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Research article

Investigation of multi-term delay fractional differential equations with integro-multipoint boundary conditions

Najla Alghamdi[1,*], **Bashir Ahmad**[2], **Esraa Abed Alharbi**[1] **and Wafa Shammakh**[1]

[1] Department of Mathematics and Statistics, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: nmalghamdi1@uj.edu.sa.

Abstract: A new class of nonlocal boundary value problems consisting of multi-term delay fractional differential equations and multipoint-integral boundary conditions is studied in this paper. We derive a more general form of the solution for the given problem by applying a fractional integral operator of an arbitrary order β_ξ instead of β_1; for details, see Lemma 2.2. The given problem is converted into an equivalent fixed-point problem to apply the tools of fixed-point theory. The existence of solutions for the given problem is established through the use of a nonlinear alternative of the Leray-Schauder theorem, while the uniqueness of its solutions is shown with the aid of Banach's fixed-point theorem. We also discuss the stability criteria, icluding Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stability, for solutions of the problem at hand. For illustration of the abstract results, we present examples. Our results are new and useful for the discipline of multi-term fractional differential equations related to hydrodynamics. The paper concludes with some interesting observations.

Keywords: delay differential equations; stability criteria; nonlocal integral boundary conditions; Caputo fractional drivative
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1. Introduction

During the last few decades, fractional calculus has been extensively investigated because of its numerous applications in natural and social sciences [1, 2]. The mathematical models based on the fractional order operators provide more insight into the characteristics of the phenomena under investigation due to their nonlocal nature, unlike the associated classical integer order operators.

Examples include fractional kinetics [3], chaos synchronization [4], relaxation filtration processes [5], blood flow in small-lumen arterial vessels [6], epithelial cells [7], neural networks [8], zooplankton-phytoplankton system [9], etc. For background details on the subject, we refer the reader to [10], while some recent works on nonlocal nonlinear fractional order boundary value problems can be found in a recent monograph [11]. One can find some interesting results on boundary-value problems involving multi-term fractional differential equations, inclusions and systems of such equations, as equipped with different kinds of boundary conditions in [12–16]. For some recent works on multi-term fractional differential equations, see [17–20]. There has also been a great surge in the study of fractional differential equations with delay [21–23]. Such equations play a key role in tracing the history of the phenomena under investigation [24]. Many researchers have also shown a keen interest in studying the stability properties of fractional differential equations; for instance, see [25–28].

In this paper, we aim to investigate a new class of boundary value problems of multi-term fractional differential equations complemented with nonlocal multipoint-integral boundary conditions. In precise terms, we explore the existence criteria for solutions of the following problem:

$$\begin{cases} \displaystyle\sum_{i=1}^{p} \alpha_i \; {}^c\mathcal{D}^{\beta_i} u(t) = h(t,u(t),u(wt)), \; \beta_\xi \in (p-1, p], \xi \neq i, 0 < \beta_i \leqslant 1, \; t \in [0,1], \\ u(0) = u_0, \quad u^{(k)}(0) = 0, \quad k = 1, 2, 3, \cdots, p-2, \\ u(1) = v \displaystyle\int_0^1 u(s)ds - \sum_{\ell=1}^{m} a_\ell u(\sigma_\ell), \;\; u_0, \; v, \; a_\ell \in \mathbb{R}, \; \sigma_\ell \in (0,1), \end{cases} \tag{1.1}$$

where ${}^c\mathcal{D}^{\beta_i}$ denotes the Caputo fractional derivative of order β_i, $w \in (0,1)$, $\alpha_i \in \mathbb{R}$, $i \in \{1, 2, \ldots, p\}$, and $h : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Here, we emphasize that the objective of the present work is to obtain a more general form of the solution by applying the fractional integral operator of order β_ξ instead of β_1, as used in [29] (for details, see the proof of Lemma 2.2). Moreover, we consider the multipoint-integral boundary conditions instead of only assuming the multipoint boundary conditions used in [29]. These conditions describe the boundary response, excluding the multipoint positions at $\sigma_\ell, \ell = 1, \ldots, m$, and give more insight into the study of fractional boundary-value problems related to multi-term fractional differential equations. We recall that differential equations containing more than one fractional-order differential operator constitute an interesting area of investigation, as such equations appear in the modeling of the motion of a rigid plate immersed in a Newtonian fluid (Bagley-Torvik equation [30]) and generalization of the Basset equation via the fractional calculus Basset equation [31]. It is imperative to mention that the results associated with purely integral boundary conditions follow as a special case (for explanation, see the Conclusions section). One of the useful techniques to obtain the existence theory for initial- and boundary-value problems relies on the tools of fixed-point theory. We adopt this methodology to develop the existence theory for the problem (1.1).

This paper is organized as follows. Section 2 contains the preliminary material related to our study. The main results, relying on the nonlinear alternative of the Leray-Schauder theorem and Banach contraction mapping principle, are presented in Section 3. The stability of solutions for the problem (1.1) is discussed in Section 4, while examples illustrating the obtained results are constructed in Section 5. Finally, in Section 6, we present some interesting observations.

2. Preliminaries

This section is devoted to the preliminary material related to our main work. Let us begin this section with some basic concepts of fractional calculus.

Definition 2.1. *([10]) Let υ be a locally integrable real-valued function on $-\infty \leq a < t < b \leq +\infty$. The Riemann-Liouville fractional integral I_a^{ω} of order $\omega \in \mathbb{R}$ $(\omega > 0)$ for the function υ is defined as follows:*

$$I_a^{\omega}\upsilon(t) = (\upsilon * K_{\omega})(t) = \int_a^t \frac{\upsilon(\vartheta)}{\Gamma(\omega)(t-\vartheta)^{1-\omega}}d\vartheta,$$

where $K_{\omega} = \frac{t^{\omega-1}}{\Gamma(\omega)}$ and Γ denotes the Euler gamma function.

Definition 2.2. *([10]) For $(p-1)$-times the absolutely continuous differentiable function $\upsilon : [a,\infty) \longrightarrow \mathbb{R}$, the Caputo derivative of fractional order ω for the function υ is defined as follows:*

$${}^{c}D^{\omega}\upsilon(t) = \int_a^t \frac{\upsilon^{(p)}(\vartheta)}{\Gamma(p-\omega)(t-\vartheta)^{1+\omega-p}}d\vartheta,\ p-1 < \omega \leq p,\ p = [\omega]+1,$$

where $[\omega]$ denotes the integer part of the real number ω.

Lemma 2.1. *([10]) For $p-1 < \omega < p$, the general solution of the fractional differential equation ${}^{c}D^{\omega}\upsilon(t) = 0$, $t \in [a,b]$, is given by*

$$\upsilon(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + ... + c_{p-1}(t-a)^{p-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,...,p-1$. Furthermore,

$$I^{\omega}\ {}^{c}D^{\omega}\upsilon(t) = \upsilon(t) + \sum_{i=0}^{p-1} c_i(t-a)^i.$$

The following lemma, as related the linear variant of the problem (1.1), is of fundamental importance in the forthcoming analysis. This result allows us to find the solution of the nonlinear problem (1.1) and then convert it into an equivalent fixed-point problem to discuss the existence and uniqueness of its solutions.

Lemma 2.2. *Let $y \in C([0,1],\mathbb{R})$. Then, the solution of the problem given by*

$$\begin{aligned}
&\sum_{i=1}^{p} \alpha_i\, {}^{c}\mathcal{D}^{\beta_i}u(t) = y(t),\ \beta_{\xi} \in (p-1,p],\ 0 < \beta_i \leq 1,\ i \in \{1,2,\ldots,p\},\ i \neq \xi,\ \alpha_i \in \mathbb{R},\ t \in [0,1],\\
&u(0) = u_0,\ u^{(k)}(0) = 0,\ \ k = 1,2,3,\ldots,p-2,\ u(1) = \nu\int_0^1 u(s)ds - \sum_{\ell=1}^{m} a_{\ell}u(\sigma_{\ell}),
\end{aligned} \tag{2.1}$$

is given by

$$\begin{aligned}
u(t) =&u_0 + \frac{t^{p-1}}{\gamma_1}\Big[u_0\Big(\nu - 1 - \sum_{\ell=1}^{m} a_{\ell}\Big) + \frac{\nu}{\alpha_{\xi}\Gamma(\beta_{\xi})}\int_0^1\int_0^s (s-z)^{\beta_{\xi}-1}y(z)dzds\\
&- \sum_{i=1,i\neq\xi}^{p} \frac{\alpha_i}{\alpha_{\xi}}\frac{\nu}{\Gamma(\beta_{\xi}-\beta_i)}\int_0^1\int_0^s (s-z)^{\beta_{\xi}-\beta_i-1}u(z)dzds
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\ell=1}^{m} \frac{a_\ell}{\alpha_\xi \Gamma(\beta_\xi)} \int_0^{\sigma_\ell} (\sigma_\ell - z)^{\beta_\xi - 1} y(z) dz \\
&+ \sum_{\ell=1}^{m} a_\ell \sum_{i=1, i \neq \xi}^{p} \frac{\alpha_i}{\alpha_\xi} \frac{1}{\Gamma(\beta_\xi - \beta_i)} \int_0^{\sigma_\ell} (\sigma_\ell - z)^{\beta_\xi - \beta_i - 1} u(z) dz \\
&- \frac{1}{\alpha_\xi \Gamma(\beta_\xi)} \int_0^1 (1 - z)^{\beta_\xi - 1} y(z) dz + \sum_{i=1, i \neq \xi}^{p} \frac{\alpha_i}{\alpha_\xi} \frac{1}{\Gamma(\beta_\xi - \beta_i)} \int_0^1 (1 - z)^{\beta_\xi - \beta_i - 1} u(z) dz \Big] \\
&+ \frac{1}{\alpha_\xi \Gamma(\beta_\xi)} \int_0^t (t - z)^{\beta_\xi - 1} y(z) dz - \sum_{i=1, i \neq \xi}^{p} \frac{\alpha_i}{\alpha_\xi} \frac{1}{\Gamma(\beta_\xi - \beta_i)} \int_0^t (t - z)^{\beta_\xi - \beta_i - 1} u(z) dz, \qquad (2.2)
\end{aligned}
$$

where

$$
\gamma_1 = 1 - \frac{v}{p} + \sum_{\ell=1}^{m} a_\ell \sigma_l^{p-1}. \qquad (2.3)
$$

Proof. Applying the fractional integral operator of order β_ξ to both sides of the multi-term fractional differential equation in (2.1), we obtain

$$
\begin{aligned}
u(t) =& E_1 + E_2 t + ... + E_p t^{p-1} + \frac{1}{\alpha_\xi} \int_0^t \frac{(t - z)^{\beta_\xi - 1}}{\Gamma(\beta_\xi)} y(z) dz \\
&- \sum_{i=1, i \neq \xi}^{p} \frac{\alpha_i}{\alpha_\xi} \frac{1}{\Gamma(\beta_\xi - \beta_i)} \int_0^t (t - z)^{\beta_\xi - \beta_i - 1} u(z) dz, \qquad (2.4)
\end{aligned}
$$

where $E_1, E_2, ..., E_p$ are arbitrary constants. Applying (2.4) with the conditions $u(0) = u_0$ and $u^{(k)}(0) = 0,\ k = 1, 2, 3, \ldots, p - 2$, we find that $E_1 = u_0$ and $E_i = 0$ for $i = 2, \ldots, (p - 1)$. Therefore, (2.4) takes the following form:

$$
\begin{aligned}
u(t) =& u_0 + E_p t^{p-1} + \frac{1}{\alpha_\xi} \int_0^t \frac{(t - z)^{\beta_\xi - 1}}{\Gamma(\beta_\xi)} y(z) dz \\
&- \sum_{i=1, i \neq \xi}^{p} \frac{\alpha_i}{\alpha_\xi} \frac{1}{\Gamma(\beta_\xi - \beta_i)} \int_0^t (t - z)^{\beta_\xi - \beta_i - 1} u(z) dz. \qquad (2.5)
\end{aligned}
$$

Applying (2.5) with the condition $u(1) = v \int_0^1 u(s) ds - \sum_{\ell=1}^{m} a_\ell u(\sigma_\ell)$, we find that

$$
\begin{aligned}
E_p =& \frac{1}{\gamma_1} \Big[u_0 \Big(v - 1 - \sum_{\ell=1}^{m} a_\ell \Big) + \frac{v}{\alpha_\xi \Gamma(\beta_\xi)} \int_0^1 \int_0^s (s - z)^{\beta_\xi - 1} y(z) dz ds \\
&- \sum_{i=1, i \neq \xi}^{p} \frac{\alpha_i}{\alpha_\xi} \frac{v}{\Gamma(\beta_\xi - \beta_i)} \int_0^1 \int_0^s (s - z)^{\beta_\xi - \beta_i - 1} u(z) dz ds \\
&- \sum_{\ell=1}^{m} \frac{a_\ell}{\alpha_\xi \Gamma(\beta_\xi)} \int_0^{\sigma_\ell} (\sigma_\ell - z)^{\beta_\xi - 1} y(z) dz
\end{aligned}
$$

$$+\sum_{\ell=1}^{m}a_\ell\sum_{i=1,i\neq\xi}^{p}\frac{\alpha_i}{\alpha_\xi}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^{\sigma_\ell}(\sigma_\ell-z)^{\beta_\xi-\beta_i-1}u(z)dz-\frac{1}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^1(1-z)^{\beta_\xi-1}y(z)dz$$
$$+\sum_{i=1,i\neq\xi}^{p}\frac{\alpha_i}{\alpha_\xi}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^1(1-z)^{\beta_\xi-\beta_i-1}u(z)dz\Big],$$

where γ_1 is given in (2.3). Substituting the above value of E_p in (2.5), we obtain the solution (2.2). The converse of this lemma follows by direct computation. This completes the proof. □

Theorem 2.1. *(Nonlinear alternative of Leray-Schauder theorem [32]) Let U be a Banach space and C a non-empty convex subset of U. Let N be a non-empty open subset of C with $0\in N$ and $\mathcal{H}:\overline{N}\to C$ be a continuous and compact operator. Then, either $\mathcal{H}$ has fixed points or there exists $u\in\partial N$ such that $u=\lambda_*\mathcal{H}(u)$ with $\lambda_*\in(0,1)$.*

3. Main results

In this section, we establish the existence and uniqueness results for the problem (1.1) by applying the standard tools of fixed-point theory.

Let us first transform the problem (1.1) into a fixed-point problem:

$$\mathcal{H}u(t)=u(t),$$

where $\mathcal{H}:C([0,1],\mathbb{R})\mapsto C([0,1],\mathbb{R})$ is the fixed operator defined by

$$\begin{aligned}\mathcal{H}u(t)=&u_0+\frac{t^{p-1}}{\gamma_1}\Big[u_0\Big(v-1-\sum_{\ell=1}^{m}a_\ell\Big)+\frac{v}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^1\int_0^s(s-z)^{\beta_\xi-1}h(z,u(z),u(\omega z))dzds\\&-\sum_{i=1,i\neq\xi}^{p}\frac{\alpha_i}{\alpha_\xi}\frac{v}{\Gamma(\beta_\xi-\beta_i)}\int_0^1\int_0^s(s-z)^{\beta_\xi-\beta_i-1}u(z)dzds\\&-\sum_{\ell=1}^{m}\frac{a_\ell}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^{\sigma_\ell}(\sigma_\ell-z)^{\beta_\xi-1}h(z,u(z),u(\omega z))dz\\&+\sum_{\ell=1}^{m}a_\ell\sum_{i=1,i\neq\xi}^{p}\frac{\alpha_i}{\alpha_\xi}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^{\sigma_\ell}(\sigma_\ell-z)^{\beta_\xi-\beta_i-1}u(z)dz\\&-\frac{1}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^1(1-z)^{\beta_\xi-1}h(z,u(z),u(\omega z))dz\\&+\sum_{i=1,i\neq\xi}^{p}\frac{\alpha_i}{\alpha_\xi}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^1(1-z)^{\beta_\xi-\beta_i-1}u(z)dz\Big]\\&+\frac{1}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^t(t-z)^{\beta_\xi-1}h(z,u(z),u(\omega z))dz\\&-\sum_{i=1,i\neq\xi}^{p}\frac{\alpha_i}{\alpha_\xi}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^t(t-z)^{\beta_\xi-\beta_i-1}u(z)dz.\end{aligned}\tag{3.1}$$

Here, $C([0,1],\mathbb{R})$ denotes the Banach space of all continuous real-valued functions defined on the interval $[0,1]$, equipped with the norm $\|u\| = \sup\left\{|u(t)|, t \in [0,1]\right\}$.

In the sequel, we need the following assumptions.

(M_1) The function $h : [0,1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is continuous.

(M_2) For each $t \in [0,1]$ and $u_1, u_2, y_1, y_2 \in \mathbb{R}$, there exist $L_1, L_2 > 0$ such that

$$|h(t,u_1,u_2) - h(t,y_1,y_2)| \leqslant L_1\|u_1 - y_1\| + L_2\|u_2 - y_2\|.$$

(M_3) $|h(z,u(z),u(wz))| \leqslant |\psi_h(t)|$ for each $(t,u_1,u_2) \in [0,1] \times \mathbb{R} \times \mathbb{R}$, where $\psi_h \in C([0,1],\mathbb{R}^+)$.

(M_4) For any non-decreasing function $\zeta \in C([0,1],\mathbb{R}^+)$, there exists a positive constant $\mathfrak{q}$ such that $I^{\beta_\xi}\zeta(t) = \mathfrak{q}\zeta(t). \quad \forall t \in [0,1]$.

In passing, we remark that the assumptions (M_1) and (M_2) will be used to establish the uniqueness of solutions for the given problem, while an existence result will be obtained with the aid of conditions (M_1) and (M_3). We need the conditions (M_1) and (M_4) to formulate Theorem 4.1, while the assumptions (M_1), (M_2), and (M_4) are used in the construction of Theorem 4.2.

In our first result, we prove the existence of a unique solution to the problem (1.1) by applying the Banach fixed-point theorem.

Theorem 3.1. *If the assumptions* $(M_1)-(M_2)$ *are satisfied, then the problem* (1.1) *has a unique solution on* $[0,1]$*, provided that* $\theta < 1$*, where*

$$\begin{aligned}\theta =&\frac{1}{|\gamma_1|}\frac{L_1+L_2}{|\alpha_\xi|\Gamma(\beta_\xi+1)}\Big(\frac{|v|}{(\beta_\xi+1)} + \sum_{\ell=1}^{m}|a_\ell|\sigma_\ell^{\beta_\xi} + 1 + |\gamma_1|\Big)\\ &+\sum_{\ell=1}^{m}\frac{|a_\ell|}{|\gamma_1|}\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i+1)}\sigma_\ell^{\beta_\xi-\beta_i}\\ &+\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i+1)}\Big(\frac{1}{|\gamma_1|} + \frac{|v|}{|\gamma_1|(\beta_\xi-\beta_i+1)} + 1\Big).\end{aligned}\tag{3.2}$$

Proof. Let us establish that the operator $\mathcal{H} : C([0,1],\ \mathbb{R}) \mapsto C([0,1],\ \mathbb{R})$ defined in (3.1) is a contraction. For $u_1, u_2 \in C([0,1],\mathbb{R})$, we have

$$\begin{aligned}&\|\mathcal{H}u_1(t) - \mathcal{H}u_2(t)\|\\ \leqslant& \sup_{t\in[0,1]}\Big\{\frac{|t^{p-1}|}{|\gamma_1|}\Big[\frac{|v|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1\int_0^s(s-z)^{\beta_\xi-1}|h(z,u_1(z),u_1(\omega z)) - h(z,u_2(z),u_2(\omega z))|dzds\\ &+\sum_{\ell=1}^{m}\frac{|a_\ell|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^{\sigma_\ell}(\sigma_\ell - z)^{\beta_\xi-1}|h(z,u_1(z),u_1(\omega z)) - h(z,u_2(z),u_2(\omega z))|dz\\ &+\frac{1}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1(1-z)^{\beta_\xi-1}|h(z,u_1(z),u_1(\omega z)) - h(z,u_2(z),u_2(\omega z))|dz\\ &+\sum_{\ell=1}^{m}|a_\ell|\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i)}\int_0^{\sigma_\ell}(\sigma_\ell - z)^{\beta_\xi-\beta_i-1}|u_1(z) - u_2(z)|dz\end{aligned}$$

$$+\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i)}\int_0^1(1-z)^{\beta_\xi-\beta_i-1}|u_1(z)-u_2(z)|dz$$
$$+\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i||\nu|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i)}\int_0^1\int_0^s(s-z)^{\beta_\xi-\beta_i-1}|u_1(z)-u_2(z)|dzds\Big]$$
$$+\frac{1}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^t(t-z)^{\beta_\xi-1}|h(z,u_1(z),u_1(\omega z))-h(z,u_2(z),u_2(\omega z))|dz$$
$$+\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i)}\int_0^t(t-z)^{\beta_\xi-\beta_i-1}|u_1(z)-u_2(z)|dz\Big\}.$$

Using (M_2) in the above inequality with $L_3=L_1+L_2$, we obtain

$$\begin{aligned}
&\|\mathcal{H}u_1(t)-\mathcal{H}u_2(t)\|\\
\leqslant&\|u_1-u_2\|\sup_{t\in[0,1]}\Big\{\frac{1}{|\gamma_1|}\Big[L_3\frac{|\nu|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1\int_0^s(s-z)^{\beta_\xi-1}dzds+\sum_{\ell=1}^{m}L_3\frac{|a_\ell|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^{\sigma_\ell}(\sigma_\ell-z)^{\beta_\xi-1}dz\\
&+\frac{L_3}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1(1-z)^{\beta_\xi-1}dz+\sum_{\ell=1}^{m}|a_\ell|\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i)}\int_0^{\sigma_\ell}(\sigma_\ell-z)^{\beta_\xi-\beta_i-1}dz\\
&+\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i)}\int_0^1(1-z)^{\beta_\xi-\beta_i-1}dz+\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i||\nu|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i)}\int_0^1\int_0^s(s-z)^{\beta_\xi-\beta_i-1}dzds\\
&+\frac{L_3}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^t(t-z)^{\beta_\xi-1}dz+\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i)}\int_0^t(t-z)^{\beta_\xi-\beta_i-1}dz\Big\}\\
\leqslant&\|u_1-u_2\|\Big[\frac{1}{|\gamma_1|}\frac{L_3}{|\alpha_\xi|\Gamma(\beta_\xi+1)}\Big(\frac{|\nu|}{(\beta_\xi+1)}+\sum_{\ell=1}^{m}|a_\ell|\sigma_\ell^{\beta_\xi}+1+|\gamma_1|\Big)\\
&+\sum_{\ell=1}^{m}\frac{|a_\ell|}{|\gamma_1|}\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i+1)}\sigma_\ell^{\beta_\xi-\beta_i}+\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|\Gamma(\beta_\xi-\beta_i+1)}\Big(\frac{1}{|\gamma_1|}+\frac{|\nu|}{|\gamma_1|(\beta_\xi-\beta_i+1)}+1\Big)\Big]\\
=&\theta\|u_1-u_2\|,
\end{aligned}$$

where θ is given in (3.2). In view of the given condition, i.e., $\theta<1$, it follows from the above inequality that the operator $\mathcal{H}:C([0,1],\mathbb{R})\mapsto C([0,1],\mathbb{R})$ is a contraction. Therefore, by the conclusion of the Banach contraction mapping principle, there exists a unique solution to the problem (1.1) on $[0,1]$. □

The following theorem deals with the existence of solutions for the problem (1.1), and it is based on the nonlinear alternative of the Leray-Schauder theorem [32].

Theorem 3.2. *If the assumptions* (M_1) *and* (M_3) *hold, then there exists at least one solution for the problem* (1.1) *on* $[0,1]$.

Proof. For a positive real number r_0, we define a set $A=\{u\in C([0,1],\mathbb{R}):\|u\|\leqslant r_0\}$. The proof will be completed in several steps.

Step 1. In the first step, we claim that $\mathcal{H}$ is uniformly bounded. Let $u \in C([0,1],\mathbb{R})$. Then, we have

$$\begin{aligned}
\|\mathcal{H}u\| \leqslant\ & |u_0| + \frac{1}{|\gamma_1|}\Bigg[\Big|u_0\Big(v - 1 - \sum_{\ell=1}^{m} a_\ell\Big)\Big| + \frac{|v|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1\int_0^s (s-z)^{\beta_\xi - 1}|h(z,u(z),u(wz))|dzds \\
&+ \sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{|v|}{\Gamma(\beta_\xi-\beta_i)}\int_0^1\int_0^s (s-z)^{\beta_\xi-\beta_i-1}|u(z)|dzds \\
&+ \sum_{\ell=1}^{m}\frac{|a_\ell|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^{\sigma_\ell}(\sigma_\ell - z)^{\beta_\xi-1}|h(z,u(z),u(wz))|dz \\
&+ \sum_{\ell=1}^{m}|a_\ell|\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^{\sigma_\ell}(\sigma_\ell - z)^{\beta_\xi-\beta_i-1}|u(z)|dz \\
&+ \frac{1}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1 (1-z)^{\beta_\xi-1}|h(z,u(z),u(wz))|dz + \sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^1(1-z)^{\beta_\xi-\beta_i-1}|u(z)|dz\Bigg] \\
&+ \sup_{t\in[0,1]}\Big\{\frac{1}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^t (t-z)^{\beta_\xi-1}|h(z,u(z),u(wz))|dz\Big\} \\
&+ \sup_{t\in[0,1]}\Big\{\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^t (t-z)^{\beta_\xi-\beta_i-1}|u(z)|dz\Big\}. \qquad (3.3)
\end{aligned}$$

In view of the definition of the set A and the assumption (M_3), (3.3) takes the following form:

$$\begin{aligned}
\|\mathcal{H}u\| \leqslant\ & |u_0| + \frac{1}{|\gamma_1|}\Big|u_0\Big(v - 1 - \sum_{\ell=1}^{m} a_\ell\Big)\Big| + \|\psi_h\|\Bigg[\frac{|v|}{|\alpha_\xi|\Gamma(\beta_\xi+2)} + \sum_{\ell=1}^{m}\frac{|a_\ell|}{|\alpha_\xi|\Gamma(\beta_\xi+1)}\sigma_\ell^{\beta_\xi} \\
&+ \frac{1}{|\alpha_\xi||\gamma_1|\Gamma(\beta_\xi+1)} + \frac{1}{|\alpha_\xi|\Gamma(\beta_\xi+1)}\Bigg] + r_0\Bigg[\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi||\gamma_1|}\frac{|v|}{\Gamma(\beta_\xi-\beta_i+2)} \\
&+ \sum_{\ell=1}^{m}|a_\ell|\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi||\gamma_1|}\frac{1}{\Gamma(\beta_\xi-\beta_i+1)}\sigma_\ell^{\beta_\xi-\beta_i} \\
&+ \sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi||\gamma_1|}\frac{1}{\Gamma(\beta_\xi-\beta_i+1)} + \sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{1}{\Gamma(\beta_\xi-\beta_i+1)}\Bigg].
\end{aligned}$$

Step 2. We now claim that $\mathcal{H}$ is continuous. To prove this, we consider a sequence $u_n \in A$ that converges to u and show that $\mathcal{H}u_n \mapsto \mathcal{H}u(t)$ as $n \mapsto \infty$. To do this, we consider the following:

$$\begin{aligned}
\|\mathcal{H}u_n - \mathcal{H}u\| \leqslant\ & \frac{1}{|\gamma_1|}\Bigg[\frac{|v|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1\int_0^s (s-z)^{\beta_\xi-1}|h(z,u_n(z),u_n(wz)) - h(z,u(z),u(wz))|dzds \\
&+ \sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{|v|}{\Gamma(\beta_\xi-\beta_i)}\int_0^1\int_0^s (s-z)^{\beta_\xi-\beta_i-1}|u_n(z)-u(z)|dzds \\
&+ \sum_{\ell=1}^{m}\frac{|a_\ell|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^{\sigma_\ell}(\sigma_\ell - z)^{\beta_\xi-1}|h(z,u_n(z),u_n(wz)) - h(z,u(z),u(wz))|dz
\end{aligned}$$

$$+\sum_{\ell=1}^{m}|a_\ell|\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^{\sigma_\ell}(\sigma_\ell-z)^{\beta_\xi-\beta_i-1}|u_n(z)-u(z)|dz$$
$$+\frac{1}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1(1-z)^{\beta_\xi-1}|h(z,u_n(z),u_n(wz))-h(z,u(z),u(wz))|dz$$
$$+\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^1(1-z)^{\beta_\xi-\beta_i-1}|u_n(z)-u(z)|dz\Bigg]$$
$$+\sup_{t\in[0,1]}\Big\{\frac{1}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^t(t-z)^{\beta_\xi-1}|h(z,u_n(z),u_n(wz))-h(z,u(z),u(wz))|dz\Big\}$$
$$+\sup_{t\in[0,1]}\Big\{\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^t(t-z)^{\beta_\xi-\beta_i-1}|u_n(z)-u(z)|dz\Big\}.$$

Hence, by the Lebesgue dominated convergent theorem, we have that $\|\mathcal{H}u_n-\mathcal{H}u\|\mapsto 0$ as $n\mapsto\infty$.

Step 3. We claim that $\wp$ maps the bounded set into a set of equicontinuous functions. For $t_1\leqslant t_2$, it follows by using (M_3) that

$$\begin{aligned}
&|\mathcal{H}u(t_1)-\mathcal{H}u(t_2)|\\
\leqslant\ &\frac{|t_1^{p-1}-t_2^{p-1}|}{|\gamma_1|}\Bigg[\Big|u_0\Big(v-1-\sum_{\ell=1}^{m}a_\ell\Big)\Big|+\|\psi_h\|\frac{|v|}{|\alpha_\xi|\Gamma(\beta_\xi+2)}+r_0\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{|v|}{\Gamma(\beta_\xi-\beta_i+2)}\\
&+\|\psi_\hbar\|\sum_{\ell=1}^{m}\frac{|a_\ell|}{|\alpha_\xi|\Gamma(\beta_\xi+1)}\sigma_\ell^{\beta_\xi}+r_0\sum_{\ell=1}^{m}|a_\ell|\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{1}{\Gamma(\beta_\xi-\beta_i+1)}\sigma_\ell^{\beta_\xi-\beta_i}\\
&+\|\psi_\hbar\|\frac{1}{|\alpha_\xi|\Gamma(\beta_\xi+1)}+r_0\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{1}{\Gamma(\beta_\xi-\beta_i+1)}\Bigg]\\
&+\|\psi_\hbar\|\frac{1}{|\alpha_\xi|\Gamma(\beta_\xi+1)}\Big[t_2^{\beta_\xi}-(t_2-t_1)^{\beta_\xi}-t_1^{\beta_\xi}+(t_2-t_1)^{\beta_\xi}\Big]\\
&+r_0\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{1}{\Gamma(\beta_\xi-\beta_i+1)}\Big[t_2^{\beta_\xi-\beta_i}-(t_2-t_1)^{\beta_\xi-\beta_i}-t_1^{\beta_\xi-\beta_i}+(t_2-t_1)^{\beta_\xi-\beta_i}\Big].
\end{aligned}$$

Clearly, $\|\mathcal{H}u(t_1)-\mathcal{H}u(t_2)\|\to 0$ as $t_1\to t_2$.

Step 4. Now, we need to prove that there exists an open set $U\subseteq C([0,1],\mathbb{R})$ such that $u\neq\lambda_*\mathcal{H}(u(t))$ for $\lambda_*\in(0,1)$ and $u\in\partial U$. Let $u\in C([0,1],\mathbb{R})$ be such that $u=\lambda_*\mathcal{H}(u(t))$ for $\lambda_*\in(0,1)$. Then, for $t\in[0,1]$, we have

$$\begin{aligned}
|u(t)|=|\lambda_*\mathcal{H}u(t)|\ \leqslant\ &|u_0|+\frac{1}{|\gamma_1|}\Big|u_0\Big(v-1-\sum_{\ell=1}^{m}a_\ell\Big)\Big|+\|\psi_h\|\Bigg[\frac{|v|}{|\alpha_\xi|\Gamma(\beta_\xi+2)}+\sum_{\ell=1}^{m}\frac{|a_\ell|}{|\alpha_\xi||\gamma_1|\Gamma(\beta_\xi+1)}\sigma_\ell^{\beta_\xi}\\
&+\frac{1}{|\alpha_\xi||\gamma_1|\Gamma(\beta_\xi+1)}+\frac{1}{|\alpha_\xi|\Gamma(\beta_\xi+1)}\Bigg]+r_0\Bigg[\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi||\gamma_1|}\frac{|v|}{\Gamma(\beta_\xi-\beta_i+2)}\\
&+\sum_{\ell=1}^{m}|a_\ell|\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi||\gamma_1|}\frac{1}{\Gamma(\beta_\xi-\beta_i+1)}\sigma_\ell^{\beta_\xi-\beta_i}
\end{aligned}$$

$$+\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi||\gamma_1|}\frac{1}{\Gamma(\beta_\xi-\beta_i+1)}+\sum_{i=1,i\neq\xi}^{p}\frac{|\alpha_i|}{|\alpha_\xi|}\frac{1}{\Gamma(\beta_\xi-\beta_i+1)}\Bigg]=\mathcal{M}.$$

Let us select $U=\{u\in C([0,1],\mathbb{R}):\|u\|\leqslant\mathcal{M}+1\}$. Then, there is no $u\in\partial U$ such that $u=\lambda_*\mathcal{H}u(t)$ for $\lambda_*\in(0,1)$. Hence, by the nonlinear alternative of the Leray-Schauder theorem [32], we deduce that $\mathcal{H}$ has at least one fixed point in $\overline{U}$, which is a solution of the problem (1.1). □

4. Stability results

In this section, we study the stability criteria, including Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stability, for solutions of the problem (1.1). Let us first define the different concepts of stability in relation to the problem at hand.

Definition 4.1. *The solution $u(t)$ of the problem* (1.1) *is called Ulam-Hyers-stable if there exist constants $A_1\geqslant 0$ and $\varepsilon\geqslant 0$ such that, for each solution $u\in C([0,1],\mathbb{R})$ of the following inequality:*

$$\Big|\sum_{i=1}^{p}\alpha_i\,{}^c\mathcal{D}^{\beta_i}u(t)-h(t,u(t),u(wt))\Big|\leqslant\varepsilon,\quad \text{where } t\in[0,1],\quad \alpha_i\in\mathbb{R},\tag{4.1}$$

we have that $|u-u^*|\leqslant A_1\varepsilon$, where $u^*(t)$ is a unique solution of the problem (1.1). *On the other hand, if there exist a positive function $\kappa:(0,\infty)\mapsto(0,\infty)$ with $\kappa(0)=0$ such that $|u-u^*|\leqslant A_1\kappa(\varepsilon)$, then the solution $u(t)$ of the problem* (1.1) *is called generalized Ulam-Hyers-stable.*

Definition 4.2. *The solution $u(t)$ of the problem* (1.1) *is Ulam-Hyers-Rassias-stable for a continuous function $\vartheta\in U$ if there exist constants $\psi,A_2>0$ and $\varepsilon>0$ such that, for each solution $u\in C([0,1],\mathbb{R})$ of the following inequality:*

$$\Big|\sum_{i=1}^{p}\alpha_i\,{}^c\mathcal{D}^{\beta_i}u(t)-h(t,u(t),u(wt))\Big|\leqslant(\vartheta(t)+\psi)\varepsilon,\tag{4.2}$$

we have that $|u-u^*|\leqslant A_2(\vartheta(t)+\psi)\varepsilon$, where $u^*\in C([0,1],\mathbb{R})$ is a unique solution of the problem (1.1).

Definition 4.3. *The solution $u(t)$ of the problem* (1.1) *is generalized Ulam-Hyers-Rassias-stable for a continuous function $\vartheta\in U$ and a positive constant ψ if we can find a constant $A_2>0$ such that, for each solution $u\in C([0,1],\mathbb{R})$ of the following differential inequality:*

$$\Big|\sum_{i=1}^{p}\alpha_i\,{}^c\mathcal{D}^{\beta_i}u(t)-h(t,u(t),u(wt))\Big|\leqslant\vartheta(t)+\psi,\tag{4.3}$$

we have that $|u-u^*|\leqslant A_2\vartheta(t)\varepsilon$, where $u^*\in C([0,1],\mathbb{R})$ is a unique solution of the problem (1.1).

Remark 4.1. *The inequality* (4.1) *has a solution $u\in C([0,1],\mathbb{R})$ if and only if there exists a function $\zeta\in C([0,1],\mathbb{R})$, depending on u, such that*

(1) $\varepsilon\geqslant\zeta(t)$, *where* $t\in[0,1]$,

(2) $\sum_{i=1}^{p}\alpha_i\,{}^c\mathcal{D}^{\beta_i}u(t)-h(t,u(t),u(wt))-\zeta(t)=0$.

Remark 4.2. *$u \in C([0,1],\mathbb{R})$ is a solution of (4.2) if and only if there exists a function $\zeta \in C([0,1],\mathbb{R})$, depending on u, such that*

(1) $\zeta(t) \leqslant \vartheta(t)\,\varepsilon$ and $\zeta(t) \leqslant \psi\,\varepsilon$, $t \in [0,1]$,

(2) $\sum_{i=1}^{p} \alpha_i^c \mathcal{D}^{\beta_i} u(t) - h(t,u(t),u(wt)) - \zeta(t) = 0$.

Remark 4.3. *$u \in C([0,1],\mathbb{R})$ is a solution of (4.3) if and only if one can find a function $\zeta \in C([0,1],\mathbb{R})$, depending on u, such that*

(1) $\zeta(t) \leqslant \vartheta(t)$ and $\vartheta \leqslant \psi$, $t \in [0,1]$,

(2) $\sum_{i=1}^{p} \alpha_i^c \mathcal{D}^{\beta_i} u(t) - h(t,u(t),u(wt)) - \zeta(t) = 0$.

Lemma 4.1. *Let $u \in C([0,1],\mathbb{R})$ be a solution of the following problem:*

$$\begin{cases} \sum_{i=1}^{p} \alpha_i \mathcal{D}^{\beta_i} u(t) = h(t,u(t),u(wt)) + \zeta(t), \ \ p-i < \beta_i \leqslant p+1-i, \ \ w \in (0,1), \alpha_i \in \mathbb{R}, t \in [0,1], \\ u(0) = u_0, \ \dfrac{d^{\ell} u(0)}{dt^{\ell}} = 0, \\ u(1) = v\int_0^1 u(s)ds - \sum_{\ell=1}^{m} a_\ell u(\sigma_\ell), \ \ a_\ell \in \mathbb{R}, \ \ \sigma_\ell \in (0,1), \ell = 1,2,\cdots,p-2 \ \ \& \ \ i = 1,\cdots,m. \end{cases}$$

Then, u satisfies the following relation:

$$|u(t) - \Im u(t)| \leqslant \mathcal{A}_1 \varepsilon,$$

where

$$\begin{aligned} \Im u(t) = & u_0 + \frac{t^{p-1}}{\gamma_1}\Big[u_0\Big(v - 1 - \sum_{\ell=1}^{m} a_\ell\Big) + \frac{v}{\alpha_\xi \Gamma(\beta_\xi)}\int_0^1\int_0^s (s-z)^{\beta_\xi - 1} h(z,u(z),u(\omega z))dzds \\ & - \sum_{i=1,i\neq\xi}^{p} \frac{\alpha_i}{\alpha_\xi}\frac{v}{\Gamma(\beta_\xi - \beta_i)}\int_0^1\int_0^s (s-z)^{\beta_\xi-\beta_i-1} u(z)dzds \\ & - \sum_{\ell=1}^{m} \frac{a_\ell}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^{\sigma_\ell} (\sigma_\ell - z)^{\beta_\xi-1} h(z,u(z),u(\omega z))dz \\ & + \sum_{\ell=1}^{m} a_\ell \sum_{i=1,i\neq\xi}^{p} \frac{\alpha_i}{\alpha_\xi}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^{\sigma_\ell} (\sigma_\ell - z)^{\beta_\xi-\beta_i-1} u(z)dz \\ & - \frac{1}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^1 (1-z)^{\beta_\xi-1} h(z,u(z),u(\omega z))dz \\ & + \sum_{i=1,i\neq\xi}^{p} \frac{\alpha_i}{\alpha_\xi}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^1 (1-z)^{\beta_\xi-\beta_i-1} u(z)dz\Big] \\ & + \frac{1}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^t (t-z)^{\beta_\xi-1} h(z,u(z),u(\omega z))dz \\ & - \sum_{i=1,i\neq\xi}^{p} \frac{\alpha_i}{\alpha_\xi}\frac{1}{\Gamma(\beta_\xi-\beta_i)}\int_0^t (t-z)^{\beta_\xi-\beta_i-1} u(z)dz, \end{aligned}$$

and

$$\mathcal{A}_1 = \frac{\varepsilon}{|\gamma_1|\alpha_\xi|\Gamma(\beta_\xi+1)}\Big(\frac{|v|}{(\beta_\xi+1)}+\sum_{\ell=1}^{m}|a_\ell|\sigma_\ell^{\beta_\xi}+1+|\gamma_1|\Big). \tag{4.4}$$

Proof. Suppose that $u \in C([0,1],\mathbb{R})$ is a solution of (1.1). Then, we have

$$\begin{aligned} u(t) &= \Im u(t) + \frac{t^{p-1}}{\gamma_1}\Big[\frac{v}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^1\int_0^s (s-z)^{\beta_\xi-1}\zeta(z)dzds \\ &\quad -\sum_{\ell=1}^{m}\frac{a_\ell}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^{\sigma_\ell}(\sigma_\ell-z)^{\beta_\xi-1}\zeta(z)dz - \frac{1}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^1(1-z)^{\beta_\xi-1}\zeta(z)dz\Big] \\ &\quad +\frac{1}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^t(t-z)^{\beta_\xi-1}\zeta(z)dz. \end{aligned}$$

Using Remark 4.1 in (4.5) and the method of calculation employed to prove the results of the last section, we get

$$\begin{aligned} |u(t)-\Im u(t)| \leqslant& \varepsilon \sup_{t\in[0,1]}\Big\{\frac{1}{|\gamma_1|}\Big[\frac{|v|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1\int_0^s(s-z)^{\beta_\xi-1}dzds \\ &+\sum_{\ell=1}^{m}\frac{|a_\ell|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^{\sigma_\ell}(\sigma_\ell-z)^{\beta_\xi-1}dz+\frac{1}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1(1-z)^{\beta_\xi-1}dz\Big] \\ &+\frac{1}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^t(t-z)^{\beta_\xi-1}dz\Big\} \\ \leqslant& \frac{\varepsilon}{|\gamma_1|\alpha_\xi|\Gamma(\beta_\xi+1)}\Big(\frac{|v|}{(\beta_\xi+1)}+\sum_{\ell=1}^{m}|a_\ell|\sigma_\ell^{\beta_\xi}+1+|\gamma_1|\Big). \end{aligned} \tag{4.5}$$

Applying (4.4) in (4.5), we obtain

$$|u(t)-\Im u(t)| \leqslant \mathcal{A}_1\varepsilon,$$

which completes the proof. □

Theorem 4.1. *Under the assumptions* (M_1) *and* (M_4)*, the problem* (1.1) *is Ulam-Hyers-stable and generalized Ulam-Hyers-stable if* $\theta < 1$, *where* θ *is defined by* (3.2).

Proof. For any solution $u \in C([0,1],\mathbb{R})$ and a unique solution u^* of the problem (1.1), we have

$$\begin{aligned} \|u-u^*\| &= \|u(t)-\Im u^*(t)\| = \|u(t)-\Im u(t)+\Im u(t)-\Im u^*(t)\| \\ &\leqslant \|u(t)-\Im u(t)\|+\|\Im u(t)-\Im u^*(t)\|. \end{aligned}$$

Using Theorem 3.1 and Lemma 4.1, and using the method of calculation employed to derive the results of the previous section, we can obtain that

$$\|u(t)-u^*(t)\| \leqslant \mathcal{A}_1\varepsilon+\theta\|u-u^*\|,$$

which can alternatively be written as

$$\|u-u^*\| \leqslant \frac{\mathcal{A}_1}{1-\theta}\varepsilon.$$

Letting $\mathfrak{B}_1 = \frac{\mathcal{A}_1}{1-\theta}$, the solution of the problem (1.1) is Ulam-Hyers-stable. Further, if we set $k(\varepsilon) = \varepsilon$, then the problem (1.1) is generalized Ulam-Hyers-stable. □

Lemma 4.2. *If the assumption* (M_4) *holds, then any solution* $u \in C([0,1],\mathbb{R})$ *of the problem given by*

$$\begin{cases} \sum_{i=1}^{p} \alpha_i \mathcal{D}^{\beta_i} u(t) = h(t,u(t),u(wt)) + \zeta(t),\ p-i<\beta_i \leqslant p+1-i,\ \ w\in(0,1),\ \alpha_i\in\mathbb{R},\ t\in[0,1], \\ u(0)=u_0,\ \dfrac{d^\ell u(0)}{dt^\ell}=0, \\ u(1) = v\displaystyle\int_0^1 u(s)ds - \sum_{\ell=1}^{m} a_\ell u(\sigma_\ell),\ \ a_\ell\in\mathbb{R},\ \ \sigma_\ell\in(0,1),\ \ell=1,2,\cdots,p-2\ \ i=1,2,\cdots,m, \end{cases}$$

satisfies the following relation:

$$|u(t)-\Im u(t)| \leqslant \left(\frac{|v|(\beta_\xi+1)^{-1} + \sum_{\ell=1}^m |a_\ell|\sigma_\ell^{\beta_\xi} + 1}{|\gamma_1||\alpha_\xi|\Gamma(\beta_\xi+1)}\psi + \frac{q}{|\alpha_\xi|}\vartheta(t)\right)\varepsilon.$$

Proof. Let $u \in C([0,1],\mathbb{R})$ be a solution of (1.1); then, we have

$$\begin{aligned} u(t) =& \Im u(t) + \frac{t^{p-1}}{\gamma_1}\Big[\frac{v}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^1\int_0^s (s-z)^{\beta_\xi-1}\zeta(z)dzds \\ &- \sum_{\ell=1}^{m}\frac{a_\ell}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^{\sigma_\ell}(\sigma_\ell-z)^{\beta_\xi-1}\zeta(z)dz - \frac{1}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^1 (1-z)^{\beta_\xi-1}\zeta(z)dz\Big] \\ &+ \frac{1}{\alpha_\xi\Gamma(\beta_\xi)}\int_0^t (t-z)^{\beta_\xi-1}\zeta(z)dz. \end{aligned} \tag{4.6}$$

Using Remark 4.2 and (M_4) in (4.6), and employing the arguments to obtain the results of the last section, we get

$$\begin{aligned} |u(t)-\Im u(t)| \leqslant& \varepsilon\Big\{\frac{1}{|\gamma_1|}\Big[\frac{|v|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1\int_0^s (s-z)^{\beta_\xi-1}\vartheta(z)dzds \\ &+ \sum_{\ell=1}^{m}\frac{|a_\ell|}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^{\sigma_\ell}(\sigma_\ell-z)^{\beta_\xi-1}\vartheta(z)dz + \frac{1}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^1(1-z)^{\beta_\xi-1}\vartheta(z)dz\Big] \\ &+ \frac{1}{|\alpha_\xi|\Gamma(\beta_\xi)}\int_0^t (t-z)^{\beta_\xi-1}\vartheta(z)dz\Big\} \\ \leqslant& \varepsilon\left(\frac{|v|(\beta_\xi+1)^{-1} + \sum_{\ell=1}^m |a_\ell|\sigma_\ell^{\beta_\xi}+1}{|\gamma_1||\alpha_\xi|\Gamma(\beta_\xi+1)}\psi + \frac{q}{|\alpha_\xi|}\vartheta(t)\right), \end{aligned}$$

which is the desired condition. □

Theorem 4.2. *If the assumptions* (M_1), (M_2)*, and* (M_4) *are satisfied, then the problem* (1.1) *is Ulam-Hyers-Rassias-stable and generalized Ulam-Hyers-Rassias-stable if* $\theta < 1$, *where* θ *is defined by* (3.2).

Proof. For any solution $u \in C([0,1],\mathbb{R})$ and a unique solution u^* of the problem (1.1), we have

$$\begin{aligned}\|u(t)-u^*(t)\| &= \|u(t)-\mathfrak{J}u^*(t)\| = \|u(t)-\mathfrak{J}u(t)+\mathfrak{J}u(t)-\mathfrak{J}u^*(t)\| \\ &\leqslant \|u(t)-\mathfrak{J}u(t)\| + \|\mathfrak{J}u(t)-\mathfrak{J}u^*(t)\|.\end{aligned}$$

Using Theorem 3.1 and Lemma 4.2 in the above inequality, we obtain

$$\|u(t)-u^*(t)\| \leqslant \Big(\mathcal{A}_2\psi + \frac{q}{|\alpha_\xi|}\vartheta(t)\Big)\varepsilon + \theta\|u-u^*\|,$$

where

$$\mathcal{A}_2 = \frac{|v|(\beta_\xi+1)^{-1} + \sum_{\ell=1}^{m}|a_\ell|\sigma_\ell^{\beta_\xi} + 1}{|\gamma_1||\alpha_\xi|\Gamma(\beta_\xi+1)}.$$

Alternatively, we have

$$\|u(t)-u^*(t)\| \leqslant \frac{1}{1-\theta}\Big(\mathcal{A}_2\psi + \frac{q}{|\alpha_\xi|}\vartheta(t)\Big)\varepsilon.$$

Letting $\mathfrak{B}_2 = \frac{1}{1-\theta}\max\Big(\mathcal{A}_2, \frac{q}{|\alpha_\xi|}\Big)$, the solution to the problem (1.1) is Ulam-Hyers-Rassias-stable. In the case that $\varepsilon = 1$, the solution to the problem (1.1) is Ulam-Hyers-Rassias-stable. The proof is complete. □

5. Examples

This section is devoted to the illustration of the results derived in the last two sections.

Example 1. *Consider the following multipoint-integral boundary value of multi-term delay fractional differential equations given by*

$$\begin{aligned}&\frac{15}{11}\,{}^c\mathcal{D}^{0.9}u(t) - \frac{5}{2}\,{}^c\mathcal{D}^{0.75}u(t) + 15\,{}^c\mathcal{D}^{4.5}u(t) - \frac{15}{16}\,{}^c\mathcal{D}^{0.56}u(t) + \frac{5}{7}\,{}^c\mathcal{D}^{0.5}u(t) \\ =\; &h(t,u(t),u(wt)),\ t\in[0,1], \\ &u(0)=2,\quad \frac{d^\ell u(0)}{dt^\ell}=0,\quad u(1) = 4\int_0^1 u(s)ds + 3u\Big(\frac{1}{5}\Big) - 2u\Big(\frac{1}{3}\Big) - u\Big(\frac{2}{5}\Big), \end{aligned} \tag{5.1}$$

where $\beta_1 = \dfrac{9}{10}$, $\beta_2 = \dfrac{3}{4}$, $\beta_{\xi=3} = \dfrac{9}{2}$, $\beta_4 = \dfrac{14}{25}$, $\beta_5 = \dfrac{19}{2}$, $\alpha_1 = \dfrac{15}{11}$, $\alpha_2 = \dfrac{-5}{2}$, $\alpha_{\xi=3} = 15$, $\alpha_4 = \dfrac{-15}{16}$, $\alpha_5 = \dfrac{5}{7}$, $a_1 = -3$, $a_2 = 2$, $a_3 = 1$, $\sigma_1 = \dfrac{1}{5}$, $\sigma_2 = \dfrac{1}{3}$, $\sigma_3 = \dfrac{2}{5}$, $v = 4$, $w = \dfrac{1}{3}$, *and*

$$h(t,u(t),u(wt)) = \frac{N}{t+e^t}\Big(\frac{2u(t)}{3t+1+u(t)} - \frac{u(\frac{t}{3})}{2+t^2e^t+u(\frac{t}{3})} - \cos(t)\Big).$$

Observe that

$$\Big|h(t,u_1(t),u_1(wt)) - h(t,u_2(t),u_2(wt))\Big| \leqslant |N|\Big(2\|u_1-u_2\| + \|u_1\Big(\frac{t}{3}\Big) - u_2\Big(\frac{t}{3}\Big)\|\Big).$$

Clearly, $L_1 + L_2 = 3|N| = L_3$. *With the given data, it is found that* $\gamma_1 = 0.245491358 \neq 0$ *and* $\theta < 1$ *if* $N < 25.92324750$. *Therefore, by the conclusion of Theorem 3.1, we deduce that the problem* (5.1) *has a unique solution on* $[0, 1]$.

Next, we verify the hypotheses of Theorem 4.1 and Theorem 4.2. Let us set $\zeta(t) = t^3 + 1$, $\forall t \in [0,1]$. *Then, we have*

$$I^{\beta_{4.5}}\zeta(t) = 0.0004275207263\, t^{15/2} + 0.01910483246\, t^{9/2} \leqslant 0.01910483246\,(t^3 + 1).$$

Thus, the condition (M_4) *is satisfied with* $\mathfrak{q} = 0.01910483246$. *Hence, the problem* (1.1) *is Ulam-Hyers- and generalized Ulam-Hyers-stable as a consequence of Theorem 4.1. Also, all of the assumptions of Theorem 4.2 are satisfied. Therefore, the solution to problem* (1.1) *is both Ulam-Hyers-Rassias- and generalized Ulam-Hyers-Rassias-stable.*

Example 2. *Consider the multi-term four-point integral fractional-order boundary-value problem given by*

$$\begin{aligned} &10\,{}^c\mathcal{D}^{2.7}u(t) + 0.1\,{}^c\mathcal{D}^{0.5}u(t) + 0.01\,{}^c\mathcal{D}^{0.33}u(t) = h(t,u(t),u(\frac{t}{3})),\ \ t \in [0, 1],\\ &u(0) = 4,\ \frac{d^\ell u(0)}{dt^\ell} = 0,\ u(1) = 2\int_0^1 u(s)ds - u\Big(\frac{1}{2}\Big) - 3u\Big(\frac{2}{3}\Big), \end{aligned} \tag{5.2}$$

where

$$h(t,u(t),u(\frac{t}{3}) = \frac{2t^3}{e^t}\Big(\frac{t}{t^3e^{-2t} + 3|u(t)| + \cos t|u\big(\frac{t}{3}\big)|} + \frac{e^t}{2te^{-t} + 4|u(t)| + |u\big(\frac{t}{3}\big)|} + \sin(t)e^{-t}\Big),$$

$a_1 = 1, a_2 = 3, \xi = 1, \sigma_1 = \dfrac{1}{2}, \sigma_2 = \dfrac{2}{3}, \beta_\xi = \dfrac{27}{10}, \beta_2 = \dfrac{1}{2}, \beta_3 = \dfrac{1}{3}, \alpha_\xi = 10, \alpha_2 = \dfrac{1}{10}, \alpha_3 = \dfrac{1}{100}, \nu = 2, \text{ʉ}_0 = 4$. *Observe that*

$$\begin{aligned} |h(t,u(t),u(\frac{t}{3})| &= \Big|\frac{2t^3}{e^t}\Big(\frac{t}{t^3e^{-2t} + 3|u(t)| + (\cos t)|u\big(\frac{t}{3}\big)|} + \frac{e^t}{2te^{-t} + 4|u(t)| + |u\big(\frac{t}{3}\big)|} + \sin(t)e^{-t}\Big)\Big|\\ &\leqslant \Big|\frac{2t^3}{e^t}\Big|\Big|\Big(\frac{t}{t^3e^{-2t} + 3|u(t)| + (\cos 1)|u\big(\frac{t}{3}\big)|} + \frac{e^t}{2te^{-t} + 4|u(t)| + |u\big(\frac{t}{3}\big)|} + |\sin(t)|e^{-t}\Big)\Big|\\ &\leqslant \frac{2t^3}{e^t}\Big(\frac{t}{t^3e^{-2t}} + \frac{e^t}{2te^{-t}} + e^{-t}\Big)\\ &\leqslant 2te^t + t^2e^t + 2t^3 = \psi_{\hbar}(t), \end{aligned}$$

where we have used the fact that $\cos t$ *is a positive decreasing function on* $[0, 1]$. *Moreover, we have that* $\|\psi_{\hbar}\| = 10.15484548$, $\gamma_1 = 0.8871381040$, *and* $r_0 > 0.03949508399$. *Clearly, the hypothesis of Theorem 3.2 is satisfied; hence, it follows by its conclusion that problem* (5.2) *has at least one solution on* $[0, 1]$.

6. Conclusions

In this study, we investigated a new class of nonlocal boundary-value problems involving multi-term fractional differential equations and multipoint-integral boundary conditions. Initially, we found an integral operator associated with the problem at hand. Then, we applied the Leray-Schauder nonlinear alternative and Banach fixed-point theorem to establish the existence and uniqueness of solutions for the given problem. We have also developed several stability criteria, including Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stability, for the problem at hand. Our results are new and enrich the literature on multi-term nonlocal boundary-value problems. For instance, our results generalize those obtained in [29]. As a special case, our results correspond to purely integral boundary conditions for all $a_\ell = 0, \ell = 1, 2, ..., m$, which are indeed new. The present work will open new avenues for investigating the inclusions and impulsive variants for the problem at hand. Moreover, we plan to extend our study to the coupled systems of multi-term fractional differentials with coupled multipoint-integral boundary conditions.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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