AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 12870–12905.
DOI: 10.3934/math.2024629
Received: 12 January 2024
Revised: 19 March 2024
Accepted: 26 March 2024
Published: 07 April 2024

Research article

Double inertial steps extragadient-type methods for solving optimal control and image restoration problems

Austine Efut Ofem[1,*], Jacob Ashiwere Abuchu[1,2], Godwin Chidi Ugwunnadi[3,4], Hossam A. Nabwey[5,6,*], Abubakar Adamu[7,8] and Ojen Kumar Narain[1]

[1] School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa

[2] Department of Mathematics, University of Calabar, Calabar, Nigeria

[3] Department of Mathematics, University of Eswatini, Private Bag 4, Kwaluseni, Eswatini

[4] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa 0204, Pretoria, South Africa

[5] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia

[6] Department of Basic Engineering, Faculty of Engineering, Menoufia University, Shibin el Kom 32511, Egypt

[7] Operational Research Center in Healthcare, Near East University, TRNC Mersin 10, Nicosia 99138, Turkey

[8] Mathematics Institute, African University of Science and Technology, Abuja 900107, Nigeria

* **Correspondence:** Email: ofemaustine@gmail.com, h.mohamed@psau.edu.sa.

Abstract: In order to approximate the common solution of quasi-nonexpansive fixed point and pseudo-monotone variational inequality problems in real Hilbert spaces, this paper presented three new modified sub-gradient extragradient-type methods. Our algorithms incorporated viscosity terms and double inertial extrapolations to ensure strong convergence and to speed up convergence. No line search methods of the Armijo type were required by our algorithms. Instead, they employed a novel self-adaptive step size technique that produced a non-monotonic sequence of step sizes while also correctly incorporating a number of well-known step sizes. The step size was designed to lessen the algorithms' reliance on the initial step size. Numerical tests were performed, and the results showed that our step size is more effective and that it guarantees that our methods require less execution time. We stated and proved the strong convergence of our algorithms under mild conditions imposed on the control parameters. To show the computational advantage of the suggested methods over some well-known methods in the literature, several numerical experiments were provided. To test the applicability

and efficiencies of our methods in solving real-world problems, we utilized the proposed methods to solve optimal control and image restoration problems.

Keywords: variational inequality problem; fixed point; pseudo-monotone operator; strong convergence; viscosity; subgradient extragradient method
Mathematics Subject Classification: 47H05, 47J20, 47J25, 65K15

1. Introduction

In this paper, let H denote a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let M, $\mathbb{R}$, and $\mathbb{N}$ stand for the nonempty closed convex subset of H, set of real numbers and set of positive integers, respectively. Let $G : H \to H$ be a mapping. The variational inequality problem (VIP) is concerned with the problem of finding a point $u^\star \in M$ such that

$$\langle Gu^\star, u - u^\star\rangle \geq 0,\ \ \forall\, u \in M. \tag{1.1}$$

We denote the solution set of VIP (1.1) by $VI(M,G)$. The VIP, which Fichera [12] and Stampacchia [38] independently examined, is a crucial tool in both the applied and pure sciences. It has attracted the attention of many authors in recent years due to its wide range of applications to issues arising from partial differential equations, optimal control problems, saddle point problems, minimization problems, economics, engineering, and mathematical programming.

On the other hand, an element $u \in M$ is said to be the fixed point of a mapping $S : M \to M$, if $Su = u$. The set of all the fixed points of S is denoted by $F(S) = \{u \in M : Su = u\}$. The study of the fixed point theory of nonexpansive mappings has been applied in several fields such as game theory, differential equations, signal processing, integral equations, convex optimization, and control theory [19]. There are several recent results in the literature on approximation of fixed points of nonexpansive mappings (see, for example, [8, 9, 26–29, 34–36] and the references therein).

It is well-known that the VIP (1.1) can be reformulated as a fixed point problem as follows:

$$u^\star = P_M(I - \eta G)u^\star, \tag{1.2}$$

where $P_M : H \to M$ is the metric projection and $\eta > 0$. The extragradient method is a prominent method that has been used by many authors over the years to solve VIP. This method was first introduced by Korpelevich [21] in 1976. Given an initial point $u_0 \in M$, the sequence $\{u_m\}$ generated by the extragradient method is as follows:

$$\begin{cases} v_m = P_M(I - \eta G)u_m, \\ u_{m+1} = P_M(u_m - \eta G v_m),\ \ \forall m \geq 0, \end{cases} \tag{1.3}$$

where $\eta \in (0, \frac{1}{L})$, and G is an operator that is L-Lipschitz continuous and monotone. For $VI(M,G) \neq \emptyset$, the author showed that the sequence $\{u_m\}$ defined by (1.3) converges weakly to an element in $VI(M,G)$.

The extragradient method's main flaw is its iterative requirement to compute two projections on the feasible set M. In fact, if M has a complex structure, this might have an impact on how efficiently the method computes. In recent years, several authors have paid a great deal of attention to overcoming this restriction (see, for example [6,7,11,16,48]). In order to address the drawback of the extragragient method, in 1997, He [16] introduced a method that requires only a single projection per each iteration. This method is known as the projection and contraction method and it is given as follows:

$$\begin{cases} v_m = P_M(u_m - \eta G u_m), \\ w_m = (u_m - v_m) - \eta(G u_m - G v_m), \\ u_{m+1} = u_m - \sigma \varpi_m w_m, \end{cases}$$

where $\sigma \in (0, 2)$, $\eta \in (0, \frac{1}{L})$ and ϖ_m is defined as

$$\varpi_m = \frac{\langle u_m - v_m, w_m \rangle}{\|w_m\|^2}. \tag{1.4}$$

The author showed that the sequence $\{u_m\}$ generated by (1.4) converges weakly to a unique solution of VIP (1.1). The subgradient extragradient method, which was developed by Censor et al. [6, 7, 11], is another effective strategy for addressing the limitation of the extragradient method and it is defined as follows:

$$\begin{cases} v_m = P_M(u_m - \eta G u_m), \\ T_m = \{u \in H | \langle u_m - \eta G u_m - v_m, u - v_m \rangle \leq 0\}, \\ u_{m+1} = P_{T_m}(u_m - \eta G v_m), \end{cases} \tag{1.5}$$

where $\eta \in (0, \frac{1}{L})$, and G is a L-Lipschitz continuous and monotone operator. The main idea in this method is that a projection onto a special contractible half-space is used to replace the second projection onto M of the extragradient method, and this significantly reduces the difficulty of calculation. The authors showed that if $VI(M, G) \neq \emptyset$, the sequence $\{u_m\}$ defined by (1.5) weakly converges to a point in $VI(M, G)$.

Furthermore, the notion of the inertial extrapolation technique is based upon a discrete analogue of a second order dissipative dynamical system and it is known as an acceleration process of iterative methods. It was first developed in [37] to solve smooth convex minimization problems. For some years now, the inertial techniques have been widely adopted by many authors to improve the convergence rate of various iterative algorithms for solving several kinds of optimization problems (see, for example, [1, 17, 30–32, 41, 44–46, 55]).

It is worthy to note that the study of the problem involving the approximation of the common solution of the fixed point problem (FPP) and VIP plays a significant role in mathematical models whose constraints can be expressed as FPP and VIP. This happens in real-world applications such as image recovery, signal processing, network resource allocation, and composite site reduction (see, for example, [2, 14, 18, 22, 24, 25, 33, 51] and the references therein).

Very recently, Thong and Hieu [43] introduced two modified subgradient extragradient methods with line search process for solving the VIP with L-Lipschitz continuous and monotone operator G and FPP involving quasi-nonexpansive mapping S, such that $I - S$ is demiclosed at zero. Under appropriate assumptions, the authors showed that the sequences generated by their algorithms weakly converge some points in $F(S) \cap VI(M, G)$.

We note that Thong and Hieu [43] only proved weak convergence results for their algorithms. According to Bauschke and Combettes [3], for the solution of optimization problems, the strong convergence of iterative methods are more desirable than their weak convergence counterparts. Furthermore, we observe that Thong and Hieu [43] employed the Armijo-type line search rule step size to their algorithms in order to enable them to operate without requiring prior knowledge of the Lipschitz constant of the operators. However, the use of Armijo-type step sizes may cause the considered methods to perform multiple calculations of the projection values per iteration on the feasible set. To overcome this limitation, Liu and Yang [23] developed an adaptive step size criterion, which only needs the use of some previously given information to complete the step size calculation.

As far as we know, there is no result in the literature involving the subgradient extragradient method with double inertial extrapolations for finding the common solution of VIP and FPP in real Hilbert spaces. Due to the importance of common solutions of VIP and FPP to some real-world problems, it is natural to ask the following question:

Is it possible to construct a double inertial subgradient extragradient-type algorithms with a new step size for finding the common solution of VIP and FPP?

One of the purposes of this article is to give an affirmative answer to the above question. Motivated by the ongoing research in these directions, we propose some modified subgradient extragradient methods with a new step size. These proposed methods are derived from the combinations of the original subgradient extragradient method, viscosity method, projection and contraction method. We prove that our new methods converge strongly to the common solutions of VIP involving pseudo-monotone mappings and FPP involving quasi-nonexpansive mappings that are demiclosed at zero in real Hilbert spaces. The following are more contributions made in this research:

- Our algorithms do not need any Armijo-type line search techniques. Rather, they use a new self-adaptive step size technique, which generates a non-monotonic sequence of step sizes. This step size is formulated such that it reduces the dependence of the algorithms on the initial step size. Conducted numerical experiments proved that the proposed step size is more efficient and ensures that our methods require less computation time than many methods in the literature that work with Armijo-type line search technique.
- Our step size properly includes those in [23, 41, 50].
- Our algorithms are constructed to approximate the common solution of VIP involving pseudo-monotone mappings and FPP involving quasi-nonexpansive mappings. Since the class of Pseudo-monotone mappings is more general than the class of monotone mappings, it means that our results improve and generalize several results in the literature for finding common solution VIP involving monotone mappings and quasi-nonexpansive mappings. Hence, our results are improvements of the results in [22, 43, 47] and several others.
- Our algorithms are embedded with double inertial terms to accelerate their convergence speed. Numerical tests showed that the proposed algorithms converge faster than the compared existing methods with single inertial term.
- We prove our strong convergence result under mild conditions imposed on the parameters. Our results are improvements on the weak convergence results in [43, 47].
- To show the computational advantage of the suggested methods over some well-known methods in the literature, several numerical experiments are provided.
- We utilize our methods to solve some real-world problems, such as optimal control and signal processing problems.
- The proofs of our strong convergence results do not require the conventional "two cases" approach

that have been employed by several authors in the literature to establish strong convergence results; see, for example, [5, 30].

The article is organized as follows: In Section 2, some useful definitions and lemmas are recalled. The proposed algorithms and their convergence results are presented in Section 3. In Section 4, we conduct some numerical experiments to show the efficiency of our proposed algorithms over several well known methods. In Section 5, we consider the application of our algorithms to the solution of optimal control problem. In Section 6, we apply our methods to image recovery problem and in Section 7, we give summary of the basic contributions in this work.

2. Preliminaries

In what follows, we denote the weak convergence of the sequence $\{u_m\}$ to u by $u_m \rightharpoonup u$ as $m \to \infty$ and the strong convergence of the sequences $\{u_m\}$ is denoted by $u_m \to u$ as $m \to \infty$.

Next, the following definitions and lemmas will be recalled. Let $G : H \to H$ be an operator, then G is called:

(a_1) contraction if there exists a constant $k \in [0, 1)$ such that

$$\|Gu - Gv\| \leq k\|u - v\|, \ \ \forall u, v \in H;$$

(a_2) L-Lipschitz continuous, if $L > 0$ exists with

$$\|Gu - Gv\| \leq L\|u - v\|, \ \ \forall u, v \in H.$$

If $L = 1$, then G becomes a nonexpansive mapping;

(a_3) Quasi-nonexpansive, if $F(G) \neq \emptyset$ such that

$$\|Gu - u^\star\| \leq \|u - u^\star\|, \ \ \forall u \in H, u^\star \in F(G);$$

(a_4) α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Gu - Gv, u - v\rangle \geq \alpha\|u - v\|^2, \ \ \forall u, v \in H;$$

(a_5) Monotone, if

$$\langle Gu - Gv, u - v\rangle \geq 0, \ \ \forall u, v \in H;$$

(a_6) Pseudo-monotone, if

$$\langle Gu, u - v\rangle \geq 0 \quad \Longrightarrow \quad \langle Gu, u - v\rangle \geq 0, \ \ \forall u, v \in H;$$

(a_7) Sequentially weakly continuous, if for any sequence $\{u_m\}$ which converges weakly to u, then the sequence $\{Gu_m\}$ weakly converges to Gu.

Lemma 2.1. *[15] Let H be a real Hilbert space and M a nonempty closed convex subset of H. Suppose $u \in H$ and $v \in M$, then $v = P_M u \iff \langle u - v, v - w\rangle \geq 0$, $\forall w \in M$.*

Lemma 2.2. *[15] Let M be a closed convex subset of a real Hilbert space H. If $u \in H$, then* *(i)* $\|P_M u - P_M v\|^2 \leq \langle P_M u - P_M v, u - v\rangle, \ \forall v \in H$;

(ii) $\langle (I - P_M)u - (I - P_M)v, u - v\rangle \geq \|(I - P_M)u - (I - P_M)v\|^2, \ \forall v \in H$;

(iii) $\|P_M u - v\|^2 \leq \|u - v\|^2 - \|u - P_M u\|^2, \ \forall v \in H$.

Lemma 2.3. *For each $u, v, w \in H$ and where $\alpha, \beta, \delta \in [0, 1]$ with $\alpha + \beta + \delta = 1$, the followings hold in Hilbert spaces:*

(a)

$$\|u + v\| \leq \|u\|^2 + 2\langle v, u + v\rangle;$$

(b)

$$\|u + v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2;$$

(c)

$$\|\alpha u + \beta v + \gamma w\|^2 = \alpha\|u\|^2 + \beta\|v\|^2 + \gamma\|w\|^2 - \alpha\beta\|u - v\|^2 - \alpha\gamma\|u - w\|^2 - \beta\gamma\|v - w\|^2.$$

Lemma 2.4. *[15] Let $G : H \to H$ be a nonlinear operator such that $F(G) \neq \emptyset$. Then $I - G$ is called demiclosed at zero if for any $u_m \in H$, the following implication holds:*

$$u_m \rightharpoonup u \text{ and } (I - G)u_m \to 0 \implies u \in F(G).$$

Lemma 2.5. *[52] Let $\{a_m\}$ be a sequence of nonnegative real numbers such that*

$$a_{m+1} \leq (1 - \nu_m)a_m + \nu_m b_m, \ \forall m \geq 1,$$

where $\{\nu_m\} \subset (0, 1)$ with $\sum_{m=0}^{\infty} \nu_m = \infty$. If $\limsup_{k\to\infty} b_{m_k} \leq 0$ for every subsequence $\{a_{m_k}\}$ of $\{a_m\}$, the following inequality holds:

$$\liminf_{k\to\infty}(a_{m_{k+1}} - a_{m_k}) \geq 0.$$

Then $\lim_{m\to\infty} a_m = 0$.

3. Main results

In this section, we introduce three new double inertial subgradient extragradient algorithm-types for solving VIP and FPP. In order to establish our main results, we assume that the following conditions are fulfilled:

(C_1) The feasible set M is nonempty, closed and convex.
(C_2) The mapping $G : H \to H$ is pseudo-monotone and L-Lipschitz continuous.
(C_3) The solution set $F(S) \cap VI(M, G) \neq \emptyset$.
(C_4) The mapping G is sequentially weak continuous on M.
(C_5) The mappings $K, J : H \to H$ are non-expansive.
(C_6) The mapping $S : H \to H$ is quasi-nonexpansive such that $I - S$ is demiclosed at zero.
(C_7) The mapping $f : H \to H$ is a contraction with constant $k \in [0, 1)$.

(C_8) Let $\{\alpha_m\} \subset (0,1)$, $\{\beta_m\}, \{\gamma_m\} \subset [a,b] \subset (0,1)$ such that $\alpha_m + \beta_m + \gamma_m = 1$, $\lim_{m\to\infty} \alpha_m = 0$, $\sum_{m=}^{\infty} \alpha_m = \infty$ and $\lim_{m\to\infty} \frac{\epsilon_m}{\alpha_m} = 0 = \lim_{m\to\infty} \frac{\xi_m}{\alpha_m}$, where $\{\epsilon_m\}$ and $\{\xi_m\}$ are positive real sequences.

(C_9) Let $\{p_m\}, \{q_m\} \subset [0,\infty)$ and $\{h_m\} \subset [1,\infty)$ such that $\sum_{m=0}^{\infty} p_m < \infty$, $\lim_{m\to\infty} q_m = 0$, and $\lim_{m\to\infty} h_m = 1$.

Algorithm 3.1.

Initialization: *Choose $\eta_1 > 0, \phi > 0, \theta > 0, \rho \in (0,2), \mu \in (0,1)$ and let $g_0, g_1 \in H$ be arbitrary.*

Iterative Steps: *Given the iterates u_{m-1} and $\{u_m\}$ ($m \geq 1$), calculate u_{m+1} as follows:*

Step 1: *Choose ϕ_m and θ_m such that $\phi_m \in [0, \bar{\phi}_m]$ and $\theta_m \in [0, \bar{\theta}_m]$, where*

$$\bar{\phi}_m = \begin{cases} \min\left\{\frac{m-1}{m+\phi-1}, \frac{\epsilon_m}{\|u_m - u_{m-1}\|}\right\}, & \text{if } u_m \neq u_{m-1}, \\ \frac{m-1}{m+\phi-1}, & \textit{otherwise}. \end{cases} \tag{3.1}$$

$$\bar{\theta}_m = \begin{cases} \min\left\{\frac{m-1}{m+\theta-1}, \frac{\xi_m}{\|u_m - u_{m-1}\|}\right\}, & \text{if } u_m \neq u_{m-1}, \\ \frac{m-1}{m+\theta-1}, & \textit{otherwise}. \end{cases} \tag{3.2}$$

Step 2: *Set*

$$s_m = u_m + \phi_m(Ku_m - Ku_{m-1}), \tag{3.3}$$

$$r_m = u_m + \theta_m(Ju_m - Ju_{m-1}), \tag{3.4}$$

and compute

$$w_m = P_M(s_m - \eta_m G s_m). \tag{3.5}$$

If $s_m = w_m$ or $Gs_m = 0$, stop; s_m is a solution of the VIP. Otherwise, do Step 3.

Step 3: *Compute*

$$z_m = P_{T_m}(s_m - \rho\eta_m\delta_m G w_m), \tag{3.6}$$

where

$$T_m = \{u \in H : \langle s_m - \eta_m G s_m - w_m, u - w_m\rangle \leq 0\}, \tag{3.7}$$

$$\delta_m = \begin{cases} \frac{\langle s_m - w_m, v_m\rangle}{\|v_m\|^2}, & \text{if } v_m \neq 0, \\ \\ 0, & \textit{otherwise}, \end{cases} \tag{3.8}$$

and

$$v_m = s_m - w_m - \eta_m(G s_m - G w_m). \tag{3.9}$$

Step 4: *Compute*

$$u_{m+1} = \alpha_m f(r_m) + \beta_m z_m + \gamma_m S z_m. \tag{3.10}$$

Update

$$\eta_{m+1} = \begin{cases} \min\left\{\frac{(q_m+h_m\mu)\|s_m-w_m\|}{\|Gs_m-Gw_m\|}, \eta_m + p_m\right\}, & \text{if } Gs_m \neq Gw_m, \\ \\ \eta_m + p_m, & \textit{otherwise}. \end{cases} \tag{3.11}$$

Set $m := m + 1$ and go back to Step 1.

Remark 3.1. *We note the following in Algorithm 3.1:*

(i) It is not hard to see from (3.1), (3.2), *and condition* (C_8) *that*

$$\lim_{m\to\infty} \phi_m\|u_m - u_{m-1}\| = \lim_{m\to\infty} \theta_m\|u_m - u_{m-1}\| = 0$$

and

$$\lim_{m\to\infty} \frac{\phi_m}{\alpha_m}\|u_m - u_{m-1}\| = \lim_{m\to\infty} \frac{\theta_m}{\alpha_m}\|u_m - u_{m-1}\| = 0.$$

(ii) In order to get larger step sizes, we introduce the sequence $\{q_m\}$ and $\{h_m\}$ in (3.11) *to relax the the parameter μ. The relaxation parameters can often improve the numerical performances of algorithms, see [10]. If $q_m = 0$ in* (3.11)*, then $\{\eta_m\}$ becomes the step size in [41]. If $h_m = 1$ in* (3.11)*, then $\{\eta_m\}$ becomes that in [50]. If $q_m = 0$ and $h_m = 1$ in* (3.11)*, then the step size $\{\eta_m\}$ reduces to that in [23]. Lastly, if $q_m = p_m = 0$ and $h_m = 1$, $\{\eta_m\}$ reduces to the step sizes used by many authors in the literature (see, for example, [13, 42, 53, 54]).*

We now establish the following lemmas that will be useful in proving our strong convergence theorems.

Lemma 3.1. *If conditions (C_3) and (C_4) are fulfilled and $\{\eta_m\}$ is the sequence generated by* (3.11)*. Then, $\{\eta_m\}$ is well-defined and $\lim_{m\to\infty} \eta_m = \eta \in \left[\min\left\{\frac{\mu}{L}, \eta_1\right\}, \eta_1 + \sum_{m=1}^{\infty} p_m\right]$.*

Proof. Since G is Lipschitz continuous with $L > 0$, $q_m \geq 0$ and $h_m \geq 1$, by (3.11), if $Gs_m \neq Gw_m$, we have

$$\eta_m \geq \frac{(q_m + h_m\mu)\|s_m - w_m\|}{\|Gs_m - Gw_m\|} \geq \frac{q_m + h_m\mu}{L} \geq \frac{\mu}{L}.$$

We omit the remaining part of the proof to avoid repetitive expressions of the proof of Lemma 3.1 in [50]. □

Lemma 3.2. *Let $\{s_m\}$ and $\{w_m\}$ be two sequences generated by Algorithm 3.1. Suppose that conditions (C_1)–(C_4) are fulfilled and if a subsequence $\{s_{m_k}\}$ of $\{s_m\}$ exists, such that $s_{m_k} \rightharpoonup v^\star \in H$ and $\lim_{k\to\infty} \|s_{m_k} - w_{m_k}\| = 0$, then $v^\star \in VI(M, G)$.*

Proof. Since $w_{m_k} = P_M(s_{m_k} - \eta_{m_k} G s_{m_k})$, then by applying Lemma 2.1, we have

$$\langle s_{m_k} - \eta_{m_k} G s_{m_k} - w_{m_k}, u - w_{m_k} \rangle \leq 0, \ \forall u \in M.$$

Equivalently, we have

$$\frac{1}{\eta_{m_k}} \langle s_{m_k} - w_{m_k}, u - w_{m_k} \rangle \leq \langle G s_{m_k}, u - w_{m_k} \rangle, \ \forall u \in M.$$

It follows that

$$\frac{1}{\eta_{m_k}} \langle s_{m_k} - w_{m_k}, u - w_{m_k} \rangle + \langle G s_{m_k}, w_{m_k} - s_{m_k} \rangle \leq \langle G s_{m_k}, u - s_{m_k} \rangle, \ \forall u \in M. \tag{3.12}$$

Since $s_{m_k} \rightharpoonup v^\star$, we know that $\{s_{m_k}\}$ is bounded and G is L-Lipschitz continuous on H, this means that $\{G s_{m_k}\}$ is also bounded. Again, since $\lim_{k \to \infty} \|s_{m_k} - w_{m_k}\| = 0$, then $\{w_{m_k}\}$ is also bounded and $\{\eta_{m_k}\} \geq \left\{\frac{\mu}{L}, \eta_1\right\}$. From (3.12), we have

$$\liminf_{k \to \infty} \langle G s_{m_k}, u - s_{m_k} \rangle \geq 0, \ \forall u \in M. \tag{3.13}$$

On the other hand, we have

$$\langle G w_{m_k}, u - w_{m_k} \rangle = \langle G w_{m_k} - G s_{m_k}, u - s_{m_k} \rangle + \langle G s_{m_k}, u - s_{m_k} \rangle + \langle G w_{m_k}, s_{m_k} - w_{m_k} \rangle, \ \forall u \in M. \tag{3.14}$$

Since $\lim_{k \to \infty} \|s_{m_k} - w_{m_k}\| = 0$ and G is L-Lpischitz continuous on H, we have

$$\lim_{k \to \infty} \|G s_{m_k} - G w_{m_k}\| = 0. \tag{3.15}$$

By $\lim_{k \to \infty} \|s_{m_k} - w_{m_k}\| = 0$, (3.13) and (3.15), (3.14) reduces to

$$\liminf_{k \to \infty} \langle G w_{m_k}, u - w_{m_k} \rangle \geq 0, \ \forall u \in M. \tag{3.16}$$

Next, we show that $v^\star \in VI(M, G)$. To show this, we choose a decreasing sequence $\{\xi_k\}$ of positive numbers which approaches zero. For each k, let N_k stand for the smallest positive integer fulfilling the following inequality:

$$\langle G w_{m_j}, u - w_{m_j} \rangle + \xi_k \geq 0, \ \ \forall j \geq N_k. \tag{3.17}$$

It is not hard to see that the sequence $\{N_k\}$ increases as $\{\xi_k\}$ decreases. Moreover, since $w_{N_k} \subset M$, for each k, we can assume that $G w_{N_k} \neq 0$ (otherwise, w_{N_k} is a solution). Putting

$$g_{N_k} = \frac{G w_{N_k}}{\|G w_{N_k}\|^2},$$

we get $\langle G w_{N_k}, g_{N_k} \rangle = 1$, for each k. We can infer from (3.17) that for each k

$$\langle G w_{N_k}, u + \xi_k g_{N_k} - w_{N_k} \rangle \geq 0.$$

Now, owing to the pseudo-monotonicity of G on H, we have

$$\langle G(u+\xi_k g_{N_k}), u+\xi_k g_{N_k} - w_{N_k}\rangle \geq 0.$$

This means that

$$\langle Gu, u - w_{N_k}\rangle \geq \langle Gu - G(u+\xi_k g_{N_k}), u+\xi_k g_{N_k} - w_{N_k}\rangle - \xi_k\langle Gu, g_{N_k}\rangle. \tag{3.18}$$

We now have to show that $\lim_{k\to\infty} \xi_k g_{N_k} = 0$. Indeed, by the fact that $s_{m_k} \rightharpoonup v^\star$ and $\lim_{k\to\infty} \|s_{m_k} - w_{m_k}\| = 0$, we have $w_{N_k} \rightharpoonup v^\star$ as $k \to \infty$. Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$0 < \|Gv^\star\| \leq \liminf_{k\to\infty} \|Gw_{m_k}\|. \tag{3.19}$$

Since $w_{N_k} \subset w_{m_k}$ and $\xi_k \to 0$ as $k \to \infty$, we have

$$0 \leq \limsup_{k\to\infty} \|\xi_k g_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\xi_k}{\|Gw_{m_k}\|}\right) \leq \frac{\lim_{k\to\infty} \xi_k}{\liminf_{k\to\infty} \|Gw_{m_k}\|} = 0, \tag{3.20}$$

which implies that $\lim_{k\to\infty} \xi_k g_{N_k} = 0$. Now, owing to the fact that G is Lipschitz continuous, $\{w_{m_k}\}$, $\{g_{N_k}\}$ are bounded, and $\lim_{k\to\infty} \xi_k g_{N_k} = 0$, then letting $k \to \infty$ in (3.18), we obtain

$$\liminf_{k\to\infty} \langle Gu, u - w_{N_k}\rangle \geq 0.$$

Thus, for all $u \in M$, we have

$$\langle Gu, u - v^\star\rangle = \lim_{k\to\infty} \langle Gu, u - w_{N_k}\rangle = \liminf_{k\to\infty} \langle Gu, u - w_{N_k}\rangle \geq 0.$$

□

Lemma 3.3. *Assume that conditions (C_1)–(C_3) hold and $\{z_m\}$ is a sequence generated by Algorithm 3.1, then, for all $u^\star \in VI(M,G)$, and for $m_0 > 0$, we have*

$$\|z_m - u^\star\|^2 \leq \|s_m - u^\star\|^2 - \|s_m - z_m - \rho\delta_m v_m\|^2 - (2-\rho)\rho\left(\frac{1 - \frac{q_m + h_m\mu}{\eta_{m+1}}}{1 + \frac{q_m + h_m\mu}{\eta_{m+1}}}\right)^2 \|s_m - w_m\|^2, \ \forall m \geq m_0. \tag{3.21}$$

Proof. From Lemma 3.1 and (3.9), we have

$$\begin{aligned} \|v_m\| &= \|s_m - w_m - \eta_m(Gs_m - Gw_m)\| \\ &\geq \|s_m - w_m\| - \eta_m\|Gs_m - Gw_m\| \\ &\geq \|s_m - w_m\| - \frac{(q_m + h_m\mu)\eta_m}{\eta_{m+1}}\|s_m - w_m\| \\ &= \left(1 - \frac{(q_m + h_m\mu)\eta_m}{\eta_{m+1}}\right)\|s_m - w_m\|. \end{aligned} \tag{3.22}$$

By Lemma 3.1, we know that $\lim_{m\to\infty} \eta_m$ exists, which together with $\lim_{m\to\infty} q_m = 0$ and $\lim_{m\to\infty} h_m = 1$ gives

$$\lim_{m\to\infty}\left(1 - \frac{(q_m + h_m\mu)\eta_m}{\eta_{m+1}}\right) = 1 - \mu > 0.$$

Thus, there exists $m_0 \in \mathbb{N}$ such that

$$1 - \frac{(q_m + h_m\mu)\eta_m}{\eta_{m+1}} > \frac{1-\mu}{2}, \ \forall m \geq m_0.$$

By (3.22), for all $m \geq m_0$, we have

$$\|v_m\| > \left(\frac{1-\mu}{2}\right)\|s_m - w_m\| \geq 0. \tag{3.23}$$

Since $u^\star \in VI(M, C) \subset M \subset T_m$, then by Lemmas 2.2 and 2.3,

$$\begin{aligned}
2\|z_m - u^\star\|^2 &= 2\|P_{T_m}(s_m - \rho\eta_m\delta_m G w_m) - P_{T_m}u^\star\|^2 \\
&\leq 2\langle z_m - u^\star, s_m - \rho\eta_m\delta_m G w_m - u^\star\rangle \\
&= \|z_m - u^\star\|^2 + \|s_m - \rho\eta_m\delta_m G w_m - u^\star\|^2 - \|z_m - s_m + \rho\eta_m\delta_m G w_m\|^2 \\
&= \|z_m - u^\star\|^2 + \|s_m - u^\star\|^2 + \rho\eta_m^2\delta_m^2\|Gw_m\|^2 - 2\langle s_m - u^\star, \rho\eta_m\delta_m G w_m\rangle \\
&\quad -\|z_m - s_m\|^2 - \rho\eta_m^2\delta_m^2\|Gw_m\|^2 - 2\langle z_m - s_m, \rho\eta_m\delta_m G w_m\rangle \\
&= \|z_m - u^\star\|^2 + \|s_m - u^\star\|^2 - \|z_m - s_m\|^2 - 2\langle z_m - u^\star, \rho\eta_m\delta_m G w_m\rangle.
\end{aligned}$$

This implies that

$$\|z_m - u^\star\|^2 \leq \|s_m - u^\star\|^2 - \|z_m - s_m\|^2 - 2\rho\eta_m\delta_m\langle z_m - u^\star, G w_m\rangle. \tag{3.24}$$

Since $w_m \in M$ and $u^\star \in VI(M, G)$, we have $\langle Gu^\star, w_m - u^\star\rangle \geq 0$. From the pseudo-monotonicity of G, we know that $\langle Gw_m, w_m - u^\star\rangle \geq 0$. This implies that

$$\langle Gw_m, z_m - u^\star\rangle = \langle Gw_m, z_m - w_m\rangle + \langle Gw_m, w_m - u^\star\rangle.$$

Thus,

$$-2\rho\eta_m\delta_m\langle Gw_m, z_m - u^\star\rangle \leq -2\rho\eta_m\delta_m\langle Gw_m, z_m - w_m\rangle. \tag{3.25}$$

On the other hand, from $z_m \in T_m$, we have

$$\langle s_m - \eta_m G s_m - w_m, z_m - w_m\rangle \leq 0.$$

It follows that

$$\langle s_m - w_m - \eta_m(G s_m - G w_m), z_m - w_m\rangle \leq \eta_m\langle G w_m, z_m - w_m\rangle.$$

Thus,

$$\langle v_m, z_m - w_m\rangle \leq \eta_m\langle G w_m, z_m - w_m\rangle.$$

Therefore,

$$-2\rho\eta_m\delta_m\langle Gw_m, z_m - w_m\rangle \leq -2\rho\delta_m\langle v_m, z_m - w_m\rangle. \tag{3.26}$$

Moreover, we have

$$-2\rho\delta_m\langle v_m, z_m - w_m\rangle = -2\rho\delta_m\langle v_m, s_m - w_m\rangle + 2\rho\delta_m\langle v_m, s_m - z_m\rangle. \tag{3.27}$$

Recalling (3.23), we have know that $v_m \neq 0$, for all $m \geq m_0$. This implies that $\delta_m = \frac{\langle s_m - w_m, v_m\rangle}{\|v_m\|^2}$. Thus, we have

$$\langle s_m - w_m, v_m\rangle = \delta_m\|v_m\|^2,\ \forall m \geq m_0. \tag{3.28}$$

On the other hand,

$$2\rho\delta_m\langle v_m, s_m - z_m\rangle = 2\langle\rho\delta_m v_m, s_m - z_m\rangle = \|s_m - z_m\|^2 + \rho^2\delta_m^2\|v_m\|^2 - \|s_m - z_m - \rho\delta_m v_m\|^2. \tag{3.29}$$

Putting (3.28) and (3.29) into (3.27), then for all $m \geq m_0$, we get

$$\begin{aligned} -2\rho\delta_m\langle v_m, z_m - w_m\rangle &\leq -2\rho\delta_m^2\|v_m\|^2 + \|s_m - z_m\|^2 + \rho^2\delta_m^2\|v_m\|^2 - \|s_m - z_m - \rho\delta_m v_m\|^2 \\ &= \|s_m - z_m\|^2 - \|s_m - z_m - \rho\delta_m v_m\|^2 - (2-\rho)\rho\delta_m^2\|v_m\|^2. \end{aligned} \tag{3.30}$$

Using (3.26) and (3.30), we get

$$\begin{aligned} -2\rho\eta_m\delta_m\langle Gw_m, z_m - w_m\rangle &\leq -2\rho\delta_m^2\|v_m\|^2 + \|s_m - z_m\|^2 + \rho^2\delta_m^2\|v_m\|^2 - \|s_m - z_m - \rho\delta_m v_m\|^2 \\ &= \|s_m - z_m\|^2 - \|s_m - z_m - \rho\delta_m v_m\|^2 - (2-\rho)\rho\delta_m^2\|v_m\|^2. \end{aligned} \tag{3.31}$$

Also, from the combination of (3.25) and (3.31), we have

$$\begin{aligned} -2\rho\eta_m\delta_m\langle Gw_m, z_m - u^\star\rangle &\leq -2\rho\delta_m^2\|v_m\|^2 + \|s_m - z_m\|^2 + \rho^2\delta_m^2\|v_m\|^2 - \|s_m - z_m - \rho\delta_m v_m\|^2 \\ &= \|s_m - z_m\|^2 - \|s_m - z_m - \rho\delta_m v_m\|^2 - (2-\rho)\rho\delta_m^2\|v_m\|^2. \end{aligned} \tag{3.32}$$

□

Putting (3.32) into (3.24), we obtain

$$\|z_m - u^\star\|^2 \leq \|s_m - u^\star\|^2 - \|s_m - z_m - \rho\delta_m v_m\|^2 - (2-\rho)\rho\delta_m^2\|v_m\|^2. \tag{3.33}$$

Now, by Lemma 3.1 and (3.9), we have

$$\begin{aligned} \|v_m\| &= \|s_m - w_m - \eta_m(Gs_m - Gw_m)\| \\ &\leq \|s_m - w_m\| + \eta_m\|Gs_m - Gw_m\| \\ &\leq \|s_m - w_m\| + \frac{(q_m + h_m\mu)\eta_m}{\eta_{m+1}}\|s_m - w_m\| \\ &= \left(1 + \frac{(q_m + h_m\mu)\eta_m}{\eta_{m+1}}\right)\|s_m - w_m\|. \end{aligned}$$

Thus,

$$\|v_m\|^2 \leq \left(1 + \frac{(q_m + h_m\mu)\eta_m}{\eta_{m+1}}\right)^2 \|s_m - w_m\|^2,$$

or equivalently

$$\frac{1}{\|v_m\|^2} \geq \frac{1}{\left(1 + \frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2 \|s_m - w_m\|^2}.$$

Again, from (3.9), we have

$$\begin{aligned}
\langle s_m - w_m, v_m\rangle &= \|s_m - w_m\|^2 - \eta_m\langle s_m - w_m, Gs_m - Gw_m\rangle \\
&\geq \|s_m - w_m\|^2 - \eta_m\|s_m - w_m\|\|Gs_m - Gw_m\| \\
&\geq \|s_m - w_m\|^2 - \frac{(q_m + h_m\mu)\eta_m}{\eta_{m+1}}\|s_m - w_m\|^2 \\
&= \left(1 - \frac{(q_m + h_m\mu)\eta_m}{\eta_{m+1}}\right)\|s_m - w_m\|^2.
\end{aligned}$$

Therefore, for all $m \geq m_0$, we have

$$\delta_m\|v_m\|^2 = \langle s_m - w_m, v_m\rangle \geq \left(1 - \frac{(q_m + h_m\mu)\eta_m}{\eta_{m+1}}\right)\|s_m - w_m\|^2 \tag{3.34}$$

and

$$\delta_m = \frac{\langle s_m - w_m, v_m\rangle}{\|v_m\|^2} \geq \frac{\left(1 - \frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)}{\left(1 + \frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}. \tag{3.35}$$

Combining (3.34) and (3.35), we have

$$\delta_m^2\|v_m\|^2 \geq \frac{\left(1 - \frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}{\left(1 + \frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}\|s_m - w_m\|^2, \ \forall m \geq m_0. \tag{3.36}$$

Putting (3.36) into (3.33), we have

$$\|z_m - u^\star\|^2 \leq \|s_m - u^\star\|^2 - \|s_m - z_m - \rho\delta_m v_m\|^2 - (2-\rho)\rho\frac{\left(1 - \frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}{\left(1 + \frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}\|s_m - w_m\|^2, \ \forall m \geq m_0.$$

Next, the strong convergence theorem of Algorithm 3.1 is established as follows:

Theorem 3.1. *Suppose the conditions (C_1)–(C_8) are performed and $\{u_m\}$ is the sequence generated by Algorithm 3.1, then $\{u_m\}$ converges strongly to an element $u^\star \in F(S) \cap VI(M,G)$, where $u^\star = P_{F(S)\cap VI(M,G)} \circ f(u^\star)$.*

Proof. We divide the proof into four parts as follows:
Claim 1. We show that $\{u_m\}$ is bounded.
Indeed, due to (3.21), we have

$$\|z_m - u^\star\| \le \|s_m - u^\star\|. \tag{3.37}$$

From (3.3), we have

$$\begin{aligned}\|s_m - u^\star\| &= \|u_m + \phi_m(Ku_m - Ku_{m-1}) - u^\star\| \\ &\le \|u_m - u^\star\| + \phi_m\|Ku_m - Ku_{m-1}\| \\ &\le \|u_m - u^\star\| + \phi_m\|u_m - u_{m-1}\| \\ &= \|u_m - u^\star\| + \alpha_m\frac{\phi_m}{\alpha_m}\|u_m - u_{m-1}\|. \end{aligned} \tag{3.38}$$

From Remark 3.1, $\lim_{m\to\infty}\frac{\phi_m}{\alpha_m}\|u_m - u_{m-1}\| = 0$. Therefore, $\{\frac{\phi_m}{\alpha_m}\|u_m - u_{m-1}\|\}$ is bounded, so, a constant $\Gamma_1 > 0$ exists such that

$$\frac{\phi_m}{\alpha_m}\|u_m - u_{m-1}\| \le \Gamma_1, \ \forall m \ge 1. \tag{3.39}$$

Combining (3.37)–(3.39), we have

$$\|z_m - u^\star\| \le \|s_m - u^\star\| \le \|u_m - u^\star\| + \alpha_m\Gamma_1. \tag{3.40}$$

Also, from (3.4), we have

$$\begin{aligned}\|r_m - u^\star\| &= \|u_m + \theta_m(Ju_m - Ju_{m-1}) - u^\star\| \\ &\le \|u_m - u^\star\| + \theta_m\|Ju_m - Ju_{m-1}\| \\ &\le \|u_m - u^\star\| + \theta_m\|u_m - u_{m-1}\| \\ &= \|u_m - u^\star\| + \alpha_m\frac{\theta_m}{\alpha_m}\|u_m - u_{m-1}\|. \end{aligned} \tag{3.41}$$

From Remark 3.1, we see that $\lim_{m\to\infty}\frac{\theta_m}{\alpha_m}\|u_m - u_{m-1}\| = 0$. Thus, a constant $\Gamma_2 > 0$ exists such that

$$\frac{\theta_m}{\alpha_m}\|u_m - u_{m-1}\| \le \Gamma_2, \ \forall m \ge 1. \tag{3.42}$$

Combining (3.41) and (3.42), we have

$$\|r_m - u^\star\| \le \|u_m - u^\star\| + \alpha_m\Gamma_2. \tag{3.43}$$

Using (3.10) and condition (C_7), we have

$$\begin{aligned}\|u_{m+1} - u^\star\| &= \|\alpha_m f(r_m) + \beta_m z_m + \gamma_m S z_m - u^\star\| \\ &= \|\alpha_m(f(r_m) - u^\star) + \beta_m(z_m - u^\star) + \gamma_m(S z_m - u^\star)\| \\ &\le \alpha_m\|f(r_m) - f(u^\star) + f(u^\star) - u^\star\| + \beta_m\|z_m - u^\star\| + \gamma_m\|S z_m - u^\star\| \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_m\|f(r_m)-f(u^\star)\|+\alpha_m\|f(u^\star)-u^\star\|+\beta_m\|z_m-u^\star\|+\gamma_m\|Sz_m-u^\star\| \\
&\leq \alpha_m k\|r_m-u^\star\|+\alpha_m\|f(u^\star)-u^\star\|+\beta_m\|z_m-u^\star\|+\gamma_m\|z_m-u^\star\| \\
&= \alpha_m k\|r_m-u^\star\|+\alpha_m\|f(u^\star)-u^\star\|+(1-\alpha_m)\|z_m-u^\star\|. \qquad (3.44)
\end{aligned}$$

Putting (3.40) and (3.43) into (3.44), we have

$$\begin{aligned}
\|u_{m+1}-u^\star\| &\leq \alpha_m k(\|u_m-u^\star\|+\alpha_m\Gamma_2)+\alpha_m\|f(u^\star)-u^\star\|+(1-\alpha_m)(\|u_m-u^\star\|+\alpha_m\Gamma_1) \\
&= (1-(1-k)\alpha_m)\|u_m-u^\star\|+\alpha_m^2 k\Gamma_2+\alpha_m(1-\alpha_m)\Gamma_1+\alpha_m\|f(u^\star)-u^\star\| \\
&\leq (1-(1-k)\alpha_m)\|u_m-u^\star\|+\alpha_m\Gamma_2+\alpha_m\Gamma_1+\alpha_m\|f(u^\star)-u^\star\| \\
&= (1-(1-k)\alpha_m)\|u_m-u^\star\|+\alpha_m\Gamma_3+\alpha_m\|f(u^\star)-u^\star\| \\
&= (1-(1-k)\alpha_m)\|u_m-u^\star\|+(1-k)\alpha_m\frac{\Gamma_3+\|f(u^\star)-u^\star\|}{1-k} \\
&\leq \max\left\{\|u_m-u^\star\|,\frac{\Gamma_3+\|f(u^\star)-u^\star\|}{1-k}\right\} \\
&\leq \cdots \\
&\leq \max\left\{\|u_{m_0}-u^\star\|,\frac{\Gamma_3+\|f(u^\star)-u^\star\|}{1-k}\right\},\ \forall m\geq m_0, \qquad (3.45)
\end{aligned}$$

where $\Gamma_3=\Gamma_1+\Gamma_2$. This means that $\{u_m\}$ is bounded. It follows that $\{z_m\}$, $\{s_m\}$, $\{w_m\}$, $\{f(r_m)\}$ and $\{f(z_m)\}$ are bounded.

Claim 2.

$$(1-\alpha_m)\|s_m-z_m-\rho\delta_m v_m\|^2+(1-\alpha_m)(2-\rho)\rho\frac{\left(1-\frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}{\left(1+\frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}\|s_m-w_m\|^2+\beta_m\gamma_m\|z_m-Sz_m\|^2$$
$$\leq\|u_m-u^\star\|^2-\|u_{m+1}-u^\star\|^2+\alpha_m\Gamma_7,\ \forall m\geq m_0,$$

for some $\Gamma_7>0$.

Indeed, from (3.40), we have

$$\|s_m-u^\star\|^2\leq(\|u_m-u^\star\|+\alpha_m\Gamma_1)^2=\|u_m-u^\star\|^2+\alpha_m(2\Gamma_1\|u_m-u^\star\|+\alpha_m\Gamma_1^2). \qquad (3.46)$$

Since $\{u_m\}$ is a bounded sequence, it therefore implies that a constant $\Gamma_4>0$ exists, such that $2\Gamma_1\|u_m-u^\star\|+\alpha_m\Gamma_1^2\leq\Gamma_4$. Hence, (3.46) becomes

$$\|s_m-u^\star\|^2\leq\|u_m-u^\star\|^2+\alpha_m\Gamma_4.$$

Also, from (3.43), we get

$$\|r_m-u^\star\|^2\leq(\|u_m-u^\star\|+\alpha_m\Gamma_2)^2=\|u_m-u^\star\|^2+\alpha_m(2\Gamma_2\|u_m-u^\star\|+\alpha_m\Gamma_2^2). \qquad (3.47)$$

Since $\{u_m\}$ is a bounded sequence, it therefore implies that a constant $\Gamma_5>0$ exists, such that $2\Gamma_2\|u_m-u^\star\|+\alpha_m\Gamma_2^2\leq\Gamma_5$. Hence, (3.47) becomes

$$\|r_m-u^\star\|^2\leq\|u_m-u^\star\|^2+\alpha_m\Gamma_5.$$

Now, from (3.10) and Lemma 2.3, we have

$$\begin{aligned}
\|u_{m+1}-u^\star\|^2 &= \|\alpha_m f(r_m)+\beta_m z_m+\gamma_m S z_m-u^\star\|^2 \\
&= \|\alpha_m(f(r_m)-u^\star)+\beta_m(z_m-u^\star)+\gamma_m(S z_m-u^\star)\|^2 \\
&\le \alpha_m\|f(r_m)-u^\star\|^2+\beta_m\|z_m-u^\star\|^2 \\
&\quad +\gamma_m\|S z_m-u^\star\|^2-\beta_m\gamma_m\|z_m-S z_m\|^2 \\
&\le \alpha_m(\|f(r_m)-f(u^\star)\|+\|f(u^\star)-u^\star\|)^2+\beta_m\|z_m-u^\star\|^2 \\
&\quad +\gamma_m\|S z_m-u^\star\|^2-\beta_m\gamma_m\|z_m-S z_m\|^2 \\
&\le \alpha_m(k\|r_m-u^\star\|+\|f(u^\star)-u^\star\|)^2+\beta_m\|z_m-u^\star\|^2 \\
&\quad +\gamma_m\|z_m-u^\star\|^2-\beta_m\gamma_m\|z_m-S z_m\|^2 \\
&= \alpha_m(k^2\|r_m-u^\star\|^2+2\|r_m-u^\star\|\|f(u^\star)-u^\star\|+\|f(u^\star)-u^\star\|^2) \\
&\quad +(1-\alpha_m)\|z_m-u^\star\|^2-\beta_m\gamma_m\|z_m-S z_m\|^2 \\
&\le \alpha_m(\|r_m-u^\star\|^2+2\|r_m-u^\star\|\|f(u^\star)-u^\star\|+\|f(u^\star)-u^\star\|^2) \\
&\quad +(1-\alpha_m)\|z_m-u^\star\|^2-\beta_m\gamma_m\|z_m-S z_m\|^2 \\
&= \alpha_m\|r_m-u^\star\|^2+\alpha_m(2\|r_m-u^\star\|\|f(u^\star)-u^\star\|+\|f(u^\star)-u^\star\|^2) \\
&\quad +(1-\alpha_m)\|z_m-u^\star\|^2-\beta_m\gamma_m\|z_m-S z_m\|^2.
\end{aligned} \tag{3.48}$$

Due to the boundedness of $\{r_m\}$, we know that a constant $\Gamma_6>0$ exists, such that $2\|r_m-u^\star\|\|f(u^\star)-u^\star\|+\|f(u^\star)-u^\star\|^2\le\Gamma_6$. Therefore, (3.48) becomes

$$\|u_{m+1}-u^\star\|^2 \le \alpha_m\|r_m-u^\star\|^2+(1-\alpha_m)\|z_m-u^\star\|^2-\beta_m\gamma_m\|z_m-S z_m\|^2+\alpha_m\Gamma_6. \tag{3.49}$$

Putting (3.21) into (3.49), we get

$$\begin{aligned}
\|u_{m+1}-u^\star\|^2 &\le \alpha_m\|r_m-u^\star\|^2+(1-\alpha_m)\|s_m-u^\star\|^2-(1-\alpha_m)\|s_m-z_m-\rho\delta_m v_m\|^2 \\
&\quad -(1-\alpha_m)(2-\rho)\rho\frac{\left(1-\frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}{\left(1+\frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}\|s_m-w_m\|^2-\beta_m\gamma_m\|z_m-S z_m\|^2+\alpha_m\Gamma_6.
\end{aligned} \tag{3.50}$$

Substituting (3.40) and (3.43) into (3.50), we have

$$\begin{aligned}
\|u_{m+1}-u^\star\|^2 &\le \alpha_m(\|u_m-u^\star\|+\alpha_m\Gamma_2)^2+(1-\alpha_m)(\|u_m-u^\star\|+\alpha_m\Gamma_1)^2 \\
&\quad -(1-\alpha_m)\|s_m-z_m-\rho\delta_m v_m\|^2 \\
&\quad -(1-\alpha_m)(2-\rho)\rho\frac{\left(1-\frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}{\left(1+\frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}\|s_m-w_m\|^2 \\
&\quad -\beta_m\gamma_m\|z_m-S z_m\|^2+\alpha_m\Gamma_6. \\
&\le \|u_m-u^\star\|^2-(1-\alpha_m)\|s_m-z_m-\rho\delta_m v_m\|^2 \\
&\quad -(1-\alpha_m)(2-\rho)\rho\frac{\left(1-\frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}{\left(1+\frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}\|s_m-w_m\|^2 \\
&\quad -\beta_m\gamma_m\|z_m-S z_m\|^2+\alpha_m\Gamma_1+\alpha_m\Gamma_2+\alpha_m\Gamma_6,
\end{aligned} \tag{3.51}$$

it follows from (3.51) that

$$(1-\alpha_m)\|s_m - z_m - \rho\delta_m v_m\|^2 + (1-\alpha_m)(2-\rho)\rho\frac{\left(1-\frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}{\left(1+\frac{(q_m+h_m\mu)\eta_m}{\eta_{m+1}}\right)^2}\|s_m - w_m\|^2 + \beta_m\gamma_m\|z_m - Sz_m\|^2$$
$$\leq \|u_m - u^\star\|^2 - \|u_{m+1} - u^\star\|^2 + \alpha_m\Gamma_7,\ \forall m \geq m_0,$$

where $\Gamma_7 = \Gamma_1 + \Gamma_2 + \Gamma_6 > 0$.

Claim 3.

$$\begin{aligned}\|u_{m+1} - u^\star\|^2 &\leq (1-(1-k)\alpha_m)\|u_m - u^\star\|^2 + (1-k)\alpha_m\left[\frac{2}{1-k}\langle f(u^\star) - u^\star, u_{m+1} - u^\star\rangle\right.\\ &\quad \left.+\frac{3\Gamma_8}{1-k}\cdot\frac{\theta_m}{\alpha_m}\|u_m - u_{m-1}\| + \frac{3\Gamma_9}{1-k}\cdot\frac{\phi_m}{\alpha_m}\|u_m - u_{m-1}\|\right],\ \forall m \geq m_0, \end{aligned} \tag{3.52}$$

for some $\Gamma_8 > 0$ and $\Gamma_9 > 0$.

Indeed, using (3.3), we have

$$\begin{aligned}\|s_m - u^\star\|^2 &= \|u_m + \phi_m(Ku_m - Ku_{m-1}) - u^\star\|^2\\ &= \|u_m - u^\star + \phi_m(Ku_m - Ku_{m-1})\|^2\\ &\leq \|u_m - u^\star\|^2 + 2\phi_m\|u_m - u^\star\|\|Ku_m - Ku_{m-1}\| + \phi_m^2\|Ku_m - Ku_{m-1}\|^2\\ &\leq \|u_m - u^\star\|^2 + 2\phi_m\|u_m - u^\star\|\|u_m - u_{m-1}\| + \phi_m^2\|u_m - u_{m-1}\|^2. \end{aligned} \tag{3.53}$$

Also, from (3.4), we get

$$\begin{aligned}\|r_m - u^\star\|^2 &= \|u_m + \theta_m(Ju_m - Ju_{m-1}) - u^\star\|^2\\ &= \|u_m - u^\star + \theta_m(Ju_m - Ju_{m-1})\|^2\\ &\leq \|u_m - u^\star\|^2 + 2\theta_m\|u_m - u^\star\|\|Ju_m - Ju_{m-1}\| + \theta_m^2\|Ju_m - Ju_{m-1}\|^2\\ &\leq \|u_m - u^\star\|^2 + 2\theta_m\|u_m - u^\star\|\|u_m - u_{m-1}\| + \theta_m^2\|u_m - u_{m-1}\|^2. \end{aligned} \tag{3.54}$$

Using (3.10) and Lemma 2.3, we have

$$\begin{aligned}\|u_{m+1} - u^\star\|^2 &= \|\alpha_m f(r_m) + \beta_m z_m + \gamma_m Sz_m - u^\star\|^2\\ &= \|\alpha_m(f(r_m) - u^\star) + \beta_m(z_m - u^\star) + \gamma_m(Sz_m - u^\star)\|^2\\ &= \|\alpha_m(f(r_m) - f(u^\star)) + \beta_m(z_m - u^\star) + \gamma_m(Sz_m - u^\star) + \alpha_m(f(u^\star) - u^\star)\|^2\\ &\leq \|\alpha_m(f(r_m) - f(u^\star)) + \beta_m(z_m - u^\star) + \gamma_m(Sz_m - u^\star)\|^2\\ &\quad +2\alpha_m\langle f(u^\star) - u^\star, u_{m+1} - u^\star\rangle\\ &\leq \alpha_m\|f(r_m) - f(u^\star)\|^2 + \beta_m\|z_m - u^\star\|^2 + \gamma_m\|Sz_m - u^\star\|^2\\ &\quad +2\alpha_m\langle f(u^\star) - u^\star, u_{m+1} - u^\star\rangle\\ &\leq \alpha_m k^2\|r_m - u^\star\|^2 + \beta_m\|z_m - u^\star\|^2 + \gamma_m\|z_m - u^\star\|^2\\ &\quad +2\alpha_m\langle f(u^\star) - u^\star, u_{m+1} - u^\star\rangle\\ &\leq \alpha_m k\|r_m - u^\star\|^2 + \beta_m\|z_m - u^\star\|^2 + \gamma_m\|z_m - u^\star\|^2\\ &\quad +2\alpha_m\langle f(u^\star) - u^\star, u_{m+1} - u^\star\rangle \end{aligned}$$

$$
\begin{aligned}
&= \alpha_m k\|r_m - u^\star\|^2 + (1-\alpha_m)\|z_m - u^\star\|^2 + 2\alpha_m\langle f(u^\star) - u^\star, u_{m+1} - u^\star\rangle \\
&\leq \alpha_m k\|r_m - u^\star\|^2 + (1-\alpha_m)\|s_m - u^\star\|^2 + 2\alpha_m\langle f(u^\star) - u^\star, u_{m+1} - u^\star\rangle. \qquad (3.55)
\end{aligned}
$$

Substituting (3.53) and (3.54) into (3.55), we obtain

$$
\begin{aligned}
\|u_{m+1} - u^\star\|^2 \leq\ & \alpha_m k[\|u_m - u^\star\|^2 + 2\theta_m\|u_m - u^\star\|\|u_m - u_{m-1}\| + \theta_m^2\|u_m - u_{m-1}\|^2] \\
&+(1-\alpha_m)[\|u_m - u^\star\|^2 + 2\phi_m\|u_m - u^\star\|\|u_m - u_{m-1}\| + \phi_m^2\|u_m - u_{m-1}\|^2] \\
&+2\alpha_m\langle f(u^\star) - u^\star, u_{m+1} - u^\star\rangle \\
\leq\ & (1-(1-k)\alpha_m)\|u_m - u^\star\|^2 + (1-k)\alpha_m\frac{2}{1-k}\langle f(u^\star) - u^\star, u_{m+1} - u^\star\rangle \\
&+\theta_m\|u_m - u_{m-1}\|[2\|u_m - u^\star\| + \theta_m\|u_m - u_{m-1}\|] \\
&+\phi_m\|u_m - u_{m-1}\|[2\|u_m - u^\star\| + \phi_m\|u_m - u_{m-1}\|] \\
\leq\ & (1-(1-k)\alpha_m)\|u_m - u^\star\|^2 + (1-k)\alpha_m\left[\frac{2}{1-k}\langle f(u^\star) - u^\star, u_{m+1} - u^\star\rangle\right. \\
&\left.+\frac{3\Gamma_8}{1-k}\cdot\frac{\theta_m}{\alpha_m}\|u_m - u_{m-1}\| + \frac{3\Gamma_9}{1-k}\cdot\frac{\phi_m}{\alpha_m}\|u_m - u_{m-1}\|\right], \ \forall m \geq m_0,
\end{aligned}
$$

where $\Gamma_8 = \sup_{m\in\mathbb{N}}\{\|u_m - u^\star\|, \theta\|u_m - u_{m-1}\|\}$ and $\Gamma_9 = \sup_{m\in\mathbb{N}}\{\|u_m - u^\star\|, \phi\|u_m - u_{m-1}\|\}$.

Claim 4. The sequence $\{\|u_m - u^\star\|^2\}$ converges to zero. Indeed, from (3.52), Remark 3.1 and Lemma 2.5, it is enough to show that $\limsup_{k\to\infty}\langle f(u^\star) - u^\star, u_{m_k+1} - u^\star\rangle \leq 0$ for any subsequence of $\{\|u_{m_k} - u^\star\|^2\}$ of $\{\|u_m - u^\star\|^2\}$ fulfilling

$$
\liminf_{k\to\infty}(\|u_{m_k+1} - u^\star\|^2 - \|u_{m_k} - u^\star\|^2) \geq 0. \qquad (3.56)
$$

Now, we assume that $\|u_{m_k} - u^\star\|^2$ is a subsequence of $\|u_m - u^\star\|^2$ such that (3.56) holds, then

$$
\begin{aligned}
&\liminf_{k\to\infty}(\|u_{m_k+1} - u^\star\|^2 - \|u_{m_k} - u^\star\|^2) \\
&= \liminf_{k\to\infty}[(\|u_{m_k+1} - u^\star\| - \|u_{m_k} - u^\star\|)(\|u_{m_k+1} - u^\star\| + \|u_{m_k} - u^\star\|)] \geq 0.
\end{aligned}
$$

By Claim 2 and condition (C_8), we get

$$
\begin{aligned}
&\limsup_{k\to\infty}\left\{\begin{array}{c}(1-\alpha_{m_k})\|s_{m_k} - z_{m_k} - \rho\delta_{m_k}v_{m_k}\|^2 \\ +(1-\alpha_{m_k})(2-\rho)\rho\dfrac{\left(1-\frac{(q_{m_k}+h_{m_k}\mu)\eta_{m_k}}{\eta_{m_k+1}}\right)^2}{\left(1+\frac{(q_{m_k}+h_{m_k}\mu)\eta_{m_k}}{\eta_{m_k+1}}\right)^2}\|s_{m_k} - w_{m_k}\|^2 \\ +\beta_{m_k}\gamma_{m_k}\|z_{m_k} - Sz_{m_k}\|^2\end{array}\right\} \\
&\leq \limsup_{k\to\infty}\{\|u_{m_k} - u^\star\|^2 - \|u_{m_k+1} - u^\star\|^2 + \alpha_{m_k}\Gamma_7\} \\
&= -\liminf_{k\to\infty}\{\|u_{m_k} - u^\star\|^2 - \|u_{m_k+1} - u^\star\|^2\},
\end{aligned}
$$

which implies that

$$
\lim_{k\to\infty}\|s_{m_k} - z_{m_k} - \rho\delta_{m_k}v_{m_k}\| = \lim_{k\to\infty}\|s_{m_k} - w_{m_k}\| = \lim_{k\to\infty}\|z_{m_k} - Sz_{m_k}\| = 0. \qquad (3.57)
$$

On the other hand,

$$\|s_{m_k} - z_{m_k}\| = \|s_{m_k} - z_{m_k} - \rho\delta_{m_k} v_{m_k} + \rho\delta_{m_k} v_{m_k}\| \leq \|s_{m_k} - z_{m_k} - \rho\delta_{m_k} v_{m_k}\| + \rho\delta_{m_k}\|v_{m_k}\|. \tag{3.58}$$

By (3.8) and (3.23), we know that

$$\delta_{m_k}\|v_{m_k}\| = \frac{\langle s_{m_k} - w_{m_k}, v_{m_k}\rangle}{\|v_{m_k}\|}. \tag{3.59}$$

Putting (3.59) into (3.58) and using the Cauchy Schwartz inequality, we have

$$\|s_{m_k} - z_{m_k}\| \leq \|s_{m_k} - z_{m_k} - \rho\delta_{m_k} v_{m_k}\| + \rho\|s_{m_k} - w_{m_k}\|.$$

Recalling (3.57), we have

$$\lim_{k\to\infty} \|s_{m_k} - z_{m_k}\| = 0. \tag{3.60}$$

Also, from (3.3), we have

$$\|s_{m_k} - u_{m_k}\| = \phi_{m_k}\|Ku_{m_k} - Ku_{m_k-1}\| \leq \phi_{m_k}\|u_{m_k} - u_{m_k-1}\| \leq \alpha_{m_k} \cdot \frac{\phi_{m_k}}{\alpha_{m_k}}\|u_{m_k} - u_{m_k-1}\|. \tag{3.61}$$

By Remark 3.1, condition (C_8) and (3.61), we have

$$\lim_{k\to\infty} \|s_{m_k} - u_{m_k}\| = 0. \tag{3.62}$$

Using (3.60) and (3.62), we have

$$\lim_{k\to\infty} \|z_{m_k} - u_{m_k}\| \leq \lim_{k\to\infty}(\|z_{m_k} - s_{m_k}\| + \|s_{m_k} - u_{m_k}\|) = 0. \tag{3.63}$$

Again, from (3.10), we have

$$\|u_{m_k+1} - z_{m_k}\| \leq \alpha_{m_k}\|f(r_m) - z_{m_k}\| + \beta_{m_k}\|z_{m_k} - z_{m_k}\| + \gamma_{m_k}\|S z_{m_k} - z_{m_k}\|. \tag{3.64}$$

From condition (C_8), (3.57) and (3.64), we obtain

$$\lim_{k\to\infty} \|u_{m_k+1} - z_{m_k}\| = 0. \tag{3.65}$$

Next, we have that

$$\|u_{m_k+1} - u_{m_k}\| \leq \|u_{m_k+1} - z_{m_k}\| + \|z_{m_k} - s_{m_k}\| + \|s_{m_k} - u_{m_k}\|. \tag{3.66}$$

Combing (3.60), (3.62), (3.65), and (3.66), we have

$$\lim_{k\to\infty} \|u_{m_k+1} - u_{m_k}\| = 0. \tag{3.67}$$

Since the sequence $\{u_{m_k}\}$ is bounded, then we know that a subsequence $\{u_{m_{k_j}}\}$ of $\{u_{m_k}\}$ exists such that $u_{m_{k_j}} \rightharpoonup q^\star$. Furthermore,

$$\limsup_{k\to\infty}\langle f(u^\star) - u^\star, u_{m_k} - u^\star\rangle = \lim_{j\to\infty}\langle f(u^\star) - u^\star, u_{m_{k_j}} - u^\star\rangle = \langle f(u^\star) - u^\star, q^\star - u^\star\rangle. \tag{3.68}$$

Thus, we have $s_{m_{k_j}} \rightharpoonup q^\star$ since $\lim_{k\to\infty} \|s_{m_k} - u_{m_k}\| = 0$. Since $\lim_{k\to\infty} \|s_{m_k} - w_{m_k}\| = 0$, it follows from Lemma 3.2 that $q^\star \in VI(M, G)$. From (3.63), it follows that $z_{m_{k_j}} \rightharpoonup q^\star$. Following the demiclosedness of $I - S$ at zero as defined in Lemma 2.4, we know that $q^\star \in F(S)$. Thus, $q^\star \in F(S) \cap VI(M, G)$. By combining (3.68), $q^\star \in F(S)$ and $u^\star = P_{F(S)\cap VI(M,G)} \circ f(u^\star)$, we get

$$\limsup_{k\to\infty} \langle f(u^\star) - u^\star, u_{m_k} - u^\star\rangle = \langle f(u^\star) - u^\star, q^\star - u^\star\rangle \le 0. \tag{3.69}$$

Using (3.67) and (3.69), we have

$$\begin{aligned}\limsup_{k\to\infty} \langle f(u^\star) - u^\star, u_{m_k+1} - u^\star\rangle &\le \limsup_{k\to\infty} \langle f(u^\star) - u^\star, u_{m_k+1} - u_{m_k}\rangle + \limsup_{k\to\infty} \langle f(u^\star) - u^\star, u_{m_k} - u^\star\rangle \\ &= \langle f(u^\star) - u^\star, q^\star - u^\star\rangle \le 0. \end{aligned} \tag{3.70}$$

By Claim 3, Remark 3.1, (3.70), and Lemma 2.5, we obtain that $\lim_{m\to\infty} \|u_m - u^\star\| = 0$, and this completes the proof of Theorem 3.1. □

Next, we propose our second and third algorithms as in Algorithms 3.2 and 3.3, which differ slightly from Algorithm 3.1.

Algorithm 3.2.

Initialization: *Choose* $\eta_1 > 0, \phi > 0, \theta > 0, \rho \in (0, 2), \mu \in (0, 1)$ *and let* $g_0, g_1 \in H$ *be arbitrary.*

Iterative Steps: *Given the iterates* u_{m-1} *and* $\{u_m\}$ $(m \ge 1)$*, calculate* u_{m+1} *as follows:*

Step 1: *Choose* ϕ_m *and* θ_m *such that* $0 \le \phi_m \le \bar{\phi}_m$ *and* $0 \le \theta_m \le \bar{\theta}_m$*, where* $\bar{\phi}_m$ *and* $\bar{\theta}_m$ *are as defined in* (3.1) *and* (3.2).

Step 2: *Set*

$$\begin{aligned} s_m &= u_m + \phi_m(Ku_m - Ku_{m-1}), \\ r_m &= u_m + \theta_m(Ju_m - Ju_{m-1}), \end{aligned}$$

and compute

$$w_m = P_M(s_m - \eta_m G s_m).$$

If $s_m = w_m$ *or* $Gs_m = 0$*, stop,* s_m *is a solution of the VIP. Otherwise, do Step 3.*

Step 3: *Compute*

$$z_m = P_{T_m}(s_m - \rho\eta_m\delta_m G w_m),$$

where T_m*,* δ_m *and* v_m *are as defined in* (3.7)–(3.9).

Step 4: *Compute*

$$u_{m+1} = \alpha_m f(u_m) + \beta_m z_m + \gamma_m S z_m.$$

Update η_{m+1} *by* (3.11).

Set $m := m + 1$ *and go back to Step 1.*

Algorithm 3.3.

Initialization: *Choose* $\eta_1 > 0, \phi > 0, \theta > 0, \rho \in (0, 2), \mu \in (0, 1)$ *and let* $g_0, g_1 \in H$ *be arbitrary.*

Iterative Steps: *Given the iterates* u_{m-1} *and* $\{u_m\}$ $(m \geq 1)$*, calculate* u_{m+1} *as follows:*

Step 1: *Choose* ϕ_m *and* θ_m *such that* $0 \leq \phi_m \leq \bar{\phi}_m$ *and* $0 \leq \theta_m \leq \bar{\theta}_m$*, where* $\bar{\phi}_m$ *and* $\bar{\theta}_m$ *are as defined in* (3.1) *and* (3.2).

Step 2: *Set*

$$\begin{aligned} s_m &= u_m + \phi_m(Ku_m - Ku_{m-1}), \\ r_m &= u_m + \theta_m(Ju_m - Ju_{m-1}), \end{aligned}$$

and compute

$$w_m = P_M(s_m - \eta_m G s_m).$$

If $s_m = w_m$ *or* $Gs_m = 0$*, stop,* s_m *is a solution of the VIP. Otherwise, do Step 3.*

Step 3: *Compute*

$$z_m = P_{T_m}(s_m - \rho\eta_m\delta_m G w_m),$$

where T_m*,* δ_m *and* v_m *are as defined in* (3.7)–(3.9).

Step 4: *Compute*

$$u_{m+1} = \alpha_m f(s_m) + \beta_m z_m + \gamma_m S z_m.$$

Update η_{m+1} *by* (3.11).

Set $m := m + 1$ *and go back to Step 1.*

Remark 3.2. *In Algorithm 3.2, we replace the term* $f(z_m)$ *in* (3.10) *of Algorithm 3.1 with* $f(u_m)$*. Also, in Algorithm 3.3, we replace the term* $f(z_m)$ *in* (3.10) *of Algorithm 3.1 with* $f(s_m)$*. Now, the strong convergence theorems of Algorithms 3.2 and 3.3 will be stated without proofs. Their proofs are very similar to that of Theorem 3.1. Hence, we leave the proofs for the reader to verify.*

Theorem 3.2. *Suppose the conditions* (C_1)–(C_8) *are performed and* $\{u_m\}$ *is the sequence generated by Algorithm 3.2, then* $\{u_m\}$ *converges strongly to an element* $u^\star \in F(S) \cap VI(M, G)$*, where* $u^\star = P_{F(T)\cap VI(M,G)} \circ f(u^\star)$.

Theorem 3.3. *Suppose the conditions* (C_1)–(C_8) *are performed and* $\{u_m\}$ *is the sequence generated by Algorithm 3.3, then* $\{u_m\}$ *converges strongly to an element* $u^\star \in F(S) \cap VI(M, G)$*, where* $u^\star = P_{F(T)\cap VI(M,G)} \circ f(u^\star)$.

4. Number experiments

In this part of the work, we consider two numerical examples to demonstrate the computational efficiency of our Algorithms 3.1–3.3 (shortly, OAUAN Algs. 3.1, 3.7 and 3.8) over some existing modified algorithms, namely, Algorithms 1 and 2 of Thong and Hieu [43] (shortly, TH Alg. 1 and TH Alg. 2), Algorithm 2 of Tian and Tong [47] (shortly, TT Alg. 2), Algorithm 3.1 of Ogwo et al. [33] (shortly, OAM Alg. 3.1), Algorithm 3.1 of Godwin et al. [14] (shortly, GAMY Alg 3.1), and Algorithm 3.1 of Maluleka et al. [24] (shortly, MUA Alg 3.1). We perform all numerical simulations using MATLAB R2020b and carried out on PC Desktop Intel® Core^{TM} i7-3540M CPU @ 3.00GHz $\times$ 4 memory 400.00GB.

Example 4.1. *Suppose that $G : \mathbb{R}^k \to \mathbb{R}^k$ ($k = 30, 50, 80, 110$) is defined by $G(u) = Qu + q$, where $q \in \mathbb{R}^k$ and $Q = AA^T + B + C$, C is a $k \times k$ diagonal matrix whose diagonal terms are nonnegative (hence Q is positive symmetric definite), B is a $k\times k$ skew-symmetric, and A is a $k\times k$ matrix. We define the feasible set M by*

$$M = \{u \in \mathbb{R}^k : -5 \le u_i \le 5,\ i = 1, \cdots k\}.$$

It is not hard to see that the mapping G is monotone and L-Lipschitz continuous with $L = \|Q\|$ (hence, G is pseudo-monotone). For $q = 0$, the solution set $VI(M, G) = \{0\}$. On the other hand, let $Su = \frac{3}{4}u \sin \|u\|$. Clearly, the only fixed point of S is 0, i.e., $F(S) = \{0\}$. The mapping S is quasi-nonexpansive but not nonexpansive. Indeed, for $k = 1$, we have

$$|Su - 0| = \left|\frac{3}{4}u \sin |u|\right| \le \left|\frac{3u}{4}\right| \le |u| = |u - 0|,\ \forall u \in M.$$

Hence, S is quasi-nonexpansive. Moreover, if we take $u = 2\pi$ and $v = \frac{3\pi}{2}$, then we have

$$|Su - Sv| = \left|\frac{6\pi}{4}\sin 2\pi - \frac{9\pi}{8}\sin\frac{3\pi}{2}\right| = \frac{9\pi}{8} > \frac{\pi}{2} = |u - v|.$$

Therefore, S is not quasinonexpansive. Notice that $I - S$ is demiclosed at 0 and $F(S) \cap VI(M, G) = \{0\} \neq \emptyset$. Furthermore, we take $Ku = \sin u$, where for $k > 1$, $\sin u = (\sin u_1, \sin u_2,\ \ldots\ , \sin u_k)^T$ and $Ju = \frac{u}{2}$.

The parameters for all the algorithms are taken as follows:

- *For Algorithms 3.1–3.3, we take $\eta_1 = 0.9$, $\mu = 0.4$, $\alpha_m = \frac{1}{2m+20}$, $\beta_m = \gamma_m = \frac{m}{2m+20}$, $p_m = \frac{1}{(m+100)^{1.1}}$, $q_m = \frac{m+1}{m}$, $h_m = \frac{1}{m+100}$, $\phi = 0.6$, $\theta = 0.9$, $\rho = 0.0001$ and $\epsilon_m = \frac{1}{(2m+1)^3}$.*
- *For TH Algs. 1 and 2 $\gamma = 2$, $l = 0.5$, $\tau_1 = 0.8$, $\alpha_m = 0.5$, $\beta_m = 0.5$, $\mu = 0.6$.*
- *For Algorithm 2 of Tian and Tong [47] (TT Alg.), we take $\alpha_m = 0.5$, $\beta_m = 0.5$, $\mu = 0.4$ and $\lambda_1 = \frac{1}{7}$.*
- *For Algorithm 3.1 of Godwin et al. [14] (GAMY Alg. 3.1), we take $\alpha = 4$, $\lambda_1 = 0.5$, $\theta_m = \bar{\theta}_m$ $\delta = 0.4$ $c'(x) = 2x$, $\phi_m = \frac{2m+1}{5m+2}$, $\beta_m = \frac{2m}{3m+2}$, $\gamma = 1$, $\gamma_m = \left(\frac{2}{3m+1}\right)^2$, $\alpha_m = (\frac{2}{3m+1}$, $\mu = 0.8$, $Dx = Tx = 0.5x$ and $f(x) = \frac{1}{3}x$.*
- *For Algorithm 3.1 of Maluleka et al. [24] (MUA Alg. 3.1), we take $\theta = 0.9$, $\lambda_1 = 3.1$, $\mu_m = \frac{1}{(m+1)^2}$ $\alpha_m = \frac{1}{m+1}$, $\beta_m = 0.5$ and $\rho = 0.5$.*
- *For Algorithm 3.2 of Ogwo et al. [33] (OAM Alg. 3.1), we take $\alpha = 3$, $\lambda_1 = 0.5$, $\alpha_m = \bar{\alpha}_m$ $\mu = 0.4$, $\beta_m = \frac{m}{m+10}$, $\gamma_1 = 0.01$, $\tau_m = (\frac{1}{(m+1)^2}$, $\theta_m = \frac{1}{m+10}$, $Dx = 0.01x$ and $f(x) = 0.01x$.*

In this example, all entries A, B and C are taken randomly from $[1, 100]$. *We consider 4 different dimensions for k, Case I: $k = 50$, Case II: $k = 100$, Case III: $k = 300$, Case IV: $k = 500$. The initial values $u_1 = u_2$ are chosen at random using $randn(k, 1)$ in MATLAB and stopping criterion is taken as $\|u_{m+1} - u_m\| \le 10^{-8}$. The results of the numerical simulations are presented in Table 1 and Figures 1 and 2.*

Table 1. Numerical Results for the four dimensions considered in Example 4.1.

Algorithms	Case I		Case II		Case III		Case IV	
	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
OUANC Alg. 3.1	15	0.0062	14	0.0043	15	0.0093	15	0.0205
OUANC Alg. 3.7	16	0.0099	16	0.0075	16	0.0096	17	0.0199
OUANC Alg. 3.8	17	0.0089	13	0.0037	14	0.0096	17	0.0242
TH Alg. 1	33	0.0194	35	0.0363	35	0.0777	39	0.1864
TH Alg. 2	38	0.0254	31	0.0413	38	0.0823	51	0.1878
TT Alg. 2	23	0.0092	30	0.0181	36	0.0146	30	0.0565
GAMY Alg. 3.1	90	0.0201	91	0.0399	99	0.0276	103	0.0712
MUA Alg. 3.1	47	0.0207	47	0.0159	44	0.0294	45	0.0453
OAM Alg. 3.1	40	0.0144	39	0.0076	41	0.0159	42	0.033

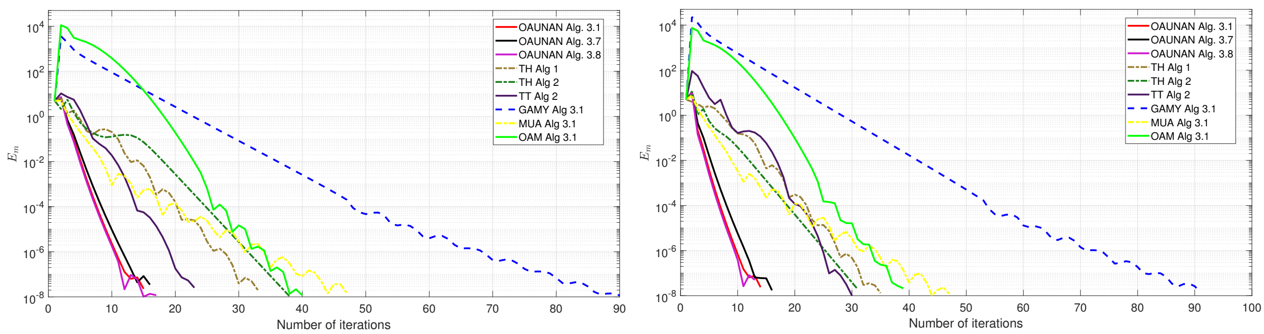

Figure 1. Graph of the iterates for Cases I and II.

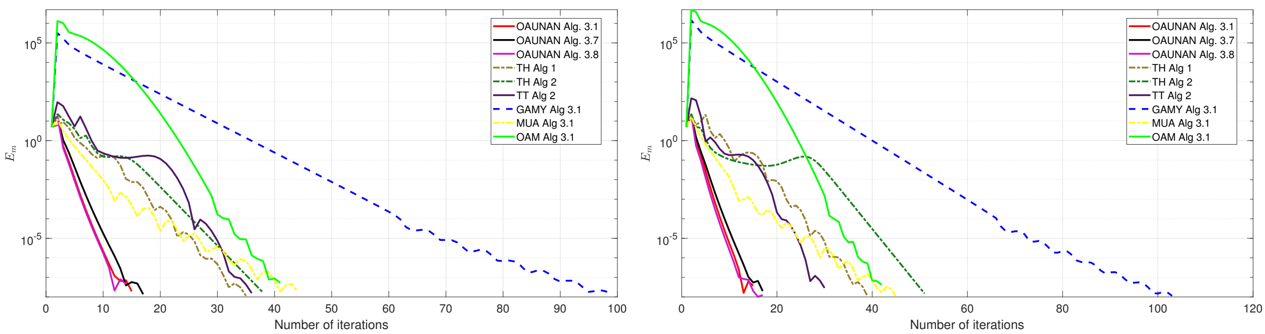

Figure 2. Graph of the iterates for Cases III and IV.

Example 4.2. *Let $H = \ell^2$, i.e., $H = \{u = (u_1, u_2, u_3, \cdots, u_i, \cdots) : \sum_{i=1}^{\infty} |u_i|^2 < +\infty\}$. Let $e, d \in \mathbb{R}$ be such that $d > e > \frac{d}{2} > 0$. Let $M = \{u \in \ell^2 : \|u\| \leq e\}$ and $Gu = (d - \|u\|)u$. Obviously, the solution set $VI(M, G) = \{0\}$. Now, we show that G is L-Lipschitz continuous on H and pseudo-monotone on M. Indeed, for any $u, v \in H$, we have*

$$\|Gu - Gv\| = \|(d - \|u\|)u - (d - \|v\|)v\|$$

$$
\begin{aligned}
&= \|d(u-v) - \|u\|(u-v) - (\|u\| - \|v\|)v\| \\
&\le d\|u-v\| + \|u\|\|u-v\| + \|\|u\| - \|v\|\|\|v\| \\
&\le d\|u-v\| + e\|u-v\| + \|u-v\|e \\
&= (d+2e)\|u-v\|.
\end{aligned}
$$

Hence, G is Lipschitz continuous with L $= d + 2e$*. Now, let* $u, v \in M$ *be such that* $\langle Gu, v-u\rangle > 0$*, then we have* $(d - \|u\|)\langle u, v-u\rangle > 0$*. Since* $\|u\| \le e \le d$*, we have* $\langle u, v-u\rangle > 0$*. Hence,*

$$\langle Mv, v-u\rangle = (d - \|v\|)\langle v, v-u\rangle \ge (d - \|v\|)(\langle v, v-u\rangle - \langle u, v-u\rangle \ge (d-e)\|u-v\|^2 \ge 0.$$

This shows that G is a pseudo-monotone mapping. If we set $e = 3$ *and* $d = 5$*, the projection formula is defined by*

$$P_M = \begin{cases} u, & \text{if } \|u\| \le 3, \\ \frac{3u}{\|u\|}, & \text{otherwise}. \end{cases} \tag{4.1}$$

Now, let $Su = \frac{u}{2}$*. It is not hard to show that the mapping S is nonexpansive (hence, quasi-nonexpansive). We see that* $F(S) = \{0\} \ne \emptyset$*. Thus,* $F(S) \cap VI(M, G)$*. We take the stopping criterion as* $\|u_{m+1} - u_m\| \le 10^{-8}$*. Furthermore more, we maintain the same control parameters as in Example 4.1. Since we cannot sum to infinity in MATLAB, we considered the subspace of* ℓ_0^2 *consisting of finite nonzero terms defined by*

$$\ell_0^2(\mathbb{R}) = \{u_1 \in \ell^2 : u_1 = (u_{1,1}, u_{1,2}, u_{1,3}, \ldots, u_{1,i}, 0, 0, \ldots)\}, \quad \text{for some } i \ge 1.$$

The first i points of the initial points are generated randomly considering the following cases for i: Case I: $i = 100$*, Case II:* $i = 1{,}000$*, Case III:* $i = 10{,}000$*, Case IV:* $i = 100{,}000$*. We use the same control parameters used in the previous example for all the algorithms. The results of the numerical simulations are presented in Table 2 and Figures 3 and 4.*

Remark 4.1. *After conducting numerical simulations in Examples 4.1 and 4.2 our proposed Algorithms 3.1–3.3 have exhibited a competitive nature and potential when compared to existing algorithms. They outperformed Algorithms 1 and 2 of Thong and Hieu [43], Algorithm 2 of Tian and Tong [47], Algorithm 3.1 of Ogwo et al. [33], Algorithm 3.1 of Godwin et al. [14], and Algorithm 3.1 of Maluleka et al. [24] in terms of computational time and the number of iterations required to meet the specified stopping criteria, highlighting their superior performance.*

Table 2. Numerical results for the four dimensions considered in Example 4.2.

Algorithms	Case I		Case II		Case III		Case IV	
	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
OUANC Alg. 3.1	13	0.0024	16	0.0042	17	0.0309	17	0.1011
OUANC Alg. 3.7	16	0.0067	17	0.0083	18	0.0220	19	0.1094
OUANC Alg. 3.8	16	0.0089	16	0.0081	17	0.0273	20	0.1105
TH Alg. 1	37	0.0065	35	0.0286	40	0.1310	45	1.1786
TH Alg. 2	34	1.0409	35	0.0190	37	0.1328	38	1.1063
TT Alg. 2	36	0.0131	37	0.0101	38	0.0256	46	0.1978
GAMY Alg. 3.1	67	0.0089	65	0.0081	69	0.0545	73	0.3740
MUA Alg. 3.1	44	0.0083	42	0.0063	45	0.0467	47	0.2787
OAM Alg. 3.1	33	0.0039	34	0.0128	37	0.0299	39	0.1892

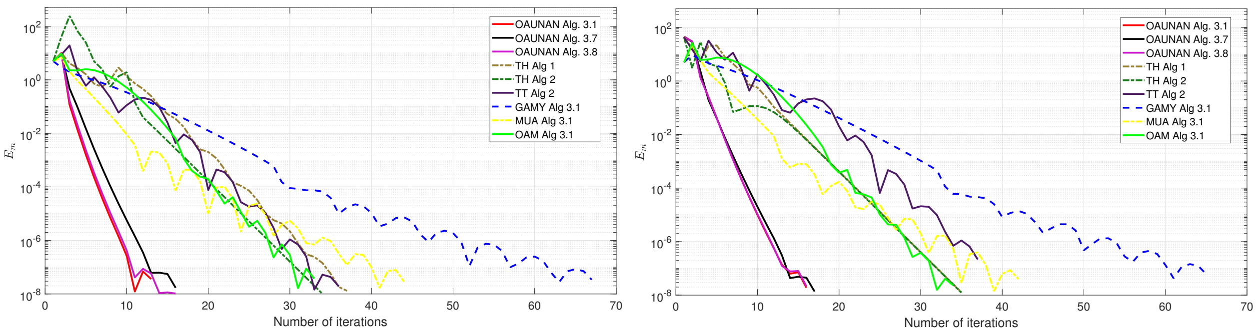

Figure 3. Graph the Iterates for Cases I and II.

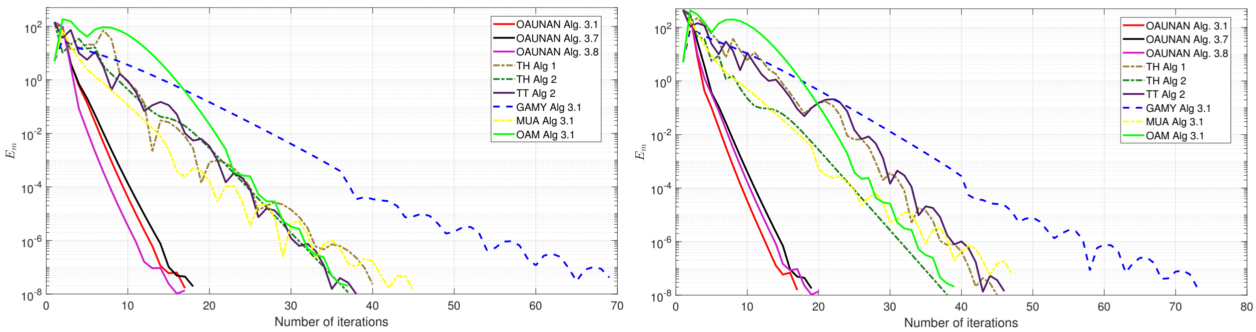

Figure 4. Graph the Iterates for Cases III and IV.

5. Application to optimal control problems

In this section, the solution of variational inequality problem arising from optimal control problem is approximated by our Algorithm 3.1. Let $0 < T \in \mathbb{R}$, then we denote the Hilbert space of the square integrable by $L_2([0,1], \mathbb{R}^k)$, measurable vector function $s : [0, T] \to \mathbb{R}^m$ induced with the inner product

$$\langle s, r\rangle = \int_0^T \langle s(g), r(g)\rangle dg,$$

and norm

$$\|s\|_2 = \sqrt{\langle s, s\rangle} < \infty.$$

Now, the following optimal control problem will be considered on [0,T]:

$$s^*(g) = argmin\{\zeta(s) : s \in S\}, \tag{5.1}$$

supposing such control exists. Note that S denotes the set of admissible controls, which takes the form an k-dimensional box and is made up of a piecewise continuous function:

$$S = \{s(g) \in L_2([0,1], \mathbb{R}^k) : s_i(g) \in [s_i^-, s_i^+],\ i = 1, 2, ..., k\}.$$

Particularly, the control can be piecewise constant function (bang-bang).

The terminal objective can be expressed as:

$$\zeta(s) = \theta(u(T)),$$

where θ is a differentiable and convex function defined on the attainability set. If the trajectory $u(z) \in L_2([0,1])$ fulfills constrains in the form of a linear differential equation system:

$$\dot{u}(g) = D(z)u(g) + B(g)s(g),\ u(0) = u_0,\ z \in [0, T], \tag{5.2}$$

where $D(g) \in \mathbb{R}^{m\times m}$ and $B(g) \in \mathbb{R}^{m\times k}$ are matrices which are continuous for all $z \in [0, T]$. Using the Pontryagin maximum principle, we know that a function $x^* \in L_2([0,1])$ exists with the triple (u^*, x^*, s^*) solving the following system for a.e. $z \in [0, T]$:

$$\begin{cases} \dot{u^*}(g) = D(g)u^*(z) + B(g)s^*(z), \\ u^*(0) = u_0, \end{cases} \tag{5.3}$$

$$\begin{cases} \dot{x^*}(g) = -D(g)^T x^*(z), \\ x^*(0) = \nabla\zeta(u(T)), \end{cases} \tag{5.4}$$

$$0 \in B(g)^T x^*(g) + N_S(s^*(g)), \tag{5.5}$$

where $N_S(s)$ is the normal cone to S at s defined by

$$N_S(s) = \begin{cases} \emptyset, & \text{if } s \notin S, \\ \{\ell \in H : \langle \ell, r - s\rangle \leq 0\ \forall\, s \in S\}, & \text{if } s \in S. \end{cases} \tag{5.6}$$

Letting $Fs(g) = B(z)^T x(g)$, where Fs is shown by Khoroshilova [20] to be the gradient of objective cost function ζ. The express (5.4) can be expressed as a variational inequality problem as follows:

$$\langle Fs^*, r - s^*\rangle \geq 0,\ \ \forall\, r \in S. \tag{5.7}$$

Next, we discretize the continuous function and also take a natural number N with the mesh size $h = \frac{T}{N}$. Furthermore, we identify any discretized control $s^N = (s_0, s_1, \cdots, s_N)$ with its piecewise constant extension:

$$s^N(g) = s_j, \quad \forall\, g \in [g_j, g_{j+1}), \;\; j = 0, 1, \cdots, N-1.$$

Again, any discretized state $u^N = (u_0, u_1, \cdots, u_N)$ is identified with its piecewise linear interpolation

$$u^N(g) = u_j + \frac{g - g_j}{h}(u_{j+1} - u_j), \; g \in [g_j, g_{j+1}), \;\; j = 0, 1, \cdots, N-1. \tag{5.8}$$

The same approach can be used to identify the co-state variable $x^N = (x_0, x_1, \cdots, x_N)$.

The system of ordinary differential equations (ODEs) (5.3) and (5.4) will be solved by the Euler method [49]

$$\begin{cases} u_{j+1}^N = u_j^N + h[D(g_i)u_j^N + B(g_j)s_j^N], \\ u(0) = 0, \end{cases} \tag{5.9}$$

$$\begin{cases} x_i^N = x_{j+1}^N + hD(g_i)^T x_{j+1}^N, \\ x(N) = \nabla\theta(u(N)). \end{cases} \tag{5.10}$$

Next, we solve use Algorithm 3.1 to solve the problem in the following example:

Example 5.1. *(see [4])*

$$\begin{aligned} \textit{minimize} \;\; & -u_1(2) + (u_2(2))^2, \\ \textit{subject to} \;\; & \dot{u}_1(g) = u_2(g), \\ & \dot{u}_2(g) = x(g), \quad \forall g \in [0, 2], \\ & \dot{u}_1(0) = 0 \;\; \dot{u}_2(0) = 0, \\ & s(g) \in [-1, 1]. \end{aligned}$$

The exact solution of the problem in Example 5.1 is

$$s^* = \begin{cases} 1, & \textit{if } g \in [0, 1.2), \\ -1, & \textit{if } g \in [1.2, 2]. \end{cases}$$

The initial controls $s_0(t) = s_1(t)$ are randomly taken in [-1,1]. For this, we use the same parameters defined in Example 4.1 and set $Su = \frac{u}{2}$. The stopping criterion for this section is $\|u_{m+1} - u_m\| \leq 10^{-7}$. The approximate optimal control and the corresponding trajectories of Algorithm 3.1 are shown in Figure 5.

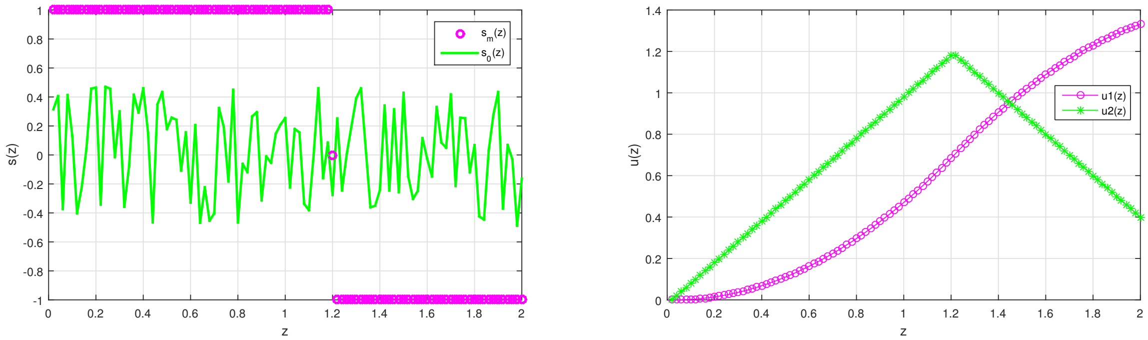

Figure 5. Random initial control (green) and optimal control (purple) on the left and optimal trajectories on the right for Example 5.1 generated by Algorithm 3.1.

6. Application to restoration problem

It is noticed that images are, in most cases distorted by the process of acquisition. The purpose of the restoration technique for distorted images is to restore the original image from the noisy observation of it. The image restoration problem can be modeled as the following undetermined system of the linear equation:

$$v = Fu + w, \tag{6.1}$$

where $F : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear operator, $u \in \mathbb{R}^N$ is an original image and $v \in \mathbb{R}^M$ is the observed image with noise w. It is well-known that the solution of the model (6.1) is equivalent the solution of the (LASSO) problem as follows [39]:

$$\min_{u \in \mathbb{R}^N} \{k\|u\|_1 + \frac{1}{2}\|v - Fu\|_2^2\}, \tag{6.2}$$

where $k > 0$. It is worthy to know that according [40], one can reconstruct the LASSO problem (6.2) as a variational inequality problem by letting $Gu = F^T(Fu - v)$. For this, G is monotone (hence G is pseudomonotone) and Lipschitz continuous with $L = \|F^T F\|$.

Now, we compare the restoration efficiency of our suggested Algorithms 3.1–3.3 (shortly, OAUAN Algs. 3.1, 3.7 and 3.8) with Algorithms 1 and 2 of Thong and Hieu [43] (shortly, TH Alg. 1 and TH Alg. 2), and Algorithm 2 of Tian and Tong [47] (shortly, TT Alg. 2), Algorithm 3.1 of Ogwo et al. [33] (shortly, OAM Alg. 3.1), Algorithm 3.1 of Godwin et al. [14] (shortly, GAMY Alg. 3.1), and Algorithm 3.1 of Maluleka et al. [24], (shortly, MUA Alg. 3.1). The test images are *Austine* and *Peacock* of sizes 289×350 and 245×245, respectively. The images went through a Gaussian blur of size 9×9 and standard deviation of $\sigma = 4$. The performances of the algorithms are measured via signal-to-noise ratio (SNR) defined by

$$SNR = 25 \log_{10} \left(\frac{\|u\|_2}{\|u - u^*\|_2} \right), \tag{6.3}$$

where u^* is the restored image and u is the original image. In this experiment, we maintain the same parameters used for all the algorithms in Example 4.1 with stopping criterion $E_m = \|u_{m+1} - u_m\| \le 10^{-5}$. The numerical results for this experiment are shown in Figures 6–9 and Tables 3–6.

It is well-known that the higher the SNR value of an algorithm, the better the quality of the image it restores. From Figures 6–9 and Tables 3–6, it is evident that our Algorithms 3.1–3.3 restored the blurred images better than Algorithms 1 and 2 of Thong and Hieu [43], and Algorithm 2 of Tian and Tong [47], Algorithm 3.1 of Ogwo et al. [33], Algorithm 3.1 of Godwin et al. [14], and Algorithm 3.1 of Maluleka et al. [24]. Hence, our algorithms are more effective and applicable than many existing methods.

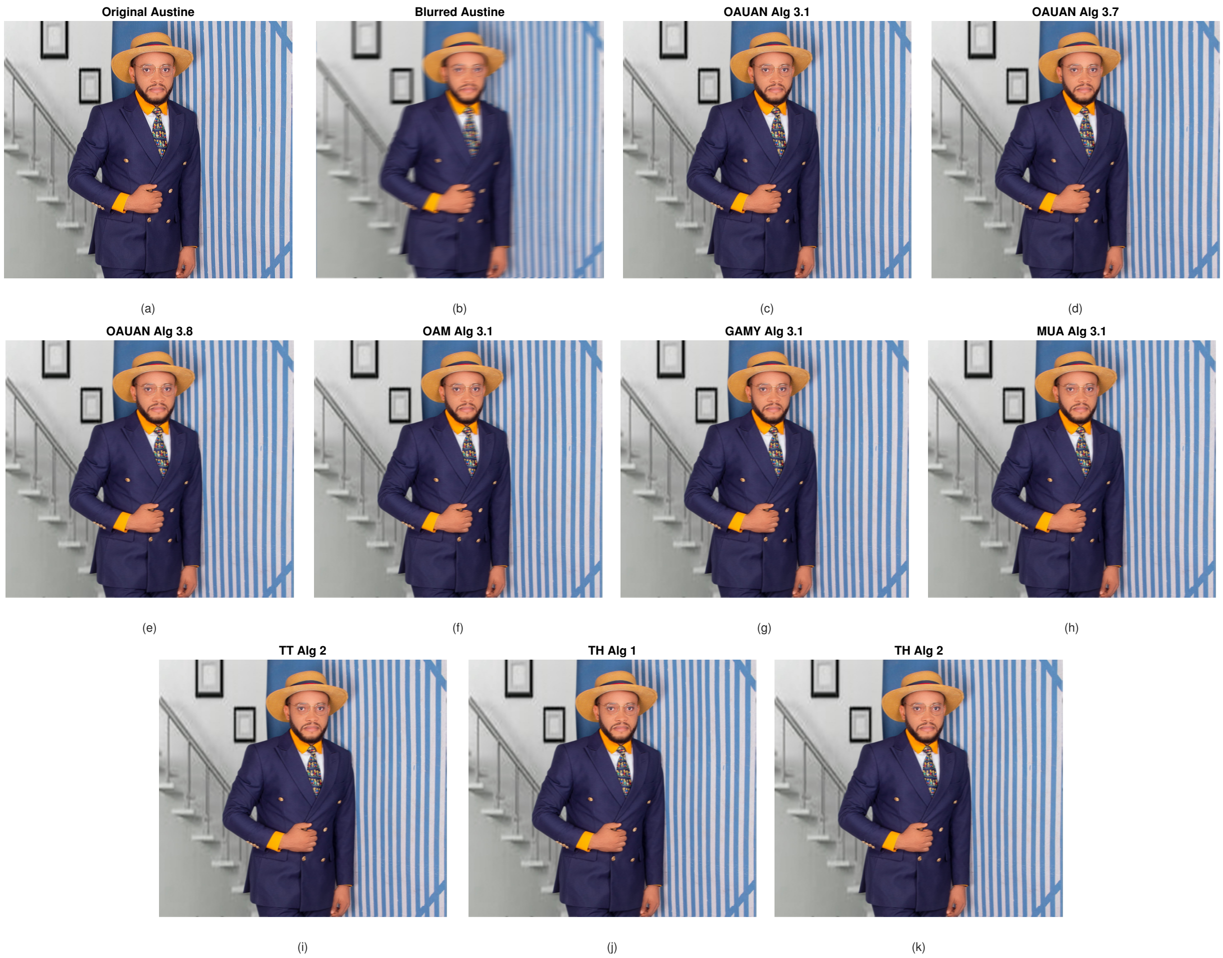

Figure 6. Austine's image deblurring by various algorithms.

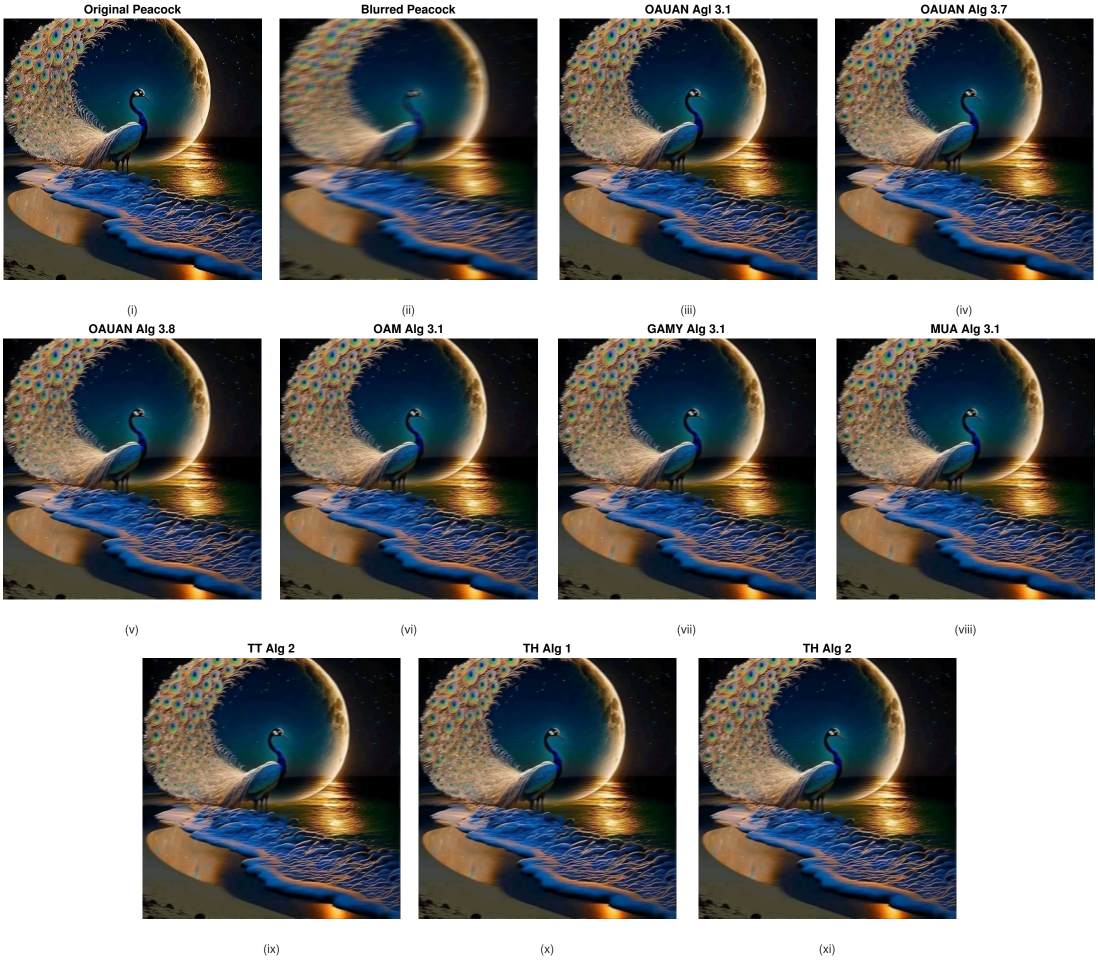

Figure 7. Peacock's image deblurring by various algorithms.

Table 3. Numerical comparison of various algorithms using their SNR values for Austine's image.

Images	m	OAUAN Alg. 3.1	OAUAN Alg. 3.7	OAUAN Alg. 3.8	OAM Alg 3.1	GAMY Alg. 3.1
Austine.png		SNR	SNR	SNR	SNR	SNR
(285 × 350)	50	54.18938	40.5451	33.1598	28.1770	26.6383
	100	54.2745	40.7152	34.2100	28.8195	26.6932
	150	55.3164	41.3918	34.8141	29.5183	27.7202
	200	55.3532	41.17770	34.5151	29.9243	27.7442

Table 4. Numerical comparison of various algorithms using their SNR values for Austine's image.

Images	m	MUA Alg. 3.1	TT Alg. 2	TH Alg. 1	TH Alg. 2
Austine.png		SNR	SNR	SNR	SRN
(285 × 350)	50	26.6726	21.18938	21.5451	13.1598
	100	26.6726	25.2745	21.7152	13.2100
	150	26.8450	25.3164	21.3918	13.8141
	200	26.9953	25.3532	21.1777	13.5151

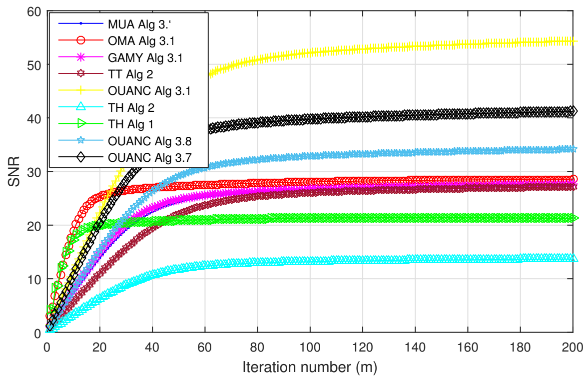

Figure 8. Graph corresponding to Tables 3 and 4.

Table 5. Numerical comparison of various algorithms using their SNR values for Peacock's image.

Images	m	OAUAN Alg. 3.1	OAUAN Alg. 3.7	OAUAN Alg. 3.8	OAM Alg. 3.1	GAMY Alg. 3.1
Peacock.png		SNR	SNR	SNR	SNR	SNR
(285 × 350)	40	53.17939	40.6452	33.2599	28.2771	26.7384
	80	54.3746	40.8153	34.3101	28.9196	26.7933
	120	55.4165	41.4919	34.9142	29.6184	27.8203
	150	55.4533	41.27771	34.6152	29.9244	27.8443

Table 6. Numerical comparison of various algorithms using their SNR values for Peacock's image.

Images	m	MUA Alg. 3.1	TT Alg. 2	TH Alg. 1	TH Alg. 2
Peacock.png		SNR	SNR	SNR	SNR
(285 × 350)	40	26.7727	21.28939	21.6452	13.2599
	80	26.8727	25.3746	21.8153	13.3101
	120	26.9451	25.4165	21.4919	13.9142
	150	26.9955	25.4533	21.2778	13.6152

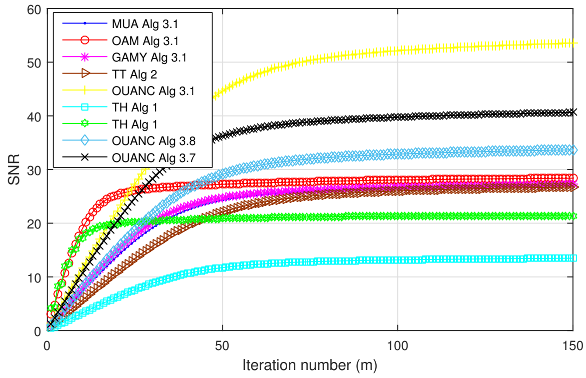

Figure 9. Graph corresponding to Tables 5 and 6.

7. Conclusions

In this work, we have introduced three novel iterative algorithms for finding the common solution of quasi-nonexpansive FPP and pseudo-monotone variational inequality problems. Our algorithms embed double inertial steps which accelerate their convergence rates. Numerical experiments have shown that our algorithms outperformed several existing algorithms with single or no inertial terms. Further, we a considered a new self-adaptive step size technique that produces a non-monotonic sequence of step sizes while also correctly incorporating a number of well-known step sizes. The step size is designed to lessen the algorithms' reliance on the initial step size. Numerical tests were performed, and the results showed that our step size is more effective and that it guarantees that our methods require less execution time. Our convergence results were obtained without the imposition of stringent conditions on the control parameters. The class of pseudo-monotone operators, which has been studied in the work, is more general than the class of monotone operators which has been studied in [43, 47] and several other articles. To test the applicability and efficiencies of our methods in solving real-world problems, we utilized the methods to solve optimal control and image restorations problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this work through the projection number (PSAU/2023/01/8980).

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. J. Abuchu, A. Ofem, G. Ugwunnadi, O. Narain, A. Hussain, Hybrid alternated inertial projection and contraction algorithm for solving bilevel variational inequality problems, *J. Math.*, **2023** (2023), 3185746. http://dx.doi.org/10.1155/2023/3185746
2. A. Adamu, A. Adam, Approximation of solutions of split equality fixed point problems with applications, *Carpathian J. Math.*, **37** (2021), 381–392. http://dx.doi.org/10.37193/CJM.2021.03.02
3. H. Bauschke, P. Combettes, A weak-to-strong convergence principle for Fejer-monotone methods in Hilbert spaces, *Math. Oper. Res.*, **26** (2001), 248–264. http://dx.doi.org/10.1287/moor.26.2.248.10558
4. A. Bressan, B. Piccoli, *Introduction to the mathematical theory of control*, San Francisco: American Institute of Mathematical Sciences, 2007.
5. G. Cai, Y. Shehu, O. Iyiola, Inertial Tseng's extragradient method for solving variational inequality problems of pseudo-monotone and non-Lipschitz operators, *J. Ind. Manag. Optim.*, **18** (2022), 2873–2902. http://dx.doi.org/10.3934/jimo.2021095
6. Y. Censor, A. Gibali, S. Reich, The subgradient extragradient method for solving variational inequalities Hilbert space, *J. Optim. Theory Appl.*, **148** (2011), 318–335. http://dx.doi.org/10.1007/s10957-010-9757-3
7. Y. Censor, A. Gibali, S. Reich, Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space, *Optim. Method. Softw.*, **26** (2011), 827–845. http://dx.doi.org/10.1080/10556788.2010.551536
8. C. Chidume, A. Adamu, L. Okereke, Strong convergence theorem for some nonexpansive-type mappings in certain Banach spaces, *Thai J. Math.*, **18** (2020), 1537–1548.
9. C. Chidume, A. Adamu, Solving split equality fixed point problem for quasi-phi-nonexpansive mappings, *Thai J. Math.*, **19** (2021), 1699–1717.
10. P. Cholamjiak, D. Hieu, Y. Cho, Relaxed forward-backward splitting methods for solving variation inclusions and aplications, *J. Sci. Comput.*, **88** (2021), 85. http://dx.doi.org/10.1007/s10915-021-01608-7
11. Y. Censor, A. Gibali, S. Reich, Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space, *Optimization*, **61** (2012), 1119–1132. http://dx.doi.org/10.1080/02331934.2010.539689
12. G. Fichera, Sul problema elastostatico di signorini con ambigue condizioni al contorno, *Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat*, **34** (1963), 138–142.
13. A. Gibali, D. Thong, Tseng type-methods for solving inclusion problems and its applications, *Calcolo*, **55** (2018), 49. http://dx.doi.org/10.1007/s10092-018-0292-1
14. E. Godwin, T. Alakoya, O. Mewomo, J. Yao, Relaxed inertial Tseng extragradient method for variational inequality and fixed point problems, *Appl. Anal.*, **102** (2023), 4253–4278. http://dx.doi.org/10.1080/00036811.2022.2107913
15. K. Goebel, S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, New York: Marcel Dekker, 1983.
16. B. He, A class of projection and contraction methods for monotone variational inequalities, *Appl. Math. Optim.*, **35** (1997), 69–76. http://dx.doi.org/10.1007/BF02683320

17. C. Izuchukwu, S. Reich, Y. Shehu, A. Taiwo, Strong convergence of forward-reflected-backward splitting methods for solving monotone inclusions with applications to image restoration and optimal control, *J. Sci. Comput.*, **94** (2023), 73. http://dx.doi.org/10.1007/s10915-023-02132-6

18. L. Jolaoso, A. Taiwo, T. Alakoya, O. Mewomo, A self adaptive inertial subgradient extragradient algorithm for variational inequality and common fixed point of multivalued mappings in Hilbert spaces, *Demonstr. Math.*, **52** (2019), 183–203. http://dx.doi.org/10.1515/dema-2019-0013

19. I. Karahan, L. Jolaso, Athree steps iterative process for approximating the fixed points of multivalued generalized α-nonexpansive mappings in uniformly convex hyperbolic spaces, *Sigma J. Eng. Nat. Sci.*, **38** (2020), 1031–1050.

20. E. Khoroshilova, Extragradient-type method for optimal control problem with linear constraints and convex ojective function, *Optim. Lett.*, **7** (2013), 1193–1214. http://dx.doi.org/10.1007/s11590-012-0496-2

21. G. Korpelevich, The extragradient method for finding saddle points and other problems, *Matecon*, **12** (1976), 747–756.

22. W. Kumam, H. Rehman, P. Kumam, A new class of computationally efficient algorithms for solving fixed-point problems and variational inequalities in real Hilbert spaces, *J. Inequal. Appl.*, **2023** (2023), 48. http://dx.doi.org/10.1186/s13660-023-02948-8

23. H. Liu, J. Yang, Weak convergence of iterative methods for solving quasimonotone variational inequalities, *Comput. Optim. Appl.*, **77** (2020), 491–508. http://dx.doi.org/10.1007/s10589-020-00217-8

24. R. Maluleka, G. Ugwunnadi, M. Aphane, Inertial subgradient extragradient with projection method for solving variational inequality and fixed point problems, *AIMS Mathematics*, **8** (2023), 30102–30119. http://dx.doi.org/10.3934/math.20231539

25. K. Muangchoo, A. Adamu, A. Ibrahim, A. Abubakar, An inertial Halpern-type algorithm involving monotone operators on real Banach spaces with application to image recovery problems, *Comp. Appl. Math.*, **41** (2022), 364. http://dx.doi.org/10.1007/s40314-022-02064-1

26. A. Ofem, D. Igbokwe, A new faster four step iterative algorithm for Suzuki generalized nonexpansive mappings with an application, *Advances in the Theory of Nonlinear Analysis and its Applications*, **5** (2021), 482–506. http://dx.doi.org/10.31197/atnaa.869046

27. A. Ofem, H. Isik, G. Ugwunnadi, R. George, O. Narain, Approximating the solution of a nonlinear delay integral equation by an efficient iterative algorithm in hyperbolic spaces, *AIMS Mathematics*, **8** (2023), 14919–14950. http://dx.doi.org/10.3934/math.2023762

28. A. Ofem, J. Abuchu, R. George, G. Ugwunnadi, O. Narain, Some new results on convergence, weak w^2–stability and data dependence of two multivalued almost contractive mappings in hyperbolic spaces, *Mathematics*, **10** (2022), 3720. http://dx.doi.org/10.3390/math10203720

29. A. Ofem, U. Udofia, D. Igbokwe, A robust iterative approach for solving nonlinear Volterra delay integro-differential equations, *Ural Mathematical Journal*, **7** (2021), 59–85. http://dx.doi.org/10.15826/umj.2021.2.005

30. A. Ofem, A. Mebawondu, G. Ugwunnadi, H. Işık, O. Narain, A modified subgradient extragradient algorithm-type for solving quasimonotone variational inequality problems with applications, *J. Inequal. Appl.*, **2023** (2023), 73. http://dx.doi.org/10.1186/s13660-023-02981-7

31. A. Ofem, A. Mebawondu, G. Ugwunnadi, P. Cholamjiak, O. Narain, Relaxed Tseng splitting method with double inertial steps for solving monotone inclusions and fixed point problems, *Numer. Algor.*, in press. http://dx.doi.org/10.1007/s11075-023-01674-y
32. A. Ofem, A. Mebawondu, C. Agbonkhese, G. Ugwunnadi, O. Narain, Alternated inertial relaxed Tseng method for solving fixed point and quasi-monotone variational inequality problems, *Nonlinear Functional Analysis and Applications*, **29** (2024), 131–164. http://dx.doi.org/10.22771/nfaa.2024.29.01.10
33. G. Ogwo, T. Alakoya, O. Mewomo, Iterative algorithm with self-adaptive step size for approximating the common solution of variational inequality and fixed point problems, *Optimization*, **72** (2023), 677–711. http://dx.doi.org/10.1080/02331934.2021.1981897
34. G. Okeke, A. Ofem, A novel iterative scheme for solving delay differential equations and nonlinear integral equations in Banach spaces, *Math. Method. Appl. Sci.*, **45** (2022), 5111–5134. http://dx.doi.org/10.1002/mma.8095
35. G. Okeke, A. Ofem, T. Abdeljawad, M. Alqudah, A. Khan, A solution of a nonlinear Volterra integral equation with delay via a faster iteration method, *AIMS Mathematics*, **8** (2023), 102–124. http://dx.doi.org/10.3934/math.2023005
36. R. Pant, R. Pandey, Existence and convergence results for a class of non-expansive type mappings in hyperbolic spaces, *Appl. Gen. Topol.*, **20** (2019), 281–295. http://dx.doi.org/10.4995/agt.2019.11057
37. B. Polyak, Some methods of speeding up the convergence of iteration methods, *USSR Comput. Math. Math. Phys.*, **4** (1964), 1–17. http://dx.doi.org/10.1016/0041-5553(64)90137-5
38. G. Stampacchia, Formes bilinearies coercitives sur les ensembles convexes, *C. R. Math. Acad. Sci. Paris*, **258** (1964), 4413.
39. Y. Shehu, P. Vuong, P. Cholamjiak, A self-adaptive projection method with an inertial technique for split feasibility problems in Banach spaces with applications to image restoration problems, *J. Fixed Point Theory Appl.*, **21** (2019), 50. http://dx.doi.org/10.1007/s11784-019-0684-0
40. Y. Shehu, O. Iyiola, F. Ogbuisi, Iterative method with inertial terms for nonexpansive mappings, Applications to compressed sensing, *Numer. Algor.*, **83** (2020), 1321–1347. http://dx.doi.org/10.1007/s11075-019-00727-5
41. B. Tan, X. Qin, J. Yao, Strong convergence of inertial projection and contraction methods for pseudomonotone variational inequalities with applications to optimal control problems, *J. Glob. Optim.*, **82** (2022), 523–557. http://dx.doi.org/10.1007/s10898-021-01095-y
42. D. Thong, D. Hieu, Some extragradient-viscosity algorithms for solving variational inequality problems and fixed point problems, *Numer. Algor.*, **82** (2019), 761–789. http://dx.doi.org/10.1007/s11075-018-0626-8
43. D. Thong, D. Hieu, Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems, *Numer. Algor.*, **80** (2019), 1283–1307. http://dx.doi.org/10.1007/s11075-018-0527-x
44. D. Thong, P. Anh, V. Dung, D. Linh, A novel method for finding minimum-norm solutions to pseudomonotone variational inequalities, *Netw. Spat. Econ.*, **23** (2023), 39–64. http://dx.doi.org/10.1007/s11067-022-09569-6

45. D. Thong, D. Hieu, T. Rassias, Self adaptive inertial subgradient extragradient algorithms for solving pseudomonotone variational inequality problems, *Optim. Lett.*, **14** (2020), 115–144. http://dx.doi.org/10.1007/s11590-019-01511-z
46. D. Thong, V. Dung, A relaxed inertial factor of the modified subgradient extragradient method for solving pseudo monotone variational inequalities in hilbert spaces, *Acta Math. Sci.*, **43** (2023), 184–204. http://dx.doi.org/10.1007/s10473-023-0112-9
47. M. Tian, M. Tong, Self-adaptive subgradient extragradient method with inertial modification for solving monotone variational inequality problems and quasinonexpansive fixed point problems, *J. Inequal. Appl.*, **2019** (2019), 7. http://dx.doi.org/10.1186/s13660-019-1958-1
48. P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, *SIAM J. Control Optim.*, **38** (2000), 431–446. http://dx.doi.org/10.1137/S0363012998338806
49. P. Vuong, Y. Shehu, Convergence of an extragradient-type method for variational inequality with applications to optimal control problems, *Numer. Algor.*, **81** (2019), 269–291. http://dx.doi.org/10.1007/s11075-018-0547-6
50. Z. Wang, Z. Lei, X. Long, Z. Chen, Tseng splitting method with double inertial steps for solving monotone inclusion problems, arXiv: 2209.11989.
51. Z. Wang, P. Sunthrayuth, A. Adamu, P. Cholamjiak, Modified accelerated Bregman projection methods for solving quasi-monotone variational inequalities, *Optimization*, in press, http://dx.doi.org/10.1080/02331934.2023.2187663
52. H. Xu, Iterative algorithms for nonlinear operators, *J. Lond. Math. Soc.*, **66** (2002), 240–256. http://dx.doi.org/10.1112/S0024610702003332
53. J. Yang, H. Liu, Strong convergence result for solving monotone variational inequalities in Hilbert space, *Numer. Algor.*, **80** (2019), 741–752. http://dx.doi.org/10.1007/s11075-018-0504-4
54. J. Yang, H. Liu, A modified projected gradient method for monotone variational inequalities, *J. Optim. Theory Appl.*, **179** (2018), 197–211. http://dx.doi.org/10.1007/s10957-018-1351-0
55. Y. Yao, O. Iyiola, Y. Shehu, Subgradient extragradient method with double inertial steps for variational inequalities, *J. Sci. Comput.*, **90** (2022), 71. http://dx.doi.org/10.1007/s10915-021-01751-1

AIMS Press