Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 12825–12851.
DOI: 10.3934/math.2024627
Received: 13 January 2024
Revised: 11 March 2024
Accepted: 19 March 2024
Published: 03 April 2024

Research article

Existence and stability results of nonlinear swelling equations with logarithmic source terms

Mohammad Kafini[1,2,*], **Mohammad M. Al-Gharabli**[1,2] **and Adel M. Al-Mahdi**[1,2]

[1] Department of Mathematics, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia

[2] The Interdisciplinary Research Center in Construction and Building Materials, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

* **Correspondence:** Email: mkafini@kfupm.edu.sa.

Abstract: We considered a swelling porous-elastic system characterized by two nonlinear variable exponent damping and logarithmic source terms. Employing the Faedo-Galerkin method, we established the local existence of weak solutions under suitable assumptions on the variable exponents functions. Furthermore, we proved the global existence utilizing the well-depth method. Finally, we established several decay results by employing the multiplier method and the Logarithmic Sobolev inequality. To the best of our knowledge, this represents the first study addressing swelling systems with logarithmic source terms.

Keywords: swelling system; Faedo-Galerkin method; well-depth method; logarithmic Sobolev inequality; variable exponents; general decay
Mathematics Subject Classification: 35B40, 93D20

1. Introduction

Swelling soils are a significant environmental issue that has garnered considerable attention from many researchers due to their potential to cause structural damage or destruction. These soils show a tendency to swell in volume when exposed to moisture, primarily due to the presence of clay minerals that naturally attract and absorb water molecules. Upon introducing water to swelling soils, the molecules are drawn into gaps between the soil plates. As the amount of absorbed water increases, the plates are forced further apart, leading to an increase in soil pore pressure. Consequently, swelling soils pose substantial geotechnical and structural challenges to the environment and society. Swelling soils are prevalent worldwide, and recent estimates from the American Society of Civil Engineers suggests that one in four homes experience some form of damage caused by swelling soils. Typically, the financial losses incurred by property owners due to these soils exceed those caused by earthquakes, floods, hurricanes, and tornadoes combined. Therefore, it is important to explore practical methods for eliminating or minimizing the damages caused by swelling soils. Therefore, studying of the asymptotic behavior of swelling porous elastic soils is important for architecture and civil engineering. For more information in the continuum theory of material, we refer the reader to [1], [2], and [3].

In this paper, we consider the following nonlinear swelling soil system with nonlinear source terms of logarithmic-type:

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + z + \gamma |z_t|^{\nu(\cdot)-2} z_t = \alpha z \ln |z|, & \text{in } \Omega \times (0, \infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + u + \beta |u_t|^{\omega(\cdot)-2} u_t = \alpha u \ln |u|, & \text{in } \Omega \times (0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \quad z(x, 0) = z_0(x), z_t(x, 0) = z_1(x), & x \in \Omega, \\ z(0, t) = z(1, t) = u(0, t) = u(1, t) = 0, & t \geq 0\ , \end{cases} \tag{1.1}$$

where the constituents z and u represent the displacement of the fluid and the elastic solid material, respectively. The positive constant coefficients ρ_u and ρ_z are the densities of each constituent. The coefficients a_1, a_2, and a_3 are positive constants satisfying specific conditions. z_0, z_1, u_0, u_1 are given data. $\gamma,\beta \geq 0$, α is a small positive constant, and $\nu(.)$ and $\omega(.)$ are the variable functions that are satisfying some specific conditions.

In the present work, our goals are to prove the existence and stability of the system (1.1). We begin by using the Faedo-Galerkin method to prove the local existence of the weak solutions to system (1.1) under suitable assumptions on the variable exponent functions and the logarithmic source terms. We also prove the global existence using the well depth method. Finally, we establish several decay results using the multiplier method and the logarithmic Sobolev inequality.

1.1. Importance and motivations

Model (1.1) describes swelling of soils with external forces given by nonlinear logarithmic functions. We dissipate this model by the frictional damping mechanism acting on the domain. These dampings of variable exponent-type employ variable exponents in this model, and significantly enhance the ability to capture spatial variations in material properties, nonlinearity, anisotropy, and other complex behaviors. This approach can lead to more accurate simulations and predictions, thereby contributing to the stability, optimization, and design of some tools for a variety of engineering applications [4–9].

The righthand sides of the system (1.1) represent nonlinear sources of logarithmic-type, which models an external force that amplifies energy and drives the system to possible instability.

We add the logarithmic source terms because they occur in some phenomena; such phenomena are common in nature such as in inflation cosmology, nuclear physics, geophysics, and optics (see [10–23]).

1.2. The novelty of our results

In the system (1.1), it is evident that the the damping terms and the source terms are the two major players in this model. Their interactions stimulate many interesting phenomena, which deserve careful investigation. To control an object means to influence its behavior so as to achieve a desired goal. In the system (1.1), the intrinsic frictional damping mechanism acting on the system is responsible for dissipation of its energy. The purpose of this line of study is to find conditions on the initial state to control the dissipations that are needed in order to obtain a decay rate of the energy. In other words, the goal is to discover an adequate choice of the controls that can drive the system from a given initial state to a final given state, in a given time.

The study of the interaction of nonlinear damping and source terms was initiated by Georgiev and Todorova [24] in the wave equation. In this line of research, an important breakthrough was made by Bociu and Lasiecka in a series of papers [25] and [26] where they provided a complete study of a wave equation with damping and supercritical sources in the interior and on the boundary of the domain. Indeed, a source term $|u|^{m-1}u$ is called subcritical if $1 \leq m < 3$, critical if $m = 3$, and supercritical if $m > 3$, in three space dimensions.

The novelty of our results can be seen from the following aspects:

(1) The source term in our model (1.1) is logarithmic. Let us note here that though the logarithmic nonlinearity is somehow weaker than polynomial nonlinearity, both the existence and stability result are not obtained by straightforward application of the method used for polynomial nonlinearity. We need to make some extra conditions on the nonlinearity coefficient.

(2) The frictional damping mechanismins are nonstandard. They are of variable exponent-type. Variable exponents in the context of swelling soils are often associated with mathematical models used to represent the relationship between soil moisture content and volume change.

(3) How to control the frictional damping mechanism to stabilize the system because the external forces may lead to instability.

1.3. The originality of the model

The fundamental field equations for the linear theory of swelling porous elastic soils were mathematically presented by Ieşan [27] and later simplified by Quintanilla [28]. These basic equations are given by

$$\begin{cases} \rho_z z_{tt} = \phi_{1x} - \chi_1 + \psi_1, \\ \rho_u u_{tt} = \phi_{2x} + \chi_2 + \psi_2, \end{cases} \tag{1.2}$$

where z and u represent the displacements of the fluid and the elastic solid material, respectively. The coefficients $\rho_z, \rho_u > 0$ and represent the densities of the constituents z and u, respectively. The functions (ϕ_1, χ_1, ψ_1) represent the partial tension, internal body forces, and external forces acting on the displacement, respectively. A similar definition holds for (ϕ_2, χ_2, ψ_2), but acts on the elastic solid. Additionally, the constitutive equations of partial tensions are given by

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}}_{\text{A}} \begin{bmatrix} z_x \\ u_x \end{bmatrix}, \tag{1.3}$$

where $a_1, a_3 > 0$ and $a_2 \neq 0$ is a real number. The coefficient matrix A is positive definite, i.e., $a_1 a_3 > a_2^2$. After that, Quintanilla [28] investigated

$$\begin{cases} \rho_z z_{tt} = a_1 z_{xx} + a_2 u_{xx} - \zeta(z_t - u_t) + a_3 z_{xxt}, \\ \rho_u u_{tt} = a_2 z_{xx} + a_3 u_{xx} + \zeta(z_t - u_t), \end{cases} \tag{1.4}$$

where ζ is a positive constant, and he obtained an exponential stability result. Similarly, Wang and Guo [29] considered

$$\begin{cases}\rho_z z_{tt} = a_1 z_{xx} + a_2 u_{xx} - \rho_z \xi(x) z_t, \\ \rho_u u_{tt} = a_2 z_{xx} + a_3 u_{xx},\end{cases} \tag{1.5}$$

where $\xi(x)$ is an internal viscous damping function with a positive mean. The authors established their exponential stability result by using the spectral method technique. Subsequently, a growing body of new research has explored the stability of system (1.2) by employing various damping mechanisms including viscoelastic damping and/ frictional damping (see, for example [30–33, 33–40]). Recently, Al-Mahdi et al. [41] established exponential and polynomial decay results for the following system with variable exponent nonlinearity

$$\begin{cases}\rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + |z_t|^{m(\cdot)-2} z_t = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} = 0, & \text{in } (0,1)\times(0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad z(x,0) = z_0(x), z_t(x,0) = z_1(x), & x \in [0,1], \\ z(0,t) = z(1,t) = u(0,t) = u(1,t) = 0, & t \geq 0,\end{cases} \tag{1.6}$$

where the constituents z and u represent the displacement of the fluid and the elastic solid material, respectively. The positive constant coefficients ρ_u and ρ_z are the densities of each constituent. The coefficients a_1, a_2, and a_3 are positive constants satisfying specific conditions. z_0, z_1, u_0, u_1 are given data, and $m(.)$ is a variable function that satisfies some specific conditions.

1.4. Comparison results

Here, we compare our problem (1.1) with other problems involving source terms of logarithmic-type and source terms of polynomial-type. Regarding swelling soils, many authors investigated the stability analysis of swelling soils problems with different damping mechanism without external forces (source terms). For example, Al-Mahdi et al. [42] and [43] proved the stability of the swelling soil problem with memory damping terms. Kafini et al. [44] studied the stability of the swelling soils problem with time delay and variable exponents without source terms.

Logarithmic sources terms have been added in the literature for some other models such as plate equations [19], [45], and [46]. For the polynomial source terms, we refer to the works [47], [48], and [49].

We notice that adding source terms does not improve the stability rate decay. In addition, the logarithmic nonlinearity is weaker than the polynomial nonlinearity. However, we include the logarithmic source terms because they occur in some phenomena. Such phenomena are common in nature such as in inflation cosmology, nuclear physics, geophysics, and optics.

2. Preliminaries

In this section, we present some preliminaries necessary for proving the stability results. Throughout the paper, Ω denotes the interval $(0,1)$ and c represents a generic positive constant.
Let $p : \Omega \to [1,\infty]$ be a measurable function. The Lebesgue space with a variable exponent $p(\cdot)$ is defined as:

$$L^{p(\cdot)}(\Omega) := \Big\{ v : \Omega \to \mathbb{R};\ \text{measurable in } \Omega :\ \varrho_{p(\cdot)}(\lambda v) < \infty,\ \text{for some } \lambda > 0 \Big\},$$

where

$$\varrho_{p(\cdot)}(v) = \int_\Omega |v(x)|^{p(x)}dx.$$

Equipped with the following Luxembourg-type norm

$$\|v\|_{p(\cdot)} := \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{v(x)}{\lambda}\Big|^{p(x)} dx < \infty\Big\},$$

the space $L^{p(\cdot)}(\Omega)$ is a Banach space (see [50]), separable if $p(\cdot)$ is bounded and reflexive if $1 < p_1 \leq p_2 < \infty$, where

$$p_1 := \text{essinf}_{x\in\Omega} p(x), \quad p_2 := \text{esssup}_{x\in\Omega} p(x).$$

The variable-exponent Sobolev space is defined as :

$$W^{1,p(\cdot)}(\Omega) = \Big\{v \in L^{p(\cdot)}(\Omega) \text{ such that } v_x \text{ exists and } v_x \in L^{p(\cdot)}(\Omega)\Big\}.$$

This is a Banach space with respect to the norm $\|v\|_{W^{1,p(\cdot)}(\Omega)} = \|v\|_{p(\cdot)} + \|v_x\|_{p(\cdot)}$ and it is separable if $p(\cdot)$ is bounded and reflexive if $1 < p_1 \leq p_2 < \infty$. Furthermore, we set $W_0^{1,p(.)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

The exponent $p(\cdot) : \Omega \to [1, \infty]$ is said to be satisfying for the log-Hölder continuity condition; that is, if there exists a constant $A > 0$ such that, for all δ with $0 < \delta < 1$,

$$|p(x) - p(y)| \leq -\frac{A}{\log|x-y|}, \text{ for all } x, y \in \Omega, \text{ with } |x-y| < \delta. \tag{2.1}$$

Lemma 2.1. *[50] (Poincaré's inequality) Let Ω be a bounded domain of $\mathbb{R}^n$ and $p(\cdot)$ satisfies* (2.2), *then*

$$\|v\|_{p(\cdot)} \leq c_\rho \|v_x\|_{p(\cdot)}, \quad \textit{for all } v \in W_0^{1,p(\cdot)}(\Omega),$$

where the positive constant c_ρ depends on p_1, p_2, and Ω only. In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has an equivalent norm given by $\|v\|_{W_0^{1,p(\cdot)}(\Omega)} = \|v_x\|_{p(\cdot)}$.

Lemma 2.2. *[50] (Embedding property) Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Assume that $p, k \in C(\overline{\Omega})$ such that*

$$1 < p_1 \leq p(x) \leq p_2 < +\infty, \qquad 1 < k_1 \leq k(x) \leq k_2 < +\infty, \quad \forall x \in \overline{\Omega},$$

and $k(x) < p^*(x)$ in $\overline{\Omega}$ with

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & if p_2 < n; \\ +\infty, & if p_2 \geq n, \end{cases}$$

then we have continuous and compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{k(.)}(\Omega)$. So, there exists $c_e > 0$ such that

$$\|v\|_k \leq c_e \|v\|_{W^{1,p(.)}}, \quad \forall v \in W^{1,p(.)}(\Omega).$$

For more details about the Lebesgue and Sobolev spaces with variable exponents, [50–52]. We consider the following hypotheses:

$(A1)$ $\nu, \omega : \overline{\Omega} \to [1, \infty)$ are measurable functions on Ω that satisfy the following conditions

$$2 \le \nu_1 \le \nu(x) \le \nu_2 < \infty, \;\; 2 \le \omega_1 \le \omega(x) \le \omega_2 < \infty,$$

where

$$\nu_1 := \text{essinf}_{x\in\Omega}\nu(x), \; \nu_2 := \text{esssup}_{x\in\Omega}\nu(x), \; \omega_1 := \text{essinf}_{x\in\Omega}\omega(x), \; \omega_2 := \text{esssup}_{x\in\Omega}\omega(x),$$

and they also satisfy the log-Hölder continuity condition; that is, for any λ with $0 < \lambda < 1$, there exists a constant $\delta > 0$ such that,

$$|f(x) - f(y)| \le -\frac{\delta}{\log|x-y|}, \text{ for all } x, y \in \Omega, \text{ with } |x-y| < \lambda. \tag{2.2}$$

$(A2)$ The coefficients of the system a_i, $i = 1, ..., 3$ satisfy $a_1 a_3 - a_2^2 > 0$.
$(A3)$ The constant α in (1.1) satisfies $0 < \alpha < \alpha_0$, where α_0 is the positive real number satisfying

$$\sqrt{\frac{2\pi\tilde{c}}{\alpha_0}} = e^{-\frac{3}{2}-\frac{1}{\alpha_0}}, \tag{2.3}$$

where $\tilde{c}$ is a positive constant appearing in (3.7).

Lemma 2.3. *[14, 53] (Logarithmic Sobolev inequality) Let v be any function in $H_0^1(\Omega)$ and $a > 0$ be any real number, then the following inequality holds:*

$$\int_\Omega v^2 \ln|v| dx \le \frac{1}{2}\|v\|_2^2 \ln\|v\|_2^2 + \frac{a^2}{2\pi}\|v_x\|_2^2 - (1+\ln a)\|v\|_2^2. \tag{2.4}$$

Remark 2.1. The function $f(s) = \sqrt{\frac{2\pi}{s}} - e^{-\frac{3}{2}-\frac{1}{s}}$ is continuous and decreasing on $(0, \infty)$, with

$$\lim_{s\to 0^+} f(s) = \infty \text{ and } \lim_{s\to\infty} f(s) = -e^{-\frac{3}{2}}.$$

Therefore, there exists a unique $\alpha_0 > 0$ such that $f(\alpha_0) = 0$, that is,

$$\sqrt{\frac{2\pi}{\alpha_0}} = e^{-\frac{3}{2}-\frac{1}{\alpha_0}}. \tag{2.5}$$

Moreover,

$$e^{-\frac{3}{2}-\frac{1}{s}} < \sqrt{\frac{2\pi\tilde{c}}{s}}, \;\; \forall s \in (0, \alpha_0). \tag{2.6}$$

Lemma 2.4. *[54] (Logarithmic Gronwall inequality) Let $c > 0$, $u \in L^1(0, T; \mathbb{R}^+)$, and assume that the function $v : [0, T] \to [1, \infty)$ satisfies*

$$v(t) \le c\left(1 + \int_0^t u(s)v(s)\ln v(s) ds\right), \;\; 0 \le t \le T, \tag{2.7}$$

then

$$v(t) \le c \exp\left(c\int_0^t u(s) ds\right), \;\; 0 \le t \le T. \tag{2.8}$$

The energy functional associated with system (1.1) is defined by

$$\begin{aligned} E(t) =&\frac{1}{2}\int_\Omega \left[\rho_z z_t^2 + \rho_u u_t^2 + a_3 u_x^2 + a_1 z_x^2 + 2a_2 z_x u_x\right] dx + \frac{\alpha+2}{4}\left[\|z\|_2^2 + \|u\|_2^2\right] \\ &- \frac{1}{2}\int_\Omega z^2 \ln|z| dx - \frac{1}{2}\int_\Omega u^2 \ln|u| dx. \end{aligned} \tag{2.9}$$

Direct differentiation, using (1.1), gives

$$E'(t) = -\gamma \int_\Omega |z_t|^{\gamma(\cdot)} dx - \beta \int_\Omega |u_t|^{\omega(\cdot)} dx \le 0. \tag{2.10}$$

Remark 2.2. The nonnegativity of the energy functional is obtained by $(A2)$ and the following identity

$$a_3 u_x^2 + a_1 z_x^2 + 2a_2 z_x u_x = \left(a_3 - \frac{a_2^2}{a_1}\right) u_x^2 + \left(\sqrt{a_1} z_x + \frac{a_2}{\sqrt{a_1}} u_x\right)^2. \tag{2.11}$$

Remark 2.3. The following inequality is needed for the proof of our main results:
There exist two positive constants c_0 and d_0 such that

$$c_0\left(A^2 + B^2\right) \le (A+B)^2 \le d_0\left(A^2 + B^2\right),\ A, B \in \mathbb{R},\ \text{such that } A + B \neq 0. \tag{2.12}$$

In fact, c_0 is the largest positive constant, which satisfies $c_0 \le \frac{(A+B)^2}{A^2+B^2}$, and d_0 is the smallest positive constant, which satisfies $d_0 \ge \frac{(A+B)^2}{A^2+B^2}$.

3. Local existence

First, we multiply the first equation in (1.1) by $\phi \in C_0^\infty(\Omega)$ and the second equation by $\psi \in C_0^\infty(\Omega)$, integrate each result over Ω, and use Green's formula and the boundary conditions to obtain the definition of the weak solution. Second, we provide a detailed proof of the local existence theorem by using the Faedo-Galerkin approximations.

Definition 3.1. The pair of functions (z, u) is called a weak solution of (P), if it satisfies the following:

$$\begin{cases} \frac{d}{dt}\int_\Omega \rho_z z_t \phi(x) dx + a_1 \int_\Omega z_x \phi_x(x) dx + a_2 \int_\Omega u_x \phi_x(x) dx, \\ + \int_\Omega z\phi(x) dx + \gamma \int_\Omega |z_t|^{\gamma(.)-2} z_t \phi(x) dx = \alpha \int_\Omega z \ln|z| \phi(x) dx, \\ \frac{d}{dt}\int_\Omega \rho_u u_t \psi(x) dx + a_3 \int_\Omega u_x \psi_x(x) dx + a_2 \int_\Omega z_x \psi_x(x) dx, \\ + \int_\Omega u\psi(x) dx + \beta \int_\Omega |u_t|^{\omega(.)-2} u_t \psi(x) dx = \alpha \int_\Omega u \ln|u| \psi(x) dx, \\ z(0) = z_0,\ z_t(0) = z_1,\ u(0) = u_0,\ u_t(0) = u_1, \end{cases} \tag{3.1}$$

for a.e. $t \in [0, T]$,

$$(z, u) \in L^\infty([0, T), H_0^1(\Omega)), \qquad z_t \in L^\infty([0, T), L^2(\Omega)) \cap L^\gamma(\Omega \times (0, T)),$$

$u_t \in L^\infty([0, T), L^2(\Omega)) \cap L^\omega(\Omega \times (0, T))$, and the test functions $\phi, \psi \in H_0^1(\Omega)$. Note that $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$. Further, the spaces $H_0^1(\Omega) \subset L^{\gamma(.)}(\Omega) \cap L^{\omega(.)}(\Omega)$.

Theorem 3.1. *Assume that* $(A1)-(A3)$ *hold and let* $(z_0, z_1), (u_0, u_1) \in H_0^1(\Omega)\times L^2(\Omega)$, *then problem* (1.1) *has a unique local weak solution* (z, u) *on* $[0, T)$ *in the sense of Definition 3.1.*

Proof. The proof of the existence of a weak solution of (1.1) consists of four steps:
Step 1. Approximate problem: In this step, we consider $\{w_j\}_{j=1}^{\infty}$ an orthogonal basis of $H_0^1(\Omega)$ and define, for all $k \geq 1$, (z^k, u^k) a sequence in the finite - dimensional subspace $(V_k \times V_k)$, where $V_k = span\{w_1, w_2, ..., w_k\}$ as follows:

$$z^k(x,t) = \sum_{j=1}^{k} a_j(t) w_j, \quad u^k(x,t) = \sum_{j=1}^{k} b_j(t) w_j,$$

for all $x \in \Omega$ and $t \in (0, T)$, satisfying the following approximate problem:

$$\begin{cases} \rho_z\langle z_{tt}^k, w_j\rangle_{L^2(\Omega)} + a_1\langle {z_x}^k, w_{j_x}\rangle_{L^2(\Omega)} + a_2\langle u_x^k, w_{j_x}\rangle_{L^2(\Omega)} \\ \ + \langle z^k, w_j\rangle_{L^2(\Omega)} + \gamma\langle |z_t^k|^{\nu(x)-2} z_t^k, w_j\rangle_{L^2(\Omega)} = \langle \alpha z^k \ln|z^k|, w_j\rangle_{L^2(\Omega)}, \quad j = 1, 2, ..., k, \\ \rho_u\langle u_{tt}^k, w_j\rangle_{L^2(\Omega)} + a_3\langle {u_x}^k, w_{j_x}\rangle_{L^2(\Omega)} + a_2\langle z_x^k, w_{j_x}\rangle_{L^2(\Omega)} \\ \ + \langle u^k, w_j\rangle_{L^2(\Omega)} + \beta\langle |u_t^k|^{\omega(x)-2} u_t^k, w_j\rangle_{L^2(\Omega)} = \langle \alpha u^k \ln|u^k|, w_j\rangle_{L^2(\Omega)}, \quad j = 1, 2, ..., k, \\ z^k(0) = z_0^k,\ z_t^k(0) = z_1^k,\ u^k(0) = u_0^k,\ u_t^k(0) = u_1^k, \end{cases} \tag{3.2}$$

where $\langle\ ,\ \rangle$ is the inner product in $L^2(\Omega)$ and

$$z_0^k = \sum_{i=1}^{k}\langle z_0, w_i\rangle w_i, u_0^k = \sum_{i=1}^{k}\langle u_0, w_i\rangle w_i, z_1^k = \sum_{i=1}^{k}\langle z_1, w_i\rangle w_i, u_1^k = \sum_{i=1}^{k}\langle u_1, w_i\rangle w_i,$$

such that

$$\begin{cases} z_0^k \to z_0 \text{ and } u_0^k \to u_0 \text{ in } H_0^1(\Omega), \\ \text{and} \\ z_1^k \to z_1 \text{ and } u_1^k \to u_1 \text{in } L^2(\Omega). \end{cases} \tag{3.3}$$

Based on standard existence theory for integro-differential equations, system (3.2) admits a unique local solution (z^k, u^k) on a maximal time interval $[0, T_k)$, $0 < T_k < T$, for each $k \in \mathbb{N}$.
Step 2. A priori estimates: In this step, we show, by priory estimates, that $T_k = T$ for each $k \in \mathbb{N}$. We multiply the first equation by $a_j'(t)$ and the second equation by $b_j'(t)$ in (3.2), sum over $j = 1, 2, ...k$, and add the two equations to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\rho_z\|z_t^k\|_2^2 + \rho_u\|u_t^k\|_2^2 + a_1\|z_x^k\|_2^2 + a_3\|u_x^k\|_2^2 + 2a_2\int_\Omega u_x^k z_x^k dx\right] \\ &+ \frac{d}{dt}\left[\frac{\alpha+2}{4}\left[\|z^k\|_2^2 + \|u^k\|_2^2\right] - \frac{1}{2}\int_\Omega (z^k)^2\ln|z^k|dx - \frac{1}{2}\int_\Omega (u^k)^2\ln|u^k|dx\right] \\ &= -\gamma\int_\Omega |z_t^k(x,t)|^{\nu(.)}dx - \beta\int_\Omega |u_t^k(x,t)|^{\omega(.)}dx. \end{aligned} \tag{3.4}$$

Integration of (3.4) over $(0,t)$ leads to

$$\begin{aligned}
&\frac{1}{2}\left(\rho_z\|z_t^k\|_2^2+\rho_u\|u_t^k\|_2^2+a_1\|z_x^k\|_2^2+a_3\|u_x^k\|_2^2+2a_2\int_\Omega u_x^k z_x^k dx\right)\\
&\frac{\alpha+2}{4}\left[\|z^k\|_2^2+\|u^k\|_2^2\right]-\frac{1}{2}\int_\Omega (z^k)^2\ln|z^k|dx-\frac{1}{2}\int_\Omega (u^k)^2\ln|u^k|dx\\
&+\gamma\int_0^t\int_\Omega |z_t^k(s)|^{\nu(.)}dxds+\beta\int_0^t\int_\Omega |u_t^k(s)|^{\omega(.)}dxds \qquad (3.5)\\
=&\frac{1}{2}\left(\rho_z\|z_1^k\|_2^2+\rho_u\|u_1^k\|_2^2+\rho_z\|z_{0x}^k\|_2^2+\rho_u\|u_{0x}^k\|_2^2+2a_2\int_\Omega z_{0x}^k u_{0x}^k dx\right)\\
&\frac{1}{2}\int_\Omega (\psi_0^k)^2\ln|z_0^k|dx+\frac{1}{2}\int_\Omega (u_0^k)^2\ln|u_0^k|dx+\frac{\alpha+2}{4}\left(\|z_0^k\|_2^2+\|u_0^k\|_2^2\right),\text{for all } t\le T_k.
\end{aligned}$$

Using (2.11), Young's inequality, and convergence (3.3), we have

$$\begin{aligned}
&\frac{1}{2}\left(\rho_z\|z_t^k\|_2^2+\rho_u\|u_t^k\|_2^2+\left(a_3-\frac{a_2^2}{a_1}\right)\|u_x^k\|_2^2+\int_\Omega\left(\sqrt{a_1}z_x^k+\frac{a_2}{\sqrt{a_1}}u_x^k\right)^2dx\right)\\
&\frac{\alpha+2}{4}\left[\|z\|_2^2+\|u\|_2^2\right]-\frac{1}{2}\int_\Omega (z^k)^2\ln|z^k|dx-\frac{1}{2}\int_\Omega (u^k)^2\ln|u^k|dx \qquad (3.6)\\
&+\gamma\int_0^t\int_\Omega |z_t^k(s)|^{\nu(.)}dxds+\beta\int_0^t\int_\Omega |u_t^k(s)|^{\omega(.)}dxds\le C_0,\ \forall t\le T_k,\ k\ge 1.
\end{aligned}$$

Using (2.12) and applying the logarithmic Sobolev inequality for (3.6), we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\rho_z\|z_t^k\|_2^2+\rho_u\|u_t^k\|_2^2+\left(\tilde{c}-\frac{\alpha a^2}{2\pi}\right)\|z_x^k\|_2^2+\left(a_3-\frac{a_2^2}{a_1}+\tilde{c}-\frac{\alpha a^2}{2\pi}\right)\|u_x^k\|_2^2+2a_2\int_\Omega u_x^k z_x^k dx\right)\\
&\left(\frac{\alpha+2}{2}+\alpha(1+\ln a)\right)\left[\|z\|_2^2+\|u\|_2^2\right]+\gamma\int_0^t\int_\Omega |z_t^k(s)|^{\nu(.)}dxds+\beta\int_0^t\int_\Omega |u_t^k(s)|^{\omega(.)}dxds \qquad (3.7)\\
\le& C_0+\frac{\alpha}{2}\left(\|z^k\|_2^2\ln\|z^k\|_2^2+\|u^k\|_2^2\ln\|u^k\|_2^2\right),\ \forall t\le T_k, k\ge 1,
\end{aligned}$$

where $\tilde{c}=\min\{c_0a_1^2, c_0\frac{a_2^2}{a_1}\}$, $C_0=cE^k(0)$. Now, we select

$$e^{-\frac{3}{2}-\frac{1}{\alpha}}<a<\sqrt{\frac{2\pi\tilde{c}}{\alpha}}, \qquad (3.8)$$

and use $(A2)$ to obtain

$$\tilde{c}-\frac{\alpha a^2}{2\pi}>0,\ a_3-\frac{a_2^2}{a_1}+\tilde{c}-\frac{\alpha a^2}{2\pi}>0 \text{ and } \frac{\alpha+2}{2}+\alpha(1+\ln a)>0. \qquad (3.9)$$

Combining (3.7) and (3.9), we have

$$\begin{aligned}
\|z_t^k\|_2^2+\|u_t^k\|_2^2\le&\|z_t^k\|_2^2+\|u_t^k\|_2^2+\|z_x^k\|_2^2+\|u_x^k\|_2^2+\|z^k\|_2^2+\|u^k\|_2^2+\frac{\gamma}{c}\int_0^t\int_\Omega |z_t^k(s)|^{\nu(.)}dxds\\
&+\frac{\beta}{c}\int_0^t\int_\Omega |u_t^k(s)|^{\omega(.)}dxds\le\frac{C_0}{c}+\frac{\alpha}{2c}\left(\|z^k\|_2^2\ln\|z^k\|_2^2+\|u^k\|_2^2\ln\|u^k\|_2^2\right).
\end{aligned} \qquad (3.10)$$

Hence,

$$\begin{aligned}\|z_t^k\|_2^2+\|u_t^k\|_2^2 &\le \frac{C_0}{c}+\frac{\alpha}{2c}\left(\|z^k\|_2^2\ln\|z^k\|_2^2+\|u^k\|_2^2\ln\|u^k\|_2^2\right)\\ &\le c\left(1+\|z^k\|_2^2\ln\|z^k\|_2^2+\|u^k\|_2^2\ln\|u^k\|_2^2\right).\end{aligned}\tag{3.11}$$

Let us note that

$$z^k(.,t)=z^k(.,0)+\int_0^t\frac{\partial z^k}{\partial s}(.,s)ds,\ \text{and}\ u^k(.,t)=u^k(.,0)+\int_0^t\frac{\partial u^k}{\partial s}(.,s)ds.$$

Thus, applying the Cauchy-Schwarz' inequality, we get

$$\begin{aligned}\|z^k(t)\|_2^2&\le 2\|z^k(0)\|_2^2+2\left\|\int_0^t\frac{\partial z^k}{\partial s}(s)ds\right\|_2^2\le 2\|z^k(0)\|_2^2+2T\int_0^t\|z_t^k(s)\|_2^2ds,\\ \|u^k(t)\|_2^2&\le 2\|u^k(0)\|_2^2+2\left\|\int_0^t\frac{\partial u^k}{\partial s}(s)ds\right\|_2^2\le 2\|u^k(0)\|_2^2+2T\int_0^t\|u_t^k(s)\|_2^2ds.\end{aligned}\tag{3.12}$$

The addition of the two estimates in (3.12) gives

$$\|z^k(t)\|_2^2+\|u^k(t)\|_2^2\le 2\|z^k(0)\|_2^2+2\|u^k(0)\|_2^2+2T\int_0^t\|z_t^k(s)\|_2^2ds+2T\int_0^t\|u_t^k(s)\|_2^2ds.\tag{3.13}$$

Combining (3.11) and (3.13) leads to

$$\begin{aligned}\|z^k\|_2^2+\|u^k\|_2^2&\le 2\|z^k(0)\|_2^2+2\|u^k(0)\|_2^2+2cT\left(1+\int_0^t\|z^k\|_2^2\ln\|z^k\|_2^2ds+\int_0^t\|u^k\|_2^2\ln\|u^k\|_2^2ds\right)\\ &\le 2C\left(1+\int_0^t\|z^k\|_2^2\ln\|z^k\|_2^2ds+\int_0^t\|u^k\|_2^2\ln\|u^k\|_2^2ds\right)\\ &\le 2C_1\Big(1+\int_0^t\left(C_1+\|z^k\|_2^2\right)\ln\left(C_1+\|z^k\|_2^2\right)ds\\ &\quad+\int_0^t\left(C_1+\|u^k\|_2^2\right)\ln\left(C_1+\|u^k\|_2^2\right)ds\Big),\end{aligned}\tag{3.14}$$

where, without loss of generality, $C_1\ge 1$. The logarithmic Gronwall inequality implies that

$$\|z^k\|_2^2+\|u^k\|_2^2\le 2C_1e^{2C_1T}:=C_2,$$

and hence,

$$\|z^k\|_2^2\ln\|z^k\|_2^2+\|u^k\|_2^2\ln\|u^k\|_2^2\le C.\tag{3.15}$$

After combining (3.10) and (3.15), we obtain

$$\sup_{(0,T_k)}\left[\|z_t^k\|_2^2+\|u_t^k\|_2^2+\|z_x^k\|_2^2+\|u_x^k\|_2^2\right]\le C.$$

Therefore, the local solution (z^k,u^k) of system (3.2) can be extended to $(0,T)$, for all $k\ge 1$. Furthermore, we have

$$z^k\ \text{and}\ u^k\ \text{are bounded in}\ L^\infty((0,T),H_0^1(\Omega)),$$

$$(z_t^k) \text{ is bounded in } L^\infty((0,T),L^2(\Omega)) \cap L^{\nu(.)}(\Omega\times(0,T)),$$

$$(u_t^k) \text{ is bounded in } L^\infty((0,T),L^2(\Omega)) \cap L^{\omega(.)}(\Omega\times(0,T)).$$

Consequently, we have, up to two subsequences,

$$\begin{aligned} &z^k \to z \text{ and } u^k \to u \text{ weakly * in } L^\infty((0,T),H_0^1(\Omega)),\\ &z_t^k \to z_t \text{ weakly * in } L^\infty((0,T),L^2(\Omega)) \text{ and weakly in } L^{\nu(.)}(\Omega\times(0,T)),\\ &u_t^k \to u_t \text{ weakly * in } L^\infty((0,T),L^2(\Omega)) \text{ and weakly in } L^{\omega(.)}(\Omega\times(0,T)).\end{aligned} \tag{3.16}$$

Step 3. The logarithmic terms: In this step, we show that the approximate solutions (z^k,u^k) satisfy for all $k\ge 1$,

$$\begin{aligned} &z^k \ln|z^k|^\alpha \to z\ln|z|^\alpha \text{ strongly in } L^2(0,T;L^2(\Omega)),\\ &u^k \ln|u^k|^\alpha \to u\ln|u|^\alpha \text{ strongly in } L^2(0,T;L^2(\Omega)).\end{aligned} \tag{3.17}$$

Making use of the arguments in (3.16) and applying the Aubin-Lions theorem, we find, up to subsequences, that

$$z^k \to z \text{ and } u^k \to u \text{ strongly in } L^2(0,T;L^2(\Omega))$$

and

$$z^k \to z \text{ and } u^k \to u \text{ a.e. in } \Omega\times(0,T). \tag{3.18}$$

Using (3.18), and the fact that the map $s \to s\ln|s|^\alpha$ is continuous on $\mathbb{R}$, then we have the convergence

$$z^k \ln|z^k|^\alpha \to z\ln|z|^\alpha \text{ a.e. in } \Omega\times(0,T).$$

Using the embedding of $H_0^1(\Omega)$ in $L^\infty(\Omega)$ (since $\Omega\subset\mathbb{R}$), it is clear that $z^k\ln|z^k|^\alpha$ is bounded in $L^\infty(\Omega\times(0,T))$. Next, taking into account the Lebesgue bounded convergence theorem (Ω is bounded), we get

$$z^k \ln|z^k|^\alpha \to z\ln|z|^\alpha \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{3.19}$$

Similarly, we can establish the second argument of (3.17).
Step 4. The nonlinear terms: In this step, we show that

$$| z_t^k |^{\nu(.)-2} z_t^k \to | z_t |^{\nu(.)-2} z_t \text{ weakly in } L^{\frac{\nu(.)}{\nu(.)-1}}(\Omega\times(0,T)),$$

$$| u_t^k |^{\omega(.)-2} u_t^k \to | u_t |^{\omega(.)-2} u_t \text{ weakly in } L^{\frac{\omega(.)}{\omega(.)-1}}(\Omega\times(0,T)),$$

and that (z,u) satisfies the partial differential equations of (1.1) on $\Omega\times(0,T)$.

Since (z_t^k) is bounded in $L^{\nu(.)}(\Omega\times(0,T))$, then $(|z_t^k|^{\nu(.)-2}z_t^k)$ is bounded in $L^{\frac{\nu(.)}{\nu(.)-1}}(\Omega\times(0,T))$. Hence, up to a subsequence,

$$|z_t^k|^{\nu(.)-2}z_t^k \rightharpoonup \chi_1 \quad \text{in } L^{\frac{\nu(.)}{\nu(.)-1}}(\Omega\times(0,T)). \tag{3.20}$$

Similarly, we have

$$|u_t^k|^{\omega(.)-2}u_t^k \rightharpoonup \chi_2 \quad \text{in } L^{\frac{\omega(.)}{\omega(.)-1}}(\Omega\times(0,T)). \tag{3.21}$$

We can show that

$$\chi_1 = |z_t|^{\nu(.)-2}z_t \quad\text{and}\quad \chi_2 = |u_t|^{\omega(.)-2}u_t,$$

by following the same steps as in [55, 56]. Now, integrate (3.2) on $(0, t)$ to obtain $\forall j < k$,

$$\int_\Omega z_t^k w_j(x)dx - \int_\Omega z_1^k w_j(x)dx + a_1\int_0^t\int_\Omega z_x^k w_{j_x}(x)dxds + a_2\int_0^t\int_\Omega u_x^k w_{j_x}(x)dxds$$
$$+\int_0^t\int_\Omega z^k w_j(x)dxds + \gamma\int_0^t\int_\Omega |z_t^k|^{\nu(.)-2} z_t^k w_j(x)dxds = \alpha\int_0^t\int_\Omega w_j z^k \ln|z^k|dxds,$$
$$\int_\Omega u_t^k w_j(x)dx - \int_\Omega u_1^k w_j(x)dx + a_3\int_0^t\int_\Omega u_x^k w_{j_x}(x)dxds + a_2\int_0^t\int_\Omega z_x^k w_{j_x}(x)dxds$$
$$+\int_0^t\int_\Omega u^k w_j(x)dxds + \beta\int_0^t\int_\Omega |u_t^k|^{\omega(.)-2} u_t^k w_j(x)dxds = \alpha\int_0^t\int_\Omega w_j u^k \ln|u^k|dxds.$$

Using all the above convergence and taking $k \to +\infty$, we easily check that $\forall j < k$,

$$\int_\Omega z_t w_j(x)dx - \int_\Omega z_1 w_j(x)dx + a_1\int_0^t\int_\Omega z_x w_{j_x}(x)dxds + a_2\int_0^t\int_\Omega u_x w_{j_x}(x)dxds$$
$$+\int_0^t\int_\Omega z w_j(x)dxds + \gamma\int_0^t\int_\Omega |z_t|^{\nu(.)-2} z_t w_j(x)dxds = \alpha\int_0^t\int_\Omega w_j z^k \ln|z^k|dxds,$$
$$\int_\Omega u_t w_j(x)dx - \int_\Omega u_1 w_j(x)dx + a_3\int_0^t\int_\Omega u_x w_{j_x}(x)dxds + a_2\int_0^t\int_\Omega z_x w_{j_x}(x)dxds$$
$$+\int_0^t\int_\Omega u w_j(x)dxds + \beta\int_0^t\int_\Omega |u_t|^{\omega(.)-2} u_t w_j(x)dxds = \alpha\int_0^t\int_\Omega w_j u^k \ln|u^k|dxds.$$

Consequently, we have $\forall w \in H_0^1(\Omega)$

$$\int_\Omega z_t w(x)dx - \int_\Omega z_1 w(x)dx + a_1\int_0^t\int_\Omega z_x w_x(x)dxds + a_2\int_0^t\int_\Omega u_x w_x(x)dxds$$
$$+\int_0^t\int_\Omega z w(x)dxds + \gamma\int_0^t\int_\Omega |z_t|^{\nu(.)-2} z_t w(x)dxds = \alpha\int_0^t\int_\Omega w z^k \ln|z^k|dxds,$$
$$\int_\Omega u_t w(x)dx - \int_\Omega u_1 w(x)dx + a_3\int_0^t\int_\Omega u_x w_x(x)dxds + a_2\int_0^t\int_\Omega z_x w_x(x)dxds$$
$$+\int_0^t\int_\Omega u w(x)dxds + \beta\int_0^t\int_\Omega |u_t|^{\omega(.)-2} u_t w(x)dxds = \alpha\int_0^t\int_\Omega w u^k \ln|u^k|dxds.$$

All terms define absolute continuous functions, so we get, for a.e. $t \in [0, T]$ and $\forall w \in H_0^1(\Omega)$,

$$\int_\Omega z_{tt} w(x)dx + a_1\int_\Omega z_x w_x(x)dx + a_2\int_\Omega u_x w_x(x)dx + \gamma\int_\Omega |z_t|^{\nu(.)-2} z_t w(x)dx$$
$$+\int_\Omega z w(x)dx = \alpha\int_\Omega w z^k \ln|z^k|dx,$$
$$\int_\Omega u_{tt} w(x)dx + a_3\int_\Omega u_x w_x(x)dx + a_2\int_\Omega z_x w_x(x)dx + \beta\int_\Omega |u_t|^{\omega(.)-2} u_t w(x)dx$$
$$+\int_\Omega u w(x)dx = \alpha\int_\Omega w z^k \ln|z^k|dx.$$

This implies that

$$\rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + z + \gamma|z_t|^{\nu(\cdot)-2} z_t = \alpha z \ln|z|, \text{ in } D'(\Omega\times(0,T)),$$
$$\rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + u + \beta|u_t|^{\omega(\cdot)-2} u_t = \alpha u \ln|u|, \text{ in } D'(\Omega\times(0,T)).$$

This implies that (z,u) satisfies the two differential equations in (1.1), on $\Omega \times (0,T)$.
Step 5. The initial conditions: We can handle the initial conditions like the one in [55]. Hence, we deduce that (z,u) is the unique local solution of (1.1). This completes the proof of Theorem 3.1. □

4. Global existence

By using the potential wells, we prove the existence of the global solution to our problem. To this end, we define the following functionals:

$$\begin{aligned} J(z,u) =&\frac{1}{2}\int_\Omega \left[a_3u_x^2 + a_1z_x^2 + 2a_2z_xu_x\right]dx + \frac{\alpha+2}{4}\left[\|z\|_2^2 + \|u\|_2^2\right] \\ &-\frac{1}{2}\int_\Omega z^2\ln|z|dx - \frac{1}{2}\int_\Omega u^2\ln|u|dx, \end{aligned} \tag{4.1}$$

$$I(z,u) = \int_\Omega \left[a_3u_x^2 + a_1z_x^2 + 2a_2z_xu_x\right]dx + \|z\|_2^2 + \|u\|_2^2 - \int_\Omega z^2\ln|z|dx - \int_\Omega u^2\ln|u|dx. \tag{4.2}$$

Remark 4.1. (1) From the above definitions, it is clear that

$$J(z,u) = \frac{1}{2}I(z,u) + \frac{\alpha}{4}\left(\|z\|_2^2 + \|u\|_2^2\right), \tag{4.3}$$

$$E(t) = \frac{1}{2}\left(\rho_z\|z_t\|_2^2 + \rho_u\|u_t\|_2^2\right) + J(z,u). \tag{4.4}$$

(2) According to the logarithmic Sobolev inequality, $J(z,u)$ and $I(z,u)$ are well-defined.

We define the potential well (stable set):

$$W = \{(z,u) \in H_0^1(\Omega)\times H_0^1(\Omega), I(z,u) > 0\} \cup \{(0,0)\}.$$

The potential well depth is defined by

$$0 < d = \inf_{(z,u)}\{\sup_{\lambda\geq 0} J(\lambda z,\lambda u) : (z,u)\in H_0^1(\Omega)\times H_0^1(\Omega), \|z_x\|_2 \neq 0 \text{ and } \|u_x\|_2 \neq 0\}, \tag{4.5}$$

and the well-known Nehari manifold is

$$\mathcal{N} = \{(z,u) : (z,u)\in H_0^1(\Omega)\times H_0^1(\Omega)/I(z,u) = 0, \|z_x\|_2 \neq 0 \text{ and } \|u_x\|_2 \neq 0\}. \tag{4.6}$$

Proceeding as in [57, 58], one has

$$0 < d = \inf_{(z,u)\in\mathcal{N}} J(z,u). \tag{4.7}$$

Lemma 4.1. *For any $(z,u) \in H_0^1(\Omega)\times H_0^1(\Omega)$, $\|z\|_2 \neq 0$, and $\|u\|_2 \neq 0$, let $g(\lambda) = J(\lambda z,\lambda u)$, then we have*

$$I(\lambda z,\lambda u) = \lambda g'(\lambda)\begin{cases} > 0, & 0 \leq \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < +\infty, \end{cases}$$

where

$$\lambda^* = \exp\left(\frac{\alpha_0\|u_x\|_2^2 + \int_\Omega\left(\sqrt{a_1}z_x + \frac{a_2}{\sqrt{a_1}}u_x\right)^2 dx + \|z\|_2^2 + \|u\|_2^2 - \int_\Omega z^2\ln|z|^\alpha dx - \int_\Omega u^2\ln|u|^\alpha dx}{\alpha(\|z\|_2^2 + \|u\|_2^2)}\right),$$

where $\alpha_0 = \left(a_3 - \frac{a_2^2}{a_1}\right) > 0$.

Proof.

$$\begin{aligned} g(\lambda) = J(\lambda z, \lambda u) =& \frac{1}{2}\lambda^2\left(\left(a_3 - \frac{a_2^2}{a_1}\right)\|u_x\|_2^2 + \int_\Omega\left(\sqrt{a_1}z_x + \frac{a_2}{\sqrt{a_1}}u_x\right)^2 dx\right) \\ &- \frac{1}{2}\lambda^2\left(\int_\Omega z^2\ln|z|^\alpha dx + \int_\Omega u^2\ln|u|^\alpha dx\right) + \lambda^2\left(\frac{\alpha+2}{4} - \frac{\alpha}{2}\ln|\lambda|\right)\left(\|z\|_2^2 + \|u\|_2^2\right). \end{aligned}$$

Since $\|z\|_2 \neq 0$ and $\|u\|_2 \neq 0$, then $g(0) = 0$, $g(+\infty) = -\infty$, and

$$\begin{aligned} I(\lambda z, \lambda u) =& \lambda\frac{dJ(\lambda z, \lambda u)}{d\lambda} = \lambda g'(\lambda) = \lambda^2\left(\left(a_3 - \frac{a_2^2}{a_1}\right)\|u_x\|_2^2 + \int_\Omega\left(\sqrt{a_1}z_x + \frac{a_2}{\sqrt{a_1}}u_x\right)^2 dx\right) \\ &- \lambda^2\left(\int_\Omega z^2\ln|z|^\alpha dx + \int_\Omega u^2\ln|u|^\alpha dx\right) + \lambda^2\left(1 - \alpha\ln|\lambda|\right)\left(\|z\|_2^2 + \|u\|_2^2\right), \end{aligned}$$

which implies that $\frac{d}{d\lambda}J(\lambda z, \lambda u)_{\lambda=\lambda^*} = 0$, $J(\lambda z, \lambda u)$ is increasing on $0 < \lambda \leq \lambda^*$, decreasing on $\lambda^* \leq \lambda < \infty$, and reaching its maximum value at $\lambda = \lambda^*$. In other words, there exists a unique $\lambda^* \in (0, \infty)$ such that $I(\lambda^* z, \lambda^* u) = 0$, which establishes the desired result. □

Lemma 4.2. *Let* $(z, u) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and* $\beta_0 = \sqrt{\frac{2\pi\tilde{c}}{\alpha}}e^{1+\frac{1}{\alpha}}$. *If* $0 < \|z\|_2 \leq \beta_0$ *and* $0 < \|u\|_2 \leq \beta_0$, *then* $I(z, u) \geq 0$.

Proof. Using the logarithmic Sobolev inequality (2.4), for any $a > 0$, we have

$$\begin{aligned} I(z, u) =& \int_\Omega\left[a_3u_x^2 + a_1z_x^2 + 2a_2z_xu_x\right]dx + \|z\|_2^2 + \|u\|_2^2 - \int_\Omega z^2\ln|z|dx - \int_\Omega u^2\ln|u|dx \\ \geq& \left(\tilde{c} - \frac{\alpha a^2}{2\pi}\right)\|u_x\|_2^2 + \left(\tilde{c} - \frac{\alpha a^2}{2\pi}\right)\|z_x\|_2^2 + \frac{1}{2}\left(1 + \alpha(1 + \ln a) - \frac{\alpha}{2}\ln\|u\|_2^2\right)\|u\|_2^2 \\ &+ \frac{1}{2}\left(1 + \alpha(1 + \ln a) - \frac{\alpha}{2}\ln\|z\|_2^2\right)\|z\|_2^2. \end{aligned} \tag{4.8}$$

Taking $a < \min\left\{\sqrt{\frac{2\pi\tilde{c}}{\alpha}}, \sqrt{\frac{2\pi\tilde{c}}{\alpha}}\right\}$ in (4.8), we obtain

$$I(z, u) \geq \frac{1}{2}\left(\frac{\alpha}{2} + \alpha(1 + \ln a) - \frac{\alpha}{2}\ln\|u\|_2^2\right)\|u\|_2^2 + \frac{1}{2}\left(\frac{\alpha}{2} + \alpha(1 + \ln a) - \frac{\alpha}{2}\ln\|z\|_2^2\right)\|z\|_2^2. \tag{4.9}$$

If $0 < \|z\|_2 \leq \beta_0$ and $0 < \|u\|_2 \leq \beta_0$, then

$$\frac{\alpha}{2} + \alpha(1 + \ln a) - \frac{\alpha}{2}\ln\|u\|_2^2 \geq 0 \text{ and } \frac{\alpha}{2} + \alpha(1 + \ln a) - \frac{\alpha}{2}\ln\|z\|_2^2 \geq 0,$$

which gives $I(z, u) \geq 0$. □

Lemma 4.3. *The potential well depth d satisfies*

$$d \geq \frac{\tilde{c}\pi}{2} e^{2+\frac{2}{\alpha}}. \tag{4.10}$$

Proof. The proof of this lemma is similar to the proof of Lemma 4.3. in [59]. □

Lemma 4.4. *Let* $(z_0, z_1), (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *such that* $0 < E(0) < d$ *and* $I(z_0, u_0) > 0$*, then any solution of* (1.1) *is* $(z, u) \in W$.

Proof. Let T be the maximal existence time of a weak solution of (ψ, φ). From (2.10) and (4.4), we have

$$\frac{1}{2}\left(\rho_z\|z_t\|^2 + \rho_u\|u_t\|^2\right) + J(z,u) \leq \frac{1}{2}\left(\rho_z\|z_1\|^2 + \rho_u\|u_1\|^2\right) + J(z_0,u_0) < d, \text{ for any } t \in [0,T), \tag{4.11}$$

then we claim that $(z(t), u(t)) \in W$ for all $t \in [0, T)$. If not, then there is a $t_0 \in (0, T)$ such that $I(z(t_0), u(t_0)) < 0$. Using the continuity of $I(z(t), u(t))$ in t, we deduce that there exists a $t_* \in (0, T)$ such that $I(z(t_*), u(t_*)) = 0$. Using the definition of d in (4.5) gives

$$d \leq J(z(t_*), u(t_*)) \leq E(z(t_*), u(t_*)) \leq E(0) < d,$$

which is a contradiction. □

5. Stability

In this section, we state and prove our main decay results. For this purpose, we present the following lemmas.

Lemma 5.1. *For any* $\eta > 0$*, we have the following:*

$$-\beta \int_\Omega u|u_t|^{\omega(\cdot)-2}u_t dx \leq c\eta\beta \int_\Omega u_x^2 dx + \beta \int_\Omega c_\eta(x)|u_t|^{\omega(x)}dx, \quad \omega_1 \geq 2, \tag{5.1}$$

and if $1 < \omega_1 < 2$*, we have*

$$-\beta \int_\Omega u|u_t|^{\omega(\cdot)-2}u_t dx \leq c\eta\beta \int_\Omega u_x^2 dx + c\left[\beta \int_\Omega |u_t|^{\omega(x)}dx + \left(\int_\Omega \beta|u_t|^{\omega(x)}dx\right)^{\omega_1-1}\right]. \tag{5.2}$$

Lemma 5.2. *For any* $\lambda > 0$*, we have the following:*

$$-\gamma \int_\Omega z|z_t|^{\nu(\cdot)-2}z_t dx \leq c\lambda\gamma \int_\Omega z_x^2 dx + \gamma \int_\Omega c_\lambda(x)|z_t|^{\nu(x)}dx, \quad \nu_1 \geq 2, \tag{5.3}$$

and if $1 < \nu_1 < 2$*, we have*

$$-\gamma \int_\Omega z|z_t|^{\nu(\cdot)-2}\varphi_t dx \leq c\lambda\gamma \int_\Omega z_x^2 dx + c\left[\gamma \int_\Omega |z_t|^{\nu(x)}dx + \left(\int_\Omega \gamma|z_t|^{\nu(x)}dx\right)^{\nu_1-1}\right]. \tag{5.4}$$

Proof. We prove Lemma 5.1, and the proof of Lemma 5.2 will be similar. We start by applying Young's inequality with $\xi(x) = \frac{\omega(x)}{\omega(x)-1}$ and $\xi'(x) = \omega(x)$. So, for a.e $x \in (0, 1)$ and any $\eta > 0$, we have

$$|u_t|^{\omega(x)-2}u_t u \leq \eta |u|^{\omega(x)} + c_\eta(x)|u_t|^{\omega(x)},$$

where

$$c_\eta(x) = \eta^{1-\omega(x)}(\omega(x))^{-\omega(x)}(\omega(x) - 1)^{\omega(x)-1}.$$

Hence,

$$-\beta \int_\Omega u|u_t|^{\omega(x)-2}u_t dx \leq \eta\beta \int_\Omega |u|^{\omega(x)}dx + \beta \int_\Omega c_\eta(x)|u_t|^{\omega(x)}dx. \tag{5.5}$$

Next, using (2.9), (2.10), (4.8), Poincaré's inequality, and the embedding property, we get

$$\begin{aligned}
\int_\Omega |u|^{\omega(x)}dx &= \int_{\Omega_+} |u|^{\omega(x)}dx + \int_{\Omega_-} |u|^{\omega(x)}dx \\
&\leq \int_{\Omega_+} |u|^{\omega_2}dx + \int_{\Omega_-} |u|^{\omega_1}dx \\
&\leq \int_\Omega |u|^{\omega_2}dx + \int_\Omega |u|^{\omega_1}dx \\
&\leq c_e^{\omega_1}\|u_x\|_2^{\omega_1} + c_e^{\omega_2}\|u_x\|_2^{\omega_2} \\
&\leq \left(c_e^{\omega_1}\|u_x\|_2^{\omega_1-2} + c_e^{\omega_2}\|u_x\|_2^{\omega_2-2}\right)\|u_x\|_2^2 \\
&\leq \left(c_e^{\omega_1}\left(\frac{2\pi}{2\pi\tilde{c} - \alpha a^2}E(0)\right)^{\omega_1-2} + c_e^{\omega_2}\left(\frac{2\pi}{2\pi\tilde{c} - \alpha a^2}E(0)\right)^{\omega_2-2}\right)\|u_x\|_2^2 \\
&\leq c_1\|u_x\|_2^2,
\end{aligned} \tag{5.6}$$

where c_e is the embedding constant,

$$\Omega_+ = \{x \in (0, L) : |u(x, t)| \geq 1\}, \quad \Omega_- = \{x \in (0, L) : |u(x, t)| < 1\}$$

and

$$c_1 = \left(c_e^{\omega_1}\left(\frac{2\pi}{2c\pi\tilde{c} - \alpha a^2}E(0)\right)^{\omega_1-2} + c_e^{\omega_2}\left(\frac{2\pi}{2c\pi\tilde{c} - \alpha a^2}E(0)\right)^{\omega_2-2}\right). \tag{5.7}$$

Thus, from (5.5) and (5.6), we find that

$$-\beta \int_\Omega u|u_t|^{\omega(x)-2}u_t dx \leq c_1\eta\beta \int_\Omega u_x^2 dx + \beta \int_\Omega c_\eta(x)|u_t|^{\omega(x)}dx. \tag{5.8}$$

Combining all the above estimations, estimate (5.1) is established. To prove (5.2), we set

$$\Omega_1 = \{x \in (0, L) : \omega(x) < 2\} \text{ and } \Omega_2 = \{x \in (0, L) : \omega(x) \geq 2\},$$

then, we have

$$-\beta \int_\Omega u|u_t|^{\omega(x)-2}u_t dx = -\beta \int_{\Omega_1} u|u_t|^{\omega(x)-2}u_t dx - \beta \int_{\Omega_2} u|u_t|^{\omega(x)-2}u_t dx. \tag{5.9}$$

We notice that on Ω_1, we have

$$2\omega(x) - 2 < \omega(x), \quad \text{and} \quad 2\omega(x) - 2 \geq 2\omega_1 - 2. \tag{5.10}$$

Therefore, by using Young's and Poincaré's inequalities and (5.10), we find that

$$\begin{aligned}
-\beta \int_{\Omega_1} u|u_t|^{\omega(x)-2}u_t dx &\leq \eta\beta \int_{\Omega_1} |u|^2 dx + \frac{\beta}{4\eta} \int_{\Omega_1} |u_t|^{2\omega(x)-2} dx \\
&\leq c\eta\beta \|u_x\|_2^2 + c_\eta \beta \Big[\int_{\Omega_1^+} |u_t|^{2\omega(x)-2} dx + \int_{\Omega_1^-} |u_t|^{2\omega(x)-2} dx \Big] \\
&\leq c\eta\beta \|u_x\|_2^2 + c_\eta \beta \Big[\int_{\Omega_1^+} |u_t|^{\omega(x)} dx + \int_{\Omega_1^-} |u_t|^{2\omega_1-2} dx \Big] \\
&\leq c\eta\beta \|u_x\|_2^2 + c_\eta \beta \Big[\int_{\Omega} |u_t|^{\omega(x)} dx + \Big(\int_{\Omega_1^-} |u_t|^2 dx \Big)^{\omega_1 - 1} \Big] \\
&\leq c\eta\beta \|u_x\|_2^2 + c_\eta \beta \Big[\int_{\Omega} |u_t|^{\omega(x)} dx + \Big(\int_{\Omega_1^-} |u_t|^{\omega(x)} dx \Big)^{\omega_1 - 1} \Big] \\
&\leq c\eta\beta \|u_x\|_2^2 + c_\eta \Big[\beta \int_{\Omega} |u_t|^{\omega(x)} dx + \beta^{2-\omega_1} \Big(\int_{\Omega} \beta |u_t|^{\omega(x)} dx \Big)^{\omega_1 - 1} \Big],
\end{aligned} \tag{5.11}$$

where

$$\Omega_1^+ = \{x \in \Omega_1 : |u_t(x,t)| \geq 1\} \text{ and } \Omega_1^- = \{x \in \Omega_1 : |u_t(x,t)| < 1\}. \tag{5.12}$$

Next, by the case of $\omega(x) \geq 2$, we have

$$-\beta \int_{\Omega_2} u|u_t|^{\omega(x)-2}u_t dx \leq c\eta\beta \int_\Omega u_x^2 dx + \beta \int_\Omega c_\eta(x)|u_t|^{\omega(x)} dx. \tag{5.13}$$

Combining (5.11) and (5.13), the proof of (5.2) is completed. □

Remark 5.1. For the stability results, we assume that the coefficients $a_i,\ i = 1, ..., 3$ satisfy

$$a_1 a_3 - 4a_2^2 > 0. \tag{5.14}$$

It is clear that (5.14) gives the condition in $(A2)$.

Lemma 5.3. *Assume that* $(A1 - A3)$ *and* (5.14) *hold and let* $(z_0, z_1), (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *Assume further that* $0 < E(0) < \ell\tau < d$, *where*

$$\tau = \frac{\tilde{c}\pi}{2} e^{2+\frac{2}{\alpha}}, \quad 0 < e^{\frac{1}{\alpha}} \sqrt{\frac{\ell\tilde{c}}{a_0}} < 1, a_0 = \min\{a_1, a_3\}, \tag{5.15}$$

then the functional

$$L(t) = NE(t) + \rho_u \int_\Omega u u_t dx + \rho_z \int_\Omega z z_t dx + \frac{1}{2}\int_\Omega u^2 dx + \frac{1}{2}\int_\Omega z^2 dx$$

satisfies, along with the solutions of (1.1) *and for a suitable choice of N,*

$$L \sim E \tag{5.16}$$

and

$$Ł'(t) \le \begin{cases} -\vartheta E(t) + c\int_\Omega z_t^2 dx + c\int_\Omega u_t^2 dx, & \nu_1, \omega_1 \ge 2, \\ -\vartheta E(t) + c\int_\Omega z_t^2 dx + c\int_\Omega u_t^2 dx - cE^{-\alpha_1}(t)E'(t), & \gamma = 0, \beta \ne 0, \text{and } 1 < \nu_1, \omega_1 < 2, \\ -\vartheta E(t) + c\int_\Omega z_t^2 dx + c\int_\Omega u_t^2 dx - cE^{-\alpha_2}(t)E'(t), & \beta = 0, \gamma \ne 0, \text{and } 1 < \nu_1, \omega_1 < 2, \\ -\vartheta E(t) + c\int_\Omega z_t^2 dx + c\int_\Omega u_t^2 dx - cE^{-\alpha_3}(t)E'(t), & \gamma \ne 0, \beta \ne 0, \text{and } 1 < \nu_1, \omega_1 < 2, \end{cases} \tag{5.17}$$

where $\alpha_1 = \frac{2-\omega_1}{\omega_1 - 1} > 0, \alpha_2 = \frac{2-\nu_1}{\nu_1-1} > 0, \alpha_3 = \frac{2-m_1}{m_1-1} > 0$*, and* $m_1 = \min\{\nu_1, \omega_1\}$.

Proof. If we want to prove all cases, the proof will be very lengthy, so we prove $(5.17)_2$ and the proofs of the other cases are very similar with minor modifications. To prove $(5.17)_2$, we differentiate $L(t)$ and use integrations by parts, to get

$$\begin{aligned} L'(t) = &-\beta\int_\Omega |u_t|^{\omega(\cdot)} dx + \int_\Omega \left(\rho_u |u_t|^2 + \rho_z |z_t|^2\right) dx - \int_\Omega \left(a_3 |u_x|^2 + a_1 |z_x|^2 + 2a_2 u_x z_x\right) dx \\ &+ \alpha\int_\Omega u^2 \ln|u| dx + \alpha\int_\Omega z^2 \ln|z| dx - \int_\Omega u^2 dx - \int_\Omega z^2 dx \\ &- \beta\int_\Omega u|u_t|^{\omega(\cdot)-2} u_t dx + \int_\Omega (\rho_u u u_t + \rho_z z z_t)\, dx \\ &+ c\left[\beta\int_\Omega |u_t|^{\omega(x)} dx + \left(\int_\Omega \beta |u_t|^{\omega(x)} dx\right)^{\omega_1 - 1}\right]. \end{aligned} \tag{5.18}$$

Using Young's inequality, we have for some positive constants λ_i,

$$2a_2 u_x z_x \le \lambda_1 u_x^2 + \frac{a_2^2}{\lambda_1} z_x^2, \tag{5.19}$$

$$\rho_u u u_t + \rho_z z z_t \le \lambda_2 (u^2 + z^2) + \frac{1}{4\lambda_2}(\rho_u^2 u_t^2 + \rho_z^2 z_t^2), \tag{5.20}$$

and

$$\rho_u u u_t + \rho_z z z_t \le \lambda_4 \left(u^2 + z^2\right) + \frac{\rho_u^2}{4\lambda_4} u_t^2 + \frac{\rho_z^2}{4\lambda_4} z_t^2. \tag{5.21}$$

Using (5.2), (5.4), and (5.18)–(5.21), we have

$$\begin{aligned} L'(t) \le &- N\beta\int_\Omega |u_t|^{\omega(\cdot)} dx + \lambda_4 \int_\Omega \left(u^2 + z^2\right) dx + \frac{c}{\lambda_4}\int_\Omega \left(u_t^2 + z_t^2\right) dx \\ &- \int_\Omega (a_3 - c\eta\beta - \lambda_1)\, u_x^2 dx - \int_\Omega \left(a_1 - \frac{a_2^2}{\lambda_1}\right) z_x^2 dx \\ &+ \alpha\int_\Omega u^2 \ln|u| dx + \alpha\int_\Omega z^2 \ln|z| dx - \int_\Omega u^2 dx - \int_\Omega z^2 dx \\ &+ c\left[\beta\int_\Omega |u_t|^{\omega(x)} dx + \left(\int_\Omega \beta|u_t|^{\omega(x)} dx\right)^{\omega_1 - 1}\right]. \end{aligned} \tag{5.22}$$

Using (2.10) and the logarithmic Sobolev inequality, (5.22) becomes

$$\begin{aligned} L'(t) \le & -\beta(N-c)\int_\Omega |u_t|^{\omega(\cdot)}dx + \frac{c}{\lambda_4}\int_\Omega \left(u_t^2+z_t^2\right)dx \\ & - \int_\Omega \left(a_3 - \frac{a^2\alpha}{2\pi} - c\eta\beta - \lambda_1\right)u_x^2dx - \int_\Omega \left(a_1 - \frac{\alpha a^2}{2\pi} - \frac{a_2^2}{\lambda_1}\right)z_x^2dx \\ & - \left(1 - \frac{\alpha}{2}\ln\|u\|_2^2 - \lambda_4 + \alpha(1+\ln a)\right)\|u\|_2^2 - \left(1 - \frac{\alpha}{2}\ln\|z\|_2^2 - \lambda_4 + \alpha(1+\ln a)\right)\|z\|_2^2 \\ & + c\Big(-E'(t)\Big)^{\omega_1-1}. \end{aligned} \tag{5.23}$$

Now, we select N large enough so that $N-c>0$, then we select $a < \sqrt{\frac{2\pi a_0}{\alpha}}$, where $a_0 = \min\{a_1, a_3\}$, which makes

$$a_3 - \frac{\alpha a^2}{2\pi} > 0, \text{ and } a_1 - \frac{\alpha a^2}{2\pi} > 0.$$

After that, we choose $\eta = \frac{a_3 - \frac{\alpha a^2}{2\pi}}{2c\beta}$, and $\frac{2a_2^2}{a_1 - \frac{\alpha a^2}{2\pi}} < \lambda_1 < \frac{a_3 - \frac{\alpha a^2}{2\pi}}{2}$, to get

$$a_3 - \frac{a^2\alpha}{2\pi} - c\eta\beta - \lambda_1 > 0,\ a_1 - \frac{\alpha a^2}{2\pi} - \frac{a_2^2}{\lambda_1} > 0.$$

This selection is possible thanks to (5.14). Using (2.9), (2.10), and the fact that $u \in W$,

$$\ln\|u\|_2^2 < \ln\left(\frac{4}{\alpha}E(t)\right) < \ln\left(\frac{4}{\alpha}E(0)\right) < \ln\left(\frac{4}{\alpha}\ell\tau\right) < \ln\left(\frac{2\ell\tilde{c}\pi e^{2+\frac{2}{\alpha}}}{\alpha}\right). \tag{5.24}$$

After taking a satisfying

$$e^{\frac{1}{\alpha}}\sqrt{\frac{2\ell\tilde{c}\pi}{\alpha}} < a < \sqrt{\frac{2\pi a_0}{\alpha}},$$

and λ_4 is small enough, we guarantee the following:

$$1 - \frac{\alpha}{2}\ln\|u\|_2^2 - \lambda_4 + \alpha(1+\ln a) > 0 \text{ and } 1 - \frac{\alpha}{2}\ln\|z\|_2^2 - \lambda_4 + \alpha(1+\ln a) > 0.$$

Then, (5.23) reduces to

$$L'(t) \le -cE(t) + c\int_\Omega z_t^2dx + c\int_\Omega u_t^2dx + c\beta(-E'(t))^{\omega_1-1}. \tag{5.25}$$

Using Young's inequality with $\zeta = \frac{1}{\omega_1-1}$ and $\zeta^* = \frac{1}{2-\omega_1}$, for any $\varepsilon > 0$, we estimate this term $E^\alpha(t)(-E'(t))^{\omega_1-1}$ as follows:

$$E^\alpha(t)(-E'(t))^{\omega_1-1} \le \varepsilon E^{\frac{\alpha}{2-\omega_1}}(t) + c_\varepsilon(-E'(t)).$$

Multiplying both sides of the last inequality by $E^{-\alpha}$, where $\alpha = \frac{2-\omega_1}{\omega_1-1}$, gives us

$$(-E'(t))^{\omega_1-1} \le \varepsilon E(t) + c_\varepsilon E^{-\alpha}(t)(-E'(t)).$$

Inserting this estimate in the last term in (5.25), we find that

$$L'(t) \le -(c-\varepsilon)E(t) + c\int_\Omega z_t^2dx + c\int_\Omega u_t^2dx + c_\varepsilon E^{-\alpha}(t)(-E'(t)). \tag{5.26}$$

By taking ε small enough and using the nonincreasing property of E, (5.17) is established. On the other hand, we can choose N even larger (if needed) so that $L \sim E$. □

Lemma 5.4. *Assume that* $(A1)$ *holds, then*

$$\begin{aligned}\int_0^1 z_t^2 dx &\leq -cE'(t), \quad \textit{if } \nu_2 = 2,\\ \int_0^1 u_t^2 dx &\leq -cE'(t), \quad \textit{if } \omega_2 = 2,\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}\int_0^1 z_t^2 dx &\leq -cE'(t) + c\left(-E'(t)\right)^{\frac{2}{\nu_2}}, \quad \textit{if } \nu_2 > 2,\\ \int_0^1 u_t^2 dx &\leq -cE'(t) + c\left(-E'(t)\right)^{\frac{2}{\omega_2}}, \quad \textit{if } \omega_2 > 2.\end{aligned} \tag{5.28}$$

Proof. By recalling (2.10), it is easy to establish (5.27). To prove the first estimate in (5.28), we set the following partitions

$$\Omega_1 = \{x \in \Omega : |z_t| \geq 1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega : |z_t| < 1\}. \tag{5.29}$$

The use of Hölder's and Young's inequalities and (2.9), give for Ω_1,

$$\int_{\Omega_1} z_t^2 dx \leq \int_\Omega |z_t|^{\omega(x)} dx \leq -cE'(t), \tag{5.30}$$

and for Ω_2, we get

$$\begin{aligned}\int_{\Omega_2} z_t^2 dx &\leq c\left(\int_{\Omega_2} |z_t|^{\nu_2} dx\right)^{\frac{2}{\nu_2}}\\ &\leq c\left(\int_{\Omega_2} |z_t|^{\nu(x)} dx\right)^{\frac{2}{\nu_2}} \leq c\left(\int_{\Omega} |z_t|^{\nu(x)} dx\right)^{\frac{2}{\nu_2}} \leq c\left(-E'(t)\right)^{\frac{2}{\nu_2}}.\end{aligned} \tag{5.31}$$

Combining (5.30) and (5.31), the first estimate in (5.28) is established, and we repeat the same steps to establish the second estimate in (5.28). □

Theorem 5.1. *Assume that* $(A1 - A3)$ *hold and* $\nu_1, \omega_1 \geq 2$, *then the energy functional* (2.9) *satisfies, for some positive constants* $\lambda_i, \sigma_i, \mu_i > 0,\ i = 1, 2, 3$, *and for any* $t \geq 0$,

$$\begin{cases} E(t) < \mu_1 e^{-\lambda_1 t}, & \textit{if } \gamma = 0,\ \beta \neq 0, \textit{ and } \omega_2 = 2;\\ E(t) < \mu_2 e^{-\lambda_2 t}, & \textit{if } \gamma \neq 0,\ \beta = 0, \textit{ and } \nu_2 = 2;\\ E(t) < \mu_3 e^{-\lambda_3 t}, & \textit{if } \gamma \neq 0,\ \beta \neq 0, \textit{ and } \nu_2 = \omega_2 = 2,\end{cases} \tag{5.32}$$

and

$$\begin{cases} E(t) < \frac{\sigma_1}{(t+1)^{\left(\frac{\omega_2-2}{2}\right)}}, & \textit{if } \gamma = 0,\ \beta \neq 0, \textit{ and } \omega_2 > 2;\\ E(t) < \frac{\sigma_2}{(t+1)^{\left(\frac{\nu_2-2}{2}\right)}}, & \textit{if } \gamma \neq 0,\ \beta = 0, \textit{ and } \nu_2 > 2;\\ E(t) < \frac{\sigma_3}{(t+1)^{\left(\frac{m_2-2}{2}\right)}}, & \textit{if } \gamma \neq 0,\ \beta \neq 0, \textit{ and } \nu_2, \omega_2 > 2,\end{cases} \tag{5.33}$$

where $m_2 = \min\{\nu_2, \omega_2\}$.

Proof. To prove $(5.32)_1$, we impose Lemma (5.4) in $(5.17)_1$ to obtain

$$L'(t) \le -cL(t) + c(-E'(t)), \tag{5.34}$$

which leads to

$$L_1'(t) \le -cL(t), \tag{5.35}$$

where $L_1 = L + cE \sim E$. Integrating (5.35) over $(0,t)$ and using the fact that $L_1, L \sim E$, the proof of $(5.32)_1$ is finished, and the remaining proofs of $(5.32)_2$ and $(5.32)_3$ can be achieved in the same way. Now, it is enough to prove the estimate given in $(5.33)_3$, and the remaining can be achieved in the same way. To this end, we also apply Lemma 5.4 in Eq $(5.17)_1$ to have

$$L'(t) \le -cL(t) + (-E'(t))^{\frac{2}{\nu_2}} + (-E'(t))^{\frac{2}{\omega_2}}. \tag{5.36}$$

By multiplying (5.36) by E^α, where $\alpha = \frac{\nu_2-2}{2} > 0$, we get

$$E^\alpha L'(t) \le -cE^\alpha L(t) + E^\alpha (-E'(t))^{\frac{2}{\nu_2}} + E^\alpha (-E'(t))^{\frac{2}{\omega_2}}. \tag{5.37}$$

Applying Young's inequality twice in (5.37), we find that for $\varepsilon > 0$,

$$E^\alpha L'(t) \le -cE^{\alpha+1}L(t) + \varepsilon E^{\frac{\alpha\nu_2}{\nu_2-2}} + \varepsilon E^{\frac{\alpha\omega_2}{\omega_2-2}} + C_\varepsilon(-E'(t)). \tag{5.38}$$

We discuss two cases:
Case 1. If $\nu_2 < \omega_2$, we will have

$$E^\alpha L'(t) \le -cE^{\alpha+1}L(t) + \varepsilon E^{\frac{\alpha\nu_2}{\nu_2-2}} + \varepsilon E^{\frac{\alpha\nu_2}{\nu_2-2}} E^{\frac{2\alpha(\nu_2-\omega_2)}{(\nu_2-2)(\omega_2-2)}} + C_\varepsilon(-E'(t)).$$

Using the fact that $E' \le 0$, we get

$$E^\alpha L'(t) \le -(c - \varepsilon - c\varepsilon) E^{\alpha+1}L(t) + C_\varepsilon(-E'(t)). \tag{5.39}$$

Choosing ε small enough, we see that (5.39) becomes

$$L_2(t) \le -cE^{\alpha+1}(t), \qquad \forall t \ge 0, \tag{5.40}$$

where $L_2 = E^\alpha L + cE \sim E$. By integrating (5.40) over $(0,t)$ and using the fact that $E \sim L_2$, we obtain

$$E(t) < \frac{c_{\nu_2}}{(t+1)^{1/\alpha}}, \ \forall\, t > 0, \tag{5.41}$$

where $\alpha = \frac{\nu_2-2}{2}$.
Case 2. If $\omega_2 < \nu_2$, in this case we get

$$E(t) < \frac{c_{\omega_2}}{(t+1)^{1/\alpha}}, \ \forall\, t > 0, \tag{5.42}$$

where $\alpha = \frac{\omega_2-2}{2}$. So, the proof of $(5.33)_3$ can be completed by taking $m_2 = \min\{\nu_2, \omega_2\}$. □

Theorem 5.2. *Assume that* $(A1-A3)$ *hold,* $1<\nu_1,\omega_1<2$ *and,* $\nu_2=\omega_2=2$*, then the energy functional* (2.9) *satisfies, for a positive constants* C_i*,* $i=1,2,3$*, and for any* $t>0$*,*

$$\begin{cases} E(t)<\frac{C_1}{(t+1)^{\left(\frac{\omega_1-1}{2-\omega_1}\right)}}, & \text{if } \gamma=0 \text{ and } \beta\neq 0,\\ E(t)<\frac{C_2}{(t+1)^{\left(\frac{\nu_1-1}{2-\nu_1}\right)}}, & \text{if } \gamma\neq 0 \text{ and } \beta=0,\\ E(t)<\frac{C_3}{(t+1)^{\left(\frac{m_1-1}{2-m_1}\right)}}, & \text{if } \gamma\neq 0 \text{ and } \beta\neq 0,\end{cases} \tag{5.43}$$

where $m_1=\min\{\nu_1,\omega_1\}$.

Proof. To prove $(5.43)_1$, we impose Lemma (5.4) in $(5.17)_2$ to get

$$L'(t)\leq -cE(t)+(-E'(t))+(-E'(t))-cE^{-\alpha_1}(t)E'(t),$$

where $\alpha_1=\frac{2-\omega_1}{\omega_1-1}>0$. By taking $L_1=L+cE\sim E$, this becomes

$$L_1'(t)\leq -cE(t)-cE^{-\alpha_1}(t)E'(t). \tag{5.44}$$

Multiplying (5.44) by E^{α_1}, we have

$$E^{\alpha_1}(t)L_1'(t)\leq -cE^{\alpha_1+1}(t)-cE'(t).$$

By taking $L_2=E^{\alpha}L_1+cE\sim E$, this becomes

$$L_2'(t)\leq -cE^{\alpha_1+1}(t).$$

Therefore, we obtain the following decay estimate

$$E(t)<\frac{c_{\omega_1}}{(t+1)^{1/\alpha_1}},\ \forall\, t>0, \tag{5.45}$$

where $\alpha_1=\frac{2-\omega_1}{\omega_1-1}$. The proof of $(5.43)_1$ is completed, and the proof of $(5.43)_2$ and $(5.43)_3$ can be achieved in the same way. □

Theorem 5.3. *Assume that* $(A1-A3)$ *hold,* $1<\nu_1,\omega_1<2$*, and* $\nu_2,\omega_2>2$*, then the energy functional* (2.9) *satisfies, for a positive constants* C_i*,* $i=1,2,3$*, and for any* $t>0$*,*

$$\begin{cases} E(t)<\frac{C_1}{(t+1)^{\left(\frac{2}{\omega_2-2}\right)}}, & \text{if } \gamma=0 \text{ and } \beta\neq 0,\\ E(t)<\frac{C_2}{(t+1)^{\left(\frac{2}{\nu_2-2}\right)}}, & \text{if } \gamma\neq 0 \text{ and } \beta=0,\\ E(t)<\frac{C_3}{(t+1)^{\left(\frac{2}{m_2-2}\right)}}, & \text{if } \gamma\neq 0 \text{ and } \beta\neq 0,\end{cases} \tag{5.46}$$

where $m_2=\min\{\nu_2,\omega_2\}$

Proof. To prove $(5.46)_1$, we impose Lemma (5.4) in $(5.17)_1$ to get

$$L'(t)\leq -cE(t)+(-E'(t))^{\frac{2}{\nu_2}}+(-E'(t))^{\frac{2}{\omega_2}}-cE^{-\alpha_1}(t)E'(t),$$

where $\alpha_1 = \frac{2-\omega_1}{\omega_1-1} > 0$. Multiplying by E^α where $\alpha = \frac{\omega_2-2}{2} > 0$, and using $\alpha - \alpha_1 > 0$ and Young's inequality twice, we obtain, for $\varepsilon > 0$,

$$E^\alpha L'(t) \leq -cE^{\alpha+1}(t) + \varepsilon E^{\frac{\alpha\nu_2}{\nu_2-2}} + \varepsilon E^{\frac{\alpha\omega_2}{\omega_2-2}} + C_\varepsilon(-E'(t)).$$

Assuming that $\omega_2 > \nu_2$,

$$E^\alpha L'(t) \leq -cE^{\alpha+1}\mathcal{L}(t) + \varepsilon E^{\frac{\alpha\omega_2}{\omega_2-2}} + \varepsilon E^{\frac{\alpha\omega_2}{\omega_2-2}} E^{\frac{2\alpha(\omega_2-\nu_2)}{(\nu_2-2)(\omega_2-2)}} + C_\varepsilon(-E'(t)).$$

Using the fact that E is nonincreasing, we obtain

$$E^\alpha L'(t) \leq -(c - \varepsilon - c\varepsilon) E^{\alpha+1}\mathcal{L}(t) + C_\varepsilon(-E'(t)).$$

Taking ε small enough, the above estimate becomes:

$$L_2(t) \leq -cE^{\alpha+1}(t), \qquad \forall t \geq 0, \tag{5.47}$$

where $L_2 = E^\alpha L + cE \sim E$. Integration (5.47) over $(0, t)$ and using $E \sim L_2$, we get

$$E(t) < \frac{c_{\omega_2}}{(t+1)^{1/\alpha}}, \ \forall\, t > 0, \tag{5.48}$$

where $\alpha = \frac{\omega_2-2}{2}$. So, the proof of $(5.46)_1$ is completed and the proofs of $(5.46)_2$ and $(5.46)_3$ will be the same. □

6. Conclusions and open problems

In this paper, we proved the local existence result of solutions of the nonlinear swelling porous-elastic system by using the Faedo-Galerkin method. Furthermore, we proved the global existence of solutions by using the well-depth method. Finally, we established several decay results by employing the multiplier method and the logarithmic Sobolev inequality. The problem will be very interesting if we consider the damping condiments γ and β as functions of x and t, i.e, $\gamma = \gamma(x, t)$ and $\beta = \beta(x, t)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to acknowledge the support provided by King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia. The support provided by the Interdisciplinary Research Center for Construction & Building Materials (IRC-CBM) at King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia, for funding this work through Project No. INCB2402, is also greatly acknowledged.

Conflict of interest

The authors declare no competing interests.

References

1. M. A. Goodman, S. C. Cowin, A continuum theory for granular materials, *Arch. Rational Mech. Anal.*, **44** (1972), 249–266. https://doi.org/10.1007/BF00284326
2. J. W. Nunziato, S. C. Cowin, A nonlinear theory of elastic materials with voids, *Arch. Rational Mech. Anal.*, **72** (1979), 175–201. https://doi.org/10.1007/BF00249363
3. A. C. Eringen, A continuum theory of swelling porous elastic soils, *Internat. J. Engrg. Sci.*, **32** (1994), 1337–1349. https://doi.org/10.1016/0020-7225(94)90042-6
4. E. Acerbi, G. Mingione, Regularity results for stationary electro-rheological fluids, *Arch. Rational Mech. Anal.*, **164** (2002), 213–259. https://doi.org/10.1007/s00205-002-0208-7
5. M. R ŭžička, *Electrorheological fluids: Modeling and mathematical theory*, Springer, 2007.
6. S. Antontsev, Wave equation with $p(x,t)$-laplacian and damping term: Existence and blow-up, *Differ. Equ. Appl.*, **3** (2011), 503–525.
7. S. Antontsev, Wave equation with $p(x,t)$-Laplacian and damping term: Blow-up of solutions, *C. R. Mecanique*, **339** (2011), 751–755. http://dx.doi.org/10.1016/j.crme.2011.09.001
8. S. A. Messaoudi, A. A. Talahmeh, A blow-up result for a nonlinear wave equation with variable-exponent nonlinearities, *Appl. Anal.*, **96** (2017), 1509–1515. https://doi.org/10.1080/00036811.2016.1276170
9. S. A. Messaoudi, A. A. Talahmeh, J. H. Al-Smail, Nonlinear damped wave equation: Existence and blow-up, *Comput. Math. Appl.*, **74** (2017), 3024–3041. https://doi.org/10.1016/j.camwa.2017.07.048
10. I. Bialynicki-Birula, J. Mycielski, Wave equations with logarithmic nonlinearities, *Bull. Acad. Pol. Sci. Cl*, **3** (1975), 461–466.
11. I. Bialynicki-Birula, J. Mycielski, Nonlinear wave mechanics, *Ann. Phys.*, **100** (1976), 62–93. https://doi.org/10.1016/0003-4916(76)90057-9
12. P. Górka, Logarithmic klein-gordon equation, *Acta Phys. Polon. B*, **40** (2009), 59–66.
13. X. Han, Global existence of weak solutions for a logarithmic wave equation arising from q-ball dynamics, *Bull. Korean Math. Soc.*, **50** (2013), 275–283.
14. H. Chen, P. Luo, G. Liu, Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity, *J. Math. Anal. Appl.*, **422** (2015), 84–98. https://doi.org/10.1016/j.jmaa.2014.08.030
15. M. M. Al-Gharabli, S. A. Messaoudi, Existence and a general decay result for a plate equation with nonlinear damping and a logarithmic source term, *J. Evol. Equ.*, **18** (2018), 105–125. https://doi.org/10.1007/s00028-017-0392-4
16. M. M. Al-Gharabli, A. Guesmia, S. Messaoudi, Existence and a general decay results for a viscoelastic plate equation with a logarithmic nonlinearity, *Commun. Pure Appl. Anal.*, **18** (2019), 159–180. http://dx.doi.org/10.3934/cpaa.2019009

17. X. Wang, Y. Chen, Y. Yang, J. Li, R. Xu, Kirchhoff-type system with linear weak damping and logarithmic nonlinearities, *Nonlinear Anal.*, **188** (2019), 475–499. https://doi.org/10.1016/j.na.2019.06.019

18. W. Lian, R. Xu, Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term, *Adv. Nonlinear Anal.*, **9** (2020), 613–632. https://doi.org/10.1515/anona-2020-0016

19. A. M. Al-Mahdi, Stability result of a viscoelastic plate equation with past history and a logarithmic nonlinearity, *Bound. Value Probl.*, **2020** (2020), 84. https://doi.org/10.1186/s13661-020-01382-9

20. M. M. Al-Gharabli, A. M. Al-Mahdi, M. Kafini, Global existence and new decay results of a viscoelastic wave equation with variable exponent and logarithmic nonlinearities, *AIMS Mathematics*, **6** (2021), 10105–10129. http://dx.doi.org/10.3934/math.2021587

21. E. Pişkin, S. Boulaaras, N. Irkil, Qualitative analysis of solutions for the p-laplacian hyperbolic equation with logarithmic nonlinearity, *Math. Methods Appl. Sci.*, **44** (2021), 4654–4672. https://doi.org/10.1002/mma.7058

22. H. Yüksekkaya, E. Piskin, Existence and exponential decay of a logarithmic wave equation with distributed delay, *Miskolc Math. Notes*, **24** (2023), 1057–1071. http://dx.doi.org/10.18514/MMN.2023.4155

23. H. Yüksekkaya, E. Piskin, M. M. Kafini, A. M. Al-Mahdi, Well-posedness and exponential stability for the logarithmic lamé system with a time delay, *Appl. Anal.*, **103** (2024), 506–518. https://doi.org/10.1080/00036811.2023.2196993

24. V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Differ. Equ.*, **109** (1994), 295–308. https://doi.org/10.1006/jdeq.1994.1051

25. L. Bociu, I. Lasiecka, Blow-up of weak solutions for the semilinear wave equations with nonlinear boundary and interior sources and damping, *Appl. Math.*, **35** (2008), 281–304. http://dx.doi.org/10.4064/am35-3-3

26. L. Bociu, I. Lasiecka, Local hadamard well-posedness for nonlinear wave equations with supercritical sources and damping, *J. Differ. Equ.*, **249** (2010), 654–683. https://doi.org/10.1016/j.jde.2010.03.009

27. D. Ieşan, On the theory of mixtures of thermoelastic solids, *J. Thermal Stresses*, **14** (1991), 389–408. https://doi.org/10.1080/01495739108927075

28. R. Quintanilla, Exponential stability for one-dimensional problem of swelling porous elastic soils with fluid saturation, *J. Comput. Appl. Math.*, **145** (2002), 525–533. https://doi.org/10.1016/S0377-0427(02)00442-9

29. J.-M. Wang, B.-Z. Guo, On the stability of swelling porous elastic soils with fluid saturation by one internal damping, *IMA J. Appl. Math.*, **71** (2006), 565–582. https://doi.org/10.1093/imamat/hxl009

30. A. J. A. Ramos, M. M. Freitas, D. S. Almeida Jr, A. S. Noé, M. J. D. Santos, Stability results for elastic porous media swelling with nonlinear damping, *J. Math. Phys.*, **61** (2020), 101505. https://doi.org/10.1063/5.0014121

31. T. A. Apalara, General decay of solutions in one-dimensional porous-elastic system with memory, *J. Math. Anal. Appl.*, **469** (2019), 457–471. https://doi.org/10.1016/j.jmaa.2017.08.007

32. A. Youkana, A. M. Al-Mahdi, S. A. Messaoudi, General energy decay result for a viscoelastic swelling porous-elastic system, *Z. Angew. Math. Phys.*, **73** (2022), 88. https://doi.org/10.1007/s00033-022-01696-x
33. A. M. Al-Mahdi, M. M. Al-Gharabli, T. A. Apalara, On the stability result of swelling porous-elastic soils with infinite memory, *Appl. Anal.*, **102** (2023), 4501–4517. https://doi.org/10.1080/00036811.2022.2120865
34. A. M. Al-Mahdi, S. A. Messaoudi, M. M. Al-Gharabli, A stability result for a swelling porous system with nonlinear boundary dampings, *J. Funct. Spaces*, **2022** (2022), 8079707. https://doi.org/10.1155/2022/8079707
35. R. Quintanilla, On the linear problem of swelling porous elastic soils with incompressible fluid, *Internat. J. Engrg. Sci.*, **40** (2002), 1485–1494. https://doi.org/10.1016/S0020-7225(02)00021-6
36. R. Quintanilla, Exponential stability of solutions of swelling porous elastic soils, *Meccanica*, **39** (2004), 139–145. https://doi.org/10.1023/B:MECC.0000005105.45175.61
37. T. A. A. Apalara, O. B. Almutairi, Well-posedness and exponential stability of swelling porous with Gurtin-Pipkin thermoelasticity, *Mathematics*, **10** (2022), 4498. https://doi.org/10.3390/math10234498
38. T. A. Apalara, A. Soufyane, Energy decay for a weakly nonlinear damped porous system with a nonlinear delay, *Appl. Anal.*, **101** (2022), 6113–6135. https://doi.org/10.1080/00036811.2021.1919642
39. T. A. Apalara, M. O. Yusuf, B. A. Salami, On the control of viscoelastic damped swelling porous elastic soils with internal delay feedbacks, *J. Math. Anal. Appl.*, **504** (2021), 125429. https://doi.org/10.1016/j.jmaa.2021.125429
40. T. A. Apalara, M. O. Yusuf, S. E. Mukiawa, O. B. Almutairi, Exponential stabilization of swelling porous systems with thermoelastic damping, *J. King Saud Univ. Sci.*, **35** (2023), 102460. https://doi.org/10.1016/j.jksus.2022.102460
41. A. M. AL-Mahdi, M. M. Al-Gharabli, I. Kissami, A. Soufyane, M. Zahri, Exponential and polynomial decay results for a swelling porous elastic system with a single nonlinear variable exponent damping: Theory and numerics, *Z. Angew. Math. Phys.*, **74** (2023), 72. https://doi.org/10.1007/s00033-023-01962-6
42. A. M. Al-Mahdi, M. M. Al-Gharabli, T. A. Apalara, On the stability result of swelling porous-elastic soils with infinite memory, *Appl. Anal.*, **102** (2023), 4501–4517. https://doi.org/10.1080/00036811.2022.2120865
43. A. M. Al-Mahdi, M. M. Al-Gharabli, M. Alahyane, Theoretical and computational results of a memory-type swelling porous-elastic system, *Math. Comput. Appl.*, **27** (2022), 27. https://doi.org/10.3390/mca27020027
44. M. M. Kafini, M. M. Al-Gharabli, A. M. Al-Mahdi, Asymptotic behavior of solutions to a nonlinear swelling soil system with time delay and variable exponents, *Math. Comput. Appl.*, **28** (2023), 94. https://doi.org/10.3390/mca28050094
45. M. M. Al-Gharabli, A. M. Al-Mahdi, S. A. Messaoudi, Decay results for a viscoelastic problem with nonlinear boundary feedback and logarithmic source term, *J. Dyn. Control Syst.*, **28** (2020), 71–89. https://doi.org/10.1007/s10883-020-09522-1

46. A. M. Al-Mahdi, The coupling system of Kirchhoff and Euler-Bernoulli plates with logarithmic source terms: Strong damping versus weak damping of variable-exponent type, *AIMS Mathematics*, **8** (2023), 27439–27459. http://dx.doi.org/10.3934/math.20231404
47. Y. Guo, M. A. Rammaha, S. Sakuntasathien, Energy decay of a viscoelastic wave equation with supercritical nonlinearities, *Z. Angew. Math. Phys.*, **69** (2018), 65. https://doi.org/10.1007/s00033-018-0961-6
48. S. A. Messaoudi, M. M. Al-Gharabli, A. M. Al-Mahdi, On the decay of solutions of a viscoelastic wave equation with variable sources, *Math. Methods Appl. Sci.*, **45** (2022), 8389–8411. https://doi.org/10.1002/mma.7141
49. M. M. Al-Gharabli, A. M. Al-Mahdi, Existence and stability results of a plate equation with nonlinear damping and source term, *Electron. Res. Arch.*, **30** (2022), 4038–4065. http://dx.doi.org/10.3934/era.2022205
50. L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Springer, 2011.
51. S. Antontsev, S. Shmarev, *Evolution PDEs with nonstandard growth conditions*, Atlantis Press Paris, 2015. https://doi.org/10.2991/978-94-6239-112-3
52. V. D. Radulescu, D. D. Repovs, *Partial differential equations with variable exponents: Variational methods and qualitative analysis*, CRC press, 2015.
53. L. Gross, Logarithmic sobolev inequalities, *Amer. J. Math.*, **97** (1975), 1061–1083. https://doi.org/10.2307/2373688
54. T. Cazenave, A. Haraux, Équations d'évolution avec non linéarité logarithmique, In: *Annales de la Faculté des sciences de Toulouse: Mathématiques*, **2** (1980), 21–51.
55. S. Messoaudi, M. Al-Gharabli, A. Al-Mahdi, On the existence and decay of a viscoelastic system with variable-exponent nonlinearity, *Discrete Contin. Dyn. Syst. Ser. S*, 2022. http://dx.doi.org/10.3934/dcdss.2022183
56. S. A. Messaoudi, M. M. Al-Gharabli, A. M. Al-Mahdi, M. A. Al-Osta, A coupled system of Laplacian and bi-Laplacian equations with nonlinear dampings and source terms of variable-exponents nonlinearities: Existence, uniqueness, blow-up and a large-time asymptotic behavior, *AIMS Mathematics*, **8** (2023), 7933–7966. http://dx.doi.org/10.3934/math.2023400
57. H. Chen, G. Liu, Global existence and nonexistence for semilinear parabolic equations with conical degeneration, *J. Pseudo-Differ. Oper. Appl.*, **3** (2012), 329–349. https://doi.org/10.1007/s11868-012-0046-9
58. Y. Liu, J. Zhao, On potential wells and applications to semilinear hyperbolic equations and parabolic equations, *Nonlinear Anal.*, **64** (2006), 2665–2687. https://doi.org/10.1016/j.na.2005.09.011
59. M. M. Al-Gharabli, S. A. Messaoudi, The existence and the asymptotic behavior of a plate equation with frictional damping and a logarithmic source term, *J. Math. Anal. Appl.*, **454** (2017), 1114–1128. https://doi.org/10.1016/j.jmaa.2017.05.030

AIMS Press