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Research article

Unveiling the dynamics of drug transmission: A fractal-fractional approach integrating criminal law perspectives

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Abstract: The excessive use of drugs has become a growing concern in the current century, with the global toll of drug-related deaths and disabilities posing a significant public health challenge in both developed and developing countries. In pursuit of continuous improvement in existing strategies, this article presented a nonlinear deterministic mathematical model that encapsulates the dynamics of drug addiction transmission while considering the legal implications imposed by criminal law within a population. The proposed model incorporated the fractal-fractional order derivative using the Atangana-Baleanu-Caputo ($\mathbb{ABC}$) operator. The objectives of this research were achieved by examining the dynamics of the drug transmission model, which stratifies the population into six compartments: The susceptible class to drug addicts, the number of individuals receiving drug misuse education, the count of mild drug addicts, the population of heavy-level drug addicts, individuals subjected to criminal law, and those who have ceased drug use. The qualitative analysis of the devised model established the existence and uniqueness of solutions within the framework of fixed-point theory. Furthermore, Ulam-Hyer's stability was established through nonlinear functional analysis. To obtain numerical solutions, the fractional Adam-Bashforth iterative scheme was employed, and the results were validated through simulations conducted using MATLAB. Additionally, numerical results were plotted for various fractional orders and fractal dimensions, with comparisons made against integer orders. The findings underscored the necessity of controlling the effective transmission rate to halt drug transmission effectively. The newly proposed strategy demonstrated a competitive advantage, providing a more nuanced understanding of the complex dynamics outlined in the model.

Keywords: drugs model; fractal-fractional derivative; fixed-point theory; Ulam-Hyers stability; numerical simulation
Mathematics Subject Classification: 92-10, 37C75, 65L07

1. Introduction

The constantly evolving body of research underscores the correlation between drug usage and numerous physical, psychological, and socio-economic concerns. Globally, drug abuse stands as a critical health and societal challenge, drawing significant governmental attention. A growing number of individuals suffer from the impact of diverse substances, with drug abuse defined by the habitual consumption of substances detrimental to physical, mental, or emotional well-being, often progressing to addiction: A persistent condition marked by relentless drug-seeking behavior despite adverse consequences. Among the commonly abused substances are prescription opioids, benzodiazepines, cocaine, methamphetamine, marijuana, and alcohol. Genetic predispositions play a substantial role in susceptibility to drug abuse, particularly among those with a familial history of substance misuse [34, 37, 38]. Additionally, environmental factors such as stress, trauma, societal influences, mental health disorders (e.g., anxiety, depression), and the availability of drugs contribute significantly to substance misuse [17, 25]. The ramifications of drug misuse encompass various physical health complications, including respiratory and cardiovascular issues, alongside infectious diseases like HIV and hepatitis. Prolonged drug abuse can culminate in life-threatening overdoses [41,55,58].

Prevention programs targeting youth, including family-based and community interventions, are effective in controlling drug transmission. Cognitive-behavioral therapy (CBT) and medication-assisted treatment (MAT) play crucial roles in identifying and modifying harmful behaviors associated with drug misuse [28], extending their scope to substance use disorders, prevention measures for HIV and other infectious diseases, and drug-resistant infections [31, 48]. Combating drug transmission requires a multifaceted approach that integrates harm reduction strategies, educational campaigns, and public health interventions. Needle exchange programs (NEPs) mitigate the spread of bloodborne infections among injectable drug users [19], while safe injection sites (SISs) offer medically supervised environments for narcotic injection, providing sterile equipment, overdose prevention measures, and referrals to health and social services [51]. MAT for opioid addiction, utilizing substances like methadone, buprenorphine, and naltrexone, aids in maintaining sobriety by alleviating cravings and withdrawal symptoms [48]. Education campaigns enhance awareness about drug use and harm reduction techniques, reducing risky behaviors [36]. Mathematical modeling helps analyze drug transmission dynamics and evaluate preventive measures. Network models are utilized to explore drug users' social and sexual networks, as well as the dissemination of bloodborne diseases like HIV and hepatitis C [35].

Mathematical modeling is widely applied to solve various real-world problems. In epidemiology, the Susceptible-Infectious-Recovered (SIR) model is a notable example used for understanding disease propagation [39]. Neuroscience increasingly relies on mathematical models to grasp brain function and behavior. Next-generation techniques have been employed to determine thresholds for illicit drug use and terrorism. Analytically demonstrating global stability, a suitable Lyapunov function established the equilibrium point of illicit drug usage and terrorism [10]. Drawing from mathematical epidemiology concepts, a synthetic drug transmission model with treatment discerns addiction rates among susceptible individuals with a history of drug abuse versus those without [46]. The paper explores the optimal control analysis of disease transmission within a community, as discussed in [14]. A seven-dimensional deterministic model examines illicit drug use and terrorism, delving into local and global stability [9]. An updated model addressing medication treatment's influence on HIV/AIDS transmission is presented [24], along with an exploration of rehabilitative therapy's impact on HIV/AIDS control in prisons. Illegal drug use (IDU) among students poses a significant public health challenge, leading to academic, physical, and mental health issues, as well as diminished development and productivity later in life. Implementing IDU prevention and control measures incurs substantial financial burdens on governments and the general public [2].

In contemporary research endeavors, fractional calculus has emerged as a versatile mathematical tool, encompassing derivatives and integrals of any real positive order, and has been utilized to represent and solve problems across various fields [43, 49]. Various fractional strategies, such as Katugumpola, Hadamard, Riemann-Liouville, and Caputo derivatives, have been employed to improve the accuracy of modeling real-world phenomena. Researchers investigating fractional-order mathematical models have adopted diverse approaches, including numerical, computational, and iterative methods [15, 23, 52]. To extend the limitations of standard Riemann-Liouville and Caputo fractional derivatives, a novel concept of fractional differentiation with nonlocal and nonsingular kernels has been introduced [57]. A significant advancement is the Caputo-Fabrizio (CF) derivative, which addresses the shortcomings of the traditional Caputo operator. Unlike conventional fractional derivatives, the CF derivative is nonsingular and features an exponential law kernel, enhancing precision under certain conditions [13, 26, 27]. Caputo further generalized the CF derivative into another Atangana-Baleanu fractional derivative, involving the Mittag-Leffler function [21]. Research has explored the mathematical modeling of COVID-19 with the CF operator [53], and a fractional mathematical model for alcohol consumption utilizing Atangana-Baleanu Caputo derivatives has been developed [54].

Atangana's recent contribution introduces a new fractal-fractional operator, merging fractional and fractal mathematics [21]. Existing literature has explored fractal-fractional derivatives based on power law, exponential, and Mittag-Leffler kernels. The selection of the kernel for studying the problem at hand is guided by its mathematical properties and relevance to real-world phenomena. This area holds promise for addressing various complex problems across different contexts. Notably, the fractal dimension and fractional order are both incorporated within this operator. Compared to traditional methods of obtaining fractal fractions, this approach proves more effective in generating them [12, 59]. Working with fractal-fractional derivatives enables the exploration of fractional operators and fractal dimensions simultaneously. This operator enables the construction of models that accurately capture memory effects in systems. Additionally, a variety of kernels and fractal-fractional operators have been utilized to address the intricacies of fractal-fractional differential equations [16]. Employing a fractional-order system model is essential for observing hereditary characteristics, memory, and crossover behavior [40]. This operator exhibits both fractional calculus properties and a form of self-similarity or fractal-like behavior. In the context of the Caputo operator, the fractal-fractional derivative represents an innovative mathematical tool that merges fractal and fractional calculus, offering enhanced capabilities for modeling and analyzing systems with memory-dependent dynamics.

Numerous studies have employed fractal-fractional models to predict the efficacy of health surveillance, highlighting interdisciplinary synergies. For example, Li [44] explored a mathematical model with bank data using a fractal-fractional Caputo derivative, while Owolabi [50] modeled dynamics of the Human epidermal growth factor receptor 2 breast cancer. Ahmad [8] analyzed tumor-immune interaction models. COVID-19 mathematical models employing fractal-fractional orders have been qualitatively examined [11], alongside the exploration of new fractal-fractional operators to model

COVID-19 spread [20]. Researchers investigating severe acute respiratory syndrome coronavirus 2 transmission dynamics and its link with Alzheimer's disease utilized the CF fractional-order [5]. Fractal-fractional models within the CF framework elucidated the dynamics of two-age structure smokers [3]. Studies have examined a fractal-fractional model and conducted sensitivity analysis of COVID-19 with quarantine and vaccination [47], while Q fever dynamics were analyzed using CF and Atangana-Baleanu-Caputo fractional derivatives [18]. A new fractal-fractional model for Zika virus propagation was proposed, including insecticide-treated nets [4]. Additionally, studies have explored infection dynamics in plants using fractal-fractional derivatives [30], and investigated a new chaotic system utilizing fractal-fractional differential operators [6]. Pine wilt disease models integrated nematodes, transmitting beetles, and symptomatic pine tree dynamics [7].

The selection of a fractional operator, along with its kernel type, should accurately reflect the system's memory effects, crossover behavior, and long-term dynamics. While criticisms exist regarding the use of nonsingular fractional operators, evaluating the choice of the Mittag-Leffler kernel is crucial based on the model's objectives and applicability. The Mittag-Leffler kernel offers advantages in capturing memory effects and long-range dependencies within the system, which are essential for drug abuse modeling. Its mathematical properties enable the representation of complex interactions among various factors influencing drug abuse dynamics, including law enforcement policies, societal attitudes, and individual behaviors. These insights provided the impetus to capture the broad significance of the findings in this research endeavor. The study delves into the dynamics of a drug transmission model incorporating aspects of criminal law, classifying it into six compartments: the susceptible class, individuals receiving drug misuse education, mild drug addicts, high-level drug addicts, individuals subjected to criminal law, and those in the quitter class. The proposed model integrates fractal-fractional order derivatives using the $\mathbb{ABC}$ operator. The investigation commences with a thorough examination of the developed model to ensure its validity, establishing the existence and uniqueness of the solution within the framework of fixed-point theory. Subsequently, Ulam-Hyres stability is applied using nonlinear functional analysis to demonstrate solution stability. Acknowledging the importance of the subject matter, a novel fractional Adam-Bashforth iterative numerical scheme is employed for numerical simulations, with results validated through MATLAB simulations. The numerical findings are then plotted for various fractional orders and fractal dimensions, comparing them to integer orders. The study notes the sensitivity of fixing and changing the fractional order and fractal dimension on the model's characteristics, underscoring the utility of the fractional approach.

We structure this article into seven sections. Section 2 outlines the model's development and its description. Section 3 covers the analytical preliminaries of the techniques and the fractional formulation of the devised model. Section 4 is dedicated to analyzing the existence and uniqueness within the framework of fixed-point theory. Stability analysis is established in Section 5. Section 6 presents numerical simulations to support theoretical results. Finally, the discussion and conclusion of the proposed model are presented in the last section.

2. Model development

2.1. Description of the model

The study underscores the crucial role of mathematical models in comprehending addiction dynamics and developing effective prevention and treatment strategies. Specifically, when constructing models for transmittable diseases such as drug addiction, it is essential to focus on establishing the epidemiology. This process involves classifying key factors decisive for controlling drug addiction. Several addiction models, based on transmission mechanisms, are documented in existing literature, significantly contributing to the formulation of strategies for preventing and treating addiction [14,29].

This study delves into exploring a mathematical model aimed at understanding the spread of drug addiction within the human population, incorporating a criminal law perspective [33]. It highlights key limitations in the model analysis, such as simplifying real-world complexities, potential oversights in parameter selection, reliance on assumed relationships, and the absence of dynamic external factors. These constraints may limit the model's ability to fully capture the complexities of human behavior and societal dynamics, potentially impacting the generalizability of the findings. Thus, caution is advised in interpreting the predictions, as real-world applicability may be constrained by the inherent simplifications in the modeling approach.

There is evidence that the drug abuse phenomenon tends toward younger individuals, and health education encompasses both family and public health education. Assumptions used in model formulation include the presence of mild-level drug addicts in populations with or without drug misuse education, the absence of mild drug addicts among strong drug users, and the imposition of criminal charges on light and heavy narcotics addicts. The model considers the return of drug addicts who faced criminal law to mild and severe addiction, and individuals who stop using drugs cannot be categorized as mild or severe addicts. The natural rates of birth and death are assumed to be consistent, and the age group facing criminal charges for drug addiction is 14 and above. At any time $t \geq 0$, the mathematical model divides the entire population $\mathbb{N}(t)$ into quintuple sets: Susceptible individuals to drug addiction $\mathbb{S}(t)$; individuals susceptible to being narcotic addicts but have received drug misuse education $\mathbb{C}(t)$; mild drug addicts $\mathbb{L}(t)$; heavy-level drug addicts $\mathbb{H}(t)$; individuals undergoing criminal law consequences $\mathbb{W}(t)$; and individuals who have stopped using drugs $\mathbb{R}(t)$.

The mathematical model aims to apprehend the intricate dynamics of drug addiction transmission, considering educational interventions, criminal consequences, and the cessation of drug use. Thus, the total human population at any given time t is expressed as:

$$\mathbb{N}(t) = \mathbb{S}(t) + \mathbb{C}(t) + \mathbb{L}(t) + \mathbb{H}(t) + \mathbb{W}(t) + \mathbb{R}(t).$$

This arrangement allows us to model the dynamics of the drug addiction model by incorporating the aspect of criminal penalties. This is achieved by tracking the transitions between these different states within the population over time. Therefore, the foundational transmission dynamic model of drug addiction in the human population is as follows:

$$
\begin{aligned}
\frac{d\mathbb{S}}{dt} &= (1-q)d\Lambda - \beta_1\mathbb{S}\mathbb{L} - \beta_2\mathbb{S}\mathbb{H} - (d+\mu)\mathbb{S},\\
\frac{d\mathbb{C}}{dt} &= qd\Lambda + \mu\mathbb{S} - \beta_1\xi\mathbb{C}\mathbb{L} - \beta_2\xi\mathbb{C}\mathbb{H} - (d+\delta)\mathbb{C},\\
\frac{d\mathbb{L}}{dt} &= \beta_1\mathbb{S}\mathbb{L} + \beta_2\mathbb{S}\mathbb{H} + \beta_1\xi\mathbb{C}\mathbb{L} + \beta_2\xi\mathbb{C}\mathbb{H} - (\gamma+\pi+d+d_1)\mathbb{L} + \alpha\mathbb{W},\\
\frac{d\mathbb{H}}{dt} &= \pi\mathbb{L} + \sigma\mathbb{W} - (\theta+d+d_2)\mathbb{H},\\
\frac{d\mathbb{W}}{dt} &= \theta\mathbb{H} + \gamma\mathbb{L} - (\alpha+\sigma+m+d)\mathbb{W},\\
\frac{d\mathbb{R}}{dt} &= m\mathbb{W} + \delta\mathbb{C} - d\mathbb{R},
\end{aligned}
\tag{2.1}
$$

in accordance with the conditions:

$$\mathbb{S}(0) = \mathbb{S}_0 \geq 0,\ \ \mathbb{C}(0) = \mathbb{C}_0 \geq 0,\ \mathbb{L}(0) = \mathbb{L}_0 \geq 0,\ \ \mathbb{H}(0) = \mathbb{H}_0 \geq 0,\ \ \mathbb{W}(0) = \mathbb{W}_0 \geq 0,\ \ \mathbb{R}(0) = \mathbb{R}_0 \geq 0.$$

Moreover, Table 1 comprehensively delineates the specifics of each parameter within the equations governing the system as depicted in model (2.1).

Table 1. The model parameters proposed are described as follows.

Parameter	Description
d	The natural birth and death rate
q	The probability of receiving drug misuse education
d_2	The death rate of heavy drug addicts
ξ	The probability of a direct control level decrease
α	The transition rate from individuals undergoing criminal law to mild drug addicts
d_1	The death rate of mild drug addicts
Λ	The size of the recruitment rate
θ	The treatment rate from heavy addicts to individuals undergoing criminal law
γ	The transition rate for mild drug addicts to individuals experiencing criminal law
β_1	The effective contact rate for susceptible individuals with light drug addicts
β_2	The effective contact rate for susceptible individuals with heavy drug addicts
μ	The rate at which susceptible individuals accept drug misuse education
δ	The progression rate from individuals with misuse education to those who stopped using drugs
π	The transformation rate from light to heavy drug addicts
m	The permanent quit drug rate after treatment
σ	The relapse rate after treatment

3. Fractional drugs addiction model

Traditional integer-order models often fall short in capturing the dynamic behavior of addiction. Fractional-order models, on the other hand, offer better suitability for real-world data, providing a more nuanced illustration of complex phenomena. In the context of modeling addiction within the given system (2.1), we enhance the framework by replacing the conventional integer-order time derivative D_t with a fractal-fractional order derivative. This modification allows for a more detailed exploration of the long-term memory impact of addiction dynamics. The nuanced approach provides a more precise illustration, especially crucial for navigating the inherent complexities and evolving patterns observed in real-world addiction dynamics.

Definition 3.1. *[20] Consider a continuous and differentiable function $\mathbb{V}(t)$ over the interval (c, d), characterized by a fractional order $0 < w \leq 1$ and a fractal dimension $0 < r \leq 1$. This function can be defined in an $\mathbb{ABC}$ sense as:*

$$ {}^{\mathbb{ABC}}D^{w,r}(\mathbb{V}(t)) = \frac{\mathbb{ABC}(w)}{(1-w)} \frac{d}{ds^r} \int_0^t \mathbb{V}(s) \mathcal{k}_w \Big[\frac{-\sigma}{(1-w)} \Big(t - sw \Big)^w \Big] ds. $$

In this context, $\mathbb{ABC}(w)$ represents a "normalization mapping" defined by $\mathbb{ABC}(0) = \mathbb{ABC}(1) = 1$, which serves as the normalization constant. Here, $\mathcal{k}_w$ denotes a well-established mapping referred to as the "Mittag-Leffler" function, encompassing the exponent mapping as a special case [49].

Definition 3.2. *[20] Consider a continuous function $\mathbb{V}(t)$ defined over the interval (c, d). The fractal-fractional order integral of the function $\mathbb{V}(t)$, characterized by a fractal order $0 < w \leq 1$ and a fractal dimension $0 < r \leq 1$, can be expressed in the $\mathbb{ABC}$ sense as:*

$$ {}^{\mathbb{ABC}}I_0^w(\mathbb{V}(t)) = \frac{(1-w)}{\mathbb{ABC}(w)} t^{r-1} \mathbb{V}(t) + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)} \int_0^t (t-s)^{w-1} s^{r-1} \mathbb{V}(s) ds. \tag{3.1} $$

Lemma 3.1. *[1] Let's express the solution to the provided problem considering $0 < r, w \leq 1$,*

$$ \begin{aligned} &{}^{\mathbb{ABC}}D_0^w(\Omega(t)) = rt^{r-1}\mathbb{Y}(t, \Omega(t)), \ \ t \in [0, T], \\ &\Omega(0) = \Omega_0, \ 0 < w, r \leq 1, \end{aligned} $$

as provided by

$$ \Omega(t) = \Omega_0 + \frac{(1-w)}{\mathbb{ABC}(w)} t^{r-1} \mathbb{Y}(t, \Omega(t)) + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)} \int_0^t (t-s)^{w-1} s^{r-1} \mathbb{Y}(s, \Omega(s)) ds. $$

Definition 3.3. *[32] (Contractions Mapping). Suppose $\mathcal{B}$ is a Banach space, then the operator $\mathcal{T}$: $\mathbb{X} \to \mathbb{X}$ is a contraction if*

$$ \|\mathcal{T}(x) - \mathcal{T}(y)\| \leq \mathcal{M}\|x - y\|, \quad \forall x, y, \in \mathbb{X}, \ 0 < \mathcal{M} < 1. $$

Lemma 3.2. *[32] (Banach's fixed point theorem). If a Banach space $\mathcal{B}$ contains a nonempty open subset $\mathcal{D}$, then any contraction mapping q from $\mathcal{D}$ into itself possesses a unique fixed point.*

Lemma 3.3. *[32] (Krasnoselskii's fixed points theorem). The Banach space $\mathcal{B}$ has a nonempty, closed, convex subset $\mathcal{D}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two operators satisfying the following conditions:*

(i) *$\mathcal{F}_1 x + \mathcal{F}_2 y \in \mathcal{D}, \forall x, y \in \mathcal{D}$.*

(ii) *The operator $\mathcal{F}_1$ is both compact and continuous.*

(*iii*) *The operator $\mathcal{F}_2$ is a contraction mapping.*
Thus, there exists $v \in \mathcal{D}$ such that $\mathcal{F}_1 v + \mathcal{F}_2 v = v$.

Theorem 3.4. *[42] Every sequence that contracts is a Cauchy sequence, thus ensuring convergence in a complete metric space.*

Theorem 3.5. *[42] Let $B \subseteq \mathcal{R}$ and $\Psi : B \to \mathcal{R}^n$ be continuously differentiable mappings, where $s \in B$, then for each compact subset $\mathcal{B}$ of B, Ψ satisfies a Lipschitz condition with a Lipschitz constant denoted by L. In this context, $L > 0$ represents the supremum of the derivative of Ψ on $\mathcal{B}$, i.e.,*

$$L = \sup_{s \in \mathcal{B}} \left| \frac{d\Psi}{ds} \right|.$$

Following this, for $t \geq 0$, the resulting nonlinear deterministic mathematical model (2.1) for drug addiction is introduced. This model integrates the fractal-fractional order derivative employing the $\mathbb{ABC}$ fractional order operator, characterized by a fractional order $0 < w \leq 1$ and dimension $0 < r \leq 1$.

$$\begin{aligned}
{}^{\mathbb{ABC}}D_t^{w,r}\mathbb{S}(t) &= (1-q)d\Lambda - \beta_1\mathbb{S}\mathbb{L} - \beta_2\mathbb{S}\mathbb{H} - (d+\mu)\mathbb{S}, \\
{}^{\mathbb{ABC}}D_t^{w,r}\mathbb{C}(t) &= qd\Lambda + \mu\mathbb{S} - \beta_1\xi\mathbb{C}\mathbb{L} - \beta_2\xi\mathbb{C}\mathbb{H} - (d+\delta)\mathbb{C}, \\
{}^{\mathbb{ABC}}D_t^{w,r}\mathbb{L}(t) &= \beta_1\mathbb{S}\mathbb{L} + \beta_2\mathbb{S}\mathbb{H} + \beta_1\xi\mathbb{C}\mathbb{L} + \beta_2\xi\mathbb{C}\mathbb{H} - m_1\mathbb{L} + \alpha\mathbb{W}, \\
{}^{\mathbb{ABC}}D_t^{w,r}\mathbb{H}(t) &= \pi\mathbb{L} + \sigma\mathbb{W} - m_2\mathbb{H}, \\
{}^{\mathbb{ABC}}D_t^{w,r}\mathbb{W}(t) &= \theta\mathbb{H} + \gamma\mathbb{L} - m_3\mathbb{W}, \\
{}^{\mathbb{ABC}}D_t^{w,r}\mathbb{R}(t) &= m\mathbb{W} + \delta\mathbb{C} - d\mathbb{R},
\end{aligned} \tag{3.2}$$

with the initial conditions

$$\mathbb{S}(0) = \mathbb{S}_0 \geq 0, \ \ \mathbb{C}(0) = \mathbb{C}_0 \geq 0, \mathbb{L}(0) = \mathbb{L}_0 \geq 0, \ \ \mathbb{H}(0) = \mathbb{H}_0 \geq 0, \mathbb{W}(0) = \mathbb{W}_0 \geq 0, \ \ \mathbb{R}(0) = \mathbb{R}_0 \geq 0, \ \ 0 < w, r \leq 1.$$

Moreover, for the sake of simplicity, we establish the following definitions:

$$\begin{aligned}
m_1 &= \gamma + \pi + d + d_1, \\
m_2 &= \theta + d + d_2, \\
m_3 &= \alpha + \sigma + m + d.
\end{aligned}$$

The system (3.2) is autonomous, allowing it to be expressed in a concise form:

$$\begin{cases} {}^{\mathbb{ABC}}D_t^{w,r}Y(t) = G(Y(t)), \quad 0 < t < t_f < +\infty, \\ Y(0) = Y_0. \end{cases} \tag{3.3}$$

Here, $Y : [0, +\infty) \to \mathcal{R}^6$ and $G : \mathcal{R}^6 \to \mathcal{R}^6$ are vector-valued functions defined as:

$$Y(t) = \begin{pmatrix} \mathbb{S}(t) \\ \mathbb{C}(t) \\ \mathbb{L}(t) \\ \mathbb{H}(t) \\ \mathbb{W}(t) \\ \mathbb{R}(t) \end{pmatrix}, \ \ Y_0 = \begin{pmatrix} \mathbb{S}_0 \\ \mathbb{C}_0 \\ \mathbb{L}_0 \\ \mathbb{H}_0 \\ \mathbb{W}_0 \\ \mathbb{R}_0 \end{pmatrix}, \ \ G(Y(t)) = \begin{pmatrix} (1-q)d\Lambda - \beta_1\mathbb{S}\mathbb{L} - \beta_2\mathbb{S}\mathbb{H} - (d+\mu)\mathbb{S} \\ qd\Lambda + \mu\mathbb{S} - \beta_1\xi\mathbb{C}\mathbb{L} - \beta_2\xi\mathbb{C}\mathbb{H} - (d+\delta)\mathbb{C} \\ \beta_1\mathbb{S}\mathbb{L} + \beta_2\mathbb{S}\mathbb{H} + \beta_1\xi\mathbb{C}\mathbb{L} + \beta_2\xi\mathbb{C}\mathbb{H} - (\gamma+\pi+d+d_1)\mathbb{L} + \alpha\mathbb{W} \\ \pi\mathbb{L} + \sigma\mathbb{W} - (\theta+d+d_2)\mathbb{H} \\ \theta\mathbb{H} + \gamma\mathbb{L} - (\alpha+\sigma+m+d)\mathbb{W} \\ m\mathbb{W} + \delta\mathbb{C} - d\mathbb{R} \end{pmatrix}.$$

Theorem 3.6. *The function $G(Y)$ in (3.3) exhibits Lipschitz continuity.*

Proof. Let E denote a convex compact subset of

$$F = \left\{(t, Y) \mid 0 \le t \le t_f,\ Y \in \mathcal{R}_+^6 \right\}.$$

Let $Y_1, Y_2 \in E$, then according to the mean value theorem, there exists $V \in (Y_1, Y_2)$ such that,

$$\frac{G(Y_1(t)) - G(Y_2(t))}{Y_1(t) - Y_2(t)} = G'(V(t)),$$

or

$$\begin{aligned} G(Y_1(t)) - G(Y_2(t)) &= G'(V(t)) \cdot (Y_1(t) - Y_2(t)), \\ \mid G(Y_1(t)) - G(Y_2(t)) \mid &= \mid G'(V(t)) \cdot (Y_1(t) - Y_2(t)) \mid, \\ &\le \| G'(V) \|_\infty \ \|Y_1 - Y_2\|_\infty. \end{aligned}$$

As $G \in C^1[0, t_f]$ over the convex compact set E, there exists a positive constant τ such that:

$$\| G'(V) \|_\infty \le \tau.$$

Hence,

$$\begin{aligned} &\mid G(Y_1(t)) - G(Y_2(t)) \mid \le \tau\|Y_1 - Y_2\|_\infty, \\ &\sup_{t\in[0, t_f]} \mid G(Y_1) - G(Y_2) \mid \le \tau\|Y_1 - Y_2\|_\infty, \\ &\| G(Y_1) - G(Y_2) \|_\infty \le \tau\|Y_1 - Y_2\|_\infty. \end{aligned}$$

Thus, $G(V)$ is Lipschitz. □

4. Theoretical insights into the devised model

In this section, we thoroughly analyze the devised model, exploring its key characteristics and ensuring its suitability for numerical approximations. To affirm the robustness of our model, we establish the existence and uniqueness of its solution using definitions and theorems from [21]. This comprehensive theoretical analysis not only sheds light on the behavior of the model, but also confirms its suitability for rigorous numerical investigations.

Now, we will investigate the existence of the proposed model. Let's define a Banach space $\mathcal{B} = \mathbb{Y}\times\mathbb{Y}\times\mathbb{Y}\times\mathbb{Y}\times\mathbb{Y}\times\mathbb{Y}$, where $\mathbb{Y} = H([0, T], \mathcal{R})$, representing the function space, and the norm is defined as $\|\Omega\| = \max_{t\in[0,T]}\left|\Omega(t)\right|$. Specifically,

$$\|\Omega\| = \|\mathbb{S}, \mathbb{C}, \mathbb{L}, \mathbb{H}, \mathbb{W}, \mathbb{R}\| = \max_{t\in[0,T]}\{|\mathbb{S}(t)| + |\mathbb{C}(t)| + |\mathbb{L}(t)| + |\mathbb{H}(t)| + |\mathbb{W}(t)| + |\mathbb{R}(t)|\}.$$

To establish the existence and uniqueness of the formulated model (3.2), we apply the fixed-point theorem. For this purpose, we note that the integral is differentiable, allowing us to express the given model (3.2) as:

$$\begin{aligned} {}^{\mathbb{ABC}}\mathcal{D}^w(\mathbb{S}(t)) &= rt^{r-1}G_1(\mathbb{S}(t), t) = (1 - q)d\Lambda - \beta_1\mathbb{S}\mathbb{L} - \beta_2\mathbb{S}\mathbb{H} - (d + \mu)\mathbb{S}, \\ {}^{\mathbb{ABC}}\mathcal{D}^w(\mathbb{C}(t)) &= rt^{r-1}G_2(\mathbb{C}(t), t) = qd\Lambda + \mu\mathbb{S} - \beta_1\xi\mathbb{C}\mathbb{L} - \beta_2\xi\mathbb{C}\mathbb{H} - (d + \delta)\mathbb{C}, \\ {}^{\mathbb{ABC}}\mathcal{D}^w(\mathbb{L}(t)) &= rt^{r-1}G_3(\mathbb{L}(t), t) = \beta_1\mathbb{S}\mathbb{L} + \beta_2\mathbb{S}\mathbb{H} + \beta_1\xi\mathbb{C}\mathbb{L} + \beta_2\xi\mathbb{C}\mathbb{H} - m_1\mathbb{L} + \alpha\mathbb{W}, \\ {}^{\mathbb{ABC}}\mathcal{D}^w(\mathbb{H}(t)) &= rt^{r-1}G_4(\mathbb{H}(t), t) = \pi\mathbb{L} + \sigma\mathbb{W} - m_2\mathbb{H}, \\ {}^{\mathbb{ABC}}\mathcal{D}^w(\mathbb{W}(t)) &= rt^{r-1}G_5(\mathbb{W}(t), t) = \theta\mathbb{H} + \gamma\mathbb{L} - m_3\mathbb{W}, \\ {}^{\mathbb{ABC}}\mathcal{D}^w(\mathbb{R}(t)) &= rt^{r-1}G_6(\mathbb{R}(t), t) = m\mathbb{W} + \delta\mathbb{C} - d\mathbb{R}. \end{aligned} \tag{4.1}$$

The system (4.1) is autonomous; hence, it can be encapsulated within a compact framework as:

$$\begin{cases} {}^{\mathbb{ABC}}\mathcal{D}_0^w(\Omega(t)) = rt^{r-1}\mathbb{Y}(t,\Omega(t)), t \in [0,T], \\ \Omega(0) = \Omega_0, \quad 0 < w, r \leq 1, \end{cases} \tag{4.2}$$

and the solution to Eq (4.2) can be expressed as:

$$\Omega(t) = \Omega_0 + \frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1}\mathbb{Y}(t,\Omega(t)) + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (t-s)^{w-1}s^{r-1}\mathbb{Y}(s,\Omega(s))ds, \tag{4.3}$$

where

$$\Omega(t) = \begin{pmatrix} \mathbb{S}(t) \\ \mathbb{C}(t) \\ \mathbb{L}(t) \\ \mathbb{H}(t) \\ \mathbb{W}(t) \\ \mathbb{R}(t) \end{pmatrix}, \quad \Omega_0 = \begin{pmatrix} \mathbb{S}_0 \\ \mathbb{C}_0 \\ \mathbb{L}_0 \\ \mathbb{H}_0 \\ \mathbb{W}_0 \\ \mathbb{R}_0 \end{pmatrix}, \quad \mathbb{Y}(t,\Omega(t)) = \begin{pmatrix} \mathcal{G}_6(\mathbb{S}, \mathbb{C}, \mathbb{L}, \mathbb{H}, \mathbb{W}, \mathbb{R}, t) \\ \mathcal{G}_6(\mathbb{S}, \mathbb{C}, \mathbb{L}, \mathbb{H}, \mathbb{W}, \mathbb{R}, t) \\ \mathcal{G}_6(\mathbb{S}, \mathbb{C}, \mathbb{L}, \mathbb{H}, \mathbb{W}, \mathbb{R}, t) \\ \mathcal{G}_6(\mathbb{S}, \mathbb{C}, \mathbb{L}, \mathbb{H}, \mathbb{W}, \mathbb{R}, t) \\ \mathcal{G}_6(\mathbb{S}, \mathbb{C}, \mathbb{L}, \mathbb{H}, \mathbb{W}, \mathbb{R}, t) \\ \mathcal{G}_6(\mathbb{S}, \mathbb{C}, \mathbb{L}, \mathbb{H}, \mathbb{W}, \mathbb{R}, t) \end{pmatrix}.$$

Let's reformat system (3.2) into a fixed-point form. We define the mapping $\mathcal{T} : \mathcal{V} \to \mathcal{V}$ as follows:

$$\mathcal{T}(\Omega)(t) = \Omega_0 + \frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1}\mathbb{Y}(t,\Omega(t)) + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (t-s)^{w-1}s^{r-1}\mathbb{Y}(s,\Omega(s))ds. \tag{4.4}$$

Assume $\mathcal{T} = \boldsymbol{F} + G$, where

$$\begin{aligned} \boldsymbol{F}(\Omega) &= \Omega_0 + \frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1}\mathbb{Y}(t,\Omega(t)), \\ G(\Omega) &= \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (t-s)^{w-1}s^{r-1}\mathbb{Y}(s,\Omega(s))ds. \end{aligned} \tag{4.5}$$

Next, we will demonstrate the qualitative analysis of the given system by employing the fixed point theory:

$(V1)$ There exists a constant $L_{\mathbb{Y}}, M_{\mathbb{Y}}$, such that

$$|\mathbb{Y}(t,\Omega(t))| \leq \ L_{\mathbb{Y}}|\Omega| + M_{\mathbb{Y}}.$$

$(V2)$ There exists a positive constant $\mathbb{L}_{\mathbb{Y}}$ such that for any $\Omega, \bar{\Omega} \in \mathcal{B}$ provided that,

$$|\mathbb{Y}(t,\Omega) - \mathbb{Y}(t,\bar{\Omega})| \leq \mathbb{L}_{\mathbb{Y}}[|\Omega - \bar{\Omega}|].$$

Theorem 4.1. *If conditions (V_1, V_2) are satisfied, the system (4.3) has at least one solution, then the system (3.2) produces an equivalent number of solutions under the condition that* $\frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1}\mathbb{L}_{\mathbb{Y}} < 1$.

Proof. We establish the theorem through two outlined steps as follows:

Step I. Let $\bar{\Omega} \in \mathcal{V}$, where $\mathcal{V} = \{\Omega \in \mathcal{B} : \|\Omega\| \leq \phi, \phi > 0\}$ represents a convex closed set. Therefore, for the operator $\boldsymbol{F}$ defined in (4.5), one has:

$$\begin{aligned} \|\boldsymbol{F}(\Omega) - \boldsymbol{F}(\bar{\Omega})\| &= \frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1}\max_{t\in[0,T]}|\mathbb{Y}(t,\Omega(t)) - \mathbb{Y}(t,\bar{\Omega}(t))|, \\ &\leq \frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1}\mathbb{L}_{\mathbb{Y}}\|\Omega - \bar{\Omega}\|. \end{aligned} \tag{4.6}$$

Hence, the operator $\boldsymbol{F}$ is closed and, therefore, a contraction.

Step II. We will now verify the relative compactness of the operator G and show its continuity and boundedness. It is apparent that the operator G is defined over the entire domain, since $\mathbb{Y}$ is continuous. Moreover, for $\Omega \in \mathcal{V}$, we have:

$$\begin{aligned}
\|G(\Omega)\| &= \max_{t\in[0,\tau]}\Big|\frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (t-s)^{w-1}s^{r-1}\mathbb{Y}(s,\Omega(s))ds\Big|, \\
&\le \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (s)^{w-1}(1-s)^{r-1}|\mathbb{Y}(s,\Omega(s))|ds, \\
&\le \frac{r[L_{\mathbb{Y}}|\Omega| + M_{\mathbb{Y}}T^{w+r-1}]}{\mathbb{ABC}(w)\Gamma(w)}[\mathbf{B}(w,r)].
\end{aligned} \tag{4.7}$$

The symbol $\mathbf{B}(w,,r)$ represents the beta function, respectively. Therefore, considering Eq (4.7), the operator G is bounded. Continuing, for "equi-continuity," suppose $t_1 > t_2 \in [0,\tau]$. We have:

$$\begin{aligned}
&|G(\Omega(t_2)) - G(\Omega(t_1))| = \\
&\frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\Big|\int_0^{t_2}(t_2-x)^{w-1}x^{r-1}\mathbb{Y}(x,\Omega(x))dx - \int_0^{t_1}(t_1-x)^{w-1}x^{r-1}\mathbb{Y}(x,\Omega(x))dx, \\
&\le \frac{r[L_{\mathbb{Y}}|\Omega| + M_{\mathbb{Y}}T^{w+r-1}]\,\mathbf{B}(w,r)}{\mathbb{ABC}(w)\Gamma(w)}[t_2^w - t_1^w].
\end{aligned} \tag{4.8}$$

As t_2 approaches t_1, the righthand side of (4.8) tends to zero. Furthermore, owing to the continuity of the operator G, we have:

$$|G(\Omega(t_2)) - G(\Omega(t_1))| \to 0, \text{as } t_2 \to t_1.$$

Therefore, we have shown that G is both bounded and continuous, making it uniformly continuous. According to Arzela-Ascoli's theorem, a subset $\Omega \in \mathcal{V}$ of G is compact if, and only if, it is closed, bounded, and equicontinuous. Since G is relatively compact and completely continuous, it fulfills these conditions. Considering Eqs (3.2) and (4.3), we conclude that the system has at least one solution. □

To establish the uniqueness of the solution for the model (3.2), we utilize the fixed-point method outlined in [32].

Theorem 4.2. *Under assumption* ($V2$) *and the uniqueness of solution for (4.3), we assert that the system (3.2) also possesses a unique solution if the condition* $\left[\frac{(1-w)t^{r-1}\mathbb{L}_{\mathbb{Y}}}{\mathbb{ABC}(w)} + \frac{r[\mathbb{L}_{\mathbb{Y}}T^{w+r+1}]\mathbf{B}(w,r)}{\mathbb{ABC}(w)\Gamma(w)}\right] < 1$ *is satisfied.*

Proof. Let the operator $\mathcal{T}: \mathcal{V} \to \mathcal{V}$ by

$$\mathcal{T}(\Omega)(t) = \Omega_0(t) + [\mathbb{Y}(t,\Omega(t)) - \mathbb{Y}_0(t)]\frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1} + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (t-x)^{w-1}t^{r-1}\mathbb{Y}(x,\Omega(x))dx,\ t\in[0,\tau].$$

Let $\Omega, \bar{\Omega} \in \mathcal{V}$, then

$$\begin{aligned}
\|\mathcal{T}(\Omega) - \mathcal{T}(\bar{\Omega})\| &\le \frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1}\max_{t\in[0,\tau]}\Big|\mathbb{Y}(t,\Omega(t)) - \mathbb{Y}(t,\bar{\Omega}(t))\Big| + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)} \\
&\max_{t\in[0,\tau]}\Big|\int_0^t (t-x)^{w-1}t^{r-1}\mathbb{Y}(x,\Omega(x))dx - \int_0^t (t-x)^{w-1}t^{r-1}\mathbb{Y}(x,\bar{\Omega}(x))dx, \\
&\le \Theta\|\Omega - \bar{\Omega}\|,
\end{aligned} \tag{4.9}$$

and

$$\Theta = \left[\frac{(1-w)t^{r-1}\mathbb{L}_{\mathbb{Y}}}{\mathbb{ABC}(w)} + \frac{r[\mathbb{L}_{\mathbb{Y}}T^{w+r+1}]\mathbf{B}(w,r)\mathbb{L}_{\mathbb{Y}}}{\mathbb{ABC}(w)\Gamma(w)}\right]. \tag{4.10}$$

Upon examining (4.9), it becomes evident that the operator $\mathcal{T}$ acts as a contraction. Consequently, Eq (4.3) possesses a unique solution. Therefore, the system (3.2) under consideration also has a unique solution. □

5. Ulam-Hyers stability

The objective of this section is to establish the Ulam-Hyers (UH) stability of the proposed model (3.2).

Definition 5.1. *The suggested model is UH stable if there exists a $\aleph_{w,r} > 0$ such that for any $\varrho > 0$ and for every $\Omega \in C([0,T],\mathcal{R})$, it fulfills the following condition:*

$$\left|{}^{\mathbb{ABC}}D_t^{w,r}\Omega(t) - \psi(t,\Omega(t))\right| \leq \varrho, \ \ t \in [0,T], \tag{5.1}$$

and there exists a unique solution $\phi \in C([0,T],\mathcal{R})$ such that

$$\left|\Omega - \phi(t)\right| \leq \aleph_{w,r}\,\varrho, \ \ t \in [0,T]. \tag{5.2}$$

Let's consider a small perturbation $\phi(t) \in C([0,T],\mathcal{R})$ such that $\phi(0) = 0$. Let's define

- $|\phi(t)| \leq \varrho$, for $\varrho > 0$;
- ${}^{\mathbb{ABC}}D_t^{w,r}\Omega(t) = \mathbb{Y}(t,\Omega(t)) + \phi(t)$.

Lemma 5.1. *The solution to the perturbed problem*

$$\begin{aligned} &{}^{\mathbb{ABC}}D_t^{w,r}\Omega(t) = \mathbb{Y}(t,\Omega(t)) + \psi(t), \\ &\Omega(0) = \Omega_0, \end{aligned} \tag{5.3}$$

satisfies the following relation:

$$\begin{aligned} &\left|\Omega(t) - \left(\Omega_0(t) + [\mathbb{Y}(t,\Omega(t)) - \psi_0(t)]\frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1} + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (t-x)^{w-1}x^{r-1}\mathbb{Y}(x,\Omega(x))dx\right)\right|, \\ &\leq \frac{\Gamma(w)t^{r-1} + rT^{w+r-1}}{\mathbb{ABC}(w\Gamma(w)}\mathbf{B}(w,r)\varrho = \varpi_{w,r}\varrho. \end{aligned} \tag{5.4}$$

Proof. For the sake of simplicity, we will not delve into the proof. □

Theorem 5.2. *Under assumption $(V2)$ and Eq (5.4), the solution to Eq (4.3) exhibits UH stability. Consequently, the analytical solution to the proposed system achieves UH stability if $\Theta < 1$.*

Proof. Let $\Omega \in \mathcal{V}$ denote a unique solution and $\bar{\Omega} \in \mathcal{V}$ be any solution of Eq (4.3), then

$$
\begin{aligned}
&|\Omega(t) - \bar{\Omega}(t)| = \bigg|\Omega(t) - \bigg(\Omega_0(t) + [\mathbb{Y}(t, \bar{\Omega}(t)) - \mathbb{Y}_0(t)]\frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1} \\
&+ \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (t-x)^{w-1}x^{r-1}\mathbb{Y}(x, \bar{\Omega}(x))dx\bigg)\bigg|, \\
&\leq \bigg|\Omega(t) - \bigg(\Omega_0(t) + [\mathbb{Y}(t, \Omega(t)) - \mathbb{Y}_0(t)]\frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1} + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (t-x)^{w-1}x^{r-1}\mathbb{Y}(x, \Omega(x))dx\bigg)\bigg| \\
&+ \bigg|\bigg(\Omega_0(t) + [\mathbb{Y}(t, \Omega(t)) - \mathbb{Y}_0(t)]\frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1} + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (t-x)^{w-1}x^{r-1}\mathbb{Y}(x, \Omega(x))dx\bigg) \\
&- \bigg(\Omega_0(t) + [\mathbb{Y}(t, \bar{\Omega}(t)) - \mathbb{Y}_0(t)]\frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1} + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (t-x)^{w-1}x^{r-1}\mathbb{Y}(x, \bar{\Omega}(x))dx\bigg)\bigg|, \\
&\leq \mathcal{Y}_{w,r} + \frac{(1-w)}{\mathbb{ABC}(w)}\mathbb{L}_{\mathbb{Y}}t^{r-1}\|\Omega - \bar{\Omega}\| + \frac{rT^{w+r-1}}{\mathbb{ABC}(w)\Gamma(w)}\mathbf{B}(w, r)\|\Omega - \bar{\Omega}\|, \\
&\leq \mathcal{Y}_{w,r} + \Theta\|\Omega - \bar{\Omega}\|.
\end{aligned}
\tag{5.5}
$$

From Eq (5.5), we can express it as follows:

$$
|\Omega(t) - \bar{\Omega}(t)| \leq \frac{\mathcal{Y}_{w,r}}{1-\Theta}|\Omega(t) - \bar{\Omega}(t)|. \tag{5.6}
$$

Based on Eq (5.6), we deduce that the solution to (4.3) exhibits UH stability, and, consequently, generalized UH stability can be established using $\mathbb{Y}_{\Omega}(\varrho) = \varpi_{w,r}\varrho$ with $\mathbb{Y}_{\Omega}(0) = 0$. This demonstrates that the solution to the proposed problem is both UH stable and generalized UH stable. □

6. Numerical technique for fractal-fractional model

In this segment, we aim to compute the numerical solutions for the system (3.2) with arbitrary fractal orders using the $\mathbb{ABC}$ derivative, a well-known technique in fractal-fractional calculus. We employ iterative schemes to approximate the solution for the given model. To achieve this, we rely on fractal-fractional Atangana-Baleanu techniques [56] to obtain an approximate solution for plotting the system (3.2). Consequently, we proceed with the expression (4.1) as follows:

$$
\begin{aligned}
{}^{\mathbb{ABC}}\mathcal{D}^{w}(\mathbb{S}(t)) &= rt^{r-1}G_1(\mathbb{S}(t), t), \\
{}^{\mathbb{ABC}}\mathcal{D}^{w}(\mathbb{C}(t)) &= rt^{r-1}G_2(\mathbb{C}(t), t), \\
{}^{\mathbb{ABC}}\mathcal{D}^{w}(\mathbb{L}(t)) &= rt^{r-1}G_3(\mathbb{L}(t), t), \\
{}^{\mathbb{ABC}}\mathcal{D}^{w}(\mathbb{H}(t)) &= rt^{r-1}G_4(\mathbb{H}(t), t), \\
{}^{\mathbb{ABC}}\mathcal{D}^{w}(\mathbb{W}(t)) &= rt^{r-1}G_5(\mathbb{W}(t), t), \\
{}^{\mathbb{ABC}}\mathcal{D}^{w}(\mathbb{R}(t)) &= rt^{r-1}G_6(\mathbb{R}(t), t).
\end{aligned}
\tag{6.1}
$$

The symbols G_i, $i = 1, 2, 3, ..., 6$ are defined in (4.1). Next, by applying the fractal-fractional integral in the $\mathbb{ABC}$ sense to the first equation of (4.1), we obtain:

$$
\mathbb{S}(t) - \mathbb{S}(0) = \frac{(1-w)}{\mathbb{ABC}(w)}t^{r-1}[G_1(\mathbb{S}(t), t)] + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^t (t-x)^{w-1}x^{r-1}G_1(\mathbb{S}(x), x)dx. \tag{6.2}
$$

Now, we present the numerical solution of Eq (6.2) using the new approach for discrete time instances $t = t_{k+1},\ k = 0, 1, 2, ...,$. The first equation of the system described above is then expressed accordingly

$$\begin{aligned}\mathbb{S}(t_{k+1}) - \mathbb{S}(0) &= \frac{(1-w)}{\mathbb{ABC}(w)}(t_{k+1}^{r-1})[G_1(\mathbb{S}(t_k), t_k)] + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\int_0^{t_{k+1}}(t_{k+1}-x)^{w-1}x^{r-1}G_1(\mathbb{S}(x), x)dx,\\ &= \frac{(1-w)}{\mathbb{ABC}(w)}(t_{k+1}^{r-1})[G_1(\mathbb{S}(t_k), t_k)] + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\sum_{p=0}^{k}\int_p^{t_{p+1}}(t_{k+1}-x)^{w-1}x^{r-1}G_1(\mathbb{S}(x), x)dx.\end{aligned} \tag{6.3}$$

Next, we estimated the function G_1 over the interval $[t_p, t_{p+1}]$ using the interpolation polynomial as follows:

$$G_1 \cong \frac{G_1}{\Delta}(t - t_{p-1}) - \frac{\mathbf{R}_1}{\Delta}(t - t_p), \tag{6.4}$$

which suggests that

$$\begin{aligned}\mathbb{S}(t_{k+1}) =&\mathbb{S}(0) + \frac{(1-w)}{\mathbb{ABC}(w)}(t_{k+1}^{r-1})[G_1(\mathbb{S}(t_k), t_k)] + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\sum_{p=0}^{k}\Big(\frac{G_1(\mathbb{S}(t_k), t_k)}{\Delta}\\ &\times\int_p^{t_{p+1}}(t-t_{p-1})(t_{p+1}-t)^{w-1}t_p^{r-1}dt - \frac{G_1(\mathbb{S}(t_k), t_k)}{\Delta}\int_p^{t_{p+1}}(t-t_p)(t_{k+1}-t)^{w-1}t_p^{r-1}dt\Big),\\ \mathbb{S}(t_{k+1}) =&\mathbb{S}(0) + \frac{(1-w)}{\mathbb{ABC}(w)}(t_{k+1}^{r-1})[G_1(\mathbb{S}(t_k), t_k)] + \frac{rw}{\mathbb{ABC}(w)\Gamma(w)}\sum_{p=0}^{k}\Big(\frac{t_p^{r-1}G_1(\mathbb{S}(t_p), t_p)}{\Delta}I_{p-1,w}\\ &- \frac{t_{p-1}^{r-1}G_1(\mathbb{S}(t_{p-1}), t_{p-1})}{\Delta}I_{p,w}\Big).\end{aligned} \tag{6.5}$$

Now, computing $I_{p-1,w}$ and $I_{p,w}$, we derive:

$$\begin{aligned}I_{p-1,w} &= \int_p^{t_{p+1}}(t-t_{p-1})(t_{k+1}-t)^{w-1}dt,\\ &= -\frac{1}{w}\Big[(t_{p+1}-t_{p-1})(t_{k+1}-t_{p+1})^{\sigma} - (t_p - t_{p-1})(t_{k+1}-t_p)^{\sigma}\Big],\\ &- \frac{1}{w(w-1)}\Big[(t_{k+1}-t_{p+1})^{w+1} - (t_{k+1}-t_p)^{w+1}\Big],\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}I_{p,w} &= \int_p^{t_{p+1}}(t-t_p)(t_{k+1}-t)^{w-1}dt,\\ &= -\frac{1}{w}\Big[(t_{p+1}-t_p)(t_{k+1}-t_{p+1})^{w}\Big]\\ &- \frac{1}{w(w-1)}\Big[(t_{k+1}-t_{p+1})^{w+1} - (t_{k+1}-t_p)^{\sigma+1}\Big].\end{aligned} \tag{6.7}$$

Putting $t_p = p\Delta$ yields

$$\begin{aligned}
I_{p-1,w} &= -\frac{\Delta^{w+1}}{w}\Big[(p+1-(p-1))(k+1-(p+1))^{w} - (p-(p-1))(k+1-p)^{w}\Big] \\
&- \frac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-(p+1))^{w+1} - (k+1-p)^{w+1}\Big], \\
&= \frac{\Delta^{w+1}}{w(w-1)}\Big[-2(w+1)(k-p)^{w} + (w+1)(k+1-p)^{w} - (k-p)^{w+1} + (k+1-p)^{w+1}\Big], \\
&= \frac{\Delta^{w+1}}{w(w-1)}\Big[(k-p)^{w}[-2(w+1)-(k-p)] + (k+1-p)^{w}[w+1+k+1-p]\Big], \\
&= \frac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^{w}(k-p+2+w) - (k-p)^{w}(k-p+2w+2)\Big].
\end{aligned} \tag{6.8}$$

Now for $I_{p,w}$, we have

$$\begin{aligned}
I_{p,w} &= -\frac{\Delta^{w+1}}{w}\Big[(p+1-p)(k+1-(p+1))^{w}\Big] - \frac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-(p+1))^{w+1} - (k+1-p)^{w+1}\Big], \\
&= \frac{\Delta^{w+1}}{w(w-1)}\Big[-(w+1)(k-p)^{w} - (k-p)^{w+1} + (k+1-p)^{w+1}\Big], \\
&= \frac{\Delta^{w+1}}{w(w-1)}\Big[(k-p)^{w}[-(w+1)-(k-p)] + (k+1-p)^{w+1}\Big], \\
&= \frac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^{w+1} - (k-p)^{w}(k-p+1+w)\Big].
\end{aligned} \tag{6.9}$$

Upon substituting the values from (6.8) and (6.9) into (6.5), we obtain:

$$\begin{cases}
\mathbb{S}(t_{k+1}) = \mathbb{S}(0) + \dfrac{(1-w)}{\mathbb{ABC}(w)}(t_{k+1}^{r-1})[G_1(\mathbb{S}(t_k), t_k)] + \dfrac{rw}{\mathbb{ABC}(w)\Gamma(w)}\displaystyle\sum_{p=0}^{k}\Big(\dfrac{t_p^{r-1}G_1(\mathbb{S}(t_p), t_p)}{\Delta} \\
\qquad \times\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^{w}(k-p+2+w) - (k-p)^{w}(k-p+2+2w)\Big]\Big] \\
\qquad - \dfrac{t_{p-1}^{r-1}G_1(\mathbb{S}(t_{p-1}), t_{p-1})}{\Delta}\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^{w+1} - (k-p)^{w}(k-p+1+w)\Big]\Big]\Big).
\end{cases} \tag{6.10}$$

Similarly, the remaining terms for the corresponding compartments of the devised model can be expressed as follows:

$$\begin{cases}
\mathbb{C}(t_{k+1}) = \mathbb{C}(0) + \dfrac{(1-w)}{\mathbb{ABC}(w)}(t_{k+1}^{r-1})[G_2(\mathbb{C}(t_k), t_k)] + \dfrac{rw}{\mathbb{ABC}(w)\Gamma(w)}\displaystyle\sum_{p=0}^{k}\Big(\dfrac{t_p^{r-1}G_2(\mathbb{C}(t_p), t_p)}{\Delta} \\
\qquad \times\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^{w}(k-p+2+w) - (k-p)^{w}(k-p+2+2w)\Big]\Big] \\
\qquad - \dfrac{t_{p-1}^{r-1}G_2(\mathbb{C}(t_{p-1}), t_{p-1})}{\Delta}\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^{w+1} - (k-p)^{w}(k-p+1+w)\Big]\Big]\Big).
\end{cases} \tag{6.11}$$

$$\begin{cases}\mathbb{L}(t_{k+1}) = \mathbb{L}(0) + \dfrac{(1-w)}{\mathbb{ABC}(w)}(t_{k+1}^{r-1})[G_3(\mathbb{L}(t_k), t_k)] + \dfrac{rw}{\mathbb{ABC}(w)\Gamma(w)}\displaystyle\sum_{p=0}^{k}\Big(\dfrac{t_p^{r-1}G_3(\mathbb{L}(t_p), t_p)}{\Delta} \\ \qquad \times\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^w(k-p+2+w)-(k-p)^w(k-p+2+2w)\Big]\Big] \\ \qquad -\dfrac{t_{p-1}^{r-1}G_3(\mathbb{L}(t_{p-1}), t_{p-1})}{\Delta}\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^{w+1}-(k-p)^w(k-p+1+w)\Big]\Big]\Big).\end{cases} \tag{6.12}$$

$$\begin{cases}\mathbb{H}(t_{k+1}) = \mathbb{H}(0) + \dfrac{(1-w)}{\mathbb{ABC}(w)}(t_{k+1}^{r-1})[G_4(\mathbb{S}(t_k), t_k)] + \dfrac{rw}{\mathbb{ABC}(w)\Gamma(w)}\displaystyle\sum_{p=0}^{k}\Big(\dfrac{t_p^{r-1}G_4(\mathbb{H}(t_p), t_p)}{\Delta} \\ \qquad \times\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^w(k-p+2+w)-(k-p)^w(k-p+2+2w)\Big]\Big] \\ \qquad -\dfrac{t_{p-1}^{r-1}G_4(\mathbb{H}(t_{p-1}), t_{p-1})}{\Delta}\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^{w+1}-(k-p)^w(k-p+1+w)\Big]\Big]\Big).\end{cases} \tag{6.13}$$

$$\begin{cases}\mathbb{W}(t_{k+1}) = \mathbb{W}(0) + \dfrac{(1-w)}{\mathbb{ABC}(w)}(t_{k+1}^{r-1})[G_5(\mathbb{W}(t_k), t_k)] + \dfrac{rw}{\mathbb{ABC}(w)\Gamma(w)}\displaystyle\sum_{p=0}^{k}\Big(\dfrac{t_p^{r-1}G_5(\mathbb{W}(t_p), t_p)}{\Delta} \\ \qquad \times\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^w(k-p+2+w)-(k-p)^w(k-p+2+2w)\Big]\Big] \\ \qquad -\dfrac{t_{p-1}^{r-1}G_5(\mathbb{W}(t_{p-1}), t_{p-1})}{\Delta}\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^{w+1}-(k-p)^w(k-p+1+w)\Big]\Big]\Big).\end{cases} \tag{6.14}$$

$$\begin{cases}\mathbb{R}(t_{k+1}) = \mathbb{R}(0) + \dfrac{(1-w)}{\mathbb{ABC}(w)}(t_{k+1}^{r-1})[G_6(\mathbb{R}(t_k), t_k)] + \dfrac{rw}{\mathbb{ABC}(w)\Gamma(w)}\displaystyle\sum_{p=0}^{k}\Big(\dfrac{t_p^{r-1}G_6(\mathbb{R}(t_p), t_p)}{\Delta} \\ \qquad \times\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^w(k-p+2+w)-(k-p)^w(k-p+2+2w)\Big]\Big] \\ \qquad -\dfrac{t_{p-1}^{r-1}G_6(\mathbb{R}(t_{p-1}), t_{p-1})}{\Delta}\Big[\dfrac{\Delta^{w+1}}{w(w-1)}\Big[(k+1-p)^{w+1}-(k-p)^w(k-p+1+w)\Big]\Big]\Big).\end{cases} \tag{6.15}$$

6.1. Numerical simulation and discussion

This section focuses on analyzing the proposed fractal-fractional model, emphasizing how model parameters interact and collectively influence the transmission dynamics of drug addiction within society. The numerical procedures begin by adopting the compartmental initial conditions outlined in [45] for the proposed model, which are as follows: $\mathbb{S}(0) = 135, \mathbb{C}(0) = 90, \mathbb{L}(0) = 77$, $\mathbb{H}(t) = 49, \mathbb{W}(t) = 15, \mathbb{R}(t) = 11$). The reliability of the study is ensured by adopting parametric values from existing literature, as outlined in Table 2. The simulation duration ranges from 0 to 150 units, representing days. Three distinct cases are investigated, exploring varying fractional orders w and fractional dimensions r of the independent variable t.

In the first case, we investigate different fractional orders (w) and fractional dimensions (r) for the independent variable t, with the constraint $w \neq r$. This implies that the fractal order and dimension are deliberately varied independently. Specifically, we consider $w = 0.65, 0.70, 0.75, 0.80$ and $r = 0.03, 0.04, 0.05, 0.06$. The trajectories of compartmental classes in the drug addiction transmission model exhibit noticeable changes due to the manipulation of these varying fractal orders and dimensions. Each combination of w and r results in distinct patterns and behaviors observed in Figures 1a–f. The time span covers $t \in [0, 150]$ units, representing days. This exploration enables us to understand how the interaction between fractional order and dimension influences the model dynamics over time.

In Figure 1a, we observe the dynamic behavior of susceptible individuals to drug addiction $\mathbb{S}(t)$ over time, considering various fractional orders and fractal dimension orders. The varying slopes associated with different fractional orders indicate the rate at which susceptibility changes over time. The initial decline in susceptibility reflects an increased vulnerability to addiction, likely due to factors such as drug exposure or social influences. As the model progresses, the decline stabilizes, indicating an equilibrium state of exposure where susceptibility reaches a relatively stable level. Higher fractional orders exhibit steeper slopes, suggesting a more rapid decline in susceptibility, while lower fractional orders show gentler slopes, indicating a slower decline. Overall, the varying slopes and stability of transmission highlight the complex interplay between fractional orders, fractal dimensions, and the progression of addiction over time. In Figure 1b, we observe the dynamics of individuals susceptible to narcotic addiction but have received drug misuse education $\mathbb{C}(t)$. The initial decline in addiction prevalence is evident, especially at higher fractional orders. This decline signifies the effectiveness of drug misuse education in reducing addiction susceptibility. As time progresses, the stability of transmission becomes apparent, indicating that the impact of drug misuse education persists over time. This stability enables a more careful utilization of available information and resources in combating narcotic addiction within the population.

Figure 1c depicts the dynamic behavior of mild drug addicts $\mathbb{L}(t)$ over time, particularly focusing on higher fractional orders. The observed pattern reveals a rapid initial growth in the population of mild drug addicts, followed by a gradual decline. However, with time, interventions and behavioral changes lead to a gradual decline in mild addiction cases, indicating progress in addressing drug-related issues. Figure 1d illustrates the dynamics of heavy-level drug addicts $\mathbb{H}(t)$ over time. The graph depicts an increase in the number of individuals overcoming addiction, with a more pronounced decline in cases for lower fractional orders. However, for higher fractional orders, the decline in addiction cases is less significant, suggesting slower progress in recovery. This variation in recovery rates highlights the influence of treatment effectiveness and societal factors on addiction dynamics.

In Figure 1e, a decline in the number of individuals facing criminal law consequences $\mathbb{W}(t)$ is observed at lower fractional orders over time. This suggests a reduction in drug-related legal repercussions. Lower fractional orders may indicate gradual changes in law enforcement policies or societal attitudes toward drug offenses. The decrease in criminal consequences at lower orders implies shifts in legal approaches or rehabilitation strategies targeting drug-related crimes. In Figure 1f, individuals who have ceased drug use $\mathbb{R}(t)$ are depicted. The graph reveals that at higher fractional orders, there is a rapid increase in the number of individuals who have stopped using drugs, while the growth is slower at lower fractional orders. This suggests that higher fractional orders may facilitate quicker cessation of drug use, potentially due to more effective intervention programs or behavioral changes, while lower fractional orders indicate a slower, more gradual process of discontinuation influenced by short-term memory effects within the system.

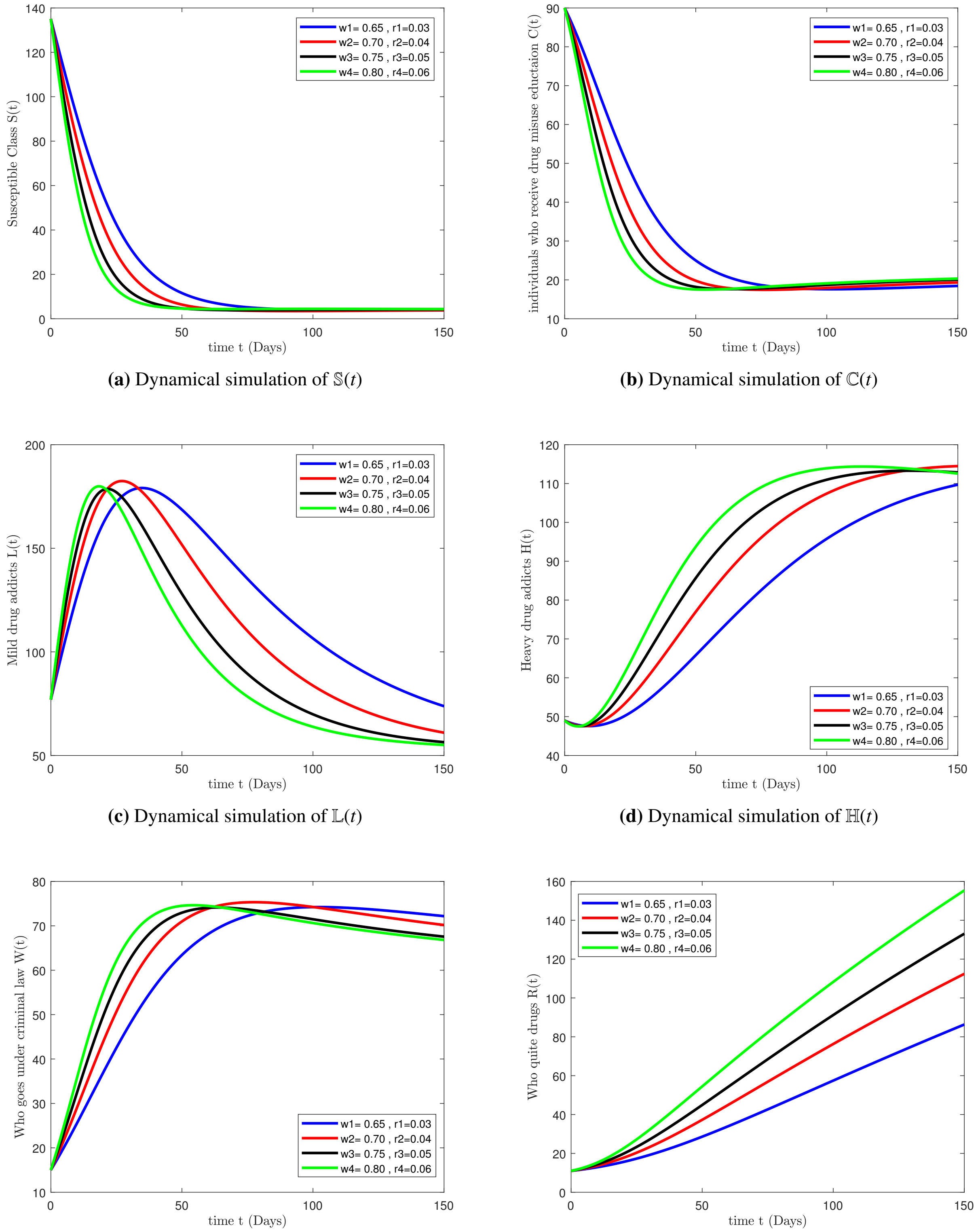
In the second scenario, we delve into the dynamics of our model by varying the fractal dimensions (r) within the range of 0.03 to 0.06, while maintaining a fixed fractal order at $w = 0.90$. The time span considered is from $t = 0$ to $t = 150$ units, as illustrated in Figures 2a–f. This exploration offers a unique perspective on how changes in fractal dimensions alone impact the dynamics of drug addiction transmission, recovery, and quit decisions. By keeping the fractal order constant, we can focus solely on the influence of fractional dimensions on addiction dynamics over an extended period. This approach allows us to isolate and better understand the specific effects of varying fractal dimensions on the behavior of the system related to drug addiction dynamics.

Table 2. Parameters and their values.

Parameters	**Values**	**Source**	**Parameters**	**Values**	**Source**
m	0.25	[45]	θ	0.421	Assumed
δ	0.01	[45]	ω	0.001	Assumed
π	0.03	[45]	γ	0.3	[45]
σ	0.7	[45]	d_2	0	[45]
ξ	0.9	[45]	d_1	0.2	[45]
β_1	0.0007	[45]	β_2	0.0008	[45]
Λ	1500	Assumed	μ	0.1	Assumed
q	0.8	[45]	d	0.02	[45]

In the last scenario, we concentrate solely on the fractional order w without considering the inclusion of the fractal dimension r. We examine a range of w values, specifically $0.75, 0.80, 0.85, 0.90, 0.95$, and 0.99, over a time span of $t \in [0, 50]$ units in days. This investigation aims to gain insights into how variations in fractional orders alone influence the dynamics of drug addiction transmission, recovery, and the decision to quit. Figures 3a–f offer a visual depiction of the dynamic behavior of compartmental classes in the proposed model for this scenario. This focused analysis allows us to discern the specific influence of changes in fractional orders on the intricate dynamics of drug addiction within the specified temporal constraints.

In summary, Figures 1–3 provide a comprehensive exploration of drug addiction dynamics across three distinct cases, considering varying fractional orders and fractal dimensions. These visualizations elucidate the intricate relationship between model parameters, offering insights into the dynamics of susceptible individuals to drug addiction ($\mathbb{S}(t)$), individuals with drug misuse education ($\mathbb{C}(t)$), mild drug addicts ($\mathbb{L}(t)$), heavy-level drug addicts ($\mathbb{H}(t)$), individuals facing criminal law consequences ($\mathbb{W}(t)$), and individuals who have ceased drug use ($\mathbb{R}(t)$). This analysis enhances our understanding of the complex dynamics of drug addiction within society and emphasizes the validity and applicability of the study's findings in understanding and addressing real-world challenges related to drug addiction.



(a) Dynamical simulation of $\mathbb{S}(t)$

(b) Dynamical simulation of $\mathbb{C}(t)$

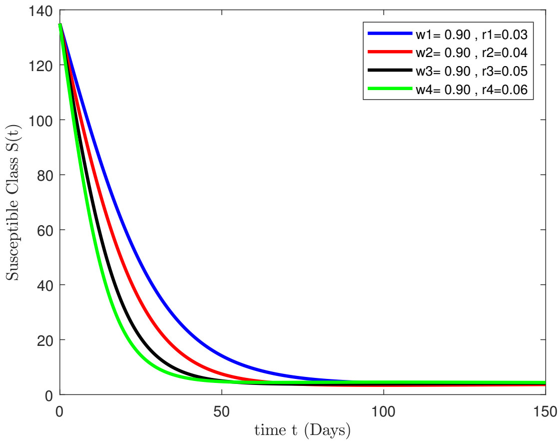
(c) Dynamical simulation of $\mathbb{L}(t)$

(d) Dynamical simulation of $\mathbb{H}(t)$

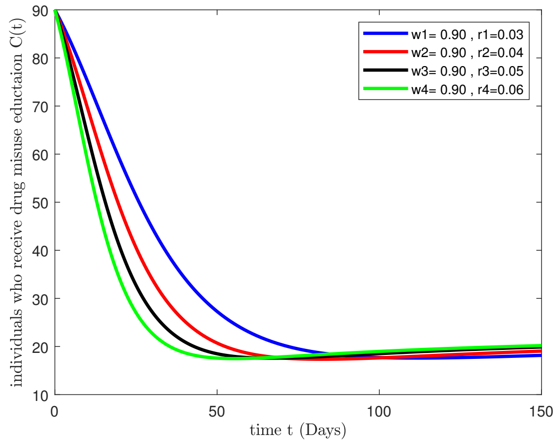
(e) Dynamical simulation of $\mathbb{W}(t)$

(f) Dynamical simulation of $\mathbb{R}(t)$

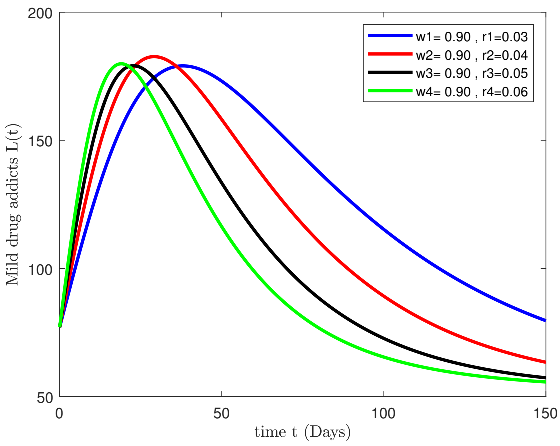
Figure 1. Numerical simulations showcase the temporal responses of all compartments within the proposed model under various fractional orders and fractal dimensions, specifically when $w \neq r$.



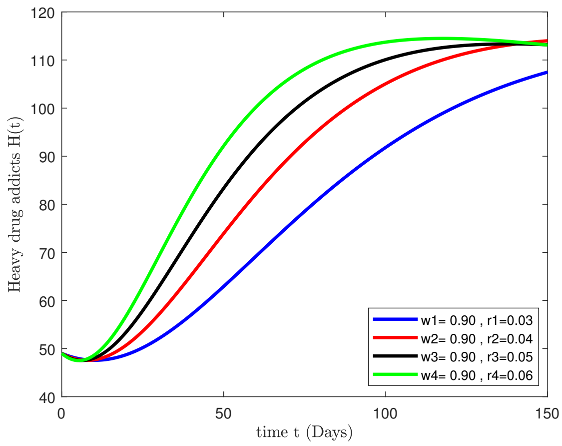
(a) Dynamical simulation of $\mathbb{S}(t)$



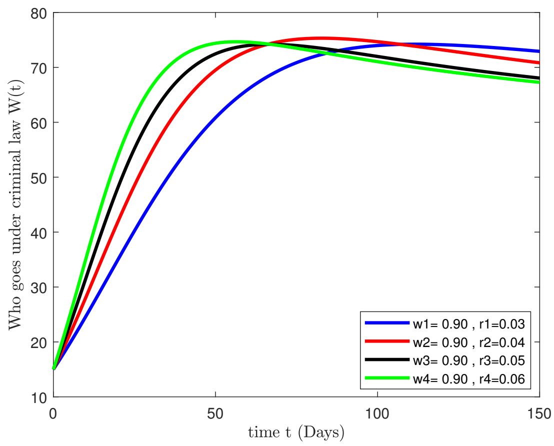
(b) Dynamical simulation of $\mathbb{C}(t)$



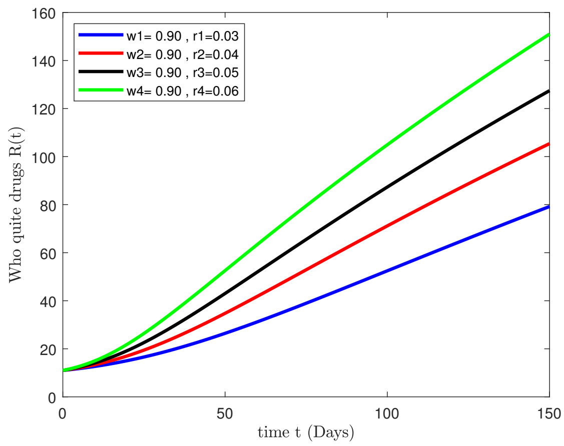
(c) Dynamical simulation of $\mathbb{L}(t)$



(d) Dynamical simulation of $\mathbb{H}(t)$

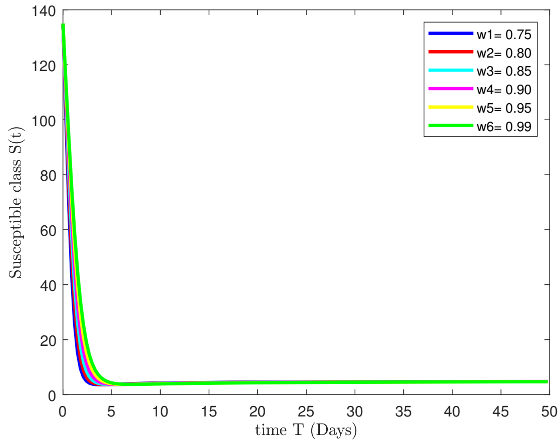


(e) Dynamical simulation of $\mathbb{W}(t)$

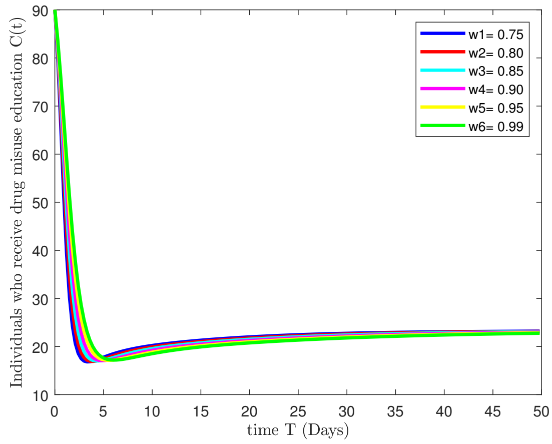


(f) Dynamical simulation of $\mathbb{R}(t)$

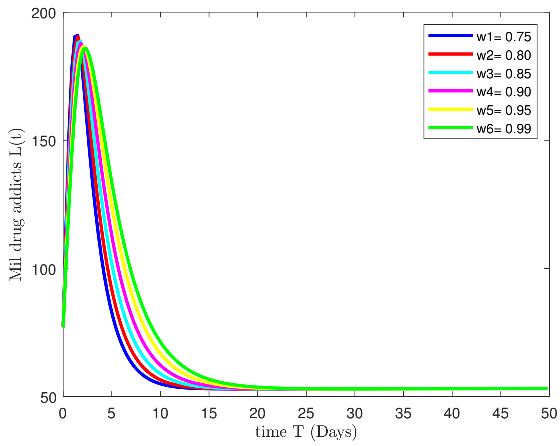
Figure 2. Numerical simulations depict the temporal responses of all classes in the devised model by varying the fractal dimensions while keeping the fractional order fixed.



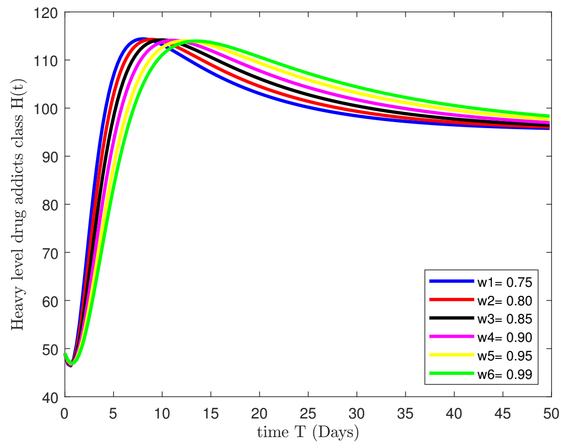
(a) Dynamical simulation of $\mathbb{S}(t)$



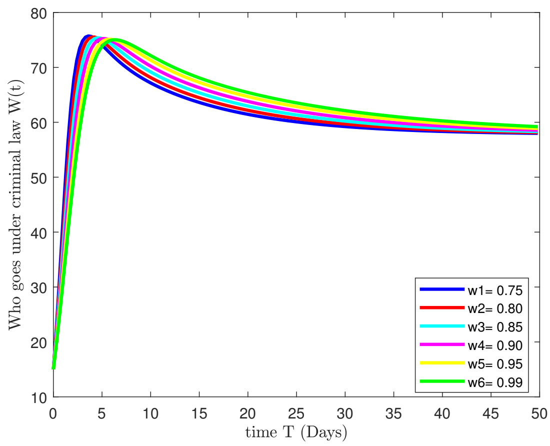
(b) Dynamical simulation of $\mathbb{C}(t)$



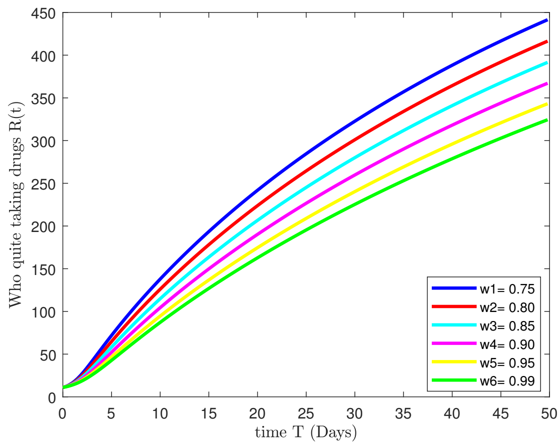
(c) Dynamical simulation of $\mathbb{L}(t)$



(d) Dynamical simulation of $\mathbb{H}(t)$



(e) Dynamical simulation of $\mathbb{W}(t)$



(f) Dynamical simulation of $\mathbb{R}(t)$

Figure 3. The numerical simulations showcase the temporal responses of all compartments in the devised model by solely considering the fractional order w .

7. Conclusions

This study focuses on the analysis of a novel deterministic mathematical model that captures the dynamics of drug addiction transmission, incorporating aspects of criminal law within a population using the fractal-fractional order derivative in the $\mathbb{ABC}$ sense. The analytical framework divides the population into six compartments: Susceptible individuals to drug addiction, those receiving drug misuse education, mild drug addicts, heavy-level drug addicts, individuals facing legal consequences, and those who have discontinued drug use. The study aims for generalization by employing a wide range of parametric settings. In the investigative process, the study establishes the existence and uniqueness of solutions using the fixed-point approach. UH stability is analyzed using nonlinear functional analysis to verify the stability of the devised model's solution. The numerical solution of the devised model is estimated using the fractal-fractional Adam Bashforth iterative scheme in the fractional order, and MATLAB validates the numerical simulation. Moreover, simulation results are depicted for various choices of fractional orders and fractal dimensions, and they are compared with integer orders, offering a comprehensive assessment of different dimensions and orders. The graphical findings demonstrate that variations in fractional orders and fractal dimensions significantly impact the dynamics of the model. The memory time for stability and convergence is shorter for lower fractional orders and fractal dimensions. Furthermore, it is anticipated that the current research will be more valuable when analyzing how drugs affect motivation and education.

This study outlines potential directions for future research in the field of drug abuse modeling and management. One avenue involves exploring the integration of behavioral dynamics into drug abuse models, considering factors such as social networks, peer influence, and psychological profiles. Additionally, integrating fractal-fractional derivatives into neural networks could optimize strategies for managing drug abuse. The anticipated improvement in effectiveness, coupled with sensitivity analyses under controlled parameters, holds promise for mitigating drug addiction transmission. Moreover, utilizing large-scale datasets and machine learning techniques can enhance the predictive accuracy of drug abuse models, aiding in identifying patterns, risk factors, and effective intervention strategies. The numerical framework developed herein serves as a robust modeling tool for addressing complex real-world problems, integrating fractional dimensions into the independent variable t.

Use of AI tools declaration

The authors confirm that they did not utilize Artificial Intelligence (AI) tools in the development of this article.

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Conflict of interest

The authors assert that they do not have any known competing financial interests or personal relationships that could have influenced the work reported in this paper.

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