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Research article

Coupled systems of nonlinear sequential proportional Hilfer-type fractional differential equations with multi-point boundary conditions

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Abstract: In this paper, we studied the existence of solutions for a coupled system of nonlinear sequential proportional ψ-Hilfer fractional differential equations with multi-point boundary conditions. By using a Burton's version of the Krasnosel'skiı̆'s fixed-point theorem we established sufficient conditions for the existence result. An example illustrating our main result was also provided.

Keywords: coupled system; Hilfer fractional proportional derivative; multi-point boundary conditions; fixed-point theorem
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1. Introduction

Fractional differential equations (FDEs) provide many mathematical models in physics, biology, economics, and chemistry, etc [1–4]. In fact, it consists of many integrals and derivative operators of non-integer orders, which generalize the theory of ordinary differentiation and integration. Hence, a more general approach is allowed to calculus and one can say that the aim of the FDEs is to consider various phenomena by studying derivatives and integrals of arbitrary orders. For intercalary specifics about the theory of FDEs, the readers are referred to the books of Kilbas et al. [2] and Podlubny [4]. In the literature, several concepts of fractional derivatives have been represented, consisting of Riemann-Liouville, Liouville-Caputo, generalized Caputo, Hadamard, Katugampola, and Hilfer derivatives. The Hilfer fractional derivative [5] extends both Riemann-Liouville and Caputo fractional derivatives. For applications of Hilfer fractional derivatives in mathematics and physics, etc see [6–11]. For recent results on boundary value problems for fractional differential equations and inclusions with the Hilfer fractional derivative see the survey paper by Ntouyas [12]. The ψ-Riemann-Liouville fractional integral and derivative operators are discussed in [1], while the ψ-Hilfer fractional derivative is discussed in [13]. Recently, the notion of a generalized proportional fractional derivative was introduced by Jarad et al. [14–16]. For some recent results on fractional differential equations with generalized proportional derivatives, see [17, 18].

In [19], an existence result was proved via Krasnosel'skiĭ's fixed-point theorem for the following sequential boundary value problem of the form

$$\begin{cases} {}^{H}\mathbb{D}^{\alpha,\varsigma,\psi}\Big[\dfrac{{}^{H}\mathbb{D}^{\beta,\varsigma,\psi}p(w)}{\phi(w,p(w))} - \displaystyle\sum_{i=1}^{n}\mathbb{I}^{\nu_i;\psi}h_i(w,p(w))\Big] = \Upsilon(w,p(w)), \quad w\in[a,b], \\ p(a)=0, \quad {}^{H}\mathbb{D}^{b,\varsigma,\psi}p(a)=0, \quad p(b)=\tau p(\zeta), \end{cases} \tag{1.1}$$

where ${}^{H}\mathbb{D}^{\omega,\varsigma,\psi}$ indicates the ψ-Hilfer fractional derivative of order $\omega\in\{\alpha,\beta\}$, with $0<\alpha\le 1$, $1<\beta\le 2$, $0\le\varsigma<1$, $\mathbb{I}^{\nu_i;\psi}$ is the ψ-Riemann–Liouville fractional integral of order $\nu_i>0$, for $i=1,2,\dots,n$, $h_i\in C([0,1]\times\mathbb{R},\mathbb{R})$, for $i=1,2,\dots,n$, $\phi\in C([0,1]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $\Upsilon\in C([0,1]\times\mathbb{R},\mathbb{R})$, $\tau\in\mathbb{R}$ and $\zeta\in(a,b)$. In [16], the consideration of Hilfer-type generalized proportional fractional derivative operators was initiated.

Coupled systems of fractional order are also significant, as such systems appear in the mathematical models in science and engineering, such as bio-engineering [20], fractional dynamics [21], financial economics [22], etc. Coupled systems of FDEs with diverse boundary conditions have been the focus of many researches. In [23], the authors studied existence and Ulam-Hyers stability results of a coupled system of ψ-Hilfer sequential fractional differential equations. Existence and uniqueness results are derived in [24] for a coupled system of Hilfer-Hadamard fractional differential equations with fractional integral boundary conditions. Recently, in [25] a coupled system of nonlinear fractional differential equations involving the (k,ψ)-Hilfer fractional derivative operators complemented with multi-point nonlocal boundary conditions were discussed. Moreover, Samadi et al. [26] have considered a coupled system of Hilfer-type generalized proportional fractional differential equations.

In this article, motivated by the above works, we study a coupled system of ψ-Hilfer sequential generalized proportional FDEs with boundary conditions generated by the problem (1.1). More precisely, we consider the following coupled system of nonlinear proportional ψ-Hilfer sequential fractional differential equations with multi-point nonlocal boundary conditions of the form

$$\begin{cases} {}^{H}D^{\nu_1,\vartheta_1;\varsigma;\psi}\Big[\dfrac{{}^{H}D^{\nu_2,\vartheta_2;\varsigma;\psi}p_1(w)}{\Phi_1(w,p_1(w),p_2(w))} - \displaystyle\sum_{i=1}^{n}{}^{p}I^{\eta_i,\varsigma,\psi}H_i(w,p_1(w),p_2(w))\Big] = \Upsilon_1(w,p_1(w),p_2(w)),\ w\in[t_1,t_2], \\ {}^{H}D^{\nu_3,\vartheta_3;\varsigma;\psi}\Big[\dfrac{{}^{H}D^{\nu_4,\vartheta_4;\varsigma;\psi}p_1(w)}{\Phi_2(w,p_1(w),p_2(w))} - \displaystyle\sum_{j=1}^{m}{}^{p}I^{\bar{\eta}_j,\varsigma,\psi}G_j(w,p_1(w),p_2(w))\Big] = \Upsilon_2(w,p_1(w),p_2(w)),\ w\in[t_1,t_2], \\ p_1(t_1) = {}^{H}D^{\nu_2,\vartheta_2;\varsigma;\psi}p_1(t_1)=0, \quad p_1(t_2)=\theta_1 p_2(\xi_1), \\ p_2(t_1) = {}^{H}D^{\nu_4,\vartheta_4;\varsigma;\psi}p_2(t_1)=0, \quad p_2(t_2)=\theta_2 p_1(\xi_2), \end{cases} \tag{1.2}$$

where ${}^{H}D^{\nu,\vartheta_1;\varsigma;\psi}$ denotes the ψ-Hilfer generalized proportional derivatives of order $\nu\in\{\nu_1,\nu_2,\nu_3,\nu_4\}$, with parameters ϑ_l, $0\le\vartheta_l\le 1$, $l\in\{1,2,3,4\}$, ψ is a continuous function on $[t_1,t_2]$, with $\psi'(w)>0$, ${}^{p}I^{\eta,\varsigma,\psi}$ is the generalized proportional integral of order $\eta>0$, $\eta\in\{\eta_i,\eta_j\}$, $\theta_1,\theta_2\in\mathbb{R}$, $\xi_1,\xi_2\in[t_1,t_2]$, $\Phi_1,\Phi_2\in C([t_1,t_2]\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $H_i,G_j,\Upsilon_1,\Upsilon_2\in C([t_1,t_2]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, for $i=1,2,\dots,n$ and $j=1,2,\dots,m$.

We emphasize that:

- We study a general system involving ψ-Hilfer proportional fractional derivatives.
- Our equations contain fractional derivatives of different orders as well as sums of fractional integrals of different orders.
- Our system contains nonlocal coupled boundary conditions.
- Our system covers many special cases by fixing the parameters involved in the problem. For example, taking $\psi(w) = w$, it will reduce to a coupled system of Hilfer sequential generalized proportional FDEs with boundary conditions, while if $\varsigma = 1$, it reduces to a coupled system of ψ-Hilfer sequential FDEs. Besides, by taking $\Phi_1, \Phi_2 = 1$ in the problem (1.2), then we obtain the following new coupled system of the form:

$$\begin{cases} {}^H D^{\nu_1,\vartheta_1;\varsigma;\psi}\left[{}^H D^{\nu_2,\vartheta_2;\varsigma;\psi} p_1(w) - \displaystyle\sum_{i=1}^{n} {}^p I^{\eta_i,\varsigma,\psi} H_i(w, p_1(w), p_2(w))\right] = \Upsilon_1(w, p_1(w), p_2(w)),\ w \in [t_1, t_2], \\ {}^H D^{\nu_3,\vartheta_3;\varsigma;\psi}\left[{}^H D^{\nu_4,\vartheta_4;\varsigma;\psi} p_1(w) - \displaystyle\sum_{j=1}^{m} {}^p I^{\bar{\eta}_j,\varsigma,\psi} G_j(w, p_1(w), p_2(w))\right] = \Upsilon_2(w, p_1(w), p_2(w)),\ w \in [t_1, t_2], \\ p_1(t_1) = {}^H D^{\nu_2,\vartheta_2;\varsigma;\psi} p_1(t_1) = 0,\ \ p_1(t_2) = \theta_1 p_2(\xi_1), \\ p_2(t_1) = {}^H D^{\nu_4,\vartheta_4;\varsigma;\psi} p_2(t_1) = 0,\ \ p_2(t_2) = \theta_2 p_1(\xi_2). \end{cases}$$

In obtaining the existence result of the problem (1.2), first the problem (1.2) is converted into a fixed-point problem and then a generalization of Krasnosel'skiǐ's fixed-point theorem due to Burton is applied.

The structure of this article has been organized as follows: In Section 2, some necessary concepts and basic results concerning our problem are presented. The main result for the problem (1.2) is proved in Section 3, while Section 4 contains an example illustrating the obtained result.

2. Preliminaries

In this section, we summarize some known definitions and lemmas needed in our results.

Definition 2.1. [17, 18] *Let $\varsigma \in (0, 1]$ and $\nu > 0$. The fractional proportional integral of order ν of the continuous function $\mathfrak{F}$ is defined by*

$${}^p I^{\nu,\varsigma,\psi}\mathfrak{F}(w) = \frac{1}{\varsigma^\nu \Gamma(\nu)} \int_{t_1}^{w} e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(s))} (\psi(w) - \psi(s))^{\nu-1} \mathfrak{F}(s)\psi'(s)ds,\ \ t_1 > w.$$

Definition 2.2. [17, 18] *Let $\varsigma \in (0, 1]$, $\nu > 0$, and $\psi(w)$ is a continuous function on $[t_1, t_2]$, $\psi'(w) > 0$. The generalized proportional fractional derivative of order ν of the continuous function $\mathfrak{F}$ is defined by*

$$({}^p D^{\nu,\varsigma,\psi}\mathfrak{F})(w) = \frac{({}^p D^{n,\varsigma,\psi})}{\varsigma^{n-\nu}\Gamma(n-\nu)} \int_{t_1}^{w} e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(s))} (\psi(w) - \psi(s))^{n-\nu-1} \mathfrak{F}(s)\psi'(s)ds,$$

where $n = [\rho] + 1$ and $[\nu]$ denotes the integer part of the real number ν, where $D^{n,\varsigma,\psi} = \underbrace{D^{\varsigma,\psi} \cdots D^{\varsigma,\psi}}_{n-times}$.

Now the generalized Hilfer proportional fractional derivative of order ν of function $\mathfrak{F}$ with respect to another function ψ is introduced.

Definition 2.3. [27] *Let $\varsigma \in (0,1]$, $\mathfrak{F}, \psi \in C^m([t_1,t_2],\mathbb{R})$ in which ψ is positive and strictly increasing with $\psi^{'}(w) \neq 0$ for all $w \in [t_1,t_2]$. The ψ-Hilfer generalized proportional fractional derivative of order ν and type ϑ for $\mathfrak{F}$ with respect to another function ψ is defined by*

$$\left({}^{H}D^{\nu,\vartheta,\varsigma,\psi}\mathfrak{F}\right)(w) = {}^{p}I^{\vartheta(n-\nu),\varsigma,\psi}[{}^{p}D^{n,\varsigma,\psi}({}^{p}I^{(1-\vartheta)(n-\nu),\varsigma,\psi}\mathfrak{F})](w),$$

where $n-1<\nu<n$ and $0 \leq \vartheta \leq 1$.

Lemma 2.4. [27] *Let $m-1<\nu<m, n \in N$, $0<\varsigma\leq 1$, $0\leq\vartheta\leq 1$ and $m-1<\gamma<m$ such that $\gamma = \nu + m\vartheta - \nu\vartheta$. If $\mathfrak{F} \in C([t_1,t_2],\mathbb{R})$ and ${}^{p}I^{(m-\gamma,\varsigma,\psi)}\mathfrak{F} \in C^m([t_1,t_2],\mathbb{R})$, then*

$$\left({}^{p}I^{\nu,\varsigma,\psi}\ {}^{H}D^{\nu,\vartheta,\varsigma,\psi}\mathfrak{F}\right)(w) = \mathfrak{F}(w) - \sum_{j=1}^{n}\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma-j}}{\varsigma^{\gamma-j}\Gamma(\gamma-j+1)}\left({}^{p}I^{j-\gamma,\varsigma,\psi}\mathfrak{F}\right)(t_1).$$

To prove the main result we need the following lemma, which concerns a linear variant of the ψ-Hilfer sequential proportional coupled system (1.2). This lemma plays a pivotal role in converting the nonlinear problem in system (1.2) into a fixed-point problem.

Lemma 2.5. *Let $0<\nu_1,\nu_3\leq 1$, $1<\nu_2,\nu_4\leq 2$, $0\leq\vartheta_i\leq 1$, $\gamma_i=\nu_i+\vartheta_i(1-\nu_i)$, $i=1,3$ and $\gamma_j=\nu_j+\vartheta_j(2-\nu_j)$, $j=2,4$, $\Theta = M_1N_2-M_2N_1\neq 0$, ψ is a continuous function on $[t_1,t_2]$, with $\psi'(w)>0$, and $Q_1,Q_2\in C([t_1,t_2],\mathbb{R})$, $\Phi_1,\Phi_2\in C([t_1,t_2]\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $H_i,G_j,Q_1,Q_2\in C([t_1,t_2]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, for $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$, and ${}^{p}I^{(1-\gamma_i,\varsigma,\psi)}Q_j\in C^m([t_1,t_2],\mathbb{R})$, $i=1,2,3,4$, $j=1,2$. Then the pair (p_1,p_2) is a solution of the system*

$$\begin{cases}{}^{H}D^{\nu_1,\vartheta_1;\varsigma;\psi}\Big[\dfrac{{}^{H}D^{\nu_2,\vartheta_2;\varsigma;\psi}p_1(w)}{\Phi_1(w,p_1(w),p_2(w))}-\displaystyle\sum_{i=1}^{n}{}^{p}I^{\eta_i,\varsigma,\psi}H_i(w,p_1(w),p_2(w))\Big]=Q_1(w),\ w\in[t_1,t_2],\\ {}^{H}D^{\nu_3,\vartheta_3;\varsigma;\psi}\Big[\dfrac{{}^{H}D^{\nu_4,\vartheta_4;\varsigma;\psi}p_1(w)}{\Phi_2(w,p_1(w),p_2(w))}-\displaystyle\sum_{j=1}^{m}{}^{p}I^{\bar{\eta}_j,\varsigma,\psi}G_j(w,p_1(w),p_2(w))\Big]=Q_2(w),\ w\in[t_1,t_2],\\ p_1(t_1)={}^{H}D^{\nu_2,\vartheta_2;\varsigma;\psi}p_1(t_1)=0,\quad p_1(t_2)=\theta_1p_2(\xi_1),\\ p_2(t_1)={}^{H}D^{\nu_4,\vartheta_4;\varsigma;\psi}p_2(t_1)=0,\quad p_2(t_2)=\theta_2p_1(\xi_2),\end{cases}$$

if and only if

$$\begin{aligned}p_1(w) \;=\;& {}^{p}I^{\nu_2,\varsigma,\psi}\Phi_1(w,p_1(w),p_2(w))\Big(\sum_{i=1}^{n}{}^{p}I^{\eta_i,\varsigma,\psi}H_i(w,p_1(w),p_2(w))+{}^{p}I^{\nu_1,\varsigma,\psi}Q_1(w)\Big)\\ &+\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\Big\{N_2\Big[\theta_1{}^{p}I^{\nu_4,\varsigma,\psi}\Phi_2(\xi_1,p_1(\xi_1),p_2(\xi_1))\\ &\times\Big(\sum_{j=1}^{m}{}^{p}I^{\bar{\eta}_j,\varsigma,\psi}G_j(\xi_1,p_1(\xi_1),p_2(\xi_1))+{}^{p}I^{\nu_3,\varsigma,\psi}Q_2(\xi_1))\Big)\\ &-{}^{p}I^{\nu_2,\varsigma,\psi}\Phi_1(t_2,p_1(t_2),p_2(t_2))\\ &\times\Big(\sum_{i=1}^{n}{}^{p}I^{\eta_i,\varsigma,\psi}H_i(t_2,p_1(t_2),p_2(t_2))+{}^{p}I^{\nu_1,\varsigma,\psi}Q_1(t_2)\Big)\Big]\end{aligned}$$

$$
\begin{aligned}
&+M_2\Big[\theta_2{}^pI^{\nu_2,\varsigma,\psi}\Phi_1(\xi_2,p_1(\xi_2),p_2(\xi_2))\\
&\times\Big(\sum_{i=1}^{n}{}^pI^{\eta_i,\varsigma,\psi}H_i(\xi_2,p_1(\xi_2),p_2(\xi_2))+{}^pI^{\nu_1,\varsigma,\psi}Q_1(\xi_2))\Big)\\
&-{}^pI^{\nu_4,\varsigma,\psi}\Phi_2(t_2,p_1(t_2),p_2(t_2))\\
&\times\Big(\sum_{j=1}^{m}{}^pI^{\bar{\eta}_j,\varsigma,\psi}G_j(t_2,p_1(t_2),p_2(t_2))+{}^pI^{\nu_3,\varsigma,\psi}Q_2(t_2)\Big)\Big]\Big\}
\end{aligned}\tag{2.1}
$$

and

$$
\begin{aligned}
p_2(w) \;=\;& {}^pI^{\nu_4,\varsigma,\psi}\Phi_2(w,p_1(w),p_2(w))\Big(\sum_{j=1}^{m}{}^pI^{\bar{\eta}_j,\varsigma,\psi}G_j(w,p_1(w),p_2(w))+{}^pI^{\nu_3,\varsigma,\psi}Q_2(w)\Big)\\
&+\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_4-1}}{\Theta\varsigma^{\gamma_4-1}\Gamma(\gamma_4)}\Big\{N_1\Big[\theta_1{}^pI^{\nu_4,\varsigma,\psi}\Phi_2(\xi_1,p_1(\xi_1),p_2(\xi_1))\\
&\times\Big(\sum_{j=1}^{m}{}^pI^{\bar{\eta}_j,\varsigma,\psi}G_j(\xi_1,p_1(\xi_1),p_2(\xi_1))+{}^pI^{\nu_3,\varsigma,\psi}Q_2(\xi_1))\Big)\\
&-{}^pI^{\nu_2,\varsigma,\psi}\Phi_2(t_2,p_1(t_2),p_2(t_2))\\
&\times\Big(\sum_{i=1}^{n}{}^pI^{\eta_i,\varsigma,\psi}H_i(t_2,p_1(t_2),p_2(t_2))+{}^pI^{\nu_1,\varsigma,\psi}Q_1(t_2)\Big)\Big]\\
&+M_1\Big[\theta_2{}^pI^{\nu_2,\varsigma,\psi}\Phi_1(\xi_2,p_1(\xi_2),p_2(\xi_2))\\
&\times\Big(\sum_{i=1}^{n}{}^pI^{\eta_i,\varsigma,\psi}H_i(\xi_2,p_1(\xi_2),p_2(\xi_2))+{}^pI^{\nu_1,\varsigma,\psi}Q_1(\xi_2))\Big)\\
&-{}^pI^{\nu_4,\varsigma,\psi}\Phi_2(t_2,p_1(t_2),p_2(t_2))\\
&\times\Big(\sum_{j=1}^{m}{}^pI^{\bar{\eta}_j,\varsigma,\psi}G_j(t_2,p_1(t_2),p_2(t_2))+{}^pI^{\nu_3,\varsigma,\psi}Q_2(t_2)\Big)\Big]\Big\},
\end{aligned}\tag{2.2}
$$

where

$$
\begin{aligned}
M_1 &= \frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(t_2)-\psi(t_1))}(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\varsigma^{\gamma_2-1}\Gamma(\gamma_2)},\quad M_2=\frac{\theta_1e^{\frac{\varsigma-1}{\varsigma}(\psi(\xi_1)-\psi(t_1))}(\psi(\xi_1)-\psi(t_1))^{\gamma_4-1}}{\varsigma^{\gamma_4-1}\Gamma(\gamma_4)},\\
N_1 &= \theta_2\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(\xi_2)-\psi(t_1))}(\psi(\xi_2)-\psi(t_1))^{\gamma_2-1}}{\varsigma^{\gamma_2-1}\Gamma(\gamma_2)},\quad N_2=\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(t_2)-\psi(t_1))}(\psi(t_2)-\psi(t_1))^{\gamma_4-1}}{\varsigma^{\gamma_4-1}\Gamma(\gamma_4)}.
\end{aligned}\tag{2.3}
$$

Proof. Due to Lemma 2.4 with $m=1$, we get

$$
\begin{aligned}
&\frac{{}^HD^{\nu_2,\vartheta_2;\varsigma;\psi}p_1(w)}{\Phi_1(w,p_1(w),p_2(w))}-\sum_{i=1}^{n}{}^pI^{\eta_i,\varsigma,\psi}H_i(w,p_1(w),p_2(w))={}^pI^{\nu_1;\varsigma;\psi}Q_1(w)\\
&+c_0\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_1-1}}{\varsigma^{\gamma_1-1}\Gamma(\gamma_1)},
\end{aligned}
$$

$$\frac{{}^{H}D^{\nu_4,\vartheta_4;\varsigma;\psi}p_2(w)}{\Phi_2(w,p_1(w),p_2(w))}-\sum_{j=1}^{m}{}^{p}I^{\bar{\eta}_j,\varsigma,\psi}G_j(w,p_1(w),p_2(w))={}^{p}I^{\nu_3;\varsigma;\psi}Q_2(w)$$
$$+d_0\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_3-1}}{\varsigma^{\gamma_3-1}\Gamma(\gamma_3)}, \tag{2.4}$$

where $c_0, d_0 \in \mathbb{R}$. Now applying the boundary conditions

$${}^{H}D^{\nu_2,\vartheta_2;\varsigma;\psi}p_1(t_1)={}^{H}D^{\nu_4,\vartheta_4;\varsigma;\psi}p_1(t_1)=0,$$

we get $c_0 = d_0 = 0$. Hence

$${}^{H}D^{\nu_2,\vartheta_2;\varsigma;\psi}p_1(w) = \Phi_1(w,p_1(w),p_2(w))\Big(\sum_{i=1}^{n}{}^{p}I^{\eta_i,\varsigma,\psi}H_i(w,p_1(w),p_2(w))+{}^{p}I^{\nu_1;\varsigma;\psi}Q_1(w)\Big),$$

$${}^{H}D^{\nu_4,\vartheta_4;\varsigma;\psi}p_2(w) = \Phi_2(w,p_1(w),p_2(w))\Big(\sum_{j=1}^{m}{}^{p}I^{\bar{\eta}_j,\varsigma,\psi}G_j(w,p_1(w),p_2(w))+{}^{p}I^{\nu_3;\varsigma;\psi}Q_2(w)\Big). \tag{2.5}$$

Now, by taking the operators ${}^{p}I^{\nu_2,\varsigma,\psi}$ and ${}^{p}I^{\nu_4,\varsigma,\psi}$ into both sides of (2.5) and using Lemma 2.4, we get

$$\begin{aligned} p_1(w) &= {}^{p}I^{\nu_2;\varsigma;\psi}\Phi_1(w,p_1(w),p_2(w))\Big(\sum_{i=1}^{n}{}^{p}I^{\eta_i,\varsigma,\psi}H_i(w,p_1(w),p_2(w)) \\ &\quad +{}^{p}I^{\nu_1;\varsigma;\psi}Q_1(w)\Big)+c_1\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_2-1}}{\varsigma^{\gamma_2-1}\Gamma(\gamma_2)} \\ &\quad +c_2\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_2-2}}{\varsigma^{\gamma_2-2}\Gamma(\gamma_2-1)}, \end{aligned}$$

$$\begin{aligned} p_2(w) &= {}^{p}I^{\nu_4;\varsigma;\psi}\Phi_2(w,p_1(w),p_2(w))\Big(\sum_{j=1}^{m}{}^{p}I^{\bar{\eta}_j,\varsigma,\psi}G_j(w,p_1(w),p_2(w)) \\ &\quad +{}^{p}I^{\nu_3;\varsigma;\psi}Q_2(w)\Big)+d_1\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_4-1}}{\varsigma^{\gamma_4-1}\Gamma(\gamma_4)} \\ &\quad +d_2\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_4-2}}{\varsigma^{\gamma_4-2}\Gamma(\gamma_4-1)}. \end{aligned} \tag{2.6}$$

Applying the conditions $p_1(t_1) = p_2(t_1) = 0$ in (2.6), we get $c_2 = d_2 = 0$ since $\gamma_2 \in [\nu_2, 2]$ and $\gamma_4 \in [\nu_4, 2]$. Thus we have

$$\begin{aligned} p_1(w) &= {}^{p}I^{\nu_2;\varsigma;\psi}\Big(\Phi_1(w,p_1(w),p_2(w))\Big(\sum_{i=1}^{n}{}^{p}I^{\eta_i,\varsigma,\psi}H_i(w,p_1(w),p_2(w)) \\ &\quad +{}^{p}I^{\nu_1;\varsigma;\psi}Q_1(w)\Big)\Big)+c_1\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_2-1}}{\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}, \end{aligned}$$

$$p_2(w) = {}^pI^{\gamma_4;\varsigma;\psi}\Big(\Phi_2(w,p_1(w),p_2(w))\Big(\sum_{j=1}^{m}{}^pI^{\bar{\eta}_j,\varsigma,\psi}G_j(w,p_1(w),p_2(w)) + {}^pI^{\nu_3;\varsigma;\psi}Q_2(w)\Big)\Big) + d_1\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_4-1}}{\varsigma^{\gamma_4-1}\Gamma(\gamma_4)}. \tag{2.7}$$

In view of (2.7) and the conditions $p_1(t_2)=\theta_1 p_2(\xi_1)$ and $p_2(t_2)=\theta_2 p_1(\xi_2)$, we get

$$\begin{aligned}
&{}^pI^{\nu_2,\varsigma,\psi}\Phi_1(t_2,p_1(t_2),p_2(t_2))\Big(\sum_{i=1}^{n}{}^pI^{\eta_i,\varsigma,\psi}H_i(t_2,p_1(t_2),p_2(t_2)) + {}^pI^{\nu_1,\varsigma,\psi}Q_1(t_2)\Big)\\
&+c_1\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(t_2)-\psi(t_1))}(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\\
= {}&\theta_1{}^pI^{\nu_4,\varsigma,\psi}\Phi_2(\xi_1,p_1(\xi_1),p_2(\xi_1))\Big(\sum_{j=1}^{m}{}^pI^{\bar{\eta}_j,\varsigma,\psi}G_j(\xi_1,p_1(\xi_1),p_2(\xi_1)) + {}^pI^{\nu_3,\varsigma,\psi}Q_2(\xi_1))\Big)\\
&+d_1\frac{\theta_1 e^{\frac{\varsigma-1}{\varsigma}(\psi(\xi_1)-\psi(t_1))}(\psi(\xi_1)-\psi(t_1))^{\gamma_4-1}}{\varsigma^{\gamma_4-1}\Gamma(\gamma_4)},
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&{}^pI^{\nu_4,\varsigma,\psi}\Phi_2(t_2,p_1(t_2),p_2(t_2))\Big(\sum_{j=1}^{m}{}^pI^{\bar{\eta}_j,\varsigma,\psi}G_j(t_2,p_1(t_2),p_2(t_2)) + I^{\nu_3,\varsigma,\psi}Q_2(t_2)\Big)\\
&+d_1\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(t_2)-\psi(t_1))}(\psi(t_2)-\psi(t_1))^{\gamma_4-1}}{\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\\
= {}&\theta_2{}^pI^{\nu_2,\varsigma,\psi}\Phi_1(\xi_2,p_1(\xi_2),p_2(\xi_2))\Big(\sum_{i=1}^{n}{}^pI^{\eta_i,\varsigma,\psi}H_i(\xi_2,p_1(\xi_2),p_2(\xi_2)) + {}^pI^{\nu_1,\varsigma,\psi}Q_1(\xi_2))\Big)\\
&+c_1\frac{\theta_2 e^{\frac{\varsigma-1}{\varsigma}(\psi(\xi_2)-\psi(t_1))}(\psi(\xi_2)-\psi(t_1))^{\gamma_2-1}}{\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}.
\end{aligned} \tag{2.9}$$

Due to (2.3), (2.8), and (2.9), we have

$$\begin{aligned}
c_1M_1 - d_1M_2 &= M,\\
-c_1N_1 + d_1N_2 &= N,
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
M = {}&\theta_1{}^pI^{\nu_4,\varsigma,\psi}\Phi_2(\xi_1,p_1(\xi_1),p_2(\xi_1))\Big(\sum_{j=1}^{m}{}^pI^{\bar{\eta}_j,\varsigma,\psi}G_j(\xi_1,p_1(\xi_1),p_2(\xi_1)) + {}^pI^{\nu_3,\varsigma,\psi}Q_2(\xi_1))\Big)\\
&-{}^pI^{\nu_2,\varsigma,\psi}\Phi_1(t_2,p_1(t_2),p_2(t_2))\Big(\sum_{i=1}^{n}{}^pI^{\eta_i,\varsigma,\psi}H_i(t_2,p_1(t_2),p_2(t_2)) + {}^pI^{\nu_1,\varsigma,\psi}Q_1(t_2)\Big),\\
N = {}&\theta_2{}^pI^{\nu_2,\varsigma,\psi}\Phi_1(\xi_2,p_1(\xi_2),p_2(\xi_2))\Big(\sum_{i=1}^{n}{}^pI^{\eta_i,\varsigma,\psi}H_i(\xi_2,p_1(\xi_2),p_2(\xi_2)) + {}^pI^{\nu_1,\varsigma,\psi}Q_1(\xi_2))\Big)\\
&-{}^pI^{\nu_4,\varsigma,\psi}\Phi_2(t_2,p_1(t_2),p_2(t_2))\Big(\sum_{j=1}^{m}{}^pI^{\bar{\eta}_j,\varsigma,\psi}G_j(t_2,p_1(t_2),p_2(t_2)) + {}^pI^{\nu_3,\varsigma,\psi}Q_2(t_2)\Big).
\end{aligned}$$

By solving the above system, we conclude that

$$c_1 = \frac{1}{\Theta}[N_2M + M_2N], \ \ d_1 = \frac{1}{\Theta}[M_1N + N_1M].$$

Replacing the values c_1 and d_1 in Eq (2.7), we obtain the solutions (2.1) and (2.2). The converse is obtained by direct computation. The proof is complete. □

3. An existence result

Let $\mathbb{Y} = C([t_1, t_2], \mathbb{R}) = \{p : [t_1, t_2] \longrightarrow \mathbb{R} \text{ is continuous}\}$. The space $\mathbb{Y}$ is a Banach space with the norm $\|p\| = \sup_{w\in[t_1,t_2]} |p(w)|$. Obviously, the space $(\mathbb{Y} \times \mathbb{Y}, \|(p_1, p_2)\|)$ is also a Banach space with the norm $\|(p_1, p_2)\| = \|p_1\| + \|p_2\|$.

Due to Lemma 2.5, we define an operator $\mathbb{V} : \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y} \times \mathbb{Y}$ by

$$\mathbb{V}(p_1, p_2)(w) = \begin{pmatrix} \mathbb{V}_1(p_1, p_2)(w) \\ \mathbb{V}_2(p_1, p_2)(w) \end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned}
\mathbb{V}_1(p_1, p_2)(w) \ = \ & {}^{p}I^{\nu_2,\varsigma,\psi}\Phi_1(w, p_1(w), p_2(w))\Big(\sum_{i=1}^{n} {}^{p}I^{\eta_i,\varsigma,\psi}H_i(w, p_1(w), p_2(w)) \\
& + {}^{p}I^{\nu_1,\varsigma,\psi}\Upsilon_1(w, p_1(w), p_2(w))\Big) \\
& + \frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w) - \psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\Bigg\{N_2\Big[\theta_1 {}^{p}I^{\nu_4,\varsigma,\psi}\Phi_2(\xi_1, p_1(\xi_1), p_2(\xi_1)) \\
& \times\Big(\sum_{j=1}^{m} {}^{p}I^{\bar{\eta}_j,\varsigma,\psi}G_j(\xi_1, p_1(\xi_1), p_2(\xi_1)) + {}^{p}I^{\nu_3,\varsigma,\psi}\Upsilon_2(\xi_1, p_1(\xi_1), p_2(\xi_1))\Big) \\
& - {}^{p}I^{\nu_2,\varsigma,\psi}\Phi_1(t_2, p_1(t_2), p_2(t_2))\Big(\sum_{i=1}^{n} {}^{p}I^{\eta_i,\varsigma,\psi}H_i(t_2, p_1(t_2), p_2(t_2)) \\
& + {}^{p}I^{\nu_1,\varsigma,\psi}\Upsilon_1(t_2, p_1(t_2), p_2(t_2))\Big)\Big] + M_2\Big[\theta_2 {}^{p}I^{\nu_2,\varsigma,\psi}\Phi_1(\xi_2, p_1(\xi_2), p_2(\xi_2)) \\
& \times\Big(\sum_{i=1}^{n} {}^{p}I^{\eta_i,\varsigma,\psi}H_i(\xi_2, p_1(\xi_2), p_2(\xi_2)) + {}^{p}I^{\nu_1,\varsigma,\psi}\Upsilon_1(\xi_2, p_1(\xi_1), p_2(\xi_2))\Big) \\
& - {}^{p}I^{\nu_4,\varsigma,\psi}\Phi_2(t_2, p_1(t_2), p_2(t_2))\Big(\sum_{j=1}^{m} {}^{p}I^{\bar{\eta}_j,\varsigma,\psi}G_j(t_2, p_1(t_2), p_2(t_2)) \\
& + {}^{p}I^{\nu_3,\varsigma,\psi}\Upsilon_2(t_2, p_1(t_2), p_2(t_2))\Big)\Big]\Bigg\}, \quad w \in [t_1, t_2],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{V}_2(p_1, p_2)(w) \ = \ & {}^{p}I^{\nu_4,\varsigma,\psi}\Phi_2(w, p_1(w), p_2(w))\Big(\sum_{j=1}^{m} {}^{p}I^{\bar{\eta}_j,\varsigma,\psi}G_j(w, p_1(w), p_2(w)) \\
& + {}^{p}I^{\nu_3,\varsigma,\psi}\Upsilon_2(w, p_1(w), p_2(w))\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_4-1}}{\Theta\varsigma^{\gamma_4-1}\Gamma(\gamma_4)}\Big\{N_1\Big[\theta_1{}^pI^{\nu_4,\varsigma,\psi}\Phi_2(\xi_1,p_1(\xi_1),p_2(\xi_1))\\
&\times\Big(\sum_{j=1}^{m}{}^pI^{\bar\eta_j,\varsigma,\psi}G_j(\xi_1,p_1(\xi_1),p_2(\xi_1))+{}^pI^{\nu_3,\varsigma,\psi}\Upsilon_2(\xi_1,p_1(\xi_1),p_2(\xi_1))\Big)\Big)\\
&-{}^pI^{\nu_2,\varsigma,\psi}\Phi_1(t_2,p_1(t_2),p_2(t_2))\Big(\sum_{i=1}^{n}{}^pI^{\eta_i,\varsigma,\psi}H_i(t_2,p_1(t_2),p_2(t_2))\\
&+{}^pI^{\nu_1,\varsigma,\psi}\Upsilon_1(t_2,p_1(t_2),p_2(t_2))\Big)\Big]+M_1\Big[\theta_2{}^pI^{\nu_2,\varsigma,\psi}\Phi_1(\xi_2,p_1(\xi_2),p_2(\xi_2))\\
&\times\Big(\sum_{i=1}^{n}{}^pI^{\eta_i,\varsigma,\psi}H_i(\xi_2,p_1(\xi_2),p_2(\xi_2))+{}^pI^{\nu_1,\varsigma,\psi}\Upsilon_1(\xi_2,p_1(\xi_2),p_2(\xi_2))\Big)\Big)\\
&-{}^pI^{\nu_4,\varsigma,\psi}\Phi_2(t_2,p_1(t_2),p_2(t_2))\Big(\sum_{j=1}^{m}{}^pI^{\bar\eta_j,\varsigma,\psi}G_j(t_2,p_1(t_2),p_2(t_2))\\
&+{}^pI^{\nu_3,\varsigma,\psi}\Upsilon_2(t_2,p_1(t_2),p_2(t_2))\Big)\Big]\Big\},\quad w\in[t_1,t_2].
\end{aligned}
$$

To prove our main result we will use the following Burton's version of Krasnosel'skiĭ's fixed-point theorem.

Lemma 3.1. [28] *Let S be a nonempty, convex, closed, and bounded set of a Banach space $(X,\|\cdot\|)$ and let $\mathcal{A}:\mathbb{X}\to\mathbb{X}$ and $\mathcal{B}:S\to\mathbb{X}$ be two operators which satisfy the following:*

(i) $\mathcal{A}$ is a contraction,
(ii) $\mathcal{B}$ is completely continuous, and
(iii) $x=\mathcal{A}x+\mathcal{B}y,\forall y\in S\Rightarrow x\in S$.

Then there exists a solution of the operator equation $x=\mathcal{A}x+\mathcal{B}x$.

Theorem 3.2. *Assume that:*

(H_1) *The functions $\Phi_k:[t_1,t_2]\times\mathbb{R}^2\to\mathbb{R}\backslash\{0\}$, $\Upsilon_k:[t_1,t_2]\times\mathbb{R}^2\to\mathbb{R}$ for $k=1,2$ and $h_i,g_j:[t_1,t_2]\times\mathbb{R}^2\to\mathbb{R}$ for $i=1,2,\dots,n$, $j=1,2,\dots,m$, are continuous and there exist positive continuous functions $\phi_k,\ \omega_k:[t_1,t_2]\to\mathbb{R}$, $k=1,2$, $h_i:[t_1,t_2]\to\mathbb{R}$, $g_j:[t_1,t_2]\to\mathbb{R}$ $i=1,2,\dots,n$ $j=1,2,\dots,m$, with bounds $\|\phi_k\|,\|\omega_k\|$, $k=1,2$, and $\|h_i\|$, $i=1,2,\dots,m$, $\|g_j\|$, $j=1,2,\dots,m$, respectively, such that*

$$
\begin{aligned}
&|\Phi_1(w,u_1,u_2)-\Phi_1(w,\bar u_1,\bar u_2)|\le\phi_1(w)\big(|u_1-\bar u_1|+|u_2-\bar u_2|\big),\\
&|\Phi_2(w,u_1,u_2)-\Phi_2(w,\bar u_1,\bar u_2)|\le\phi_2(w)\big(|u_1-\bar u_1|+|u_2-\bar u_2|\big),\\
&|\Upsilon_1(w,u_1,u_2)-\Upsilon_1(w,\bar u_1,\bar u_2|\le\omega_1(w)(|u_1-\bar u_1|+|u_2-\bar u_2|),\\
&|\Upsilon_2(w,u_1,u_2)-\Upsilon_2(w,\bar u_1,\bar u_2|\le\omega_2(w)(|u_1-\bar u_1|+|u_2-\bar u_2|),\\
&|H_i(w,u_1,u_2)-H_i(w,\bar u_1,\bar u_2)|\le h_i(w)\big(|u_1-\bar u_1|+|u_2-\bar u_2|\big),\\
&|G_j(w,u_1,u_2)-G_j(w,\bar u_1,\bar u_2)|\le g_j(w)\big(|u_1-\bar u_1|+|u_2-\bar u_2|\big),
\end{aligned}\tag{3.2}
$$

for all $w\in[t_1,t_2]$ and $u_i,\bar u_i\in\mathbb{R}$, $i=1,2$.

(H_2) *There exist continuous functions $F_k, L_k, k = 1, 2,\ \lambda_i, \mu_j, i = 1, 2, \ldots, n,\ \ j = 1, 2, \ldots, m$ such that*

$$\begin{aligned}
&|\Phi_1(w, u_1, u_2)| \leq F_1(w), \quad |\Phi_2(w, u_1, u_2)| \leq F_2(w),\\
&|H_i(w, u_1, u_2)| \leq \lambda_i(w), \quad |G_j(w, u_1, u_2)| \leq \mu_j(w),\\
&|\Upsilon_1(w, u_1, u_2)| \leq L_1(w), \quad |\Upsilon_2(w, u_1, u_2)| \leq L_2(w),
\end{aligned} \tag{3.3}$$

for all $w \in [t_1, t_2]$ and $u_1, u_2 \in \mathbb{R}$.

(H_3) *Assume that*

$$\begin{aligned}
K := & \left\{\frac{(\psi(t_2) - \psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2 + 1)}\left[1 + (N_2 + M_2|\theta_2|)\frac{(\psi(t_2) - \psi(t_1))^{\gamma_2 - 1}}{\Theta\varsigma^{\gamma_2 - 1}\Gamma(\gamma_2)}\right]\right.\\
& \left. +(N_1 + M_1|\theta_2|)\frac{(\psi(t_2) - \psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2 + 1)}\frac{(\psi(t_2) - \psi(t_1))^{\gamma_4 - 1}}{\Theta\varsigma^{\gamma_4 - 1}\Gamma(\gamma_4)}\right\}\\
& \times\left[\|F_1\|\sum_{i=1}^{n}\|h_i\|\frac{(\psi(t_2) - \psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i + 1)} + \sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2) - \psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i + 1)}\|\phi_1\|\right]\\
& +\left\{(N_2|\theta_1| + M_2)\frac{(\psi(t_2) - \psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4 + 1)}\frac{(\psi(t_2) - \psi(t_1))^{\gamma_2 - 1}}{\Theta\varsigma^{\gamma_2 - 1}\Gamma(\gamma_2)}\right.\\
& \left. +\frac{(\psi(t_2) - \psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4 + 1)}\left[1 + (N_1|\theta_1| + M_1)\frac{(\psi(t_2) - \psi(t_1))^{\gamma_4 - 1}}{\Theta\varsigma^{\gamma_4 - 1}\Gamma(\gamma_4)}\right]\right\}\\
& \times\left[\|F_2\|\sum_{j=1}^{m}\|g_j\|\frac{(\psi(t_2) - \psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}_j}\Gamma(\overline{\eta}_j + 1)} + \sum_{j=1}^{m}\|\mu_j\|\frac{(\psi(t_2) - \psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}_j}\Gamma(\overline{\eta}_j + 1)}\|\phi_2\|\right] < 1,
\end{aligned}$$

where $\|F_k\| = \sup_{t\in[t_1,t_2]}|F_k(t)|$, $\|L_k\| = \sup_{t\in[t_1,t_2]}, k = 1, 2$, $\|\lambda_i\| = \sup_{t\in[t_1,t_2]}$, $i = 1, 2, \ldots, n$, and $\|\mu_j\| = \sup_{t\in[t_1,t_2]}$, $j = 1, 2, \ldots, m$.

Then the ψ-Hilfer sequential proportional coupled system (1.2) has at least one solution on $[t_1, t_2]$.

Proof. First, we consider a subset S of $\mathbb{Y}\times\mathbb{Y}$ defined by $S = \{(p_1, p_2) \in \mathbb{Y}\times\mathbb{Y} : \|(p_1, p_2)\| \leq r\}$, where r is given by

$$r = R_1 + R_2 \tag{3.4}$$

where

$$\begin{aligned}
R_1 = & \left[1 + \frac{(\psi(t_2) - \psi(t_1))^{\gamma_2 - 1}}{\Theta\varsigma^{\gamma_2 - 1}\Gamma(\gamma_2)}(N_2 + M_2|\theta_2|)\right]\|F_1\|\frac{(\psi(t_2) - \psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2 + 1)}\\
& \times\left(\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2) - \psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i + 1)} + \sum_{i=1}^{n}\|L_1\|\frac{(\psi(t_2) - \psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1 + 1)}\right)\\
& +[N_2|\theta_1| + M_2]\frac{(\psi(t_2) - \psi(t_1))^{\gamma_2 - 1}}{\Theta\varsigma^{\gamma_2 - 1}\Gamma(\gamma_2)}\|F_2\|\frac{(\psi(t_2) - \psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4 + 1)}\\
& \times\left(\sum_{j=1}^{m}\|\mu_j\|\frac{(\psi(t_2) - \psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}}\Gamma(\overline{\eta} + 1)} + \sum_{j=1}^{m}\|L_2\|\frac{(\psi(t_2) - \psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3 + 1)}\right)
\end{aligned}$$

and

$$
\begin{aligned}
R_2 &= \left[1+\frac{(\psi(t_2)-\psi(t_1))^{\gamma_4-1}}{\Theta\varsigma^{\gamma_4-1}\Gamma(\gamma_4)}(N_1|\theta_1|+M_1)\right]\|F_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\\
&\times\left(\sum_{j=1}^{m}\|\mu_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}}\Gamma(\overline{\eta}+1)}+\sum_{j=1}^{m}\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)}\right)\\
&+[N_1+M_1|\theta_2|]\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\|F_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\\
&\times\left(\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}+\sum_{i=1}^{n}\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)}\right).
\end{aligned}
$$

Let us define the operators:

$$\mathcal{H}_i(p_1,p_2)(w)=\sum_{i=1}^{n}{}^{p}I^{\eta_i,\varsigma,\psi}H_i(w,p_1(w),p_2(w)),\quad w\in[t_1,t_2],$$

$$\mathcal{G}_j(p_1,p_2)(w)=\sum_{j=1}^{m}{}^{p}I^{\overline{\eta}_j,\varsigma,\psi}G_j(w,p_1(w),p_2(w)),\quad w\in[t_1,t_2],$$

$$
\begin{aligned}
\mathcal{Y}_1(p_1,p_2)(w)&={}^{p}I^{\nu_1,\varsigma,\psi}\Upsilon_1(w,p_1(w),p_2(w)),\quad w\in[t_1,t_2],\\
\mathcal{Y}_2(p_1,p_2)(w)&={}^{p}I^{\nu_3,\varsigma,\psi}\Upsilon_2(w,p_1(w),p_2(w)),\quad w\in[t_1,t_2],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}_1(p_1,p_2)(w)&=\Phi_1(w,p_1(w),p_2(w)),\quad w\in[t_1,t_2],\\
\mathcal{F}_2(p_1,p_2)(w)&=\Phi_2(w,p_1(w),p_2(w)),\quad w\in[t_1,t_2].
\end{aligned}
$$

Then we have

$$
\begin{aligned}
|\mathcal{H}_i(\overline{p}_1,\overline{p}_2)(w)-\mathcal{H}_i(p_1,p_2)(w)| &\le \sum_{i=1}^{n}{}^{p}I^{\eta_i,\varsigma,\psi}|H_i(w,\overline{p}_1(w),\overline{p}_2(w))-H_i(w,p_1(w),p_2(w))|\\
&\le \sum_{i=1}^{n}\|h_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|)
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathcal{H}_i(p_1,p_2)(w)| &\le \sum_{i=1}^{n}{}^{p}I^{\eta_i,\varsigma,\psi}|H_i(w,p_1(w),p_2(w))|\\
&\le \sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}.
\end{aligned}
$$

Also, we obtain

$$
\begin{aligned}
|\mathcal{G}_j(\overline{p}_1,\overline{p}_2)(w)-\mathcal{G}_j(p_1,p_2)(w)| &\le \sum_{j=1}^{m}{}^{p}I^{\overline{\eta}_j,\varsigma,\psi}|G_j(w,\overline{p}_1(w),\overline{p}_2(w))-G_j(w,p_1(w),p_2(w))|\\
&\le \sum_{j=1}^{m}\|g_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\eta_i}\Gamma(\overline{\eta}_j+1)}(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|)
\end{aligned}
$$

and

$$\begin{aligned}|\mathcal{G}_j(p_1,p_2)(w)| &\leq \sum_{j=1}^{m} {}^{p}I^{\overline{\eta}_i,\varsigma,\psi}|H_i(w,p_1(w),p_2(w))| \\ &\leq \sum_{j=1}^{m} \|\mu_j\| \frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_i}}{\varsigma^{\overline{\eta}_i}\Gamma(\overline{\eta}_i+1)}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}|\mathcal{Y}_1(\overline{p}_1,\overline{p}_2)(w)-\mathcal{Y}_1(p_1,p_2)(w)| &\leq {}^{p}I^{\nu_1,\varsigma,\psi}|\Upsilon_1(w,\overline{p}_1(w),\overline{p}_2(w))-\Upsilon_1(w,p_1(w),p_2(w))| \\ &\leq \|\omega_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)}(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|),\end{aligned}$$

$$\begin{aligned}|\mathcal{Y}_1(p_1,p_2)(w)| &\leq {}^{p}I^{\nu_1,\varsigma,\psi}|\Upsilon_1(w,p_1(w),p_2(w))| \\ &\leq \|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)},\end{aligned}$$

and

$$\begin{aligned}|\mathcal{Y}_2(\overline{p}_1,\overline{p}_2)(w)-\mathcal{Y}_2(p_1,p_2)(w)| &\leq {}^{p}I^{\nu_3,\varsigma,\psi}|\Upsilon_2(w,\overline{p}_1(w),\overline{p}_2(w))-\Upsilon_2(w,p_1(w),p_2(w))| \\ &\leq \|\omega_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)}(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|),\end{aligned}$$

$$\begin{aligned}|\mathcal{Y}_2(p_1,p_2)(w)| &\leq {}^{p}I^{\nu_1,\varsigma,\psi}|\Upsilon_2(w,p_1(w),p_2(w))| \\ &\leq \|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)}.\end{aligned}$$

Finally, we get

$$\begin{aligned}|\mathcal{F}_1(\overline{p}_1,\overline{p}_2)(w)-\mathcal{F}_1(p_1,p_2)(w)| &\leq |\Phi_1(w,\overline{p}_1(w),\overline{p}_2(w))-\Phi_1(w,p_1(w),p_2(w))| \\ &\leq \|\phi_1\|(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|),\end{aligned}$$

$$\begin{aligned}|\mathcal{F}_1(p_1,p_2)(w)| &\leq |\Phi_1(w,p_1(w),p_2(w))| \\ &\leq \|F_1\|,\end{aligned}$$

and

$$\begin{aligned}|\mathcal{F}_2(\overline{p}_1,\overline{p}_2)(w)-\mathcal{F}_2(p_1,p_2)(w)| &\leq |\Phi_2(w,\overline{p}_1(w),\overline{p}_2(w))-\Phi_2(w,p_1(w),p_2(w))| \\ &\leq \|\phi_2\|(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|),\end{aligned}$$

$$\begin{aligned}|\mathcal{F}_2(p_1,p_2)(w)| &\leq |\Phi_2(w,p_1(w),p_2(w))| \\ &\leq \|F_2\|.\end{aligned}$$

Now we split the operator $\mathbb{V}$ as

$$\begin{aligned}\mathbb{V}_1(p_1,p_2)(w) &= \mathbb{V}_{1,1}(p_1,p_2)(w)+\mathbb{V}_{1,2}(p_1,p_2)(w),\\ \mathbb{V}_2(p_1,p_2)(w) &= \mathbb{V}_{2,1}(p_1,p_2)(w)+\mathbb{V}_{2,2}(p_1,p_2)(w),\end{aligned}$$

with

$$\begin{aligned}\mathbb{V}_{1,1}(p_1,p_2)(w) &= {}^{p}I^{\gamma_2,\varsigma,\psi}\mathcal{F}_1(p_1,p_2)(w)\mathcal{H}_i(p_1,p_2)(w)\\ &+\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\\ &\times\bigg\{N_2\Big[\theta_1{}^{p}I^{\gamma_4,\varsigma,\psi}\mathcal{F}_2(p_1,p_2)(w)\mathcal{G}_j(p_1,p_2)(w)\\ &-{}^{p}I^{\gamma_2,\varsigma,\psi}\mathcal{F}_1(p_1,p_2)(w)\mathcal{H}_i(p_1,p_2)(w)\Big]\\ &+M_2\Big[\theta_2{}^{p}I^{\gamma_2,\varsigma,\psi}\mathcal{F}_1(p_1,p_2)(w)\mathcal{H}_i(p_1,p_2)(w)\\ &-{}^{p}I^{\gamma_4,\varsigma,\psi}\mathcal{F}_2(p_1,p_2)(w)\mathcal{G}_j(p_1,p_2)(w)\Big]\bigg\},\end{aligned}$$

$$\begin{aligned}\mathbb{V}_{1,2}(p_1,p_2)(w) &= {}^{p}I^{\gamma_2,\varsigma,\psi}\mathcal{F}_1(p_1,p_2)(w)\mathcal{Y}_1(p_1,p_2)(w)\\ &+\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\\ &\times\bigg\{N_2\Big[\theta_1{}^{p}I^{\gamma_4,\varsigma,\psi}\mathcal{F}_2(p_1,p_2)(w)\mathcal{Y}_2(p_1,p_2)(w)\\ &-{}^{p}I^{\gamma_2,\varsigma,\psi}\mathcal{F}_1(p_1,p_2)(w)\mathcal{Y}_1(p_1,p_2)(w)\Big]\\ &+M_2\Big[\theta_2{}^{p}I^{\gamma_2,\varsigma,\psi}\mathcal{F}_1(p_1,p_2)(w)\mathcal{Y}_1(p_1,p_2)(w)\\ &-{}^{p}I^{\gamma_4,\varsigma,\psi}\mathcal{F}_2(p_1,p_2)(w)\mathcal{Y}_2(p_1,p_2)(w)\Big]\bigg\},\end{aligned}$$

$$\begin{aligned}\mathbb{V}_{2,1}(p_1,p_2)(w) &= {}^{p}I^{\gamma_4,\varsigma,\psi}\mathcal{F}_2(p_1,p_2)(w)\mathcal{G}_j(p_1,p_2)(w)\\ &+\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_4-1}}{\Theta\varsigma^{\gamma_4-1}\Gamma(\gamma_4)}\\ &\times\bigg\{N_1\Big[\theta_1{}^{p}I^{\gamma_4,\varsigma,\psi}\mathcal{F}_2(p_1,p_2)(w)\mathcal{G}_j(p_1,p_2)(w)\\ &-{}^{p}I^{\gamma_2,\varsigma,\psi}\mathcal{F}_1(p_1,p_2)(w)\mathcal{H}_i(p_1,p_2)(w)\Big]\\ &+M_1\Big[\theta_2{}^{p}I^{\gamma_2,\varsigma,\psi}\mathcal{F}_1(p_1,p_2)(w)\mathcal{H}_i(p_1,p_2)(w)\\ &-{}^{p}I^{\gamma_4,\varsigma,\psi}\mathcal{F}_2(p_1,p_2)(w)\mathcal{G}_j(p_1,p_2)(w)\Big]\bigg\},\end{aligned}$$

and

$$\begin{aligned}
\mathbb{V}_{2,2}(p_1,p_2)(w) &= {}^pI^{\nu_4,\varsigma,\psi}\mathcal{F}_2(p_1,p_2)(w)\mathcal{Y}_2(p_1,p_2)(w)\\
&+\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_4-1}}{\Theta\varsigma^{\gamma_4-1}\Gamma(\gamma_4)}\\
&\times\Bigg\{N_1\Big[\theta_1{}^pI^{\nu_4,\varsigma,\psi}\mathcal{F}_2(p_1,p_2)(w)\mathcal{Y}_2(p_1,p_2)(w)\\
&-{}^pI^{\nu_2,\varsigma,\psi}\mathcal{F}_1(p_1,p_2)(w)\mathcal{Y}_1(p_1,p_2)(w)\Big]\\
&+M_1\Big[\theta_2{}^pI^{\nu_2,\varsigma,\psi}\mathcal{F}_1(p_1,p_2)(w)\mathcal{Y}_1(p_1,p_2)(w)\\
&-{}^pI^{\nu_4,\varsigma,\psi}\mathcal{F}_2(p_1,p_2)(w)\mathcal{Y}_2(p_1,p_2)(w)\Big]\Bigg\}.
\end{aligned}$$

In the following, we will show that the operators $\mathbb{V}_1$ and $\mathbb{V}_2$ fulfill the assumptions of Lemma 3.1. We divide the proof into three steps:

Step 1. The operators $\mathbb{V}_{1,1}$ and $\mathbb{V}_{2,1}$ are contraction mappings. For all $(p_1,p_2),(\overline{p}_1,\overline{p}_2)\in\mathbb{Y}\times\mathbb{Y}$ we have

$$\begin{aligned}
&|\mathbb{V}_{1,1}(\overline{p}_1,\overline{p}_2)(w)-\mathbb{V}_{1,1}(p_1,p_2)(w)|\\
\le\ & \frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}|\mathcal{F}_1(\overline{p}_1,\overline{p}_2)(w)\mathcal{H}_i(\overline{p}_1,\overline{p}_2)(w)-\mathcal{F}_1(p_1,p_2)(w)\mathcal{H}_i(p_1,p_2)(w)|\\
&+\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\Bigg\{N_1|\theta_1|\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}|\mathcal{F}_2(\overline{p}_1,\overline{p}_2)(w)\mathcal{G}_j(\overline{p}_1,\overline{p}_2)(w)\\
&-\mathcal{F}_2(p_1,p_2)(w)\mathcal{G}_j(p_1,p_2)(w)|\\
&+(N_2+M_2|\theta_2|)\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}|\mathcal{F}_1(\overline{p}_1,\overline{p}_2)(w)\mathcal{H}_i(\overline{p}_1,\overline{p}_2)(w)\\
&-\mathcal{F}_1(p_1,p_2)(w)\mathcal{H}_i(p_1,p_2)(w)|\\
&+M_2\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}|\mathcal{F}_2(\overline{p}_1,\overline{p}_2)(w)\mathcal{G}_j(\overline{p}_1,\overline{p}_2)(w)-\mathcal{F}_2(p_1,p_2)(w)\mathcal{G}_j(p_1,p_2)(w)|\Bigg\}\\
\le\ & \frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\left[1+(N_2+M_2|\theta_2|)\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\right]\\
&\times|\mathcal{F}_1(\overline{p}_1,\overline{p}_2)(w)\mathcal{H}_i(\overline{p}_1,\overline{p}_2)(w)-\mathcal{F}_1(p_1,p_2)(w)\mathcal{H}_i(p_1,p_2)(w)|\\
&+(N_2|\theta_1|+M_2)\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}}\\
&\times|\mathcal{F}_2(\overline{p}_1,\overline{p}_2)(w)\mathcal{G}_j(\overline{p}_1,\overline{p}_2)(w)-\mathcal{F}_2(p_1,p_2)(w)\mathcal{G}_j(p_1,p_2)(w)|\\
\le\ & \frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\left[1+(N_2+M_2|\theta_2|)\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\right]\\
&\times\Big[|\mathcal{F}_1(\overline{p}_1,\overline{p}_2)(w)||\mathcal{H}_i(\overline{p}_1,\overline{p}_2)(w)-\mathcal{H}_i(p_1,p_2)(w)|\\
&+|\mathcal{H}_i(p_1,p_2)(w)||\mathcal{F}_1(\overline{p}_1,\overline{p}_2)(w)-\mathcal{F}_1(p_1,p_2)(w)|\Big]
\end{aligned}$$

$$
\begin{aligned}
&+(N_2|\theta_1|+M_2)\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}}\\
&\times\Bigg[|\mathcal{F}_2(\overline{p}_1,\overline{p}_2)(w)||\mathcal{G}_j(\overline{p}_1,\overline{p}_2)(w)-\mathcal{G}_j(p_1,p_2)(w)|\\
&+|\mathcal{G}_j(p_1,p_2)(w)||\mathcal{F}_2(\overline{p}_1,\overline{p}_2)(w)-\mathcal{F}_2(p_1,p_2)(w)|\Bigg]\\
\leq\ & \frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\left[1+(N_2+M_2|\theta_2|)\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\right]\\
&\times\Bigg[\|F_1\|\sum_{i=1}^{n}\|h_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|)\\
&+\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}\|\phi_1\|(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|)\Bigg]\\
&+(N_2|\theta_1|+M_2)\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}}\\
&\times\Bigg[\|F_2\|\sum_{j=1}^{m}\|g_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\eta_i}\Gamma(\overline{\eta}_j+1)}(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|)\\
&+\sum_{j=1}^{m}\|\mu_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}_j}\Gamma(\overline{\eta}_j+1)}\|\phi_2\|(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|)\Bigg]\\
=\ & \Bigg\{\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\left[1+(N_2+M_2|\theta_2|)\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\right]\\
&\times\Bigg[\|F_1\|\sum_{i=1}^{n}\|h_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}+\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}\|\phi_1\|\Bigg]\\
&+(N_2|\theta_1|+M_2)\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\\
&\times\Bigg[\|F_2\|\sum_{j=1}^{m}\|g_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}_i}\Gamma(\overline{\eta}_j+1)}+\sum_{j=1}^{m}\|\mu_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}_j}\Gamma(\overline{\eta}_j+1)}\|\phi_2\|\Bigg]\Bigg\}\\
&\times(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|).
\end{aligned}
$$

Similarly we can find

$$
\begin{aligned}
&|\mathbb{V}_{2,1}(\overline{p}_1,\overline{p}_2)(w)-\mathbb{V}_{2,1}(p_1,p_2)(w)|\\
\leq\ & \Bigg\{\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\left[1+(N_1|\theta_1|+M_1)\frac{(\psi(t_2)-\psi(t_1))^{\gamma_4-1}}{\Theta\varsigma^{\gamma_4-1}\Gamma(\gamma_4)}\right]\\
&\times\Bigg[\|F_2\|\sum_{j=1}^{m}\|g_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\eta_i}\Gamma(\overline{\eta}_j+1)}+\sum_{j=1}^{n}\|\mu_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}_j}\Gamma(\overline{\eta}_j+1)}\|\phi_2\|\Bigg]\\
&+(N_1+M_1|\theta_2|)\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\frac{(\psi(t_2)-\psi(t_1))^{\gamma_4-1}}{\Theta\varsigma^{\gamma_4-1}\Gamma(\gamma_4)}
\end{aligned}
$$

$$\times\left[\|F_1\|\sum_{i=1}^{n}\|h_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}+\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}\|\phi_1\|\right]\Bigg\}$$
$$\times(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|).$$

Consequently, we get

$$\|(\mathbb{V}_{1,1},\mathbb{V}_{2,1})(\overline{p}_1,\overline{p}_2)-(\mathbb{V}_{1,1},\mathbb{V}_{2,1})(p_1,p_2)\|\leq K(\|\overline{p}_1-p_1\|+\|\overline{p}_2-p_2\|),$$

which means that $(\mathbb{V}_{1,1},\mathbb{V}_{2,1})$ is a contraction.

Step 2. The operator $\mathbb{V}_2=(\mathbb{V}_{1,2},\mathbb{V}_{2,2})$ is completely continuous on S. For continuity of $\mathbb{V}_{1,2}$, take any sequence of points (p_n,q_n) in S converging to a point $(p,q)\in S$. Then, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{V}_{1,2}(p_n,q_n)(w) &= {}^{p}I^{\nu_2,\varsigma,\psi}\lim_{n\to\infty}\mathcal{F}_1(p_n,q_n)(w)\lim_{n\to\infty}\mathcal{Y}_1(p_n,q_n)(w)\\
&\quad+\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\\
&\quad\times\Bigg\{N_2\Big[\theta_1{}^{p}I^{\nu_4,\varsigma,\psi}\lim_{n\to\infty}\mathcal{F}_2(p_n,q_n)(w)\lim_{n\to\infty}\mathcal{Y}_2(p_n,q_n)(w)\\
&\quad-{}^{p}I^{\nu_2,\varsigma,\psi}\lim_{n\to\infty}\mathcal{F}_1(p_n,q_n)(w)\lim_{n\to\infty}\mathcal{Y}_1(p_n,q_n)(w)\Big]\\
&\quad+M_2\Big[\theta_2{}^{p}I^{\nu_2,\varsigma,\psi}\lim_{n\to\infty}\mathcal{F}_1(p_n,q_n)(w)\lim_{n\to\infty}\mathcal{Y}_1(p_n,q_n)(w)\\
&\quad-{}^{p}I^{\nu_4,\varsigma,\psi}\lim_{n\to\infty}\mathcal{F}_2(p_n,q_n)(w)\lim_{n\to\infty}\mathcal{Y}_2(p_n,q_n)(w)\Big]\Bigg\}\\
&= {}^{p}I^{\nu_2,\varsigma,\psi}\mathcal{F}_1(p,q)(w)\mathcal{Y}_1(p,q)(w)\\
&\quad+\frac{e^{\frac{\varsigma-1}{\varsigma}(\psi(w)-\psi(t_1))}(\psi(w)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\\
&\quad\times\Bigg\{N_2\Big[\theta_1{}^{p}I^{\nu_4,\varsigma,\psi}\mathcal{F}_2(p,q)(w)\mathcal{Y}_2(p,q)(w)\\
&\quad-{}^{p}I^{\nu_2,\varsigma,\psi}\mathcal{F}_1(p,q)(w)\mathcal{Y}_1(p,q)(w)\Big]\\
&\quad+M_2\Big[\theta_2{}^{p}I^{\nu_2,\varsigma,\psi}\mathcal{F}_1(p,q)(w)\mathcal{Y}_1(p,q)(w)\\
&\quad-{}^{p}I^{\nu_4,\varsigma,\psi}\mathcal{F}_2(p,q)(w)\mathcal{Y}_2(p,q)(w)\Big]\Bigg\}\\
&= \mathbb{V}_{1,2}(p,q)(w),
\end{aligned}$$

for all $w\in[t_1,t_2]$. Similarly, we prove $\lim_{n\to\infty}\mathbb{V}_{2,2}(p_n,q_n)(w)=\mathbb{V}_{2,2}(p,q)(w)$ for all $w\in[t_1,t_2]$. Thus $\mathbb{V}_2(p_n,q_n)=(\mathbb{V}_{1,2}(p_n,q_n),\mathbb{V}_{2,2}(p_n,q_n))$ converges to $\mathbb{V}_2(p,q)$ on $[t_1,t_2]$, which shows that $\mathbb{V}_2$ is continuous.

Next, we show that the operator $(\mathbb{V}_{1,2},\mathbb{V}_{2,2})$ is uniformly bounded on S. For any $(p_1,p_2)\in S$ we have

$$
\begin{aligned}
|\mathbb{V}_{1,2}(p_1,p_2)(w)| \leq\ & {}^{p}I^{\nu_2,\varsigma,\psi}|\mathcal{F}_1(p_1,p_2)(w)\mathcal{Y}_1(p_1,p_2)(w)| \\
& + \frac{(\psi(w)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\Big\{N_2\Big[|\theta_1|{}^{p}I^{\nu_4,\varsigma,\psi}|\mathcal{F}_2(p_1,p_2)(w)\mathcal{Y}_2(p_1,p_2)(w)| \\
& + {}^{p}I^{\nu_2,\varsigma,\psi}|\mathcal{F}_1(p_1,p_2)(w)\mathcal{Y}_1(p_1,p_2)(w)|\Big] \\
& + M_2\Big[|\theta_2|{}^{p}I^{\nu_2,\varsigma,\psi}|\mathcal{F}_1(p_1,p_2)(w)\mathcal{Y}_1(p_1,p_2)(w)| \\
& + {}^{p}I^{\nu_4,\varsigma,\psi}|\mathcal{F}_2(p_1,p_2)(w)\mathcal{Y}_2(p_1,p_2)(w)|\Big]\Big\} \\
\leq\ & \frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\|F_1\|\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)} + \frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)} \\
& \times\Big\{N_2|\theta_1|\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\|F_2\|\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)} \\
& + N_2\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\|F_1\|\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)} \\
& + M_2|\theta_2|\|F_1\|\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)} \\
& + M_2\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\|F_2\|\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)}\Big\} := \Lambda_1.
\end{aligned}
$$

Similarly we can prove that

$$
\begin{aligned}
|\mathbb{V}_{2,2}(p_1,p_2)(w)| \leq\ & \frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\|F_2\|\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)} + \frac{(\psi(t_2)-\psi(t_1))^{\gamma_4-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)} \\
& \times\Big\{N_1|\theta_1|\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\|F_2\|\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)} \\
& + N_1\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\|F_1\|\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)} \\
& + M_1|\theta_2|\|F_1\|\sum_{i=1}^{n}\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)} \\
& + M_1\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\|F_2\|\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)}\Big\} := \Lambda_2.
\end{aligned}
$$

Therefore $\|\mathbb{V}_{1,2}\| + \|\mathbb{V}_{2,2}\| \leq \Lambda_1 + \Lambda_2, (p_1, p_2) \in S$, which shows that the operator $(\mathbb{V}_{1,2}, \mathbb{V}_{2,2})$ is uniformly bounded on S. Finally we show that the operator $(\mathbb{V}_{1,2}, \mathbb{V}_{2,2})$ is equicontinuous. Let $\tau_1 < \tau_2$ and $(p_1, p_2) \in S$. Then, we have

$$
\begin{aligned}
& |\mathbb{V}_{1,2}(p_1,p_2)(\tau_2) - \mathbb{V}_{1,2}(p_1,p_2)(\tau_1)| \\
\leq\ & \Bigg|\frac{1}{\varsigma^{\nu_2}\Gamma(\nu_2)}\int_{t_1}^{\tau_1}\psi'(s)\left[(\psi(\tau_2)-\psi(s))^{\nu_2-1} - (\psi(\tau_1)-\psi(s))^{\nu_2-1}\right] \\
& \times|\mathcal{F}_1(p_1,p_2)(s)\mathcal{Y}_1(p_1,p_2)(s)|ds \\
& + \frac{1}{\varsigma^{\nu_2}\Gamma(\nu_2)}\int_{\tau_1}^{\tau_2}\psi'(s)\,(\psi(\tau_2)-\psi(s))^{\nu_2-1}|\mathcal{F}_1(p_1,p_2)(s)\mathcal{Y}_1(p_1,p_2)(s)|ds\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{\left|(\psi(\tau_2)-\psi(t_1))^{\gamma_2-1}-(\psi(\tau_1)-\psi(t_1))^{\gamma_2-1}\right|}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\mathbb{W} \\
\leq\ & \frac{1}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}\Big[\,|(\psi(\tau_2)-\psi(t_1))^{\nu_2}-(\psi(\tau_1)-\psi(t_1))^{\nu_2}| \\
& +2(\psi(\tau_2)-\psi(\tau_1))^{\nu_2}\Big]+\frac{\left|(\psi(\tau_2)-\psi(t_1))^{\gamma_2-1}-(\psi(\tau_1)-\psi(t_1))^{\gamma_2-1}\right|}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\mathbb{W},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{W} \ =\ & N_2|\theta_1|\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\|F_2\|\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)} \\
& +N_2\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\|F_1\|\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)} \\
& +M_2|\theta_2|\|F_1\|\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)} \\
& +M_2\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\|F_2\|\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)}.
\end{aligned}
$$

As $\tau_2-\tau_1\to 0$, the right-hand side of the above inequality tends to zero, independently of (p_1,p_2). Similarly we have $|\mathbb{V}_{2,2}(p_1,p_2)(\tau_2)-\mathbb{V}_{2,2}(p_1,p_2)(\tau_1)|\to 0$ as $\tau_2-\tau_1\to 0$. Thus $(\mathbb{V}_{1,2},\mathbb{V}_{2,2})$ is equicontinuous. Therefore, it follows by the Arzelá-Ascoli theorem that $(\mathbb{V}_{1,2},\mathbb{V}_{2,2})$ is a completely continuous operator on S.

Step 3. We show that the third condition (iii) of Lemma 3.1 is fulfilled. Let $(p_1,p_2)\in\mathbb{Y}\times\mathbb{Y}$ be such that, for all $(\overline{p}_1,\overline{p}_2)\in S$

$$
(p_1,p_2)=(\mathbb{V}_{1,1}(p_1,p_2),\mathbb{V}_{2,1}(p_1,p_2))+(\mathbb{V}_{1,2}(\overline{p}_1,\overline{p}_2,\mathbb{V}_{2,2}(\overline{p}_1,\overline{p}_2)).
$$

Then, we have

$$
\begin{aligned}
|p_1(w)| \ \leq\ & |\mathbb{V}_{1,1}(p_1,p_2)(w)|+|\mathbb{V}_{1,2}(\overline{p}_1,\overline{p}_2)(w)| \\
\leq\ & {}^{p}I^{\nu_2,\varsigma,\psi}|\mathcal{F}_1(p_1,p_2)(w)\mathcal{H}_i(p_1,p_2)(w)| \\
& +\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\Bigg\{N_2\Bigg[|\theta_1|{}^{p}I^{\nu_4,\varsigma,\psi}|\mathcal{F}_2(p_1,p_2)(w)\mathcal{G}_j(p_1,p_2)(w)| \\
& +{}^{p}I^{\nu_2,\varsigma,\psi}|\mathcal{F}_1(p_1,p_2)(w)\mathcal{H}_i(p_1,p_2)(w)|\Bigg] \\
& +M_2\Bigg[|\theta_2|{}^{p}I^{\nu_2,\varsigma,\psi}|\mathcal{F}_1(p_1,p_2)(w)\mathcal{H}_i(p_1,p_2)(w)| \\
& +{}^{p}I^{\nu_4,\varsigma,\psi}|\mathcal{F}_2(p_1,p_2)(w)\mathcal{G}_j(p_1,p_2)(w)|\Bigg]\Bigg\} \\
& +{}^{p}I^{\nu_2,\varsigma,\psi}|\mathcal{F}_1(\overline{p}_1,\overline{p}_2)(w)\mathcal{Y}_1(\overline{p}_1,\overline{p}_2)(w)| \\
& +\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\Bigg\{N_2\Bigg[|\theta_1|{}^{p}I^{\nu_4,\varsigma,\psi}|\mathcal{F}_2(\overline{p}_1,\overline{p}_2)(w)\mathcal{Y}_2(\overline{p}_1,\overline{p}_2)(w)| \\
& +{}^{p}I^{\nu_2,\varsigma,\psi}|\mathcal{F}_1(\overline{p}_1,\overline{p}_2)(w)\mathcal{Y}_1(\overline{p}_1,\overline{p}_2)(w)|\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
& +M_2\Big[|\theta_2|^p I^{\nu_2,\varsigma,\psi}|\mathcal{F}_1(\overline{p}_1,\overline{p}_2)(w)\mathcal{Y}_1(\overline{p}_1,\overline{p}_2)(w)| \\
& +{}^p I^{\nu_4,\varsigma,\psi}|\mathcal{F}_2(\overline{p}_1,\overline{p}_2)(w)\mathcal{Y}_2(\overline{p}_1,\overline{p}_2)(w)|\Big]\Big\} \\
\leq\; & \frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\|F_1\|\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}+\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)} \\
& \times\Big\{N_2|\theta_1|\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\sum_{j=1}^{m}\|\mu_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}}\Gamma(\overline{\eta}+1)}\|F_2\| \\
& +N_2\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\|F_1\|\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)} \\
& +M_2|\theta_2|\|F_1\|\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)} \\
& +M_2\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\|F_2\|\sum_{j=1}^{m}\|\mu_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}}\Gamma(\overline{\eta}+1)}\Big\} \\
& +\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\|F_1\|\sum_{i=1}^{n}\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)}+\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)} \\
& \times\Big\{N_2|\theta_1|\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\|F_2\|\sum_{j=1}^{m}\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)} \\
& +N_2\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\|F_1\|\sum_{i=1}^{n}\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)} \\
& +M_2|\theta_2|\|F_1\|\sum_{i=1}^{n}\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)} \\
& +M_2\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\|F_2\|\sum_{j=1}^{m}\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)}\Big\} \\
=\; & \left[1+\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}(N_2+M_2|\theta_2|)\right]\|F_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)} \\
& \times\left(\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}+\sum_{i=1}^{n}\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)}\right) \\
& +[N_2|\theta_1|+M_2]\frac{(\psi(t_2)-\psi(t_1))^{\gamma_2-1}}{\Theta\varsigma^{\gamma_2-1}\Gamma(\gamma_2)}\|F_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)} \\
& \times\left(\sum_{j=1}^{m}\|\mu_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}}\Gamma(\overline{\eta}+1)}+\sum_{j=1}^{m}\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)}\right)=R_1.
\end{aligned}
$$

In a similar way, we find

$$
|p_2(w)| \leq |\mathbb{V}_{2,1}(p_1,p_2)(w)|+|\mathbb{V}_{2,2}(\overline{p}_1,\overline{p}_2)(w)|
$$

$$
\begin{aligned}
\leq\ & \left[1+\frac{(\psi(t_2)-\psi(t_1))^{\gamma_4-1}}{\Theta\varsigma^{\gamma_4-1}\Gamma(\gamma_4)}(N_1|\theta_1|+M_1)\right]\frac{(\psi(t_2)-\psi(t_1))^{\nu_4}}{\varsigma^{\nu_4}\Gamma(\nu_4+1)}\|F_2\| \\
& \times\left(\sum_{j=1}^{m}\|\mu_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}}\Gamma(\overline{\eta}+1)}+\sum_{j=1}^{m}\|L_2\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_3}}{\varsigma^{\nu_3}\Gamma(\nu_3+1)}\right) \\
& +[N_1+M_1|\theta_2|]\frac{(\psi(t_2)-\psi(t_1))^{\nu_2}}{\varsigma^{\nu_2}\Gamma(\nu_2+1)}\|F_1\|\frac{(\psi(t_2)-\psi(t_1))^{\gamma_4-1}}{\varsigma^{\gamma_4-1}\Gamma(\gamma_4)} \\
& \times\left(\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)}+\sum_{i=1}^{n}\|L_1\|\frac{(\psi(t_2)-\psi(t_1))^{\nu_1}}{\varsigma^{\nu_1}\Gamma(\nu_1+1)}\right)=R_2.
\end{aligned}
$$

Adding the previous inequalities, we obtain

$$\|p_1\|+\|p_2\|\leq R_1+R_2=r.$$

As $\|(p_1,p_2)\|=\|p_1\|+\|p_2\|$, we have that $\|(p_1,p_2)\|\leq r$ and so condition (iii) of Lemma 3.1 holds.

By Lemma 3.1, the ψ-Hilfer sequential proportional coupled system (1.2) has at least one solution on $[t_1,t_2]$. The proof is finished. □

4. An example

Let us consider the following coupled system of nonlinear sequential proportional Hilfer fractional differential equations with multi-point boundary conditions:

$$
\begin{cases}
{}^{H}D^{\frac{1}{3},\frac{1}{5};\frac{3}{7};\log w}\left[\dfrac{{}^{H}D^{\frac{5}{4},\frac{2}{5};\frac{3}{7};\log w}p_1(w)}{\Phi_1(w,p_1(w),p_2(w))}-\displaystyle\sum_{i=1}^{2}{}^{p}I^{\eta_i,\varsigma,\psi}H_i(w,p_1(w),p_2(w))\right]=\Upsilon_1(w,p_1(w),p_2(w)),\ w\in\left[\dfrac{1}{2},\dfrac{7}{2}\right], \\
{}^{H}D^{\frac{2}{3},\frac{3}{5};\frac{3}{7};\log w}\left[\dfrac{{}^{H}D^{\frac{7}{4},\frac{4}{5};\frac{3}{7};\log w}p_1(w)}{\Phi_2(w,p_1(w),p_2(w))}-\displaystyle\sum_{j=1}^{2}{}^{p}I^{\overline{\eta}_j,\varsigma,\psi}G_j(w,p_1(w),p_2(w))\right]=\Upsilon_2(w,p_1(w),p_2(w)),\ w\in\left[\dfrac{1}{2},\dfrac{7}{2}\right], \\
p_1\left(\dfrac{1}{2}\right)={}^{H}D^{\frac{5}{4},\frac{2}{5};\frac{3}{7};\log w}p_1\left(\dfrac{1}{2}\right)=0,\quad p_1\left(\dfrac{7}{2}\right)=\dfrac{2}{5}p_2\left(\dfrac{3}{2}\right), \\
p_2\left(\dfrac{1}{2}\right)={}^{H}D^{\frac{7}{4},\frac{4}{5};\frac{3}{7};\log w}p_2\left(\dfrac{1}{2}\right)=0,\quad p_2\left(\dfrac{7}{2}\right)=\dfrac{2}{3}p_1\left(\dfrac{5}{2}\right),
\end{cases}
\tag{4.1}
$$

where

$$
\begin{aligned}
\sum_{i=1}^{2}{}^{p}I^{\eta_i,\varsigma,\psi}H_i(w,p_1,p_2) &= \sum_{i=1}^{2}{}^{p}I^{\frac{2(i+1)}{5},\frac{3}{7},\log w}\left(\frac{|p_1|}{(w+i^2)(i+|p_1|)}+\frac{|p_2|}{(w+i^3)(i+|p_2|)}\right), \\
\sum_{j=1}^{2}{}^{p}I^{\overline{\eta}_j,\varsigma,\psi}G_j(w,p_1,p_2) &= \sum_{j=1}^{2}{}^{p}I^{\frac{2(j+1)}{7},\frac{3}{7},\log w}\left(\frac{|p_1|}{(w^2+j^2)(j+|p_1|)}+\frac{|p_2|}{(w^2+j^3)(j+|p_2|)}\right), \\
\Phi_1(w,p_1,p_2) &= \frac{1}{100(10w+255)}\left(\frac{|p_1|}{1+|p_1|}+\frac{|p_2|}{1+|p_2|}+\frac{1}{2}\right), \\
\Phi_2(w,p_1,p_2) &= \frac{2}{5(2w+99)^2}\left(\frac{|p_1|}{1+|p_1|}+\frac{|p_2|}{1+|p_2|}+\frac{1}{4}\right),
\end{aligned}
$$

$$\begin{aligned}\Upsilon_1(w,p_1,p_2) &= \frac{1}{\sqrt{w}+2}\left(\frac{|p_1|}{3+|p_1|}\right)+\frac{1}{2(\sqrt{w}+1)}\sin|p_2|+\frac{1}{3},\\ \Upsilon_2(w,p_1,p_2) &= \frac{1}{w^2+4}\left(\frac{1}{2}\tan^{-1}|p_1|+\frac{|p_2|}{2+|p_2|}\right)+\frac{1}{5}.\end{aligned}$$

Next, we can choose $\nu_1 = 1/3$, $\nu_2 = 5/4$, $\nu_3 = 2/3$, $\nu_4 = 7/4$, $\vartheta_1 = 1/5$, $\vartheta_2 = 2/5$, $\vartheta_3 = 3/5$, $\vartheta_4 = 4/5$, $\varsigma = 3/7$, $\psi(w) := \log w = \log_e w$, $t_1 = 1/2$, $t_2 = 7/2$, $\theta_1 = 2/5$, and $\theta_2 = 2/3$. Then, we have $\gamma_1 = 7/15$, $\gamma_2 = 31/20$, $\gamma_3 = 13/15$, $\gamma_4 = 39/20$, $M_1 \approx 0.1930945138$, $M_2 \approx 0.2307306625$, $N_1 \approx 0.1816223751$, $N_2 \approx 0.3208292984$, and $\Theta \approx 0.02004452646$. Now, we analyse the nonlinear functions in the fractional integral terms. We have

$$|H_i(w,p_1,p_2)-H_i(w,\overline{p}_1,\overline{p}_2)| \le \frac{1}{i(w+i^2)}(|p_1-\overline{p}_1|+|p_2-\overline{p}_2|)$$

and

$$|G_j(w,p_1,p_2)-G_j(w,\overline{p}_1,\overline{p}_2)| \le \frac{1}{j(w^2+j^2)}(|p_1-\overline{p}_1|+|p_2-\overline{p}_2|),$$

from which $h_i(w) = 1/(i(w+i^2))$ and $g_j(w) = 1/(j(w^2+j^2))$, respectively. Both of them are bounded as

$$|H_i(w,p_1,p_2)| \le \frac{2}{w+i^2} \quad \text{and} \quad |G_j(w,p_1,p_2)| \le \frac{2}{w^2+j^2}.$$

Therefore $\lambda_i(w) = 2/(w+i^2)$ and $\mu_j = 2/(w^2+j^2)$. Moreover, we have

$$\sum_{i=1}^{n}\|h_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)} \approx 3.021061781,$$

$$\sum_{i=1}^{n}\|\lambda_i\|\frac{(\psi(t_2)-\psi(t_1))^{\eta_i}}{\varsigma^{\eta_i}\Gamma(\eta_i+1)} \approx 7.281499952,$$

$$\sum_{j=1}^{m}\|g_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}_j}\Gamma(\overline{\eta}_i+1)} \approx 2.776491121$$

and

$$\sum_{j=1}^{m}\|\mu_j\|\frac{(\psi(t_2)-\psi(t_1))^{\overline{\eta}_j}}{\varsigma^{\overline{\eta}_j}\Gamma(\overline{\eta}_i+1)} \approx 7.220966978.$$

For the two non-zero functions Φ_1 and Φ_2 we have

$$\begin{aligned}|\Phi_1(w,p_1,p_2)-\Phi_1(w,\overline{p}_1,\overline{p}_2)| &\le \frac{1}{100(10w+255)}\left(|p_1-\overline{p}_1|+|p_2-\overline{p}_2|\right),\\ |\Phi_2(w,p_1,p_2)-\Phi_2(w,\overline{p}_1,\overline{p}_2)| &\le \frac{2}{5(2w+99)^2}\left(|p_1-\overline{p}_1|+|p_2-\overline{p}_2|\right),\end{aligned}$$

$$|\Phi_1(w,p_1,p_2)| \le \frac{1}{40(10w+255)}, \quad \text{and} \quad |\Phi_2(w,p_1,p_2)| \le \frac{9}{10(2w+99)^2},$$

from which we get $\|\phi_1\| = 1/26000$, $\|\phi_2\| = 1/25000$, $\|F_1\| = 1/10400$, $\|F_2\| = 9/100000$, by setting $\phi_1(w) = 1/(100(10w + 255))$, $\phi_2(w) = 2/(5(2w + 99)^2)$, $F_1(w) = 1/(40(10w + 255))$, and $F_2(w) = 9/(10(2w + 99)^2)$, respectively.

Finally, for the nonlinear functions of the right sides in problem (4.1) we have

$$\begin{aligned}
|\Upsilon_1(w, p_1, p_2) - \Upsilon_1(w, \overline{p}_1, \overline{p}_2)| &\leq \frac{1}{2(\sqrt{w}+1)}\left(|p_1 - \overline{p}_1| + |p_2 - \overline{p}_2|\right), \\
|\Upsilon_2(w, p_1, p_2) - \Upsilon_2(w, \overline{p}_1, \overline{p}_2)| &\leq \frac{1}{2(w^2+4)}\left(|p_1 - \overline{p}_1| + |p_2 - \overline{p}_2|\right),
\end{aligned}$$

which give $\omega_1(w) = 1/(2(\sqrt{w} + 1))$, $\omega_2(w) = 1/(2(w^2 + 4))$ and

$$|\Upsilon_1(w, p_1, p_2)| \leq \frac{1}{\sqrt{w}+2} + \frac{1}{2(\sqrt{w}+1)} + \frac{1}{3} := L_1(w),$$

and

$$|\Upsilon_2(w, p_1, p_2)| \leq \frac{1}{w^2+4}\left(\frac{\pi}{4} + 1\right) + \frac{1}{5} := L_2(w).$$

Therefore, using all of the information to compute a constant K in assumption (H_3) of Theorem 3.2, we obtain

$$K \approx 0.9229566975 < 1.$$

Hence, the given coupled system of nonlinear proportional Hilfer-type fractional differential equations with multi-point boundary conditions (4.1), satisfies all assumptions in Theorem 3.2. Then, by its conclusion, there exists at least one solution $(p_1, p_2)(w)$ to the problem (4.1) where $w \in [1/2, 7/2]$.

5. Conclusions

In this paper, we have presented the existence result for a new class of coupled systems of ψ-Hilfer proportional sequential fractional differential equations with multi-point boundary conditions. The proof of the existence result was based on a generalization of Krasnosel'skiĭ's fixed-point theorem due to Burton. An example was presented to illustrate our main result. Some special cases were also discussed. In future work, we can implement these techniques on different boundary value problems equipped with complicated integral multi-point boundary conditions.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

Professor Sotiris K. Ntouyas is an editorial board member for AIMS Mathematics and was not involved in the editorial review or the decision to publish this article. The authors declare no conflicts of interest.

References

1. K. Diethelm, *The analysis of fractional differential equations*, Berlin, Heidelberg: Springer, 2010. https://doi.org/10.1007/978-3-642-14574-2
2. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of the fractional differential equations*, New York: Elsevier, 2006. https://doi.org/10.1016/s0304-0208(06)x8001-5
3. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Wiley, 1993.
4. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.
5. R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World Scientific, 2000. https://doi.org/10.1142/3779
6. T. T. Soong, *Random differential equations in science and engineering*, New York: Academic Press, 1973. https://doi.org/10.1016/s0076-5392(08)x6011-4
7. K. Kavitha, V. Vijayakumar, R. Udhayakumar, K. S. Nisar, Results on the existence of Hilfer fractional neutral evolution equations with infinite delay via measures of noncompactness, *Math. Methods Appl. Sci.*, **44** (2021), 1438–1455. https://doi.org/10.1002/mma.6843
8. R. Subashini, K. Jothimani, K. S. Nisar, C. Ravichandran, New results on nonlocal functional integro-differential equations via Hilfer fractional derivative, *Alex. Eng. J.*, **59** (2020), 2891–2899. https://doi.org/10.1016/j.aej.2020.01.055
9. D. Luo, Q. Zhu, Z. Luo, A novel result on averaging principle of stochastic Hilfer-type fractional system involving non-Lipschitz coefficients, *Appl. Math. Lett.*, **122** (2021), 107549. https://doi.org/10.1016/j.aml.2021.107549
10. K. Ding, Q. Zhu, Impulsive method to reliable sampled-data control for uncertain fractional-order memristive neural networks with stochastic sensor faults and its applications, *Nonlinear Dyn.*, **100** (2020), 2595–2608. https://doi.org/10.1007/s11071-020-05670-y
11. H. M. Ahmed, Q. Zhu, The averaging principle of Hilfer fractional stochastic delay differential equations with Poisson jumps, *Appl. Math. Lett.*, **112** (2021), 106755. https://doi.org/10.1016/j.aml.2020.106755
12. S. K. Ntouyas, A survey on existence results for boundary value problems of Hilfer fractional differential equations and inclusions, *Foundations*, **1** (2021), 63–98. https://doi.org/10.3390/foundations1010007
13. J. V. C. Sousa, E. C. de Oliveira, On the ψ-Hilfer fractional derivative, *Commun. Nonlinear Sci. Numer. Simul.*, **60** (2018), 72–91. https://doi.org/10.1016/j.cnsns.2018.01.005
14. F. Jarad, T. Abdeljawad, S. Rashid, Z. Hammouch, More properties of the proportional fractional integrals and derivatives of a function with respect to another function, *Adv. Differ. Equ.*, **2020** (2020), 303. https://doi.org/10.1186/s13662-020-02767-x

15. R. Pandurangan, S. Shanmugam, M. Rhaima, H. Ghoudi, The generalized discrete proportional derivative and its applications, *Fractal Fract.*, **7** (2023), 838. https://doi.org/10.3390/fractalfract7120838
16. I. Ahmed, P. Kumam, F. Jarad, P. Borisut, W. Jirakitpuwapat, On Hilfer generalized proportional fractional derivative, *Adv. Differ. Equ.*, **2020** (2020), 329. https://doi.org/10.1186/s13662-020-02792-w
17. F. Jarad, T. Abdeljawad, J. Alzabut, Generalized fractional derivatives generated by a class of local proportional derivatives, *Eur. Phys. J. Spec. Top.*, **226** (2017), 3457–3471. https://doi.org/10.1140/epjst/e2018-00021-7
18. F. Jarad, M. A. Alqudah, T. Abdeljawad, On more general forms of proportional fractional operators, *Open Math.*, **18** (2020), 167–176. https://doi.org/10.1515/math-2020-0014
19. C. Kiataramkul, S. K. Ntouyas, J. Tariboon, An existence result for ψ-Hilfer fractional integro-differential hybrid three-point boundary value problems, *Fractal Fract.*, **5** (2021), 136. https://doi.org/10.3390/fractalfract5040136
20. R. L. Magin, Fractional calculus in bioengineering, *Crit. Rev. Biomed. Eng.*, **32** (2004). https://doi.org/10.1615/critrevbiomedeng.v32.i1.10
21. G. M. Zaslavsky, *Hamiltonian chaos and fractional dynamics*, Oxford: Oxford University Press, 2004. https://doi.org/10.1093/oso/9780198526049.001.0001
22. H. A. Fallahgoul, S. M. Focardi, F. J. Fabozzi, *Fractional calculus and fractional processes with applications to financial economics: Theory and application*, London: Academic Press, 2017.
23. M. A. Almalahi, M. S. Abdo, S. K. Panchal, Existence and Ulam-Hyers stability results of a coupled system of ψ-Hilfer sequential fractional differential equations, *Results Appl. Math.*, **10** (2021), 100142. https://doi.org/10.1016/j.rinam.2021.100142
24. B. Ahmad, S. Aljoudi, Investigation of a coupled system of Hilfer-Hadamard fractional differential equations with nonlocal coupled Hadamard fractional integral boundary conditions, *Fractal Fract.*, **7** (2023), 178. https://doi.org/10.3390/fractalfract7020178
25. A. Samadi, S. K. Ntouyas, J. Tariboon, Nonlocal coupled system for (k, φ)-Hilfer fractional differential equations, *Fractal Fract.*, **6** (2022), 234. https://doi.org/10.3390/fractalfract6050234
26. A. Samadi, S. K. Ntouyas, J. Tariboon, On a nonlocal coupled system of Hilfer generalized proportional fractional differential equations, *Symmetry*, **14** (2022), 738. https://doi.org/10.3390/sym14040738
27. I. Mallah, I. Ahmed, A. Akgul, F. Jarad, S. Alha, On ψ-Hilfer generalized proportional fractional operators, *AIMS Mathematics*, **7** (2022), 82–103. https://doi.org/10.3934/math.2022005
28. T. A. Burton, A fixed point theorem of Krasnoselskii, *Appl. Math. Lett.*, **11** (1998) , 85–88. https://doi.org/10.1016/s0893-9659(97)00138-9

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