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Research article

A new reciprocity formula of Dedekind sums and its applications†

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†Dedicated to Professor Taekyun Kim on the occasion of his sixtieth birthday

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Abstract: Our main purpose of this article was using the analytic methods and the properties of Dirichlet L-functions to study the properties of Dedekind sums and give a new reciprocity formula for it. As its applications, some exact calculating formula for one kind mean square value of Dirichlet L-fuctions with the weight of the character sums were obtained.

Keywords: Dedekind sums; reciprocity formula; Dirichlet L-functions; the mean square value of L-function
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1. Introduction

In order to describe the results of this paper, we first introduce the definition of the Dedekind sums $S(r,q)$. For any integers $q \geq 2$ and r, the classical Dedekind sums $S(r,q)$ is defined as follows (see [1]):

$$S(r,q) = \sum_{c=1}^{q} \left(\left(\frac{c}{q}\right)\right)\left(\left(\frac{rc}{q}\right)\right),$$

where $((u))$ is usually defined as

$$((u)) = \begin{cases} u - [u] - \frac{1}{2}, & \text{if } u \text{ is not an integer;} \\ 0, & \text{if } u \text{ is an integer.} \end{cases}$$

Usually, we know that $S(r,q)$ describes the behavior of the logarithm for the eta-function (see [2,3]) under modular transformations. Because of the importance of $S(r,q)$ in analytic number theory, many authors have studied the various arithmetical properties of $S(r,q)$ and obtained a series of meaningful results (some of them can be found in [4–16]). To avoid complexity, we do not want to list them one by one. However, it is worth mentioning that Girstmair acquired an interesting result in [17], and it should be noted that perhaps the most important property of $S(r,q)$ is its reciprocity theorem (see [1,4,7]). That is, for any positive integers u and v with $(u,v)=1$, one has the following identity

$$S(u,v)+S(v,u)=\frac{u^2+v^2+1}{12uv}-\frac{1}{4}. \tag{1.1}$$

Obviously, this formula is not only looks very beautiful, but also reveals the profound properties between $S(u,v)$ and $S(v,u)$. Rademacher and Grosswald [3] also obtained a three-term formula similar to (1.1). Besides, there are many properties of Dedekind sums that are worth studying, and many scholars have achieved rich results. In particular, some of the new papers related to Dedekind sums can also be found in the references [18–22].

Our main purpose of this paper is using the analytic methods and the properties of Dirichlet L-functions to study the properties of $S(r,q)$ and give a new reciprocity formula for it, which is Lemma 3 in the paper. As its applications, we deduced several new calculating formula for the mean square value of Dirichlet L-functions with the weight of the character sums. In other words, we have the following three results:

Theorem 1. For any positive integer $q>1$ and $(q,6)=1$, we have the following identities

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(3)\cdot\overline{\chi}(2)\cdot|L(1,\chi)|^2=\frac{\pi^2}{18}\cdot\frac{\phi^2(q)}{q^2}\cdot\left[\frac{q}{4}\cdot\prod_{p|q}\left(1+\frac{1}{p}\right)-\frac{9}{2}+\left(\frac{q}{3}\right)\cdot\prod_{p|q}\frac{p-\left(\frac{p}{3}\right)}{p-1}\right],$$

where $\sum\limits_{\substack{\chi \bmod q\\ \chi(-1)=-1}}$ denotes the summation over all odd characters modulo q and $\prod\limits_{p|q}$ represents the product over all distinct prime divisors of q, $L(s,\chi)$ is the Dirichlet L-function corresponding to character χ mod q, $\phi(q)$ is the Euler function and $\left(\frac{*}{3}\right)$ is the Legendre symbol modulo 3.

Theorem 2. For any positive integer $q>1$ with $(q,6)=1$, we also have the identity

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}|L(1,\chi\lambda_3)|^2=\frac{\pi^2}{27}\cdot\frac{\phi^2(q)}{q^2}\left[2q\cdot\prod_{p|q}\left(1+\frac{1}{p}\right)-\left(\frac{q}{3}\right)\cdot\prod_{p|q}\frac{p-\left(\frac{p}{3}\right)}{p-1}\right],$$

where $\sum\limits_{\substack{\chi \bmod q\\ \chi(-1)=1}}$ denotes the summation over all even characters modulo q, and $\lambda_3=\left(\frac{*}{3}\right)$ denotes the Legendre symbol modulo 3.

Theorem 3. For any positive integer $q>1$ with $(q,6)=1$, we can obtain the identity

$$\sum_{\substack{\lambda \bmod q\\ \lambda(-1)=1}}\lambda(2)\cdot|L(1,\lambda\psi_3)|^2=-\frac{\pi^2}{27}\cdot\frac{\phi^2(q)}{q^2}\left[q\cdot\prod_{p|q}\left(1+\frac{1}{p}\right)-2\left(\frac{q}{3}\right)\cdot\prod_{p|q}\frac{p-\left(\frac{p}{3}\right)}{p-1}\right],$$

where

$$\psi_3=\left(\frac{*}{3}\right)$$

denotes the Legendre symbol modulo 3.

To illustrate the interest of this article, here we provide several numerical examples to justify our obtained results. If we take $q = 5$ in Theorems 1 and 2, we could obtain the following two corollaries:

Corollary 1. Let ψ be any non-real character modulo 5. We obtain the identity

$$|L(1,\psi)| = \frac{\sqrt{2}}{5} \cdot \pi.$$

Corollary 2. Let $\lambda = \left(\frac{*}{3}\right) \cdot \left(\frac{*}{5}\right)$ modulo 15. We have the identity

$$|L(1,\lambda)| = \frac{2\pi}{\sqrt{15}}.$$

If we take $q = 7$ in Theorem 2 and $q = 35$, Theorems 2 and 3, we can also have the following corollaries:

Corollary 3. Let χ be any non-real even character modulo 7. Then we have the identity

$$|L(1,\chi)| = \frac{2\pi}{\sqrt{21}}.$$

Corollary 4. Let ψ_3 denote the Legendre symbol modulo 3. Then we have

$$\sum_{\substack{\lambda \bmod 35 \\ \lambda(-1)=1}} \lambda(2) \cdot |L(1,\lambda\psi_3)|^2 = -\frac{9792}{11025} \cdot \pi^2.$$

Corollary 5. Let ψ_3 denote the Legendre symbol modulo 3. Then we have

$$\sum_{\substack{\chi \bmod 35 \\ \chi(-1)=1}} |L(1,\chi\psi_3)|^2 = \frac{416}{245} \cdot \pi^2.$$

Some notes: It is clear that by replacing $\chi(3)$ with $\chi(5)$ or $\chi(7)$ in Theorem 1 we can also get some similar results, but the situation is more complicated and we do not list them.

In addition, Theorems 2 and 3 show two interesting results. In fact, for the mean square value of Dirichlet L-functions with the even characters at point $s = 1$, there are so far only various asymptotic formulas, without any exact identities. For Theorems 2 and 3, we obtained two exact calculating formulae for them just by turning all characters χ with $\chi(-1) = 1$ into $\chi\lambda_3$ with $\chi\lambda_3(-1) = -1$.

2. Several lemmas

In this section, we will give three simple lemmas that are necessary in the proofs of three theorems. Hereinafter, we shall combine some knowledge of analytic number theory and the properties of the Dirichlet L-functions and Dedekind sums, which can be found in references [1,23,24], so we will not repeat them here. First, we have the following:

Lemma 1. Let $q > 2$ be an integer. Then for any integer v with $(v, q) = 1$, we have the following equation:

$$S(v,q) = \frac{1}{\pi^2 q} \sum_{h|q} \frac{h^2}{\phi(h)} \sum_{\substack{\chi \bmod h \\ \chi(-1)=-1}} \chi(v)|L(1,\chi)|^2,$$

where $L(s,\chi)$ denotes the Dirichlet L-function corresponding to χ mod h.

Proof. See [6, Lemma 2]. □

Lemma 2. Let q be a positive odd number. For any positive odd number r with $(r, q) = 1$, we can calculate the identity

$$S(r,2q) = 3 \cdot S(r,q) - S(2r,q) - S\left(\overline{2}r, q\right),$$

where $2 \cdot \overline{2} \equiv 1 \bmod q$.

Proof. Let χ_0 denote the principal character modulo 2. Note that $2 \nmid q$ and for any $h \mid q$, we know that

$$\phi(2h) = \phi(h)$$

and

$$(r, 2q) = 1.$$

According to the Lemma 1, we can obtain

$$\begin{aligned} S(r,2q) &= \frac{1}{2\pi^2 q} \sum_{h|2q} \frac{h^2}{\phi(h)} \sum_{\substack{\chi \bmod h \\ \chi(-1)=-1}} \chi(r)|L(1,\chi)|^2 \\ &= \frac{1}{2\pi^2 q} \sum_{h|q} \frac{h^2}{\phi(h)} \sum_{\substack{\chi \bmod h \\ \chi(-1)=-1}} \chi(r)|L(1,\chi)|^2 + \frac{1}{2\pi^2 q} \sum_{h|q} \frac{(2h)^2}{\phi(2h)} \sum_{\substack{\chi \bmod 2h \\ \chi(-1)=-1}} \chi(r)|L(1,\chi)|^2 \qquad (2.1) \\ &= \frac{1}{2} \cdot S(r,q) + \frac{2}{\pi^2 q} \sum_{h|q} \frac{h^2}{\phi(h)} \sum_{\substack{\chi \bmod h \\ \chi(-1)=-1}} \chi(r)|L(1,\chi\chi_0)|^2. \end{aligned}$$

For any non-principal character χ mod h with $h \mid q$, using the properties of Dirichlet L-functions and the Euler product formula (see [23, Theorem 11.6]), we can calculate that

$$\begin{aligned} |L(1,\chi\chi_0)|^2 &= \left| \prod_p \left(1 - \frac{\chi(p)\chi_0(p)}{p}\right)^{-1} \right|^2 \\ &= \left|1 - \frac{\chi(2)}{2}\right|^2 \cdot \left| \prod_p \left(1 - \frac{\chi(p)}{p}\right)^{-1} \right|^2 \qquad (2.2) \\ &= \left(\frac{5}{4} - \frac{\chi(2)}{2} - \frac{\overline{\chi}(2)}{2}\right) \cdot |L(1,\chi)|^2. \end{aligned}$$

Combining (2.1), (2.2) and Lemma 1, we have the identity

$$\begin{aligned}
S\,(r,2q) =&\frac{1}{2}\cdot S\,(r,q)+\frac{2}{\pi^2 q}\sum_{h|q}\frac{h^2}{\phi(h)}\sum_{\substack{\chi \bmod h\\ \chi(-1)=-1}}\chi(r)\left(\frac{5}{4}-\frac{\chi(2)}{2}-\frac{\overline{\chi}(2)}{2}\right)\cdot|L(1,\chi)|^2\\
=&\frac{1}{2}\cdot S\,(r,q)+\frac{5}{2}\cdot S\,(r,q)-\frac{1}{\pi^2 q}\sum_{h|q}\frac{h^2}{\phi(h)}\sum_{\substack{\chi \bmod h\\ \chi(-1)=-1}}\chi(2r)|L(1,\chi)|^2\\
&-\frac{1}{\pi^2 q}\sum_{h|q}\frac{h^2}{\phi(h)}\sum_{\substack{\chi \bmod h\\ \chi(-1)=-1}}\chi\left(\overline{2}r\right)|L(1,\chi)|^2\\
=&3\cdot S\,(r,q)-S\,(2r,q)-S\left(\overline{2}r,q\right).
\end{aligned}$$

This completes the proof of Lemma 2. □

Lemma 3. Let h and q be two positive odd numbers with $(h,q)=1$. Then we have the reciprocity formula

$$S\left(\overline{2}q,h\right)+S\left(\overline{2}h,q\right)=\frac{h^2+q^2+4}{24hq}-\frac{1}{4},$$

where $\overline{2}$ in $S\left(\overline{2}h,q\right)$ and $S\left(\overline{2}q,h\right)$ are $\frac{q+1}{2}$ and $\frac{h+1}{2}$, respectively.

Proof. For any positive odd numbers h and q with $(h,q)=1$, according to the Lemma 2, we can obtain

$$S\,(h,2q)=3\cdot S\,(h,q)-S\,(2h,q)-S\left(\overline{2}h,q\right) \tag{2.3}$$

and

$$S\,(q,2h)=3\cdot S\,(q,h)-S\,(2q,h)-S\left(\overline{2}q,h\right). \tag{2.4}$$

Applying (2.3), (2.4) and the reciprocity formula (1.1), we have the following identity:

$$S\,(h,2q)+S\,(2q,h)+S\,(q,2h)+S\,(2h,q)=3\cdot S\,(h,q)+3\cdot S\,(q,h)-S\left(\overline{2}q,h\right)-S\left(\overline{2}h,q\right)$$

or

$$\frac{4q^2+h^2+1}{24hq}-\frac{1}{4}+\frac{4h^2+q^2+1}{24qh}-\frac{1}{4}=3\cdot\left(\frac{h^2+q^2+1}{12hq}-\frac{1}{4}\right)-S\left(\overline{2}q,h\right)-S\left(\overline{2}h,q\right)$$

or

$$S\left(\overline{2}q,h\right)+S\left(\overline{2}h,q\right)=\frac{h^2+q^2+4}{24hq}-\frac{1}{4}.$$

This proves Lemma 3. □

It is clear that our Lemma 3 gave a new reciprocity formula for Dedekind sums.

3. Proofs of the theorems

In this section, we use the three simple lemmas in Section 2 and the reciprocity formula (1.1) to prove our three results. We will first prove Theorem 1.

Proof. Taking $h = 3$ in Lemma 3 and because of $2 \cdot 2 \equiv 1 \bmod 3$, if $q = 6k + 1$, from the definition of $S(h, q)$, we can obtain

$$S\left(\overline{2}q, 3\right) = S\left(2q, 3\right) = S(2, 3) = -S(1, 3) = -\frac{1}{18}. \tag{3.1}$$

If $q = 6k - 1$, from the definition of $S(h, q)$, we can also obtain

$$S\left(\overline{2}q, 3\right) = S\left(2q, 3\right) = S(-2, 3) = S(1, 3) = \frac{1}{18}. \tag{3.2}$$

Combining (3.1) and (3.2), for any odd positive integer q with $(q, 3) = 1$, we have

$$S\left(\overline{2}q, 3\right) = S\left(2q, 3\right) = -\frac{\left(\frac{q}{3}\right)}{18}. \tag{3.3}$$

Note that $S(r + hq, q) = S(r, q)$, from (3.3) and Lemma 3 we have

$$S\left(\overline{2} \cdot 3, q\right) = S\left(\frac{3q+3}{2}, q\right) = S\left(\frac{q+3}{2}, q\right) = \frac{q^2+13}{72q} - \frac{1}{4} + \frac{\left(\frac{q}{3}\right)}{18}. \tag{3.4}$$

According to the formula (3.1)–(3.3) and the properties of the Möbius function, we calculate

$$\begin{aligned}\sum_{d|q} \mu(d) \cdot \frac{q}{d} \cdot S\left(\frac{q}{d}, 3\right) &= \sum_{d|q} \mu(d) \cdot \frac{q}{d} \cdot \frac{\left(\frac{q/d}{3}\right)}{18} = \frac{q}{18} \cdot \left(\frac{q}{3}\right) \cdot \sum_{d|q} \frac{\mu(d) \cdot \left(\frac{d}{3}\right)}{d} \\ &= \phi(q) \cdot \frac{\left(\frac{q}{3}\right)}{18} \cdot \prod_{p|q} \frac{p - \left(\frac{p}{3}\right)}{p-1},\end{aligned} \tag{3.5}$$

where $\mu(n)$ denotes the Möbius function.

From the formula (3.4) and Lemma 1 we have

$$\frac{q}{72} - \frac{1}{4} + \frac{13}{72q} + \frac{\left(\frac{q}{3}\right)}{18} = \frac{1}{\pi^2 q} \sum_{k|q} \frac{k^2}{\phi(k)} \sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} \chi(3) \cdot \overline{\chi}(2) \cdot |L(1, \chi)|^2. \tag{3.6}$$

Note that (3.5), applying Möbius inversion formula (see [23, Theorem 2.9]) for (3.6) we have

$$\begin{aligned}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(3) \cdot \overline{\chi}(2) \cdot |L(1, \chi)|^2 &= \frac{\pi^2 \cdot \phi(q)}{q^2} \cdot \sum_{d|q} \mu(d) \left(\frac{q^2}{72d^2} - \frac{1}{4} \cdot \frac{q}{d} + \frac{\left(\frac{q/d}{3}\right)}{18} + \frac{13}{72}\right) \\ &= \pi^2 \cdot \frac{\phi^2(q)}{q^2} \cdot \left[\frac{1}{72} \cdot q \cdot \prod_{p|q}\left(1 + \frac{1}{p}\right) - \frac{1}{4} + \frac{\left(\frac{q}{3}\right)}{18} \cdot \prod_{p|q} \frac{p - \left(\frac{p}{3}\right)}{p-1}\right].\end{aligned}$$

The proof of Theorem 1 is finished. □

Next we prove Theorem 2.

Proof. For any odd number $q > 3$ with $(q, 3) = 1$, we have

$$S(1,q) = \sum_{a=1}^{q-1}\left(\frac{a}{q}-\frac{1}{2}\right)^2 = \frac{q}{12}-\frac{1}{4}+\frac{1}{6q}. \tag{3.7}$$

On the basis of (3.7), Lemma 1 and the Möbius inversion formula, we can obtain

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{12}\cdot\frac{\phi^2(q)}{q^2}\left[q\cdot\prod_{p|q}\left(1+\frac{1}{p}\right)-3\right] \tag{3.8}$$

and

$$\sum_{\substack{\chi \bmod 3q \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{27}\cdot\frac{\phi^2(q)}{q^2}\left[4q\cdot\prod_{p|q}\left(1+\frac{1}{p}\right)-3\right]. \tag{3.9}$$

According to the formulae (1.1), (3.3), (3.5) and the Möbius inversion formula, we also calculate the

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(3)\cdot|L(1,\chi)|^2 = \frac{\pi^2}{36}\cdot\frac{\phi^2(q)}{q^2}\left[q\cdot\prod_{p|q}\left(1+\frac{1}{p}\right)-9-2\left(\frac{q}{3}\right)\cdot\prod_{p|q}\frac{p-\left(\frac{p}{3}\right)}{p-1}\right]. \tag{3.10}$$

On the other hand, let $\lambda_3 = \left(\frac{*}{3}\right)$ denote the Legendre symbol modulo 3, and λ_3^0 denote the principal character modulo 3. Then note that

$$\begin{aligned}
\sum_{\substack{\chi \bmod 3q \\ \chi(-1)=-1}} |L(1,\chi)|^2 &= \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L\left(1,\chi\lambda_3^0\right)|^2 + \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} |L(1,\chi\lambda_3)|^2 \\
&= \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left|1-\frac{\chi(3)}{3}\right|^2\cdot|L(1,\chi)|^2 + \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} |L(1,\chi\lambda_3)|^2 \\
&= \frac{10}{9}\cdot\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} |L(1,\chi)|^2 - \frac{2}{3}\cdot\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(3)\cdot|L(1,\chi)|^2 + \sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} |L(1,\chi\lambda_3)|^2.
\end{aligned} \tag{3.11}$$

Then combining (3.8)–(3.11) we have

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} |L(1,\chi\lambda_3)|^2 = \frac{\pi^2}{27}\cdot\frac{\phi^2(q)}{q^2}\left[2q\cdot\prod_{p|q}\left(1+\frac{1}{p}\right)-\left(\frac{q}{3}\right)\cdot\prod_{p|q}\frac{p-\left(\frac{p}{3}\right)}{p-1}\right].$$

The proof of Theorem 2 is finished. □

Last, we will prove Theorem 3.

Proof. Combining with the formula (1.1), Lemma 1 and the Möbius inversion formula, we have

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(2) \cdot |L(1,\chi)|^2 = \frac{\pi^2}{24} \cdot \frac{\phi^2(q)}{q^2} \left[q \cdot \prod_{p|q} \left(1 + \frac{1}{p}\right) - 6 \right] \tag{3.12}$$

and

$$\sum_{\substack{\chi \bmod 3q \\ \chi(-1)=-1}} \chi(2) \cdot |L(1,\chi)|^2 = \frac{\pi^2}{27} \cdot \frac{\phi^2(q)}{q^2} \left[2q \cdot \prod_{p|q} \left(1 + \frac{1}{p}\right) - 3 \right]. \tag{3.13}$$

For any positive integer q with $(q,6) = 1$, note that the identity

$$S(q,6) = \frac{5}{18} \cdot \left(\frac{q}{3}\right).$$

We can calculate

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(6) \cdot |L(1,\chi)|^2 = \frac{\pi^2}{72} \cdot \frac{\phi^2(q)}{q^2} \left[q \cdot \prod_{p|q} \left(1 + \frac{1}{p}\right) - 18 - 20\left(\frac{q}{3}\right) \cdot \prod_{p|q} \frac{p - \left(\frac{p}{3}\right)}{p-1} \right]. \tag{3.14}$$

On the other hand, let χ_q^0 denote the principal character modulo q and $\psi_3 = \left(\frac{*}{3}\right)$. Then $\psi_3(2) = -1$, so we also obtain

$$\begin{aligned}
\sum_{\substack{\chi \bmod 3q \\ \chi(-1)=-1}} \chi(2) \cdot |L(1,\chi)|^2 &= \sum_{\substack{\psi \bmod 3 \\ \psi(-1)=1}} \sum_{\substack{\lambda \bmod q \\ \lambda(-1)=-1}} \lambda(2)\psi(2) \cdot |L(1,\lambda\psi)|^2 \\
&\quad + \sum_{\substack{\psi \bmod 3 \\ \psi(-1)=-1}} \sum_{\substack{\lambda \bmod q \\ \lambda(-1)=1}} \chi(2)\psi(2) \cdot |L(1,\lambda\psi)|^2 \\
&= \sum_{\substack{\lambda \bmod q \\ \lambda(-1)=-1}} \chi(2) \cdot \left| L\left(1, \lambda\chi_3^0\right) \right|^2 - \sum_{\substack{\lambda \bmod q \\ \lambda(-1)=1}} \lambda(2) \, |L(1,\lambda\psi_3)|^2 \\
&= \sum_{\substack{\lambda \bmod q \\ \lambda(-1)=-1}} \lambda(2) \cdot \left| 1 - \frac{\lambda(3)}{3} \right|^2 \cdot |L(1,\lambda)|^2 - \sum_{\substack{\lambda \bmod q \\ \lambda(-1)=1}} \lambda(2) \cdot |L(1,\lambda\psi_3)|^2 \\
&= \frac{10}{9} \sum_{\substack{\lambda \bmod q \\ \lambda(-1)=-1}} \lambda(2) \cdot |L(1,\lambda)|^2 - \frac{1}{3} \sum_{\substack{\lambda \bmod q \\ \lambda(-1)=-1}} \lambda(6) \cdot |L(1,\lambda)|^2 \\
&\quad - \frac{1}{3} \sum_{\substack{\lambda \bmod q \\ \lambda(-1)=-1}} \lambda(2)\overline{\lambda}(3) \cdot |L(1,\lambda)|^2 - \sum_{\substack{\lambda \bmod q \\ \lambda(-1)=1}} \lambda(2) \, |L(1,\lambda\psi_3)|^2 .
\end{aligned} \tag{3.15}$$

Now, combining Theorem 1 and (3.12)–(3.15), we have the identity

$$\sum_{\substack{\lambda \bmod q \\ \lambda(-1)=1}} \lambda(2) \cdot |L(1,\lambda\psi_3)|^2 = -\frac{\pi^2}{27} \cdot \frac{\phi^2(q)}{q^2} \left[q \cdot \prod_{p|q} \left(1 + \frac{1}{p}\right) - 2\left(\frac{q}{3}\right) \cdot \prod_{p|q} \frac{p - \left(\frac{p}{3}\right)}{p-1} \right].$$

This completes the proof of Theorem 3. □

4. Conclusions

Our main purpose of this paper is to give a new reciprocity theorem for Dedekind sums (see Lemma 3). As an application of this result, we give a new calculating formula for one kind mean square value of Dirichlet L-fuctions with the weight of the character sums. That is, we proved that for any positive integer $q > 1$ and $(q, 6) = 1$, one has the identity

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(3) \cdot \overline{\chi}(2) \cdot |L(1,\chi)|^2 = \frac{\pi^2}{18} \cdot \frac{\phi^2(q)}{q^2} \cdot \left[\frac{q}{4} \cdot \prod_{p|q}\left(1 + \frac{1}{p}\right) - \frac{9}{2} + \left(\frac{q}{3}\right) \cdot \prod_{p|q} \frac{p - \left(\frac{p}{3}\right)}{p - 1}\right],$$

where $\left(\frac{*}{3}\right)$ denote the Legendre symbol modulo 3.

In addition, according to the reciprocity formula (1.1) and the Lemma 1, we may immediately deduce that for any two distinct odd primes p and q, one has the identity

$$\frac{q}{q-1} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(p) \cdot |L(1,\chi)|^2 + \frac{p}{p-1} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(q) \cdot |L(1,\chi)|^2 = \frac{\pi^2}{12} \cdot \frac{p^2 + q^2 - 3pq + 1}{pq}. \tag{4.1}$$

Whether there exists a direct proof of (4.1) (using only the properties of Dirichlet L-function, the reciprocity formula (1.1) cannot be used) is an open problem. It is believed that the methods used in this paper will contribute to further research in relevant fields.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. T. M. Apostol, *Modular functions and Dirichlet series in number theory*, Springer-Verlag, 1990. https://doi.org/10.1007/978-1-4612-0999-7
2. H. Rademacher, On the transformation of $\log\eta(\tau)$, *J. Indian Math. Soc.*, **19** (1955), 25–30. https://doi.org/10.1215/S0012-7094-74-04132-5

3. H. Rademacher, E. Grosswald, Dedekind sums, Carus mathematical monographs, *Math. Assoc. Amer.*, 1972.
4. L. Carlitz, The reciprocity theorem for Dedekind sums, *Pacific J. Math.*, **3** (1953), 523–527. https://doi.org/10.2140/PJM.1953.3.523
5. J. B. Conrey, E. Fransen, R. Klein, C. Scott, Mean values of Dedekind sums, *J. Number Theory*, **56** (1996), 214–226. https://doi.org/10.1006/jnth.1996.0014
6. W. P. Zhang, On the mean values of Dedekind sums, *J. Theor. Nombres Bordeaux*, **8** (1996), 429–442. https://doi.org/10.5802/JTNB.179
7. L. J. Mordell, The reciprocity formula for Dedekind sums, *Amer. J. Math.*, **73** (1951), 593–598. https://doi.org/10.2307/2372310
8. Y. N. Liu, W. P. Zhang, A hybrid mean value related to the Dedekind sums and Kloosterman sums, *Acta Math. Sin. Engl. Ser.*, **27** (2011), 435–440. https://doi.org/10.1007/s10114-010-9192-2
9. W. P. Zhang, On a note of one class mean square value of L-functions, *J. Northwest Univ.*, **20** (1990), 9–12.
10. W. P. Zhang, A sum analogous to Dedekind sums and its hybrid mean value formula, *Acta Arith.*, **107** (2003), 1–8. https://doi.org/10.4064/aa107-1-1
11. E. Tsukerman, Fourier-Dedekind sums and an extension of Rademacher reciprocity, *Ramanujan J.*, **37** (2015), 421–460. https://doi.org/10.1007/s11139-014-9555-x
12. W. Kohnen, A short note on Dedekind sums, *Ramanujan J.*, **45** (2018), 491–495. https://doi.org/10.1007/s11139-016-9851-8
13. Y. W. Chen, N. Dunn, S. Silas, Dedekind sums $s(a, b)$ and inversions modulo b, *Int. J. Number Theory*, **11** (2015), 2325–2339. https://doi.org/10.1142/S1793042115501067
14. W. P. Zhang, *On the fourth power mean of Dirichlet L-functions*, Science Press, 1989, 173–179.
15. D. Han, W. P. Zhang, Some new identities involving Dedekind sums and the Ramanujan sum, *Ramanujan J.*, **35** (2014), 253–262. https://doi.org/10.1007/s11139-014-9591-6
16. S. Macourt, An integer-valued expression of Dedekind sums, *Int. J. Number Theory*, **13** (2017), 1253–1259. https://doi.org/10.1142/S1793042117500683
17. K. Girstmair, On a recent reciprocity formula for Dedekind sums, *Int. J. Number Theory*, **15** (2019), 1469–1472. https://doi.org/10.1142/S1793042119500842
18. Z. Y. Zheng, M. Chen, J. Xu, On Gauss sums over Dedekind domains, *Int. J. Number Theory*, **18** (2022), 1047–1063. https://doi.org/10.1142/S1793042122500543
19. Y. K. Ma, L. L. Luo, T. Kim, H. Z. Li, A study on a type of poly-Dedekind type DC sums, *J. Northwest Univ. Natl. Sci. Edit.*, **53** (2023), 438–442.
20. M. C. Dagli, On the hybrid mean value of generalized Dedekind sums, generalized Hardy sums and Ramanujan sum, *Bull. Math. Soc. Sci. Math. Roum.*, **63** (2020), 325–333.
21. E. Nguyen, J. J. Ramirez, M. P. Young, The kernel of newform Dedekind sums, *J. Number Theory*, **223** (2021), 53–63. https://doi.org/10.1016/j.jnt.2020.10.005

22. M. Majure, Algebraic properties of the values of newform Dedekind sums, *J. Number Theory*, **250** (2023), 35–48. https://doi.org/10.1016/j.jnt.2023.03.004

23. T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, 1976. https://doi.org/10.1007/978-1-4757-5579-4

24. K. Ireland, M. Rosen, *A classical introduction to modern number theory*, Springer-Verlag, 1990. https://doi.org/10.1007/978-1-4757-2103-4