AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 13043–13057.
DOI: 10.3934/math.2024636
Received: 16 January 2024
Revised: 21 March 2024
Accepted: 22 March 2024
Published: 08 April 2024

Research article

On CR-lightlike submanifolds in a golden semi-Riemannian manifold

Mohammad Aamir Qayyoom[1], **Rawan Bossly**[2] **and Mobin Ahmad**[1,*]

[1] Department of Mathematics and Statistics, Faculty of Science, Integral University, Lucknow-226026, India

[2] Department of Mathematics, College of Science, Jazan University, 82817 Jazan, Saudi Arabia

* **Correspondence:** Email: mobinahmad68@gmail.com.

Abstract: CR-lightlike submanifolds of a golden semi-Riemannian manifold are the focus of the research presented in this work, which aims to define and investigate these structures. Under the context of a golden semi-Riemannian manifold, we study the properties of geodesic CR-lightlike submanifolds as well as umbilical CR-lightlike submanifolds. In addition, on a golden semi-Riemannian manifold, we find numerous intriguing findings for entirely geodesic and totally umbilical CR-lightlike subamnifolds. Also, we construct an example of a CR-lightlike submanifold of a golden semi-Riemannian manifold.

Keywords: golden structure; semi-Riemannian manifold; CR-lightlike submanifolds; totally umbilical; geodesic
Mathematics Subject Classification: 53C15, 53C20, 53C40

1. Introduction

The concept of lightlike submanifolds in geometry was initially established and expounded upon in a work produced by Duggal and Bejancu [1]. A nondegenerate screen distribution was employed in order to produce a nonintersecting lightlike transversal vector bundle of the tangent bundle. They defined the CR-lightlike submanifold as a generalization of lightlike real hypersurfaces of indefinite Kaehler manifolds and showed that CR-lightlike submanifolds do not contain invariant and totally real lightlike submanifolds. Further, they defined and studied GCR-lightlike submanifolds of Kaehler manifolds as an umbrella of invariant submanifolds, screen real submanifolds, and CR-lightlike and SCR-lightlike submanifolds in [2, 3], respectively. Subsequently, B. Sahin and R. Gunes investigated geodesic property of CR-lightlike submanifolds [4] and the integrability of distributions in CR-lightlike submanifolds [5] . In the year 2010, Duggal and Sahin published a book [6] pertaining to the field of differential geometry, specifically focusing on the study of lightlike submanifolds. This book provides a comprehensive examination of recent advancements in lightlike geometry, encompassing novel geometric findings, accompanied by rigorous proofs, and exploring their practical implications in the field of mathematical physics. The investigation of the geometric properties of lightlike hypersurfaces and lightlike submanifolds has been the subject of research in several studies (see [7–14]).

Crasmareanu and Hretcanu [15] created a special example of polynomial structure [16] on a differentiable manifold, and it is known as the golden structure $(\overline{M}, g)$. Hretcanu C. E. [17] explored Riemannian submanifolds with the golden structure. M. Ahmad and M. A. Qayyoom studied geometrical properties of Riemannian submanifolds with golden structure [18–21] and metallic structure [22, 23]. The integrability of golden structures was examined by A. Gizer et al. [24]. Lightlike hypersurfaces of a golden semi-Riemannian manifold was investigated by N. Poyraz and E. Yasar [25]. The golden structure was also explored in the studies [26–29].

In this research, we investigate the CR-lightlike submanifolds of a golden semi-Riemannian manifold, drawing inspiration from the aforementioned studies. This paper has the following outlines: Some preliminaries of CR-lightlike submanifolds are defined in Section 2. We establish a number of properties of CR-lightlike submanifolds on golden semi-Riemannian manifolds in Section 3. In Section 4, we look into several CR-lightlike submanifolds characteristics that are totally umbilical. We provide a complex illustration of CR-lightlike submanifolds of a golden semi-Riemannian manifold in the final section.

2. Opening statements

Assume that $(\overline{\aleph}, g)$ is a semi-Riemannian manifold with $(k+j)$-dimension, $k, j \geq 1$, and g as a semi-Riemannian metric on $\overline{\aleph}$. We suppose that $\overline{\aleph}$ is not a Riemannian manifold and the symbol q stands for the constant index of g.

[15] Let $\overline{\aleph}$ be endowed with a tensor field ψ of type $(1, 1)$ such that

$$\psi^2 = \psi + I, \tag{2.1}$$

where I represents the identity transformation on $\Gamma(\Upsilon\overline{\aleph})$. The structure ψ is referred to as a golden structure. A metric g is considered ψ-compatible if

$$g(\psi\gamma, \zeta) = g(\gamma, \psi\zeta) \tag{2.2}$$

for all γ, ζ vector fields on $\Gamma(\Upsilon\overline{\aleph})$, then $(\overline{\aleph}, g, \psi)$ is called a golden Riemannian manifold. If we substitute $\psi\gamma$ into γ in (2.2), then from (2.1) we have

$$g(\psi\gamma, \psi\zeta) = g(\psi\gamma, \zeta) + g(\gamma, \zeta). \tag{2.3}$$

for any $\gamma, \zeta \in \Gamma(\Upsilon\overline{\aleph})$.

If $(\overline{\aleph}, g, \psi)$ is a golden Riemannian manifold and ψ is parallel with regard to the Levi-Civita connection $\overline{\nabla}$ on $\overline{\aleph}$:

$$\overline{\nabla}\psi = 0, \tag{2.4}$$

then $(\overline{\aleph}, g, \psi)$ is referred to as a semi-Riemannian manifold with locally golden properties.

The golden structure is the particular case of metallic structure [22, 23] with $p = 1,\ q = 1$ defined by

$$\psi^2 = p\psi + qI,$$

where p and q are positive integers.

[1] Consider the case where $\aleph$ is a lightlike submanifold of k of $\overline{\aleph}$. There is the radical distribution, or $Rad(\Upsilon\aleph)$, on $\aleph$ that applies to this situation such that $Rad(\Upsilon\aleph) = \Upsilon\aleph \cap \Upsilon\aleph^{\perp},\ \forall\ p \in \aleph$. Since $Rad\Upsilon\aleph$ has rank $r \geq 0$, $\aleph$ is referred to as an r-lightlike submanifold of $\overline{\aleph}$. Assume that $\aleph$ is a submanifold of $\aleph$ that is r-lightlike. A screen distribution is what we refer to as the complementary distribution of a Rad distribution on $\Upsilon\aleph$, then

$$\Upsilon\aleph = Rad\Upsilon\aleph \perp S(\Upsilon\aleph).$$

As $S(\Upsilon\aleph)$ is a nondegenerate vector sub-bundle of $\Upsilon\overline{\aleph}|_{\aleph}$, we have

$$\Upsilon\overline{\aleph}|_{\aleph} = S(\Upsilon\aleph) \perp S(\Upsilon\aleph)^{\perp},$$

where $S(\Upsilon\aleph)^{\perp}$ consists of the orthogonal vector sub-bundle that is complementary to $S(\Upsilon\aleph)$ in $\Upsilon\overline{\aleph}|_{\aleph}$. $S(\Upsilon\aleph), S(\Upsilon\aleph^{\perp})$ is an orthogonal direct decomposition, and they are nondegenerate.

$$S(\Upsilon\aleph)^{\perp} = S(\Upsilon\aleph^{\perp}) \perp S(\Upsilon\aleph^{\perp})^{\perp}.$$

Let the vector bundle

$$tr(\Upsilon\aleph) = ltr(\Upsilon\aleph) \perp S(\Upsilon\aleph^{\perp}).$$

Thus,

$$\Upsilon\overline{\aleph} = \Upsilon\aleph \oplus tr(\Upsilon\aleph) = S(\Upsilon\aleph) \perp S(\Upsilon\aleph^{\perp}) \perp (Rad(\Upsilon\aleph) \oplus ltr(\Upsilon\aleph)).$$

Assume that the Levi-Civita connection is $\overline{\nabla}$ on $\overline{\aleph}$. We have

$$\overline{\nabla}_{\gamma}\zeta = \nabla_{\gamma}\zeta + h(\gamma, \zeta),\ \ \forall\, \gamma, \zeta \in \Gamma(\Upsilon\aleph) \tag{2.5}$$

and

$$\overline{\nabla}_{\gamma}\zeta = -A_h\zeta + \nabla^{\perp}_{\gamma}h,\ \ \forall\gamma \in \Gamma(\Upsilon\aleph)\ and\ h \in \Gamma(tr(\Upsilon\aleph)), \tag{2.6}$$

where $\{\nabla_{\gamma}\zeta, A_h\gamma\}$ and $\{h(\gamma, \zeta), \nabla^{\perp}_{\gamma}h\}$ belongs to $\Gamma(\Upsilon\aleph)$ and $\Gamma(tr(\Upsilon\aleph))$, respectively.

Using projection $L : tr(\Upsilon\aleph) \to ltr(\Upsilon\aleph)$, and $S : tr(\Upsilon\aleph) \to S(\Upsilon\aleph^{\perp})$, we have

$$\overline{\nabla}_{\gamma}\zeta = \nabla_{\gamma}\zeta + h^l(\gamma, \zeta) + h^s(\gamma, \zeta), \tag{2.7}$$

$$\overline{\nabla}_{\gamma}\aleph = -A_{\aleph}\gamma + \nabla^l_{\gamma}\aleph + \lambda^s(\gamma, \aleph), \tag{2.8}$$

and

$$\overline{\nabla}_{\gamma}\chi = -A_{\chi}\gamma + \nabla^s_{\gamma} + \lambda^l(\gamma, \chi) \tag{2.9}$$

for any $\gamma, \zeta \in \Gamma(\Upsilon\aleph),\ \aleph \in \Gamma(ltr(\Upsilon\aleph))$, and $\chi \in \Gamma(S(\Upsilon\aleph^{\perp}))$, where $h^l(\gamma, \zeta) = Lh(\gamma, \zeta),\ h^s(\gamma, \zeta) = Sh(\gamma, \zeta),\ \nabla^l_{\gamma}\aleph,\ \lambda^l(\gamma, \chi) \in \Gamma(ltr(\mathcal{T}\aleph)),\ \nabla^s_{\gamma}\ \lambda^s(\gamma, \aleph) \in \Gamma(S(\Upsilon\aleph^{\perp}))$, and $\nabla_{\gamma}\zeta,\ A_{\aleph}\gamma,\ A_{\chi}\gamma \in \Gamma(\Upsilon\aleph)$.

The projection morphism of $\Upsilon\aleph$ on the screen is represented by P, and we take the distribution into consideration.

$$\nabla_{\gamma}P\zeta = \nabla^*_{\gamma}P\zeta + h^*(\gamma, P\zeta),$$

$$\nabla_\gamma \xi = -A^*_\xi \gamma + \nabla^{*t}_\gamma \xi, \tag{2.10}$$

where $\gamma, \zeta \in \Gamma(\Upsilon\aleph)$, $\xi \in \Gamma(Rad(\Upsilon\aleph))$.

Thus, we have the subsequent equation.

$$g(h^*(\gamma, P\zeta), \aleph) = g(A_\aleph \gamma, P\zeta), \tag{2.11}$$

Consider that $\overline{\nabla}$ is a metric connection. We get

$$(\nabla_\gamma g)(\zeta, \eta) = g(h^l(\gamma, \zeta), \eta) + g(h^l(\gamma, \zeta\eta), \zeta). \tag{2.12}$$

Using the characteristics of a linear connection, we can obtain

$$(\nabla_\gamma h^l)(\zeta, \eta) = \nabla^l_\gamma (h^l(\zeta, \eta)) - h^l(\overline{\nabla}_\gamma \zeta, \eta) - h^l(\zeta, \overline{\nabla}_\gamma \eta), \tag{2.13}$$

$$(\nabla_\gamma h^s)(\zeta, \eta) = \nabla^s_\gamma (h^s(\zeta, \eta)) - h^s(\overline{\nabla}_\gamma \zeta, \eta) - h^s(\zeta, \overline{\nabla}_\gamma \eta). \tag{2.14}$$

Based on the description of a CR-lightlike submanifold in [4], we have

$$\Upsilon\aleph = \lambda \oplus \lambda',$$

where $\lambda = Rad(\Upsilon\aleph) \perp \psi Rad(\Upsilon\aleph) \perp \lambda_0$.

S and Q stand for the projection on λ and λ', respectively, then

$$\psi\gamma = f\gamma + w\gamma$$

for $\gamma, \zeta \in \Gamma(\Upsilon\aleph)$, where $f\gamma = \psi S\gamma$ and $w\gamma = \psi Q\gamma$.

On the other hand, we have

$$\psi\zeta = B\zeta + C\zeta$$

for any $\zeta \in \Gamma(tr(\Upsilon\aleph))$, $B\zeta \in \Gamma(\Upsilon\aleph)$ and $C\zeta \in \Gamma(tr(\Upsilon\aleph))$, unless $\aleph_1$ and $\aleph_2$ are denoted as ψL_1 and ψL_2, respectively.

Lemma 2.1. Assume that the screen distribution is totally geodesic and that $\aleph$ is a CR-lightlike submanifold of the golden semi-Riemannian manifold, then $\nabla_\gamma \zeta \in \Gamma(S(\Upsilon N))$, where $\gamma, \zeta \in \Gamma(S(\Upsilon\aleph))$.

Proof. For $\gamma, \zeta \in \Gamma(S(\Upsilon\aleph))$,

$$\begin{aligned} g(\nabla_\gamma \zeta, \aleph) &= g(\overline{\nabla}_\gamma \zeta - h(\gamma, \zeta), \aleph) \\ &= -g(\zeta, \overline{\nabla}_\gamma \aleph). \end{aligned}$$

Using (2.8),

$$\begin{aligned} g(\nabla_\gamma \zeta, \aleph) &= -g(\zeta, -A_\aleph \gamma + \nabla^\perp_\gamma \aleph) \\ &= g(\zeta, A_\aleph \gamma). \end{aligned}$$

Using (2.11),

$$g(\nabla_\gamma \zeta, \aleph) = g(h^*(\gamma, \zeta), \aleph).$$

Since screen distribution is totally geodesic, $h^*(\gamma, \zeta) = 0$,

$$g(\overline{\nabla}_\gamma \zeta, \aleph) = 0.$$

Using Lemma 1.2 in [1] p.g. 142, we have

$$\nabla_\gamma \zeta \in \Gamma(S(\Upsilon\aleph)),$$

where $\gamma, \zeta \in \Gamma(S(\Upsilon\aleph))$.

Theorem 2.2. Assume that $\aleph$ is a locally golden semi-Riemannian manifold $\overline{\aleph}$ with CR-lightlike properties, then $\nabla_\gamma \psi\gamma = \psi\nabla_\gamma\gamma$ for $\gamma \in \Gamma(\lambda_0)$.

Proof. Assume that $\gamma, \zeta \in \Gamma(\lambda_0)$. Using (2.5), we have

$$\begin{aligned} g(\nabla_\gamma \psi\gamma, \zeta) &= g(\overline{\nabla}_\gamma \psi\gamma - h(\gamma, \psi\gamma), \zeta) \\ g(\nabla_\gamma \psi\gamma, \zeta) &= g(\psi(\overline{\nabla}_\gamma \gamma), \zeta) \\ g(\nabla_\gamma \psi\gamma, \zeta) &= g(\psi(\nabla_\gamma \gamma), \zeta), \\ g(\nabla_\gamma \psi\gamma - \psi(\nabla_\gamma\gamma), \zeta) &= 0. \end{aligned}$$

Nondegeneracy of λ_0 implies

$$\nabla_\gamma \psi\gamma = \psi(\nabla_\gamma\gamma),$$

where $\gamma \in \Gamma(\lambda_0)$.

3. Geodesic CR-lightlike submanifolds

Definition 3.1. *[4]* A CR-lightlike submanifold of a golden semi-Riemannian manifold is mixed geodesic if h satisfies

$$h(\gamma, \alpha) = 0,$$

where h stands for second fundamental form, $\gamma \in \Gamma(\lambda)$, and $\alpha \in \Gamma(\lambda')$.

For $\gamma, \zeta \in \Gamma(\lambda)$ and $\alpha, \beta \in \Gamma(\lambda')$ if

$$h(\gamma, \zeta) = 0$$

and

$$h(\alpha, \beta) = 0,$$

then it is known as λ-geodesic and λ'-geodesic, respectively.

Theorem 3.2. Assume $\aleph$ is a CR-lightlike submanifold of $\overline{\aleph}$, which is a golden semi-Riemannian manifold. $\aleph$ is totally geodesic if

$$(L_g)(\gamma, \zeta) = 0$$

and

$$(L_\chi g)(\gamma, \zeta) = 0$$

for $\alpha, \beta \in \Gamma(\Upsilon\aleph)$, $\xi \in \Gamma(Rad(\Upsilon\aleph))$, and $\chi \in \Gamma(S(\Upsilon\aleph^\perp))$.

Proof. Since $\aleph$ is totally geodesic, then

$$h(\gamma, \zeta) = 0$$

for $\gamma, \zeta \in \Gamma(\Upsilon\aleph)$.

We know that $h(\gamma, \zeta) = 0$ if

$$g(h(\gamma, \zeta), \xi) = 0$$

and

$$g(h(\gamma, \zeta), \chi) = 0.$$

$$\begin{aligned} g(h(\gamma, \zeta), \xi) &= g(\overline{\nabla}_\gamma \zeta - \nabla_\gamma \zeta, \xi) \\ &= -g(\zeta, [\gamma, \xi] + \overline{\nabla}_\xi \gamma \\ &= -g(\zeta, [\gamma, \xi]) + g(\gamma, [\xi, \zeta]) + g(\overline{\nabla}_\zeta \xi, \gamma) \\ &= -(L_\xi g)(\gamma, \zeta) + g(\overline{\nabla}_\zeta \xi, \gamma) \\ &= -(L_\xi g)(\gamma, \zeta) - g(\xi, h(\gamma, \zeta))) \\ 2g(h(\gamma, \zeta) &= -(L_\xi g)(\gamma, \zeta). \end{aligned}$$

Since $g(h(\gamma, \zeta), \xi) = 0$, we have

$$(L_\xi g)(\gamma, \zeta) = 0.$$

Similarly,

$$\begin{aligned} g(h(\gamma, \zeta), \chi) &= g(\overline{\nabla}_\gamma \zeta - \nabla_\gamma \zeta, \chi) \\ &= -g(\zeta, [\gamma, \chi]) + g(\gamma, [\chi, \zeta]) + g(\overline{\nabla}_\zeta \chi, \gamma) \\ &= -(L_\chi g)(\gamma, \zeta) + g(\overline{\nabla}_\zeta \chi, \gamma) \\ 2g(h(\gamma, \zeta), \chi) &= -(L_\chi g)(\gamma, \zeta). \end{aligned}$$

Since $g(h(\gamma, \zeta), \chi) = 0$, we get

$$(L_\chi g)(\gamma, \zeta) = 0$$

for $\chi \in \Gamma(S(\Upsilon\aleph^\perp))$.

Lemma 3.3. Assume that $\overline{\aleph}$ is a golden semi-Riemannian manifold whose submanifold $\aleph$ is CR-lightlike, then

$$g(h(\gamma, \zeta), \chi) = g(A_\chi \gamma, \zeta)$$

for $\gamma \in \Gamma(\lambda)$, $\zeta \in \Gamma(\lambda')$ and $\chi \in \Gamma(S(\Upsilon\aleph^\perp))$.

Proof. Using (2.5), we get

$$\begin{aligned} g(h(\gamma, \zeta), \chi) &= g(\overline{\nabla}_\gamma \zeta - \nabla_\gamma \zeta, \chi) \\ &= g(\zeta, \overline{\nabla}_\gamma \chi). \end{aligned}$$

From (2.9), it follows that

$$g(h(\gamma, \zeta), \chi) = -g(\zeta, -A_\chi \gamma + \nabla^s_\gamma \chi + \lambda^s(\gamma, \chi))$$

$$
\begin{aligned}
&= g(\zeta, A_\chi\gamma) - g(\zeta, \nabla^s_\gamma\chi) - g(\zeta, \lambda^s(\gamma,\chi)) \\
g(h(\gamma,\zeta),\chi) &= g(\zeta, A_\chi\gamma),
\end{aligned}
$$

where $\gamma \in \Gamma(\lambda)$, $\zeta \in \Gamma(\lambda')$, $\chi \in \Gamma(S(\Upsilon\aleph^\perp))$.

Theorem 3.4. Assume that $\aleph$ is a CR-lightlike submanifold of the golden semi-Riemannian manifold and $\overline{\aleph}$ is mixed geodesic if

$$A^*_\xi\gamma \in \Gamma(\lambda_0 \perp \psi L_1)$$

and

$$A_\chi\gamma \in \Gamma(\lambda_0 \perp Rad(\Upsilon\aleph) \perp \psi L_1)$$

for $\gamma \in \Gamma(\lambda)$, $\xi \in \Gamma(Rad(\Upsilon\aleph))$, and $\chi \in \Gamma(S(\Upsilon\aleph^\perp))$.

Proof. For $\gamma \in \Gamma(\lambda)$, $\zeta \in \Gamma(\lambda')$, and $\chi \in \Gamma(S(\Upsilon\aleph^\perp))$, we get
Using (2.5),

$$
\begin{aligned}
g(h(\gamma,\zeta),\xi) &= g(\overline{\nabla}_\gamma\zeta - \nabla_\gamma\zeta, \xi) \\
&= -g(\zeta, \overline{\nabla}_\gamma\xi).
\end{aligned}
$$

Again using (2.5), we obtain

$$
\begin{aligned}
g(h(\gamma,\zeta),\xi) &= -g(\zeta, \nabla_\gamma\xi + h(\gamma,\xi)) \\
&= -g(\zeta, \nabla_\gamma\xi).
\end{aligned}
$$

Using (2.10), we have

$$
\begin{aligned}
g(h(\gamma,\zeta),\xi) &= -g(\zeta, -A^*_\xi\gamma + \nabla^{*t}_\gamma\xi) \\
g(\zeta, A^*_\xi\gamma) &= 0.
\end{aligned}
$$

Since the CR-lightlike submanifold $\aleph$ is mixed geodesic, we have

$$g(h(\gamma,\zeta),\xi) = 0$$

$$\Rightarrow g(\zeta, A^*_\xi\gamma) = 0$$

$$\Rightarrow A^*_\xi\gamma \in \Gamma(\lambda_0 \perp \psi L_1),$$

where $\gamma \in \Gamma(\lambda), \zeta \in \Gamma(\lambda')$.

From (2.5), we get

$$
\begin{aligned}
g(h(\gamma,\zeta),\chi) &= g(\overline{\nabla}_\gamma\zeta - \nabla_\gamma\zeta, \chi) \\
&= -g(\zeta, \overline{\nabla}_\gamma\chi).
\end{aligned}
$$

From (2.9), we get

$$
\begin{aligned}
g(h(\gamma,\zeta),\chi) &= -g(\zeta, -A_\chi\gamma + \nabla^s_\gamma\chi + \lambda^l(\gamma,\chi)) \\
g(h(\gamma,\zeta),\chi) &= g(\zeta, A_\chi\gamma).
\end{aligned}
$$

Since, $\aleph$ is mixed geodesic, then $g(h(\gamma,\zeta),\chi) = 0$

$$\Rightarrow g(\zeta, A_\chi\gamma) = 0.$$
$$A_\chi\gamma \in \Gamma(\lambda_0 \perp Rad(\Upsilon\aleph)\perp\psi_1).$$

Theorem 3.5. Suppose that $\aleph$ is a CR-lightlike submanifold of a golden semi-Riemannian manifold $\overline{\aleph}$, then $\aleph$ is λ'-geodesic if $A_\chi\eta$ and $A^*_\xi\eta$ have no component in $\aleph_2\perp\psi Rad(\Upsilon\aleph)$ for $\eta \in \Gamma(\lambda')$, $\xi \in \Gamma(Rad(\Upsilon\aleph))$, and $\chi \in \Gamma(S(\Upsilon\aleph^\perp))$.

Proof. From (2.5), we obtain

$$\begin{aligned} g(h(\eta,\beta),\chi) &= g(\overline{\nabla}_\eta\beta - \nabla_\gamma\zeta,\chi) \\ &= -\overline{g}(\nabla_\gamma\zeta,\chi), \end{aligned}$$

where $\chi,\beta \in \Gamma(\lambda')$.

Using (2.9), we have

$$\begin{aligned} g(h(\eta,\beta),\chi) &= -g(\beta, -A_\chi\eta + \nabla^s_\eta + \lambda^l(\eta,\chi)) \\ g(h(\eta,\beta),\chi) &= g(\beta, A_\chi\eta). \end{aligned} \tag{3.1}$$

Since $\aleph$ is λ'-geodesic, then $g(h(\eta,\beta),\chi) = 0$.

From (3.1), we get

$$g(\beta, A_\chi\eta) = 0.$$

Now,

$$\begin{aligned} g(h(\eta,\beta),\xi) &= g(\overline{\nabla}_\eta\beta - \nabla_\eta\beta,\xi) \\ &= g(\overline{\nabla}_\eta\beta,\xi) = -g(\beta,\overline{\nabla}_\eta\xi). \end{aligned}$$

From (2.10), we get

$$\begin{aligned} g(h(\eta,\beta),\xi) &= -g(\eta, -A^*_\xi\eta + \nabla^{*t}_\eta\xi) \\ g(h(\eta,\beta),\xi) &= g(A^*_\xi\beta,\eta). \end{aligned}$$

Since $\aleph$ is λ'- geodesic, then

$$g(h(\eta,\beta),\xi) = 0$$
$$\Rightarrow g(A^*_\xi\beta,\eta) = 0.$$

Thus, $A_\chi\eta$ and $A^*_\xi\eta$ have no component in $M_2\perp\psi Rad(\Upsilon\aleph)$.

Lemma 3.6. Assume that $\overline{\aleph}$ is a golden semi-Riemannian manifold that has a CR-lightlike submanifold $\aleph$. Due to the distribution's integrability, the following criteria hold.

(i) $\psi g(\lambda^l(\psi\gamma,\chi),\zeta) - g(\lambda^l(\gamma,\chi),\psi\zeta) = g(A_\chi\psi\gamma,\zeta) - g(A_\chi\gamma,\psi\zeta)$,

(ii) $g(\lambda^l(\psi\gamma),\xi) = g(A_\chi\gamma,\psi\xi)$,

(iii) $g(\lambda^l(\gamma,\chi),\xi) = g(A_\chi\psi\gamma,\psi\xi) - g(A_\chi\gamma,\psi\xi)$,

where $\gamma,\zeta \in \Gamma(\Upsilon\aleph)$, $\xi \in \Gamma(Rad(\Upsilon\aleph))$, and $\chi \in \Gamma(S(\Upsilon\aleph^\perp))$.

Proof. From Eq (2.9), we obtain

$$\begin{aligned} g(\lambda^l(\psi\gamma,\chi),\zeta) &= g(\overline{\nabla}_{\psi\gamma}\chi + A_\chi\psi\gamma - \nabla^s_{\psi\gamma}\chi,\zeta) \\ &= -g(\chi,\overline{\nabla}_{\psi\gamma}\zeta) + g(A_\chi\psi\gamma,\zeta). \end{aligned}$$

Using (2.5), we get

$$\begin{aligned} g(\lambda^l(\psi\gamma,\chi),\zeta) &= -g(\chi,\nabla_{\psi\gamma}\zeta + h(\psi\gamma,\zeta)) + g(A_\chi\psi\gamma,\zeta) \\ &= -g(\chi,h(\gamma,\psi\zeta)) + g(A_\chi\psi\gamma,\zeta). \end{aligned}$$

Again, using (2.5), we get

$$\begin{aligned} g(\lambda^l(\psi\gamma,\chi),\zeta) &= -g(\chi,\overline{\nabla}_\gamma\psi\zeta - \nabla_\gamma\psi\zeta) + g(A_\chi\psi\gamma,\zeta) \\ &= g(\overline{\nabla}_\gamma\chi,\psi\zeta) + g(A_\chi\psi\gamma,\zeta). \end{aligned}$$

Using (2.9), we have

$$\begin{aligned} g(\lambda^l(\psi\gamma,\chi),\zeta) &= g(-A_\chi\gamma + \nabla^s_\gamma\chi + \lambda^l(\gamma,\chi),\psi\zeta) + \\ g(\lambda^l(\psi\gamma,\chi),\zeta) - g(\lambda^l(\gamma,\chi),\psi\zeta) &= g(A_\chi\psi\gamma,\zeta) - g(A_\chi\gamma,\psi\zeta). \end{aligned}$$

(ii) Using (2.9), we have

$$\begin{aligned} g(\lambda^l(\psi\gamma,\chi),\xi) &= g(A_\chi\psi\gamma - \nabla^s_{\psi\gamma}\chi + \nabla_{\psi\gamma}\chi,\xi) \\ &= g(A_\chi\psi\gamma,\xi) - g(\chi,\overline{\nabla}_{\psi\gamma}\xi). \end{aligned}$$

Using (2.10), we get

$$\begin{aligned} g(\lambda^l(\psi\gamma,\chi),\xi) &= g(A_\chi\psi\gamma,\xi) + g(\chi,A^*_\xi\psi\gamma) - g(\chi,\nabla^{*t}_{\psi\gamma},\xi) \\ g(\lambda^l(\psi\gamma),\xi) &= g(A_\chi\gamma,\psi\xi). \end{aligned}$$

(iii) Replacing ζ by $\psi\xi$ in (i), we have

$$\psi g(\lambda^l(\psi\gamma,\chi),\psi\xi) - g(\lambda^l(\gamma,\chi),\psi^2\xi) = g(A_\chi\psi\gamma,\psi\xi) - g(A_\chi\gamma,\psi^2\xi).$$

Using Definition 2.1 in [18] p.g. 9, we get

$$\begin{aligned} \psi g(\lambda^l(\psi\gamma,\chi),\psi\xi) - g(\lambda^l(\gamma,\chi),(\psi+I)\xi) &= g(A_\chi\psi\gamma,\psi\xi) - g(A_\chi\gamma,(\psi+I)\xi) \\ \psi g(\lambda^l(\psi\gamma,\chi),\psi\xi) - g(\lambda^l(\gamma,\chi),\psi\xi) - g(\lambda^l(\gamma,\chi),\xi) &= g(A_\chi\psi\gamma,\psi\xi) - g(A_\chi\gamma,\psi\xi) - \\ &\quad g(A_\chi\gamma,\xi). \\ g(\lambda^l(\gamma,\chi),\xi) &= g(A_\chi\psi\gamma,\psi\xi) - g(A_\chi\gamma,\psi\xi). \end{aligned}$$

4. Totally umbilical CR-lightlike submanifolds

Definition 4.1. *[12]* A CR-lightlike submanifold of a golden semi-Riemannian manifold is totally umbilical if there is a smooth transversal vector field $H \in tr\,\Gamma(\Upsilon\aleph)$ that satisfies

$$h(\chi,\eta) = Hg(\chi,\eta),$$

where h is stands for second fundamental form and $\chi,\ \eta \in\ \Gamma(\Upsilon\aleph)$.

Theorem 4.2. Assume that the screen distribution is totally geodesic and that $\aleph$ is a totally umbilical CR-lightlike submanifold of the golden semi-Riemannian manifold $\overline{\aleph}$, then

$$A_{\psi\eta}\chi = A_{\psi\chi}\eta,\ \forall\ \chi, \eta \in \Gamma\lambda'.$$

Proof. Given that $\overline{\aleph}$ is a golden semi-Riemannian manifold,

$$\psi\overline{\nabla}_{\eta}\chi = \overline{\nabla}_{\eta}\psi\chi.$$

Using (2.5) and (2.6), we have

$$\psi(\nabla_{\eta}\chi) + \psi(h(\eta, \chi)) \quad = \quad -A_{\psi\chi}\eta + \nabla^{t}_{\eta}\psi\chi. \tag{4.1}$$

Interchanging η and χ, we obtain

$$\psi(\nabla_{\chi}\eta) + \psi(h(\chi, \eta)) = -A_{\psi\eta}\chi + \nabla^{t}_{\chi}\psi\eta. \tag{4.2}$$

Subtracting Eqs (4.1) and (4.2), we get

$$\psi(\nabla_{\eta}\chi - \nabla_{\chi}\eta) - \nabla^{t}_{\eta}\psi\chi + \nabla^{t}_{\chi}\psi\eta \quad = \quad A_{\psi\eta}\chi - A_{\psi\chi}\eta. \tag{4.3}$$

Taking the inner product with $\gamma \in \Gamma(\lambda_0)$ in (4.3), we have

$$\begin{aligned} g(\psi(\nabla_{\chi}\eta, \gamma) - g(\psi(\nabla_{\chi}\eta, \gamma) \quad &= \quad g(A_{\psi\eta}\chi, \gamma) \\ &\quad\ -g(A_{\psi\chi}\eta, \gamma). \end{aligned}$$

$$g(A_{\psi\eta}\chi - A_{\psi\chi}\eta, \gamma) \quad = \quad g(\nabla_{\chi}\eta, \psi\gamma) - g(\nabla_{\chi}\eta, \psi\gamma). \tag{4.4}$$

Now,

$$\begin{aligned} g(\nabla_{\chi}\eta, \psi\gamma) \quad &= \quad g(\overline{\nabla}_{\chi}\eta - h(\chi, \eta), \psi\gamma) \\ g(\nabla_{\chi}\eta, \psi\gamma) \quad &= \quad -g(\eta, (\overline{\nabla}_{\chi}\psi)\gamma - \psi(\overline{\nabla}_{\chi}\gamma)). \end{aligned}$$

Since ψ is parallel to $\overline{\nabla}$, i.e., $\overline{\nabla}_{\gamma}\psi = 0$,

$$g(\nabla_{\chi}\eta, \psi\gamma) = -\psi(\overline{\nabla}_{\chi}\gamma)).$$

Using (2.7), we have

$$\begin{aligned} g(\nabla_{\chi}\eta, \psi\gamma) \quad &= \quad -g(\psi\eta, \nabla_{\chi}\gamma + h^{s}(\chi, \gamma) + h^{l}(\chi, \gamma)) \\ g(\nabla_{\chi}\eta, \psi\gamma) \quad &= \quad -g(\psi\eta, \nabla_{\chi}\gamma) - g(\psi\eta, h^{s}(\chi, \gamma)) - g(\psi\eta, h^{l}(\chi, \gamma)). \end{aligned} \tag{4.5}$$

Since $\aleph$ is a totally umbilical CR-lightlike submanifold and the screen distribution is totally geodesic,

$$h^{s}(\chi, \gamma) = H^{s}g(\chi, \gamma) = 0$$

and

$$h^l(\chi,\gamma) = H^l g(\chi,\gamma) = 0,$$

where $\chi \in \Gamma(\lambda')$ and $\gamma \in \Gamma(\lambda_0)$.

From (4.5), we have

$$g(\nabla_\chi \eta, \psi\gamma) = -g(\psi\eta, \nabla_\chi \gamma).$$

From Lemma 2.1, we get

$$g(\nabla_\chi \eta, \psi\gamma) = 0.$$

Similarly,

$$g(\nabla_\eta \chi, \psi\gamma) = 0$$

Using (4.4), we have

$$g(A_{\psi\eta}\chi - A_{\psi\chi}\eta, \gamma) = 0.$$

Since λ_0 is nondegenerate,

$$A_{\psi\eta}\chi - A_{\psi\chi}\eta = 0$$
$$\Rightarrow A_{\psi\eta}\chi = A_{\psi\chi}\eta.$$

Theorem 4.3. Let $\aleph$ be the totally umbilical CR-lightlike submanifold of the golden semi-Riemannian manifold $\overline{\aleph}$. Consequently, $\aleph$'s sectional curvature, which is CR-lightlike, vanishes, resulting in $\overline{K}(\pi) = 0$, for the entire CR-lightlike section π.

Proof. We know that $\aleph$ is a totally umbilical CR-lightlike submanifold of $\overline{\aleph}$, then from (2.13) and (2.14),

$$(\nabla_\gamma h^l)(\zeta,\omega) = g(\zeta,\omega)\nabla^l_\gamma H^l - H^l\{(\nabla_\gamma g)(\zeta,\omega)\}, \tag{4.6}$$
$$(\nabla_\gamma h^s)(\zeta,\omega) = g(\zeta,\omega)\nabla^s_\gamma H^s - H^s\{(\nabla_\gamma g)(\zeta,\omega)\} \tag{4.7}$$

for a CR-lightlike section $\pi = \gamma \wedge \omega,\ \gamma \in \Gamma(\lambda_0), \omega \in \Gamma(\lambda')$.

From (2.12), we have $(\nabla_U g)(\zeta,\omega) = 0$. Therefore, from (4.6) and (4.7), we get

$$(\nabla_\gamma h^l)(\zeta,\omega) = g(\zeta,\omega)\nabla^l_\gamma H^l, \tag{4.8}$$
$$(\nabla_\gamma h^s)(\zeta,\omega) = g(\zeta,\omega)\nabla^s_\gamma H^s. \tag{4.9}$$

Now, from (4.8) and (4.9), we get

$$\begin{aligned}\{\overline{R}(\gamma,\zeta)\omega\}^{tr} \ = \ & g(\zeta,\omega)\nabla^l_\gamma H^l - g(\gamma,\omega)\nabla^l_\zeta H^l + g(\zeta,\omega)\lambda^l(\gamma,H^s) \\ & -g(\gamma,\omega)\lambda^l(\zeta,H^s) + g(\zeta,\omega)\nabla^s_\gamma H^s - g(\gamma,\omega)\nabla^s_\zeta H^s \\ & +g(\zeta,\omega)\lambda^s(\gamma,H^l) - g(\gamma,\omega)\lambda^s(\zeta,H^l).\end{aligned} \tag{4.10}$$

For any $\beta \in \Gamma(tr(\Upsilon\aleph))$, from Equation (4.10), we get

$$\begin{aligned}\overline{R}(\gamma,\zeta,\omega,\beta) \ = \ & g(\zeta,\omega)g(\nabla^l_\gamma H^l,\beta) - g(\gamma,\omega)g(\nabla^l_\zeta H^l,\beta) + \\ & g(\zeta,\omega)g(\lambda^l(\gamma,H^s),\zeta) - g(\gamma,\omega)g(\lambda^l(\zeta,H^s),\beta) +\end{aligned}$$

$$g(\zeta,\omega)g(\nabla^s_\gamma H^s,\beta)-g(\gamma,\omega)g(\nabla^s_\zeta H^s,\beta)$$
$$+g(\zeta,\omega)g(\lambda^s(\gamma,H^l),\beta)-g(\gamma,\omega)g(\lambda^s(\zeta,H^l,\beta).$$

$$\begin{aligned}R(\gamma,\omega,\psi\gamma,\psi\omega) &= g(\omega,\psi\gamma)g(\nabla^l_\gamma H^l,\psi\omega)-g(\gamma,\psi\gamma)g(\nabla^l_\omega H^l,\psi\omega)+\\ &\quad g(\omega,\psi\gamma)g(\lambda^l(\gamma,H^s),\psi\omega)-g(\gamma,\psi\gamma)g(\lambda^l(\omega,H^s),\psi\omega)+\\ &\quad g(\omega,\psi\gamma)g(\nabla^s_\gamma H^s,\psi\omega)-g(\gamma,\psi\gamma)g(\nabla^s_\omega H^s,\psi\omega)+\\ &\quad g(\omega,\psi\gamma)g(\lambda^s(\gamma,H^l),\psi\omega)-g(\gamma,\psi\gamma)g(\lambda^s(\omega,H^l,\psi U).\end{aligned}$$

For any unit vectors $\gamma\in\Gamma(\lambda)$ and $\omega\in\Gamma(\lambda')$, we have

$$\overline{R}(\gamma,\omega,\psi\gamma,\psi\omega)=\overline{R}(\gamma,\omega,\gamma,\omega)=0.$$

We have

$$K(\gamma)=K_N(\gamma\wedge\zeta)=g(\overline{R}(\gamma,\zeta)\zeta,\gamma),$$

where

$$\overline{R}(\gamma,\omega,\gamma,\omega)=g(\overline{R}(\gamma,\omega)\gamma,\omega)$$

or

$$\overline{R}(\gamma,\omega,\psi\gamma,\psi\omega)=g(\overline{R}(\gamma,\omega)\psi\gamma,\psi\omega)$$

i.e.,

$$\overline{K}(\pi)=0$$

for all CR-sections π.

5. Example

Example 5.1. We consider a semi-Riemannian manifold R^6_2 and a submanifold $\aleph$ of co-dimension 2 in R^6_2, given by equations

$$\upsilon_5=\upsilon_1\cos\alpha-\upsilon_2\sin\alpha-\upsilon_3 z_4\tan\alpha,$$

$$\upsilon_6=\upsilon_1\sin\alpha-\upsilon_2\cos\alpha-\upsilon_3\upsilon_4,$$

where $\alpha\in R-\{\frac{\pi}{2}+k\pi;\ k\in z\}$. The structure on R^6_2 is defined by

$$\psi(\frac{\partial}{\partial\upsilon_1},\frac{\partial}{\partial\upsilon_2},\frac{\partial}{\partial\upsilon_3},\frac{\partial}{\partial\upsilon_4},\frac{\partial}{\partial\upsilon_5},\frac{\partial}{\partial\upsilon_6})=(\overline{\phi}\ \frac{\partial}{\partial\upsilon_1},\overline{\phi}\frac{\partial}{\partial\upsilon_2},\phi\frac{\partial}{\partial\upsilon_3},\phi\frac{\partial}{\partial\upsilon_4},\phi\frac{\partial}{\partial\upsilon_5},\phi\frac{\partial}{\partial\upsilon_6}).$$

Now,

$$\psi^2(\frac{\partial}{\partial\upsilon_1},\frac{\partial}{\partial\upsilon_2},\frac{\partial}{\partial\upsilon_3},\frac{\partial}{\partial\upsilon_4},\frac{\partial}{\partial\upsilon_5},\frac{\partial}{\partial\upsilon_6})=((\overline{\phi}+1)\ \frac{\partial}{\partial\upsilon_1},(\overline{\phi}+1)\frac{\partial}{\partial\upsilon_2},(\phi+1)\frac{\partial}{\partial\upsilon_3},(\phi+1)\frac{\partial}{\partial\upsilon_4},$$

$$(\phi+1)\frac{\partial}{\partial\upsilon_5},(\phi+1)\frac{\partial}{\partial\upsilon_6})$$

$$\psi^2=\psi+I.$$

It follows that (R_2^6, ψ) is a golden semi-Reimannian manifold.

The tangent bundle $\Upsilon\aleph$ is spanned by

$$Z_0 = -\sin\alpha \frac{\partial}{\partial \upsilon_5} - \cos\alpha \frac{\partial}{\partial \upsilon_6} - \phi \frac{\partial}{\partial \upsilon_2},$$

$$Z_1 = -\phi \sin\alpha \frac{\partial}{\partial \upsilon_5} - \phi \cos\alpha \frac{\partial}{\partial \upsilon_6} + \frac{\partial}{\partial \upsilon_2},$$

$$Z_2 = \frac{\partial}{\partial \upsilon_5} - \bar{\phi} \sin\alpha \frac{\partial}{\partial \upsilon_2} + \frac{\partial}{\partial \upsilon_1},$$

$$Z_3 = -\bar{\phi} \cos\alpha \frac{\partial}{\partial \upsilon_2} + \frac{\partial}{\partial \upsilon_4} + i\frac{\partial}{\partial \upsilon_6}.$$

Thus, $\aleph$ is a 1-lightlike submanifold of R_2^6 with $Rad\Upsilon\aleph = \mathrm{Span}\{X_0\}$. Using golden structure of R_2^6, we obtain that $X_1 = \psi(X_0)$. Thus, $\psi(Rad\Upsilon\aleph)$ is a distribution on $\aleph$. Hence, the $\aleph$ is a CR-lightlike submanifold.

6. Conclusions

In general relativity, particularly in the context of the black hole theory, lightlike geometry finds its uses. An investigation is made into the geometry of the $\aleph$ golden semi-Riemannian manifolds that are CR-lightlike in nature. There are many intriguing findings on completely umbilical and completely geodesic CR-lightlike submanifolds that are examined. We present a required condition for a CR-lightlike submanifold to be completely geodesic. Moreover, it is demonstrated that the sectional curvature $\bar{K}$ of an entirely umbilical CR-lightlike submanifold $\aleph$ of a golden semi-Riemannian manifold $\bar{\aleph}$ disappears.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The present work (manuscript number IU/R&D/2022-MCN0001708) received financial assistance from Integral University in Lucknow, India as a part of the seed money project IUL/IIRC/SMP/2021/010. All of the authors would like to express their gratitude to the university for this support. The authors are highly grateful to editors and referees for their valuable comments and suggestions for improving the paper. The present manuscript represents the corrected version of preprint 10.48550/arXiv.2210.10445. The revised version incorporates the identities of all those who have made contributions, taking into account their respective skills and understanding.

Conflict of interest

Authors have no conflict of interests.

References

1. K. L. Duggal, A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Dordricht: Springer, 1996. https://doi.org/10.1007/978-94-017-2089-2
2. K. L. Duggal, B. Sahin, Generalized Cauchy Riemann lightlike submanifolds, *Acta Math. Hung.*, **112** (2006), 107–130. https://doi.org/10.1007/s10474-006-0068-y
3. K. L. Duggal, B. Sahin, Screen Cauchy Riemann lightlike submanifolds, *Acta Math. Hung.*, **106** (2005), 137–165. https://doi.org/10.1007/s10474-005-0011-7
4. B. Sahin, R. Gunes, Geodesic CR-lightlike submanifolds, *Beitr. Algebra Geom.*, **42** (2001), 583–594.
5. B. Sahin, R. Gunes, Integrability of distributions in CR-lightlike submanifolds, *Tamkang J. Math.*, **33** (2002), 209–222. https://doi.org/10.5556/j.tkjm.33.2002.209-222
6. K. L. Duggal, B. Sahin, *Differential geometry of lightlike submanifolds*, 2010.
7. K. L. Duggal, A. Bejancu, Lightlike submanifolds of semi-Riemannian manifolds, In: *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Dordrecht: Springer, 1996. https://doi.org/10.1007/978-94-017-2089-2_5
8. A. Bejancu, K. L. Duggal, Lightlike submanifolds of semi-Riemannian manifolds, *Acta Appl. Math.*, **38** (1995), 197–215. https://doi.org/10.1007/BF00992847
9. B. E. Acet, S. Y. Perktas, E. Kilic, Lightlike submanifolds of a para-Sasakian manifold, *Gen. Math. Noets*, **22** (2014), 22–45.
10. D. N. Kupeli, *Singular semi-Riemannian geometry*, Springer Science & Business Media, 1996. https://doi.org/10.1007/978-94-015-8761-7
11. D. H. Jin, J. W. Lee, Generic lightlike submanifolds of an indefinite Kahler manifold, *Int. J. Pure Appl. Math.*, **101** (2015), 543–560.
12. K. L. Duggal, D. H. Jin, Totally umbilical lightlike submanifolds, *Kodai Math. J.*, **26** (2003), 49–68. https://doi.org/10.2996/kmj/1050496648
13. K. L. Duggal, A. Bejancu, Lightlike hypersurfaces of indefinite Kaehler manifolds, *Acta. Appl. Math.*, **31** (1993), 171–190. https://doi.org/10.1007/BF00990541
14. K. L. Duggal, B. Sahin, Lightlike hypersurfaces of indefinite Sasakian manifolds, *Int. J. Math. Sci.*, **2007** (2007), 057585. https://doi.org/10.1155/2007/57585
15. M. Crasmareanu, C. E. Hretcanu, Golden differential geometry, *Chaos Soliton. Fract.*, **38** (2008), 1229–1238. https://doi.org/10.1016/j.chaos.2008.04.007
16. S. I. Goldberg, K. Yano, Polynomial structures on manifolds, *Kodai Math. Sem. Rep.*, **22** (1970), 199–218. https://doi.org/10.2996/kmj/1138846118
17. C. E. Hretcanu, Submanifolds in Riemannian manifold with golden structure, In: *Workshop on Finsler geometry and its applications*, Hungary, 2007.
18. M. Ahmad, M. A. Qayyoom, On submanifolds in a Riemannian manifold with golden structure, *Turk. J. Math. Comput. Sci.*, **11** (2019), 8–23.

19. M. Ahmad, M. A. Qayyoom, CR-submanifolds of a golden Riemannian manifold, *Palestine J. Math.*, **12** (2023), 689–696.

20. M. Ahmad, M. A. Qayyoom, Warped product skew semi-invariant submanifolds of locally golden Riemannian manifold, *Honam Math. J.*, **44** (2022), 1–16.

21. M. Ahmad, M. A. Qayyoom, Skew semi-invariant submanifolds in a golden Riemannian manifold, *J. Math. Control Sci. Appl.*, **7** (2021), 45–56.

22. M. Ahmad, J. B. Jun, M. A. Qayyoom, Hypersurfaces of a metallic Riemannian manifold, In: *Springer Proceedings in Mathematics and Statistics*, Singapore: Springer, 2020. https://doi.org/10.1007/978-981-15-5455-1_7

23. M. Ahmad, M. A. Qayyoom, Geometry of submanifolds of locally metallic Riemannian manifolds, *Ganita*, **71** (2021), 125–144.

24. A. Gezer, N. Cengiz, A. Salimov, On integrability of golden Riemannian structures, *Turk. J. Math.* **37** (2013), 693–703. https://doi.org/10.3906/mat-1108-35

25. N. O. Poyraz, E. Yasar, Lightlike hypersurfaces of a golden semi-Riemannianmanifold, *Mediterr. J. Math.*, **14** (2017), 204. https://doi.org/10.1007/s00009-017-0999-2

26. B. Gherici, Induced structure on golden Riemannian manifolds, *Beitr Algebra Geom.*, **59** (2018), 761–777. https://doi.org/10.1007/s13366-018-0392-8

27. F. Etayo, R. Santamaria, A. Upadhyay, On the geometry of almost golden Riemannian manifolds, *Mediterr. J. Math.*, **14** (2017), 187. https://doi.org/10.1007/s00009-017-0991-x

28. M. A. Qayyoom, M. Ahmad, Hypersurfaces immersed in a golden Riemannian manifold, *Afr. Mat.*, **33** (2022), 3. https://doi.org/10.1007/s13370-021-00954-x

29. M. Gok, S. Keles, E. Kilic, Some characterization of semi-invariant submanifolds of golden Riemannian manifolds, *Mathematics*, **7** (2019), 1209. https://doi.org/10.3390/math7121209