin{aligned}(Tp)(t) &= \sum_{k=0}^{n-2}\frac{c_k}{k!}(t-\varrho)^k + \left(\frac{c_\theta}{(n-1)!}+\frac{\mathcal{G}(\varrho,p(\varrho),\mu_1)(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\ &\quad - \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{G}(s,p(s),\mu_1)ds + \frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{G}(s,p(s),\mu_1)ds\end{aligned}$$

and

$$\begin{aligned}\left(\bar{T}\bar{p}\right)(t) &= \sum_{k=0}^{n-2}\frac{d_k}{k!}(t-\varrho)^k + \left(\frac{\bar{c}_\theta}{(n-1)!}+\frac{\mathcal{G}(\varrho,\bar{p}(\varrho),\mu_2)(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\right)(t-\varrho)^{n-1} \\ &\quad - \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{G}(s,\bar{p}(s),\mu_2)ds + \frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{G}(s,\bar{p}(s),\mu_2)ds.\end{aligned}$$

Assume that,

$$\|\mathcal{G}(t,u_1,\mu_1)-\mathcal{G}(t,v_1,\mu_1)\| \leq \bar{\mathcal{K}}(t)\,\|u_1-v_1\|$$

and

$$\|\mathcal{G}(t,u_1,\mu_1)-\mathcal{G}(t,u_1,\mu_2)\| \le r(t)\,\|\mu_1-\mu_2\|\,,$$

where, $\bar{\mathcal{K}}, r \in C(\mathbb{R}_+)$. The following result establishes the continuous dependency of solutions on parameters.

Theorem 3.7. *Consider the sequences $\{p_n\}_{n=1}^{\infty}$ and $\{\bar{p}_n\}_{n=1}^{\infty}$ as in the previous Theorem and satisfy the assumptions given above. If the sequence $\{\bar{p}_n\}_{n=1}^{\infty}$ converges to $\bar{p}$, then we have*

$$\|p-\bar{p}\|_B \le \frac{5\Big[M+\Big(\frac{r(\varrho)\left|\mu_1-\mu_2\right|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\Big)+\frac{\left|\mu_1-\mu_2\right|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\left|\mu_1-\mu_2\right|I_{\varrho}^{\zeta}r(t)\Big]}{1-\bar{\Phi}}$$

where,

$$\bar{\Phi}=\Big[\frac{(\theta-\varrho)^{\zeta}\bar{\mathcal{K}}(\varrho)}{(n-2)!\Gamma(\zeta-n+2)}+\frac{(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}\bar{\mathcal{K}}(\theta)+I_{\varrho}^{\zeta}\bar{\mathcal{K}}(t)\Big]<1.$$

Proof. Consider,

$$\begin{aligned}
\|p_{n+1}(t)-\bar{p}_{n+1}(t)\| &= \left\|(Tq_n)(t)-(\bar{T}\bar{q}_n)(t)\right\| \\
&= \Bigg\|\sum_{n=0}^{n-2}\frac{c_k}{k!}(t-\varrho)^k+\Big(\frac{c_\theta}{(n-1)!}+\frac{\mathcal{G}(\varrho,q_n(\varrho),\mu_1)(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\Big)(t-\varrho)^{n-1} \\
&-\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{G}(s,q_n(s),\mu_1)ds+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{G}(s,q_n(s),\mu_1)ds \\
&-\sum_{n=0}^{n-2}\frac{d_k}{k!}(t-\varrho)^k-\Big(\frac{\bar{c}_\theta}{(n-1)!}+\frac{\mathcal{G}(\varrho,\bar{q}_n(\varrho),\mu_2)(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\Big)(t-\varrho)^{n-1} \\
&+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\mathcal{G}(s,\bar{q}_n(s),\mu_2)ds-\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\mathcal{G}(s,\bar{q}_n(s),\mu_2)ds\Bigg\| \\
&\le \sum_{j=0}^{n-2}\frac{\|c_k-d_k\|}{k!}(\theta-\varrho)^k+\frac{\|c_\theta-\bar{c}_\theta\|}{(n-1)!}(\theta-\varrho)^{n-1} \\
&+\Big(\frac{\|\mathcal{G}(\varrho,q_n(\varrho),\mu_1)-\mathcal{G}(\varrho,\bar{q}_n(\varrho),\mu_2)\|\,(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\Big)(t-\varrho)^{n-1} \\
&+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\,\|\mathcal{G}(s,q_n(s),\mu_1)-\mathcal{G}(s,\bar{q}_n(s),\mu_2)\|\,ds \\
&+\frac{1}{\Gamma(\zeta)}\int_{\varrho}^{t}(t-s)^{(\zeta-1)}\,\|\mathcal{G}(s,q_n(s),\mu_1)-\mathcal{G}(s,\bar{q}_n(s),\mu_2)\|\,ds \\
&\le \sum_{j=0}^{n-2}\frac{\|c_k-d_k\|}{k!}(\theta-\varrho)^k+\frac{\|c_\theta-\bar{c}_\theta\|}{(n-1)!}(\theta-\varrho)^{n-1} \\
&+\Big(\frac{\|\mathcal{G}(\varrho,\bar{q}_n(\varrho),\mu_1)-\mathcal{G}(\varrho,\bar{q}_n(\varrho),\mu_2)\|\,(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\Big)(t-\varrho)^{n-1} \\
&+\Big(\frac{\|\mathcal{G}(\varrho,q_n(\varrho),\mu_1)-\mathcal{G}(\varrho,\bar{q}_n(\varrho),\mu_1)\|\,(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)}\Big)(t-\varrho)^{n-1} \\
&+\frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)}\int_{\varrho}^{\theta}(\theta-s)^{\zeta-n}\,\|\mathcal{G}(s,\bar{q}_n(s),\mu_1)-\mathcal{G}(s,\bar{q}_n(s)\mu_2)\|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)} \int_{\varrho}^{\theta} (\theta-s)^{\zeta-n} \left\| \mathcal{G}(s, q_n(s), \mu_1) - \mathcal{G}(s, \bar{q}_n(s), \mu_1) \right\| ds \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\varrho}^{t} (t-s)^{(\zeta-1)} \left\| \mathcal{G}(s, \bar{q}_n(s), \mu_1) - \mathcal{G}(s, \bar{q}_n(s)\mu_2) \right\| ds \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\varrho}^{t} (t-s)^{(\zeta-1)} \left\| \mathcal{G}(s, q_n(s), \mu_1) - \mathcal{G}(s, \bar{q}_n(s), \mu_1) \right\| ds \\
&\leq M + \left(\frac{r(\varrho)\left|\mu_1-\mu_2\right|(\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)} \right) (t-\varrho)^{n-1} \\
&+ \left(\frac{\mathcal{K}(\varrho) \left\| q_n(\varrho) - \bar{q}_n(\varrho) \right\| (\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)} \right) (t-\varrho)^{n-1} \\
&+ \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)} \int_{\varrho}^{\theta} (\theta-s)^{\zeta-n} r(s) \left|\mu_1-\mu_2\right| ds \\
&+ \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)} \int_{\varrho}^{\theta} (\theta-s)^{\zeta-n} \mathcal{K}(s) \left\| q_n(s) - \bar{q}_n(s) \right\| ds \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\varrho}^{t} (t-s)^{(\zeta-1)} r(s) \left|\mu_1-\mu_2\right| ds + \frac{1}{\Gamma(\zeta)} \int_{\varrho}^{t} (t-s)^{(\zeta-1)} \mathcal{K}(s) \left\| q_n(s) - \bar{q}_n(s) \right\| ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\| p_{n+1}(t) - \bar{p}_{n+1}(t) \right\| &\leq M + \frac{r(\varrho)\left|\mu_1-\mu_2\right|(\theta-\varrho)^{\zeta-n+1}(\theta-\varrho)^{n-1}}{(n-2)!\Gamma(\zeta-n+2)} \\
&+ \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)} \int_{\varrho}^{\theta} (\theta-s)^{\zeta-n} r(s) \left|\mu_1-\mu_2\right| ds \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\varrho}^{t} (t-s)^{(\zeta-1)} r(s) \left|\mu_1-\mu_2\right| ds + \left(\frac{\mathcal{K}(\varrho) \left\| q_n(\varrho) - \bar{q}_n(\varrho) \right\| (\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)} \right) (t-\varrho)^{n-1} \\
&+ \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)} \int_{\varrho}^{\theta} (\theta-s)^{\zeta-n} \mathcal{K}(s) \left\| q_n(s) - \bar{q}_n(s) \right\| ds \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\varrho}^{t} (t-s)^{(\zeta-1)} \mathcal{K}(s) \left\| q_n(s) - \bar{q}_n(s) \right\| ds \\
&\leq M + \frac{r(\varrho)\left|\mu_1-\mu_2\right|(\theta-\varrho)^{\zeta-n+1}(\theta-\varrho)^{n-1}}{(n-2)!\Gamma(\zeta-n+2)} + \frac{r(\theta)\left|\mu_1-\mu_2\right|(\theta-\varrho)^{n-1}(\theta-\varrho)^{\zeta-n+1}}{(n-1)!\Gamma(\zeta-n+2)} \\
&+ \frac{r(t)\left|\mu_1-\mu_2\right|(\theta-\varrho)^{\zeta}}{\Gamma(\zeta+1)} + \left(\frac{\mathcal{K}(\varrho) \left\| q_n(\varrho) - \bar{q}_n(\varrho) \right\| (\theta-\varrho)^{\zeta-n+1}}{(n-2)!\Gamma(\zeta-n+2)} \right) (t-\varrho)^{n-1} \\
&+ \frac{(t-\varrho)^{n-1}}{(n-1)!\Gamma(\zeta-n+1)} \int_{\varrho}^{\theta} (\theta-s)^{\zeta-n} \mathcal{K}(s) \left\| q_n(s) - \bar{q}_n(s) \right\| ds \\
&+ \frac{1}{\Gamma(\zeta)} \int_{\varrho}^{t} (t-s)^{(\zeta-1)} \mathcal{K}(s) \left\| q_n(s) - \bar{q}_n(s) \right\| ds.
\end{aligned}
$$

Taking supremum norm on both sides and simplifying, we have

$$\|p_{n+1} - \bar{p}_{n+1}\|_B \leq M + \frac{r(\varrho)\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{\zeta-n+1}(\theta-\varrho)^{n-1}}{(n-2)!\Gamma(\zeta-n+2)} + \frac{r(\theta)\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{n-1}(\theta-\varrho)^{\zeta-n+1}}{(n-1)!\Gamma(\zeta-n+2)}$$
$$+ \frac{r(t)\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{\zeta}}{\Gamma(\zeta+1)} + \bar{\Phi}\,\|q_n - \bar{q}_n\|_B\,.$$

Following arguments similar to those given in the Theorem 3.5, we get

$$\|q_n - \bar{q}_n\|_B = \left\|Tu_n - \bar{T}\bar{u}_n\right\|_B \leq M + \left(\frac{r(\varrho)\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)$$
$$+ \frac{\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!} I_\varrho^{\zeta-n+1} r(\theta) + \Big|\mu_1 - \mu_2\Big| I_\varrho^{\zeta} r(t) + \bar{\Phi}\,\|u_n - \bar{u}_n\|_B\,,$$

and

$$\|u_n - \bar{u}_n\|_B \leq (1-k_n)\left\|Ts_n - \bar{T}\bar{s}_n\right\|_B + k_n\left\|Tr_n - \bar{T}\bar{r}_n\right\|_B\,.$$

Also,

$$\begin{aligned}
\left\|Tr_n - \bar{T}\bar{r}_n\right\|_B &\leq M + \left(\frac{r(\varrho)\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right) + \frac{\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!} I_\varrho^{\zeta-n+1} r(\theta) + \Big|\mu_1 - \mu_2\Big| I_\varrho^{\zeta} r(t) + \bar{\Phi}\,\|r_n - \bar{r}_n\|_B \\
&= M + \left(\frac{r(\varrho)\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right) + \frac{\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!} I_\varrho^{\zeta-n+1} r(\theta) + \Big|\mu_1 - \mu_2\Big| I_\varrho^{\zeta} r(t) + \bar{\Phi}\left\|Tv_n - \bar{T}\bar{v}_n\right\|_B \\
&= M + \left(\frac{r(\varrho)\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right) + \frac{\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!} I_\varrho^{\zeta-n+1} r(\theta) + \Big|\mu_1 - \mu_2\Big| I_\varrho^{\zeta} r(t) \\
&+ \bar{\Phi}\Big[M + \left(\frac{r(\varrho)\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right) + \frac{\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!} I_\varrho^{\zeta-n+1} r(\theta) + \Big|\mu_1 - \mu_2\Big| I_\varrho^{\zeta} r(t) + \bar{\Phi}\,\|v_n - \bar{v}_n\|_B\Big] \\
&= M + \left(\frac{r(\varrho)\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right) + \frac{\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!} I_\varrho^{\zeta-n+1} r(\theta) + \Big|\mu_1 - \mu_2\Big| I_\varrho^{\zeta} r(t) \\
&+ \bar{\Phi}\Big[M + \left(\frac{r(\varrho)\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right) + \frac{\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!} I_\varrho^{\zeta-n+1} r(\theta) + \Big|\mu_1 - \mu_2\Big| I_\varrho^{\zeta} r(t) \\
&+ \bar{\Phi}\Big[o_n\Big(M + \left(\frac{r(\varrho)\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right) + \frac{\Big|\mu_1 - \mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!} I_\varrho^{\zeta-n+1} r(\theta) + \Big|\mu_1 - \mu_2\Big| I_\varrho^{\zeta} r(t) \\
&+ (1 + o_n(1-\bar{\Phi}))\,\|s_n - \bar{s}_n\|_B\Big)\Big]\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}\|u_n-\bar{u}_n\|_B &\leq (1-k_n)\left\|Ts_n-\bar{T}\bar{s}_n\right\|_B+k_n\left\|Tr_n-\bar{T}\bar{r}_n\right\|_B\\ &=(1-k_n)\Bigg(M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\ &+\bar{\Phi}\Bigg(\Big(w_nM+(1-w_n(1-\bar{\Phi}))\Big)\|p_n-\bar{p}_n\|_B\Bigg)\Bigg)\\ &+k_n\Bigg(M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\ &+\bar{\Phi}\Bigg[M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\ &+\bar{\Phi}\Bigg[o_n\Bigg(M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\ &+(1+o_n(1-\bar{\Phi}))\|s_n-\bar{s}_n\|_B\Bigg)\Bigg]\Bigg]\Bigg).\end{aligned}$$

Therefore,

$$
\begin{aligned}
\|p_{n+1}-\bar{p}_{n+1}\|_B &\leq M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)+\bar{\Phi}\,\|q_n-\bar{q}_n\|_B\\
&\leq M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\
&+\bar{\Phi}\Big(M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\
&+\bar{\Phi}\,\|u_n-\bar{u}_n\|_B\Big)\\
&\leq M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\
&+\bar{\Phi}\Big(M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\
&+\bar{\Phi}\Big[(1-k_n)\Big(M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\
&+\bar{\Phi}\Big(\Big(w_nM+(1-w_n(1-\bar{\Phi}))\Big)\|p_n-\bar{p}_n\|_B\Big)\Big)\\
&+k_n\Big(M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\
&+\bar{\Phi}\Big[M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\
&+\bar{\Phi}\Big[o_n\Big(M+\left(\frac{r(\varrho)\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\Big|\mu_1-\mu_2\Big|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\\
&+(1+o_n(1-\bar{\Phi}))\,\|s_n-\bar{s}_n\|_B\Big)\Big]\Big]\Big)\Big)\\
&\leq[1-k_no_n(1-\bar{\Phi})]\Big(w_nM+(1-w_n(1-\bar{\Phi}))\Big)\|p_n-\bar{p}_n\|_B+3+k_n\\
&+o_n(1-\bar{\Phi})\frac{5\Big[M+\left(\frac{r(\varrho)\left|\mu_1-\mu_2\right|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\left|\mu_1-\mu_2\right|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\Big]}{1-\bar{\Phi}}.
\end{aligned}
$$

By setting $\mu_n=k_no_n(1-\bar{\Phi})$, we have

$$
\|p_{n+1}-\bar{p}_{n+1}\|_B\leq(1-\mu_n)\,\|p_n-\bar{p}_n\|_B+\mu_n\frac{5\Big[M+\left(\frac{r(\varrho)\left|\mu_1-\mu_2\right|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\left|\mu_1-\mu_2\right|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\Big]}{1-\bar{\Phi}}.
$$

As $n\to\infty$, then by assumptions and from the Lemma 3.2, we get that

$$
\|p-\bar{p}\|_B\leq\frac{5\Big[M+\left(\frac{r(\varrho)\left|\mu_1-\mu_2\right|(\theta-\varrho)^{\zeta}}{(n-2)!\Gamma(\zeta-n+2)}\right)+\frac{\left|\mu_1-\mu_2\right|(\theta-\varrho)^{n-1}}{(n-1)!}I_{\varrho}^{\zeta-n+1}r(\theta)+\Big|\mu_1-\mu_2\Big|I_{\varrho}^{\zeta}r(t)\Big]}{1-\bar{\Phi}}.
$$

4. Numerical experiment

We now present a numerical example that not only shows the applicability but also substantiate our results presented herein. We have used MATLAB version R2018a. For a given contraction mapping, we compare our iteration scheme with other existing comparable iterative schemes. Additionally, we compare various hypotheses and parameters.

Example 4.1. *Consider the set R of all real numbers with the usual norm, that is, $\|p\| = |p|$. Define a mapping $f : R \to R$ by $f(p) = (p^2 + 2p + 5)^{\frac{1}{2}}$. Note that, f is a contraction mapping. We plot the behavior of the convergence of different iterative schemes for f. It is evident from Figures 1–4 that the AA-iteration not only converges faster but is also more stable than the comparable iteration schemes in case of contraction mapping. Note that the other iterative schemes change their convergence behaviors as the parameters are changed.*

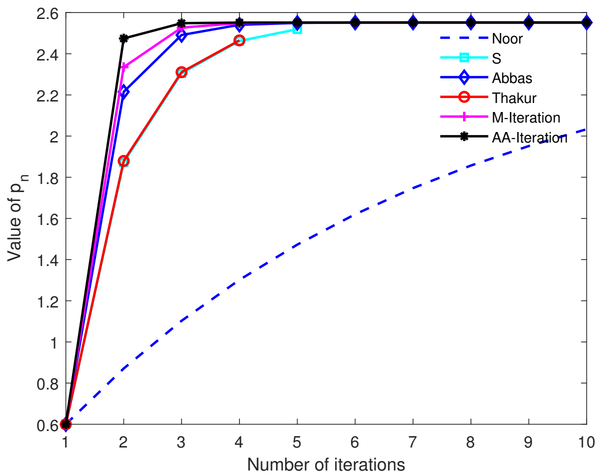

Figure 1. $k_n = \frac{2n^2+n-1}{n^3+4n^2-1}$, $o_n = \frac{(n-1)^2}{n^3+2n+1}$, $w_n = \frac{2n^4-1}{n^7+n^4+3}$.

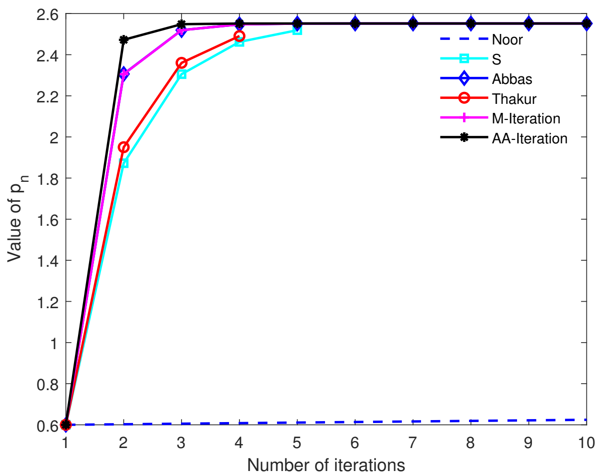

Figure 2. $k_n = \frac{2n^2+n}{n^5+7n+11}$, $o_n = \frac{n-1}{n^2+2n+1}$, $w_n = \frac{2n^2-1}{n^3+2n+1}$.

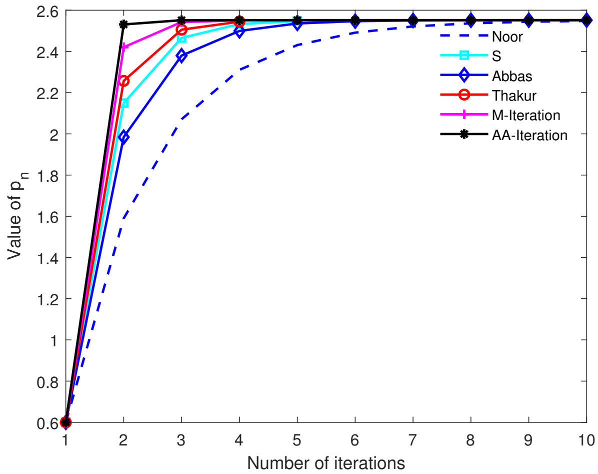

Figure 3. $k_n = \frac{n^2+n}{n^2+4n+7}$, $o_n = \frac{n+1}{n^2+n+1}$, $w_n = \frac{n^2+1}{n^2+n+7}$.

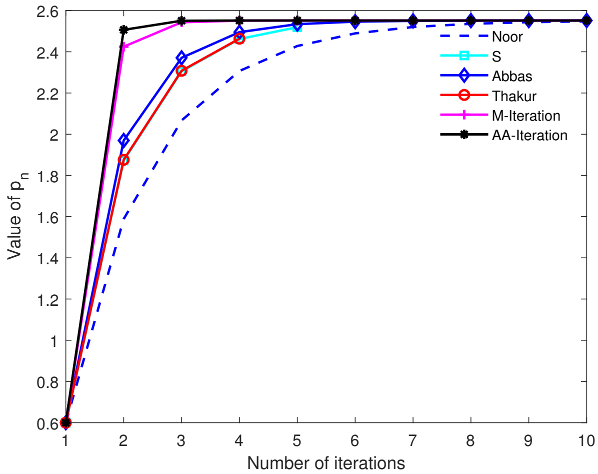

Figure 4. $k_n = \frac{n^2+n}{n^2+4n+2}$, $o_n = \frac{n+1}{n^3+n-1}$, $w_n = \frac{n^2+1}{n^5+n+7}$.

In order to compare convergence rates between two iterative processes, we use the following Definition from [9].

Definition 4.2. *Suppose that sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converge to the same point* l^* *with the following error estimates*

$$\|\alpha_n - l^*\| \le p_n,$$
$$\|\beta_n - l^*\| \le q_n.$$

If $\lim_{n\to\infty} \frac{p_n}{q_n} = 0$, *then* $\{\alpha_n\}$ *converges faster than* $\{\beta_n\}$.

Example 4.3. *We consider the following boundary value problem:*

$$(D_*^{\alpha})\, p(t) = \frac{3t}{5}\left[\frac{t - \sin(p(t))}{2}\right], t \in [0, 1], n - 1 < \alpha \le n, n \in \mathbb{N} \tag{4.1}$$

with the given boundary conditions

$$p^{(j)}(0) = 0, j = 0, 1, 2, \cdots, n-2, p^{(n-1)}(1) = 1.$$

Comparing this equation with the equation (1.1)*, we get*

$$\mathcal{G} \in C(\mathcal{J} \times \mathbb{R}, \mathbb{R}), \text{ with } \mathcal{G}(t, p(t)) = \frac{3t}{5}\left[\frac{t - \sin(p(t))}{2}\right]. \tag{4.2}$$

Now, we have

$$\begin{aligned} |\mathcal{G}(t, p(t)) - \mathcal{G}(t, \bar{p}(t))| &\leq \left|\frac{3t}{5}\right| \left|\frac{t - \sin(p(t)))}{2} - \frac{t - \sin(\bar{p}(t))}{2}\right| \\ &\leq \frac{3t}{10}|\sin(p(t)) - \sin(\bar{p}(t))| \\ &\leq \frac{3t}{10}|p(t) - \bar{p}(t)|, \end{aligned} \tag{4.3}$$

where $\mathcal{K}(t) = \frac{3t}{10}$.
Note that,

$$\begin{aligned} \Phi &= \left[\frac{(\theta - \varrho)^{\zeta}\mathcal{K}(\varrho)}{(n-2)!\Gamma(\zeta - n + 2)} + \frac{(\theta - \varrho)^{n-1}}{(n-1)!} I_{\varrho}^{\zeta - n + 1}\mathcal{K}(\theta) + I_{\varrho}^{\zeta}\mathcal{K}(t)\right] \\ &= \left[\frac{p(0)}{(n-2)!\Gamma(\zeta - n + 2)} + \frac{1}{(n-1)!} I^{\zeta - n + 1} p(1) + I^{\zeta} p(t)\right] \\ &= \left[\frac{0}{(n-2)!\Gamma(\zeta - n + 2)} + \frac{1}{(n-1)!} I^{\zeta - n + 1} p(1) + I^{\zeta} p(t)\right] (p(0) = 0) \\ &= \frac{3}{10}\left[\frac{1}{(n-1)!\Gamma(\zeta - n + 1)} \int_0^1 (1 - s)^{\zeta - n} s ds + \frac{1}{\Gamma(\zeta)} \int_0^t (t - s)^{\zeta - 1} s ds\right] \\ &\leq \frac{3}{10}\left[\frac{1}{(n-1)!\Gamma(\zeta - n + 3)} + \frac{1}{\Gamma(\zeta + 2)}\right] (t \leq 1). \end{aligned} \tag{4.4}$$

If $= \frac{3}{10}\left[\frac{1}{(n-1)!\Gamma(\zeta-n+3)} + \frac{1}{\Gamma(\zeta+2)}\right] < 1$, *then* $\Phi < 1$. *If we set,* $\zeta = \frac{5}{2}$, *then* $n = [\zeta] + 1 = \left[\frac{5}{2}\right] + 1 = 2 + 1 = 3$ *and*

$$\begin{aligned} \Phi &\leq \frac{3}{10}\left[\frac{1}{(3-1)!\Gamma\left(\frac{5}{2} - 3 + 3\right)} + \frac{1}{\Gamma\left(\frac{5}{2} + 2\right)}\right] \\ &= \frac{3}{10}\left[\frac{1}{2\Gamma\left(\frac{5}{2}\right)} + \frac{1}{\Gamma\left(\frac{9}{2}\right)}\right] \\ &= \frac{3}{5\sqrt{\pi}}\left[\frac{1}{3} + \frac{8}{105}\right] \\ &= \frac{43}{175\sqrt{\pi}} \\ &\simeq 0.1387 \\ &< 1. \end{aligned} \tag{4.5}$$

Define the operator $T : B \to B$ as follows

$$(Tp)(t) = \frac{t^2}{2} - \frac{t^2}{2}\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1 (1-s)^{-\frac{1}{2}}\frac{3s}{5}\left[\frac{s-\sin(p(s))}{2}\right]ds + \frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^t (t-s)^{\frac{3}{2}}\frac{3s}{5}\left[\frac{s-\sin(p(s))}{2}\right]ds, t \in \mathcal{J}. \tag{4.6}$$

Since all conditions of Theorem 2.1 are satisfied, we get the sequence $\{p_n\}$ generated by AA-iteration (1.3) *converges to the solution of BVP for the operator T defined in* (4.6)*, which converges to a unique solution $p \in B$. Moreover, the Table 1 shows that the convergence of AA-iteration scheme is faster than the Picard, Mann, Ishikawa, and S-iteration processes.*

Indeed, from Eqs (16) and (17) from [1] and by [7, 13, 21], we get that

(i) $\alpha_n = \varepsilon^n[1-(1-\varepsilon)kw]^n \|p_1 - p^*\|$,
(ii) $\beta_n = \varepsilon^n \|p_1 - p^*\|$,
(iii) $\gamma_n = [1-(1-\varepsilon)k]^n \|p_1 - p^*\|$,
(iv) $\nu_n = \left[1-(1-\varepsilon)^2 k\right]^n \|p_1 - p^*\|$,
(v) $\lambda_n = \varepsilon^{3n}[1-(1-\varepsilon)(k+w-kw)]^n \|p_1 - p^*\|$,

where $\varepsilon \in [0,1)$ is contraction constant. The convergence of sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\nu_n\}$, and $\{\lambda_n\}$ depend only on $\Phi_1 = \varepsilon^n[1-(1-\varepsilon)kw]^n$, $\Phi_2 = \varepsilon^n$, $\Phi_3 = [1-(1-\varepsilon)k]^n$, and $\Phi_4 = \left[1-(1-\varepsilon)^2 k\right]^n$ and $\Phi_5 = \varepsilon^{3n}[1-(1-\varepsilon)(k+w-kw)]^n$, respectively.

Table 1 shows the respective iteration for the example discussed above with $\varepsilon = \Phi = 0.138629441$ and $k_n = w_n = \frac{1}{2}$.

Table 1. Comparison of different iterative schemes.

Iteration (n)	S-iteration (Φ_1)	Picard (Φ_2)	Mann (Φ_3)	Ishikawa (Φ_4)	AA-iteration (Φ_5)
1	0.108776611	0.138629441	0.569314720	0.629020380	0.000943051
2	0.011832351	0.019218122	0.324119251	0.395666639	0.000000889
3	0.001287083	0.002664197	0.184525861	0.248882379	0.000000000
4	0.000140004	0.000369336	0.105053289	0.156552089	0.000000000
5	0.000015229	0.000051201	0.059808384	0.098474454	0.000000000
6	0.000001656	0.000007098	0.034049793	0.061942439	0.000000000
7	0.000000180	0.000000984	0.019385049	0.038963056	0.000000000
8	0.000000019	0.000000136	0.011036193	0.024508557	0.000000000
9	0.000000002	0.000000019	0.006283067	0.015416382	0.000000000
10	0.000000000	0.000000003	0.003577043	0.009697218	0.000000000

According to the Definition 4.2 and by the Table 1, the AA-iteration process converges faster than the Picard, Mann, Ishikawa, and S-iteration processes.

Error estimate. *Now, from* (2.3) *we have*

$$\|p_{n+1} - p\|_B \leq \Phi^{n+1}\|p_0 - p\|_B$$

$$\leq [\frac{43}{175\sqrt{\pi}}]^{n+1}\|p_0 - p\|_B. \tag{4.7}$$

The estimate obtained in the Eq (4.7) *is called a bound for the error.*

5. Conclusions

In this paper, we approximated the unique solution of boundary value problem (1.1) using the AA-iteration scheme. Moreover, the properties of solutions, such as continuous dependence on the boundary data, closeness of solutions, dependence of solutions on parameters, and functions involved therein, have also been discussed. Finally, we presented a numerical examples, comparing the behavior of the AA-iteration with other known iteration schemes. The simulations show that the AA-iteration converges faster than the M-iteration, S-iteration, Abbas-iteration, Thakur, and Noor-iterations. Thus, our results are generalizations and improvements of comparable results in the existing literature. In the future, one could explore the extension of the proposed AA-iterative scheme to nonlinear and multi-dimensional systems to obtain vast applications in science and engineering.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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