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Research article

Extended existence results of solutions for FDEs of order $1 < \gamma \leq 2$

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Abstract: The focus of our investigation was on determining the existence of solutions for fractional differential equations (FDEs) of order $1 < \gamma \leq 2$ involving the boundary conditions $\kappa_0\phi(0) + \eta_0\phi(v) = \mu_0$, and $\kappa_1\phi^{'}(0) + \eta_1\phi^{'}(v) = \mu_1$, for $\kappa_i, \eta_i, \mu_i \in \mathbb{R}^+$. The existence results were based on the Schauder fixed point theorem and the nonlinear alternative of the Leray-Schauder type. Examples were provided to illustrate the results.

Keywords: fractional derivatives; differential equations; fractional differential equations (FDEs); existence of solutions; fixed-point theorem
Mathematics Subject Classification: 26A33, 34A08, 34K37

1. Introduction

This paper is devoted to the following fractional differential equation (FDE):

$$\begin{cases} {}^cD^\gamma\phi(\omega) = \Phi(\omega, \phi(\omega)), \quad \omega \in [0, v], \quad 1 < \gamma \leq 2, \\ \kappa_0\phi(0) + \eta_0\phi(v) = \mu_0, \ \kappa_1\phi^{'}(0) + \eta_1\phi^{'}(v) = \mu_1, \ for\ \kappa_i, \eta_i, \mu_i \in \mathbb{R}^+, \end{cases} \tag{1.1}$$

where, ${}^cD^\gamma$ is the Caputo fractional derivative of order γ and $\Phi : [0, v] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

Fractional calculus, an area of mathematical analysis dealing with derivatives and integrals of non-integer orders, has been widely recognized in the fields of science and engineering due to its potential in providing more precise models for complex systems that exhibit memory and hereditary properties. In recent years, FDEs have emerged as a principal mathematical paradigm for describing a wide range of natural phenomena encountered in physics, engineering, biology, and finance [1, 2]. The study of FDEs is motivated by the inadequacies of classical integer-order calculus in describing systems with fractional dynamics. The traditional derivatives based on integer-order calculus assume instantaneous responses, disregarding memory effects and long-range interactions that are often present in real-world processes. By incorporating fractional-order derivatives, FDEs offer a mechanism for modeling systems with memory, nonlocal interactions, and anomalous diffusion [3–6].

In [7], the authors studied the existence and uniqueness of a solution for FDEs with antiperiodic boundary conditions of order $1 < \gamma \leq 2$ with boundary conditions $\phi(0) = -\phi(v),\ \phi^{'}(0) = -\phi^{'}(v)$. They provided the importance of fractional order models in describing physical models with more accuracy than the regular models. The existence results were presented with the aid of the Leray-Schauder degree theory.

In [8], we discussed the existence of solutions of the following:

$$\begin{cases} {}^{c}D^{\gamma}\phi(\omega) = \Phi(\omega, \phi(\omega)), \quad \omega \in [0, v], \quad 1 < \gamma \leq 2, 0 < c < v, \\ \kappa_0\phi(c) = -\eta_0\phi(v),\ \kappa_1\phi^{'}(c) = -\eta_1\phi^{'}(v), \end{cases}$$

where $\Phi : [0, v] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\kappa_i,\ \eta_i \in \mathbb{R}^{+}$, by using the Krasnoselskii fixed-point theorem and the contraction principle.

A. Bashir and V. Otero-Espinar in [9] proved the existence results of:

$$\begin{cases} {}^{c}D^{\gamma}\phi(\omega) \in \Phi(\omega, \phi(\omega)), \quad \omega \in [0, v], \quad 1 < \gamma \leq 2, \\ \phi(0) = -\phi(v),\ \phi^{'}(0) = -\phi^{'}(v), \end{cases}$$

where $\Phi : [0, v] \times \mathbb{R} \longrightarrow \mathbb{R}$, by applying the Bohnenblust-Karlin's fixed point theorem. In this problem, boundary conditions establish connections between the solution function's values and derivatives at the boundary points. In some applications, the conditions are nonuniform and vary along the boundaries such as porous media varying cross-sectional areas.

Ahmad, Nieto, and Alsaedi in [10] obtained the existence results using standard fixed point theorems with the following boundary conditions $\phi(0) - \eta_0\phi(v) = \mu_0 \int_0^v g(\alpha, \phi(\alpha))d\alpha,\ \phi^{'}(0) - \eta_1\phi^{'}(v) = \mu_1 \int_0^v h(\alpha, \phi(\alpha))d\alpha$, where $g, h : [0, v] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and $\eta_0,\ \eta_1,\ \mu_0,\ \mu_1 \in \mathbb{R}$ with $\eta_0 \neq 1$ and $\eta_1 \neq 1$. For greater accuracy, $\eta_i - 1 \neq 0$; otherwise, the expected outcomes will not be attained. Despite that the problem in [10] seems more general but may fail in specific applications, such as heat conduction on a road or fluid flow in a pipe. For instance, in physical systems such as heat conduction, the regular boundary conditions align better than the integral boundary conditions. The integral boundary conditions can be applied when the whole heat flows along the entire rod rather than describing it in a specific location.

In [11], the authors extended the work of [10] and studied the following problem:

$$\begin{cases} {}^{c}D^{\gamma}\phi(\omega) = \Phi(\omega, \phi(\omega), {}^{c}D^{\zeta}\phi(\omega)), \quad \omega \in [0, v], \quad 1 < \gamma \leq 2,\ 0 < \zeta \leq 1, \\ \phi(0) - \eta_0\phi(v) = \mu_0 \int_0^v g(\alpha, \phi(\alpha))d\alpha,\ \phi^{'}(0) - \eta_1\phi^{'}(v) = \mu_1 \int_0^v h(\alpha, \phi(\alpha))d\alpha, \end{cases}$$

where ${}^{c}D^{\gamma}$ is the Caputo fractional derivative of order γ, $\Phi \in C([0, v] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ $g, h : [0, v] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions, and $\eta_0,\ \eta_1,\ \mu_0,\ \mu_1 \in \mathbb{R}$ with $\eta_0 \neq 1$ and $\eta_1 \neq 1$, by using the contraction principal, nonlinear alternative of Leray-Schauder type, and Schauder fixed point theorem. For more interesting results, see [8, 12–14].

These cited papers have discussed a range of methods and theorems for proving the existence of FDEs. Although these studies have offered valuable insights, there are still opportunities for improvement and enhancement, particularly in extending the scope of applicability and enhancing the robustness of existence results. Our paper makes unique contributions compared to previous literature. First, we broaden the scope of existence results for FDEs by using the Schauder fixed point theorem and the nonlinear alternative of the Leray-Schauder type. This enables us to incorporate nonlinear and nonlocal terms in the equations, making our modeling more realistic. Additionally, our approach yields insights into the qualitative properties of solutions, enhancing our understanding of the dynamics of FDEs in various contexts.

The subsequent sections of the paper are organized in the following manner: Section 2 is dedicated to establishing fundamental theorems and basic definitions. The primary results were presented in Section 3, based on the Schauder fixed point theorem and nonlinear alternative of the Leray-Schauder type. Section 4 provides examples that illustrate the concepts discussed in the previous sections. The last section concludes the paper.

2. Preliminaries

Definition 2.1. For $\chi(\omega) \in C^n([0,\infty],\mathbb{R})$, we define that the Caputo fractional derivative of order $\gamma > 0$, denoted by ${}^cD^\gamma$, is defined by

$$ {}^cD^\gamma\chi(\omega) = \frac{1}{\Gamma(r-\gamma)}\int_0^\omega (\omega-\alpha)^{r-\gamma-1}\chi^{(r)}(\alpha)d\alpha, \quad r-1<\gamma<r, r=[\gamma]+1, $$

where $[\gamma]$ denotes the integer part of the real number γ.

Definition 2.2. For any order $\gamma > 0$, the Riemann–Liouville fractional integral of a function $\chi(\omega)$, denoted by I^γ, is defined by

$$ I^\gamma\chi(\omega) = \frac{1}{\Gamma(\gamma)}\int_0^\omega (\omega-\alpha)^{\gamma-1}\chi(\alpha)d\alpha. $$

Lemma 2.1. *For an $\gamma > 0$, the solution for ${}^cD^\gamma\phi(\omega) = 0$ is given by*

$$ \chi(\omega) = \sum_{i=0}^{i=n} \tau_i\omega^{i-1}, \ \tau_i \in \mathbb{R}. \tag{2.1} $$

Lemma 2.2. *For any $\beta \in C[0,v]$ and $\kappa_i,\eta_i > 0$, $\mu_i \in \mathbb{R}$, $for\ i = 0,1$, the unique solution of the following problem:*

$$ \begin{cases} {}^cD^\gamma\phi(\omega) = \beta(\omega), \ \ \omega\in[0,v], \ \ 1<\gamma\le 2, \\ \kappa_0\phi(0)+\eta_0\phi(v)=\mu_0, \ \kappa_1\phi'(0)+\eta_1\phi'(v)=\mu_1, \end{cases} \tag{2.2} $$

is given by

$$ \begin{aligned} \phi(\omega) = {} & \int_0^\omega \frac{(\omega-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\beta(\alpha)d\alpha - \frac{\eta_0}{\kappa_0+\eta_0}\int_0^v \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\beta(\alpha)d\alpha \\ & + \frac{\eta_0\eta_1 v - \eta_1(\kappa_0+\eta_0)\omega}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\int_0^v \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}\beta(\alpha)d\alpha \\ & + \frac{\mu_1[(\kappa_0+\eta_0)\omega-\eta_0 v]+\mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}. \end{aligned} \tag{2.3} $$

Proof. In a view of Lemma 2.1, it follows $\phi(\omega) = I^{\gamma}\beta(\omega) - \tau_1 - \tau_2\omega$ for some $\tau_i \in \mathbb{R},\ i = 1, 2$ that

$$\phi(\omega) = \int_0^{\omega} \frac{(\omega-\alpha)^{\alpha-1}}{\Gamma(\gamma)}\beta(\alpha)d\alpha - \tau_1 - \tau_2\omega. \tag{2.4}$$

$$\phi^{'}(\omega) = \int_0^{\omega} \frac{(\omega-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}\beta(\alpha)d\alpha - \tau_2.$$

Using the conditions in (2.2), we get

$$\begin{aligned}\tau_1 =& \frac{\eta_0}{\kappa_0+\eta_0}\int_0^{v} \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\beta(\alpha)d\alpha - \frac{\eta_0\eta_1 v}{(\kappa_0+\kappa_0)(\kappa_1+\eta_1)}\int_0^{v} \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}\beta(\alpha)d\alpha \\ &+ \frac{\mu_1\eta_0 v - \mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}.\end{aligned}$$

$$\tau_2 = \frac{\eta_1}{\kappa_1+\eta_1}\int_0^{v} \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}\beta(\alpha)d\alpha - \frac{\mu_1}{\kappa_1+\eta_1}.$$

Replacing the quantities of τ_1, τ_2 in (2.4) completes the solution (2.3). □

Remark 2.1. The solution of (2.2) when $\kappa_0 = \kappa_1 = \eta_0 = \eta_1 = 1$ and $\mu_0 = \mu_1 = 0$ is given by

$$\phi(\omega) = \int_0^{\omega} \frac{(\omega-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\beta(\alpha)d\alpha - \frac{1}{2}\int_0^{v} \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\beta(\alpha)d\alpha + \frac{v-2\omega}{4}\int_0^{v} \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}\beta(\alpha)d\alpha. \tag{2.5}$$

We see that, Lemma 2.2 reduces to Lemma 2.5 in [7].

Theorem 2.3. *[15] Let $\mathcal{B}$ be a Banach space, $\mathcal{S} \subset \mathcal{B}$ be a nonempty, closed, and convex subset, and let $\mathcal{V} : \mathcal{S} \longrightarrow \mathcal{S}$ be a continuous mapping such that $\mathcal{V}(\mathcal{S})$ is relatively compact in $\mathcal{B}$, then $\mathcal{V}$ has at least one fixed point.*

Theorem 2.4. *[15] Let $\mathcal{B}$ be a Banach space, and suppose $\mathcal{S} \subset \mathcal{B}$ is a closed and convex. Let $U \subset \mathcal{S}$ be open with $0 \in U$. Assume $\mathcal{V} : \overline{U} \longrightarrow \mathcal{S}$ is continuous and compact, then $\mathcal{V}$ has a fixed point in $\overline{U}$ or $\omega = \rho\mathcal{V}(\omega)$ for an $\omega \in \partial U$ and $\rho \in (0, 1)$.*

3. Main results

Let $C = C([0, v], \mathbb{R})$and $\mathcal{V} : C \longrightarrow C$ be the operator defined as

$$\begin{aligned}(\mathcal{V}\phi)(\omega) =& \int_0^{\omega} \frac{(\omega-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\Phi(\alpha,\phi(\alpha))d\alpha - \frac{\eta_0}{\kappa_0+\eta_0}\int_0^{v} \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\Phi(\alpha,\phi(\alpha))d\alpha \\ &+ \frac{\eta_0\eta_1 v - \eta_1(\kappa_0+\eta_0)\omega}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\int_0^{v} \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}\Phi(\alpha,\phi(\alpha))d\alpha \\ &+ \frac{\mu_1[(\kappa_0+\eta_0)\omega - \eta_0 v] + \mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}.\end{aligned} \tag{3.1}$$

Notice that the fractional differential problem (1.1) has a solution iff the operator $\mathcal{V}$ has a fixed point.

The following assumptions are required in the subsequent theorems:

(h_1) $\exists$ a $\sigma \in L^\infty([0,v],\mathbb{R}^+)$ and a nondecreasing function δ, such that $|\Phi(\omega,\phi)| \leq \sigma(\omega)\delta(|\phi|)$ for $\omega \in [0,v], \phi \in \mathbb{R}$.

(h_2) $\exists A > 0$, such that

$$A > \|\sigma\|_{L^\infty}\delta(A)\frac{v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}[\kappa_0 + 2\eta_0 + \frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}] + \frac{\kappa_0\mu_0 v + \mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}.$$

Theorem 3.1. *Let* $\Phi : [0,v]\times\mathbb{R} \longrightarrow \mathbb{R}$ *be continuous with* $|\Phi(\omega,\phi_1) - \Phi(\omega,\phi_2)| \leq L|\phi_1-\phi_2|$ *for* $\omega \in [0,v],\ \ \phi_1,\phi_2 \in \mathbb{R}$, $L > 0$ *and satisfying*

$$\frac{Lv^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\left[\kappa_0 + 2\eta_0 + \frac{\eta_0\eta_1\alpha}{\kappa_1+\eta_1}\right] < 1,$$

then problem (1.1) has a unique solution.

Proof. For simplicity of the calculation, we are going to introduce the following notations

$$\xi = L\frac{v^\alpha}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\left[\kappa_0 + 2\eta_0 + \frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}\right] < 1.$$

For any $\phi_1,\phi_2 \in C$ and $\omega \in [0,v]$, we have:

$$\begin{aligned}
\|(\mathcal{V}\phi_1)\omega - (\mathcal{V}\phi_2)\omega\| \leq & \int_0^\omega \frac{(\omega-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\|\Phi(\alpha,\phi_1(\alpha)) - \Phi(\alpha,\phi_2(\alpha))\|d\alpha \\
& + \frac{\eta_0}{\kappa_0+\eta_0}\int_0^v \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\|\Phi(\alpha,\phi_2(\alpha)) - \Phi(\alpha,\phi_1(\alpha))\|d\alpha \\
& + \frac{|\eta_0\eta_1 v - \eta_1(\kappa_0+\eta_0)\omega|}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\int_0^v \frac{(v-\omega)^{\gamma-2}}{\Gamma(\gamma-1)}\|\Phi(\alpha,\phi_1(\alpha)) - \Phi(\alpha,\phi_2(\alpha))\|d\alpha \\
\leq & L\frac{v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\left[\kappa_0 + 2\eta_0 + \frac{\kappa_0\kappa_1\gamma}{\kappa_1+\eta_1}\right]\|\phi_1-\phi_2\|.
\end{aligned}$$

Therefore,

$$\|(\mathcal{V}\phi_1)\omega - (\mathcal{V}\phi_2)\omega\| \leq \underbrace{L\frac{v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\left[\kappa_0 + 2\eta_0 + \frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}\right]}_{\text{i}}\|\phi_1-\phi_2\| = \xi\|\phi_1-\phi_2\|,$$

together with $\xi < 1$ shows that $\mathcal{V}$ is a contraction mapping. Thus, the contraction mapping principle implies the unique solution of (1.1) since $\mathcal{V}$ has a unique fixed point.

We have to mention that (i) depends on the parameters $L, \kappa_0, \kappa_1, \eta_0, \eta_1, \gamma, v$ in the problem. □

Theorem 3.2. *Let* $\Phi : [0,v]\times\mathbb{R} \longrightarrow \mathbb{R}$ *be continuous with*

$$|\Phi(\omega,\phi_1) - \Phi(\omega,\phi_2)| \leq g(\omega)|\phi_1-\phi_2|, \ \ for\ \ \omega \in [0,v],\ \ \phi_1,\phi_2 \in \mathbb{R}\ \ with\ \ g \in L^\infty([0,v],\mathbb{R}^+),$$

and satisfying

$$\|g\|_{L^\infty}\frac{v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\left[\kappa_0 + 2\eta_0 + \frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}\right] < 1,$$

then problem (1.1) has a unique solution.

Proof. For $\phi_1, \phi_2 \in C$ and $\omega \in [0, v]$, we have

$$\begin{aligned}\|(\mathcal{V}\phi_1)\omega - (\mathcal{V}\phi_2)\omega\| \leq & \int_0^\omega \frac{(\omega-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\|\Phi(\alpha,\phi_1(\alpha)) - \Phi(\alpha,\phi_2(\alpha))\|d\alpha \\ & + \frac{\eta_0}{\kappa_0+\eta_0}\int_0^v \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\|\Phi(\alpha,\phi_2(\alpha)) - \Phi(\alpha,\phi_1(\alpha))\|d\alpha \\ & + \frac{|\eta_0\eta_1 v - \eta_1(\kappa_0+\eta_0)\omega|}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\int_0^v \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}\|\Phi(\alpha,\phi_1(\alpha)) - \Phi(\alpha,\phi_2(\alpha))\|d\alpha \\ \leq & \frac{\|g\|_{L^\infty} v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\left[\kappa_0 + 2\eta_0 + \frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}\right]\|\phi_1 - \phi_2\|.\end{aligned}$$

Therefore,

$$\|(\mathcal{V}_1)\omega - (\mathcal{V}\phi_2)\omega\| \leq \frac{\|g\|_{L^\infty} v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\left[\kappa_0 + 2\eta_0 + \frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}\right]\|\phi_1 - \phi_2\|,$$

together with $\|g\|_{L^\infty}\frac{v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\left[\kappa_0 + 2\eta_0 + \frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}\right] < 1$ shows that $\mathcal{V}$ is a contraction mapping. Thus, the contraction mapping principle implies the unique solution of (1.1) since $\mathcal{V}$ has a unique fixed point. □

Theorem 3.3. *Let $\Phi : [0, v] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying h_1 and h_2, then (1.1) has at least one solution.*

Proof. Let $\overline{\mathcal{M}} \subset \mathcal{S}$ be bounded. Assume that for any $\phi \in \overline{\mathcal{M}}$, $\|\phi\| \leq r$. Let $\mathcal{V}$ be the operator defined in 3.1.

$$\begin{aligned}\left|(\mathcal{V}\phi)(\omega)\right| =& \Big|\int_0^\omega \frac{(\omega-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\Phi(\alpha,\phi(\alpha))d\alpha - \frac{\eta_0}{\kappa_0+\eta_0}\int_0^v \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\Phi(\alpha,\phi(\alpha))d\alpha \\ & + \frac{\eta_0\eta_1 v - \eta_1(\kappa_0+\eta_0)\omega}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\int_0^v \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}\Phi(\alpha,\phi(\alpha))d\alpha + \frac{\mu_1[(\kappa_0+\eta_0)\omega - \eta_0 v] + \mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\Big| \\ \leq & \int_0^\omega \frac{(\omega-\alpha)^{\gamma-1}}{\Gamma(\gamma)}|\Phi(\alpha,\phi(\alpha))|d\alpha + \Big|\frac{\eta_0}{\kappa_0+\eta_0}\Big|\int_0^v \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}|\Phi(\alpha,\phi(\alpha))|d\alpha \\ & + \Big|\frac{\eta_0\eta_1 v - \eta_1(\kappa_0+\eta_0)\omega}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\Big|\int_0^v \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}|\Phi(\alpha,\phi(\alpha))|d\alpha + \frac{\kappa_0\mu_0 v + \mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)} \\ \leq & \|\sigma\|_{L^\infty}\delta(r)\frac{v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}[\kappa_0 + 2\eta_0 + \frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}] + \frac{\kappa_0\mu_0 v + \mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)} < A. \end{aligned} \tag{3.2}$$

Therefore, we proved that $\mathcal{V}(\overline{\mathcal{M}})$ is bounded in $\overline{\mathcal{M}}$.

$$\begin{aligned}\left|(\mathcal{V}\phi)'(\omega)\right| \leq & \Big|\int_0^\omega \frac{(\omega-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}|\Phi(\alpha,\phi(\alpha))|d\alpha + \Big|\frac{\eta_0}{\kappa_0+\eta_0}\Big|\int_0^v \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}|\Phi(\alpha,\phi(\alpha))|d\alpha \\ & + \Big|\frac{\eta_0\eta_1 v - \eta_1(\kappa_0+\eta_0)\omega}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\Big|\int_0^v \frac{(v-\alpha)^{\gamma-3}}{\Gamma(\gamma-2)}|\Phi(\alpha,\phi(\alpha))|d\alpha\Big| \\ \leq & \|\sigma\|_{L^\infty}\delta(r)\frac{v^{\gamma-1}}{\Gamma(\gamma)(\kappa_0+\eta_0)}[\kappa_0 + 2\eta_0 + \frac{\eta_0\eta_1(\gamma-1)}{\kappa_1+\eta_1}] = N.\end{aligned}$$

Therefore, for any ω_1 and $\omega_2 \in [0, v]$, we have

$$\left|(\mathcal{V}\phi)(\omega_2) - (\mathcal{V}\phi)(\omega_2)\right| \leq \int_{\omega_1}^{\omega_2}\left|(\mathcal{V}\phi)'(\alpha)\right|d\alpha \leq N(\omega_2 - \omega_1). \tag{3.3}$$

Equations (3.2) and (3.3) imply that $\mathcal{V}$ is equicontinuous on bounded subsets of $\mathcal{S}$.

Now for any $\rho \in (0,1)$ and $\omega \in [0,v]$, let $\phi = \rho\mathcal{V}\phi$. We have,

$$\begin{aligned}|\phi(\omega)| \leq & |\rho(\mathcal{V}\phi)(\omega)| \leq \int_0^\omega \frac{(\omega-\alpha)^{\gamma-1}}{\Gamma(\gamma)}|\Phi(\alpha,\phi(\alpha))|d\alpha + \Big|\frac{\eta_0}{\kappa_0+\eta_0}\Big|\int_0^v \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}|\Phi(\alpha,\phi(\alpha))|d\alpha \\ & + \Big|\frac{\eta_0\eta_1 v - \eta_1(\kappa_0+\eta_0)\omega}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\Big|\int_0^v \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}|\Phi(\alpha,\phi(\alpha))|d\alpha + \Big|\frac{\mu_1[(\kappa_0+\eta_0)\omega-\eta_0 v]+\mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\Big| \\ \leq & \|\sigma\|_{L^\infty}\delta(|\phi|)\frac{v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\Big[\kappa_0+2\eta_0+\frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}\Big] + \frac{\kappa_0\mu_0 v+\mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}.\end{aligned}$$

Thus,

$$\|\phi\| \leq \|\sigma\|_{L^\infty}\delta(\|\phi\|)\frac{v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\Big[\kappa_0+2\eta_0+\frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}\Big] + \frac{\kappa_0\mu_0 v+\mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)},$$

by h_2, $\exists A > 0$ such that $|\phi| \neq A$.

We see that, $\mathcal{V} : \overline{U} \longrightarrow \mathcal{S}$ is completely continuous, where $U = \{\phi \in \mathcal{S} : \|\phi\| < A\}$. By the choice of U and Theorem 2.4, any $\phi \in \partial U$, $\phi \neq \rho\mathcal{V}\phi$, for $\rho \in (0,1)$. Therefore, $\mathcal{V}\phi = \phi$ for some $\phi \in \overline{U}$, completing the proof. □

Theorem 3.4. *Let* $\Phi : [0,v]\times\mathbb{R} \longrightarrow \mathbb{R}$. *Suppose that* $|\Phi(\omega,\phi)| \leq \lambda + \epsilon|\phi|$, *where* $0 \leq \epsilon < \frac{1}{\rho}$, $\lambda > 0$, *and* $\rho = \frac{v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\Big[\kappa_0 + 2\eta_0 + \frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}\Big]$, *then problem (1.1) has at least a solution in* $[0,v]$.

Proof. Let $\mathcal{V}$ be the operator in (3.1) and define a fixed point problem $\phi = \mathcal{V}\phi$. Define a ball B_r in $C([0,v])$ with a radius $r > 0$, which will be fixed later, as $B_r = \{\phi \in C([0,v]) : \|\phi\| < r\}$ for all $\omega \in [0,v]$.

Set $\Psi(\sigma,\phi) = \sigma\mathcal{V}\phi$, for $\sigma \in [0.1]$ and $\phi \in C(\mathbb{R})$.

Thus, $\psi_\sigma = \phi - \sigma\mathcal{V}\phi$ is completely continuous by the Arzela–Ascoli theorem. We want to show that for the operator $\mathcal{V} : \overline{B} \longrightarrow C([0,v])$ we have

$$\phi \neq \sigma\mathcal{V}\phi, \quad \forall\phi \in \partial B_r \ \textit{and} \ \forall\sigma \in [0.1]. \tag{3.4}$$

If (3.4) is true, then $deg(\psi_\sigma, B_r, 0) = deg(\psi_1, B_r, 0) = deg(\psi_0, B_r, 0) = 1 \neq 0$ and $0 \in B_r$.

Now, at least one $\phi \in B_r$ satisfies $\psi_1 = \phi - \sigma\mathcal{V}\phi$. To prove (3.4) we assume that for some $\sigma \in [0,1]$ and all $\omega \in [0,v]$, $\phi = \sigma\mathcal{V}\phi$ such that

$$\begin{aligned}|\phi| = & \big|\sigma\mathcal{V}\phi(\omega)\big| \leq \Big|\int_0^\omega \frac{(\omega-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\Phi(\alpha,\phi(\alpha))d\alpha - \frac{\eta_0}{\kappa_0+\eta_0}\int_0^v \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}\Phi(\alpha,\phi(\alpha))d\alpha \\ & + \frac{\eta_0\eta_1 v - \eta_1(\kappa_0+\eta_0)\omega}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\int_0^v \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}\Phi(\alpha,\phi(\alpha))d\alpha\Big| + \Big|\frac{\mu_1[(\kappa_0+\eta_0)\omega-\eta_0 v]+\mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\Big| \\ \leq & \int_0^\omega \frac{(\omega-\alpha)^{\gamma-1}}{\Gamma(\gamma)}|\Phi(\alpha,\phi(\alpha))|d\alpha + \Big|\frac{\eta_0}{\kappa_0+\eta_0}\Big|\int_0^v \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}|\Phi(\alpha,\phi(\alpha))|d\alpha \\ & + \Big|\frac{\eta_0\eta_1 v - \eta_1(\kappa_0+\eta_0)\omega}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\Big|\int_0^v \frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}|\Phi(\alpha,\phi(\alpha))|d\alpha + \frac{\kappa_0\mu_0 v+\mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)} \\ \leq & (\lambda+\epsilon|\phi|)\Big[\int_0^\omega \frac{(\omega-\alpha)^{\gamma-1}}{\Gamma(\gamma)}d\alpha + \Big|\frac{\eta_0}{\kappa_0+\eta_0}\Big|\int_0^v \frac{(v-\alpha)^{\gamma-1}}{\Gamma(\gamma)}d\alpha\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{\eta_0\eta_1 v-\eta_1(\kappa_0+\eta_0)\omega}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\right|\int_0^v\frac{(v-\alpha)^{\gamma-2}}{\Gamma(\gamma-1)}d\alpha\Big]+\frac{\kappa_0\mu_0 v+\mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\\
\leq&(\lambda+\epsilon|\phi|)\frac{v^\gamma}{(\kappa_0+\eta_0)\Gamma(\gamma+1)}\left[\kappa_0+2\eta_0+\frac{\eta_0\eta_1\gamma}{\kappa_1+\eta_1}\right]+\frac{\kappa_0\mu_0 v+\mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}\\
=&(\lambda+\epsilon|\phi|)\rho+\frac{\kappa_0\mu_0 v+\mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}.
\end{aligned}\tag{3.5}
$$

For simplicity of the calculation, let

$$c=\frac{\kappa_0\mu_0 v+\mu_0(\kappa_1+\eta_1)}{(\kappa_0+\eta_0)(\kappa_1+\eta_1)}.$$

Therefore,

$$\|\phi\|\leq\frac{\lambda\rho+c}{1-\epsilon\rho}.$$

Choosing $r>\frac{\lambda\rho+c}{1-\epsilon\rho}$ proves (3.4), which completes this proof. □

Remark 3.1. Theorem 3.4 can be reduced to Theorem 3.1 in [7].

4. Examples

Example 4.1. Consider the following FDE:

$$
\begin{cases}
{}^cD^{\frac{3}{2}}\phi(\omega)=\frac{1}{(\omega+3)^3}tan^{-1}(\phi)+ln(\omega+1), & \omega\in[0,1],\\
\phi(0)=-\frac{1}{2}\phi(1),\ \phi'(0)=-\frac{1}{2}\phi'(1).
\end{cases}\tag{4.1}
$$

Clearly, $|\Phi(\omega,\phi_2)-\Phi(\omega,\phi_1)|\leq\frac{1}{27}|\phi_2-\phi_1|$, with $L=\frac{1}{27}$.

Here, $\Phi(\omega,\phi)=\frac{1}{(\omega+3)^3}tan^{-1}(\phi)+ln(\omega+1)$.

Also, since

$$\frac{\frac{1}{27}}{(1+\frac{1}{27})\Gamma(\frac{3}{2}+1)}\left[1+2\frac{1}{2}+\frac{\frac{1}{2}\frac{1}{2}\frac{3}{2}}{1+\frac{1}{2}}\right]\approx 0.0418<1,$$

then, Theorem 3.1 implies that problem (4.1) has at least one solution on [0,1].

Example 4.2. Consider the following classical FDE:

$$
\begin{cases}
{}^cD^{\frac{3}{2}}\phi(\omega)=\frac{1}{2\pi}sin(4\pi\phi)+\frac{|\phi|}{1+|\phi|}, & \omega\in[0,1],\\
\phi(0)+\phi(1)=0,\ \phi'(0)+\phi'(1)=0.
\end{cases}\tag{4.2}
$$

Clearly, $|\Phi(\omega,\phi)|\leq\frac{1}{2}|\phi|+1$, where $\Phi(\omega,\phi)=\frac{1}{4\pi}sin(2\pi\phi)+\frac{|\phi|}{1+|\phi|}$, with $0<\epsilon=\frac{1}{2}<\frac{2\sqrt{\pi}}{5}$, and $\rho=1$. Thus, Theorem 3.4 implies that problem (4.2) has at least one solution on [0,1].

5. Conclusions

In this paper, we examine the solution existence for problem (1.1) under the boundary conditions $\kappa_0\phi(0) + \eta_0\phi(\nu) = \mu_0$, $\kappa_1\phi'(0) + \eta_1\phi'(\nu) = \mu_1$ for $\kappa_i, \eta_i, \mu_i \in \mathbb{R}^+$. Extra components are incorporated into the solution of (1.1). The results in this paper are obtained by using the Schauder fixed point theorem, nonlinear alternative of the Leray-Schauder type, and the contraction mapping principle, which can be reduced to the existence results of [7]. In fact, the existence results in this study can extend and generalize the results in FDE problems of order $\gamma \in (1, 2]$ under the boundary conditions $\phi(0) + \phi(\nu) = 0$ and $\phi'(0) + \phi'(\nu) = 0$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author does not have any conflict of interest.

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