AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 12800–12813.
DOI: 10.3934/math.2024625
Received: 19 February 2024
Revised: 23 March 2024
Accepted: 01 April 2024
Published: 03 April 2024

*Research article*

# Initial value problems for fractional p-Laplacian equations with singularity

**Mahir Hasanov***

Istanbul Beykent University, Faculty of Arts-Sciences, Department of Mathematics, Istanbul, Turkiye

* **Correspondence:** Email: hasanov61@yahoo.com.

**Abstract:** We have studied initial value problems for Caputo fractional differential equations with singular nonlinearities involving the p-Laplacian operator. We have given a precise mathematical analysis of the equivalence of the fractional differential equations and Volterra integral equations studied in this paper. A theorem for the global existence of the solution was proven. In addition, an example was given at the end of the article as an application of the results found in this paper.

**Keywords:** initial value problems; Caputo fractional differential equations; fractional derivatives; fractional p-Laplacian; existence theorem
**Mathematics Subject Classification:** 26A33, 34A08, 34A12

## 1. Introduction and preliminaries

Interest in the subject of fractional differential equations has increased greatly over the past decades. Fractional differential equations appear in many fields such as physics, aerodynamics, electro-dynamics, and control theory (see [1, 14, 17, 21, 24, 31–33]).

We studied the following singular nonlinear initial value problem involving p-Laplacian

$$\begin{cases} D_C^{\beta}\phi_p(D_C^{\alpha}x(t)) = t^{-\gamma}f(t, x(t)), \\ x(0) = x_0,\ D_C^{\alpha}x(0) = x_{\alpha},\ \ x_0,\ x_{\alpha} \in \mathbb{R}, \\ 0 \le \gamma < \alpha, \beta \le 1, \\ x \in AC[0, 1],\ D_C^{\alpha}x \in AC[0, 1]. \end{cases} \tag{1.1}$$

$f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\phi_p(t) = |t|^{p-2}t,\ p > 1$, and $D_C^{\alpha}x(t)$ denotes the Caputo fractional derivative which is defined by $D_C^{\alpha}x(t) := I^{1-\alpha}Dx(t)$ for $0 < \alpha \le 1$, where $I^{\alpha}$ stands for the Riemann-Liouville fractional integral of order $\alpha > 0$ defined by

$$I^{\alpha}x(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds, \quad x \in L^1,$$

where $L^p := L^p[0,1] = \{x(t) : \int_0^1 |x(t)|^p < \infty,\ 1 \leq p < \infty\}$ and $\Gamma(\alpha) := \int_0^\infty s^{\alpha-1} exp(-s)\, ds$, $\alpha > 0$ (see [14, 31–33]). A literature review will be given at the end of this section. Since the problem studied in this paper contains a singular term, we believe that the obtained results are new.

We denote the domain of an operator $T$ by $Dom(T)$. The domain of the Caputo derivative is defined by $Dom(D_C^\alpha) = \{x(t) : x \in AC[0,1]\}$, where $AC[0,1]$ is the set of absolutely continuous functions on the interval $[0,1]$.

In general, for any $m \in \mathbb{N}$, and $0 < \alpha \leq 1$, $D_C^{m+\alpha}x(t) = I^{1-\alpha}D^{m+1}x(t)$ with $Dom(D_C^{m+\alpha}) = \{x(t) : D^m x \in AC[0,1]\}$.

Besides the Caputo derivative, we will also use the Riemann-Liouville and generalized Caputo derivatives.

For $0 < \alpha \leq 1$, the Riemann-Liouville derivative is defined by $D^\alpha x(t) := DI^{1-\alpha}x(t)$, with $Dom(D^\alpha) = \{x(t) : I^{1-\alpha}x \in AC[0,1]\}$. In general, for any $m \in \mathbb{N}$, and $0 < \alpha \leq 1$, $D^{m+\alpha}x(t) = D^{m+1}I^{1-\alpha}x(t)$, with $Dom(D^{m+\alpha}) = \{x(t) : D^m(I^{1-\alpha}x) \in AC[0,1]\}$.

For $0 < \alpha \leq 1$, the generalized Caputo derivative is defined by $D_*^\alpha x(t) := D^\alpha(x(t) - x_0)$, with $Dom(D_*^\alpha) = \{x(t) : I^{1-\alpha}(x - x_0) \in AC[0,1]\}$, where $x_0 = x(0)$.

We studied initial value problems on the interval $[0,1]$, but all results in this paper are also valid for any interval $[0,T]$.

Let us give some properties of $D^\alpha$, $D_C^\alpha$, $D_*^\alpha$, and $I^\alpha$ that we will use frequently in this paper.

**Proposition 1.1.** *a)* $I^\alpha : L^p[0,1] \to L^p[0,1]$, $1 \leq p \leq \infty$ *is a bounded operator for all* $\alpha > 0$ *( [36], Proposition 3.2).*

*b)* $I^\alpha : C[0,1] \to C[0,1]$ *is a bounded operator for all* $\alpha > 0$ *( [36], Proposition 3.2), where* $C[0,1]$ *denotes the normed space of all continuous functions defined on the interval* $[0,1]$*, with the norm* $\|x\| = max_{t\in[0,1]}|x(t)|$.

*c) If* $\alpha, \beta > 0$ *and* $x \in L^1[0,1]$*, then* $I^\alpha I^\beta x(t) = I^{\alpha+\beta}x(t)$ *for a.e.* $t \in [0,1]$*. Moreover, if* $\alpha + \beta \geq 1$ *and* $x \in L^1[0,1]$*, then* $I^\alpha I^\beta x(t) = I^{\alpha+\beta}x(t)$ *for all* $t \in [0,1]$ *( [36], Lemma 3.4).*

*d) For* $0 < \alpha \leq 1$ *and* $m \in \mathbb{N}$*, the operator* $D^{m+\alpha}$ *is the left inverse of the operator* $I^{m+\alpha}$ *in* $L^1[0,1]$*, i.e.,* $D^{m+\alpha}I^{m+\alpha}x(t) = x(t)$*, for all* $x \in L^1[0,1]$ *( [36], Lemma 4.2).*

*e) Let* $0 < \alpha < 1$*. The operator* $D_*^\alpha$ *is the left inverse of the operator* $I^\alpha$ *in the space* $C[0,1]$*, i.e.,* $D_*^\alpha I^\alpha x(t) = x(t)$*, for all* $x \in C[0,1]$ *( [36], Lemma 4.5).*

*f) If* $0 < \alpha < 1$ *and* $x(t) \in AC[0,1]$*, then* $D_*^\alpha x(t) = D^\alpha(x(t) - x(0)) = D_C^\alpha x(t)$ *( [36], Proposition 4.4). Hence, the operator* $D_C^\alpha$ *is the left inverse of the operator* $I^\alpha$ *in the space* $AC[0,1]$*, i.e.,* $D_C^\alpha I^\alpha x(t) = x(t)$*, for all* $x \in AC[0,1]$.

*But, in general,* $D_C^\alpha$ *is not a right inverse for* $I^\alpha$*. However, the following formula holds.*

$$I^\alpha D_C^\alpha x(t) = x(t) - x(0), \ \text{for all } x \in AC[0,1].$$

In general, fractional differential equations include the Caputo fractional derivative, generalized Caputo fractional derivative, and Riemann-Liouville fractional derivative. Initial and boundary value problems containing Riemann-Liouville fractional derivatives were studied by many authors (see [3, 6, 7, 12, 20, 22, 38] and references therein). The application areas of Riemann-Liouville fractional differential equations have been gradually expanding in recent years (see [26, 27]).

In the past decades, different aspects of Caputo fractional differential equations were studied by a number of researchers. For example, initial value problems were studied in [13, 15, 23, 28, 36, 37]. The articles [2,4,5,8,16,25,30,34,35] deal with boundary value problems for Caputo fractional differential equations. However, none of these articles contain the fractional p-Laplacian.

Chen et al. [9] studied the existence of antiperiodic solutions for the Lienard-type p-Laplacian equation. Chen et al. [10] studied the solvability of periodic boundary value problem for the fractional p-Laplacian equation in the following form:

$$\begin{gathered} D_C^{\beta}\phi_p(D_C^{\alpha}x(t)) = f(t, x(t), D_C^{\alpha}x(t)), \\ x(0) = x(T),\ D_C^{\alpha}x(0) = D_C^{\alpha}x(T), \\ f \in C([0,T] \times \mathbb{R}),\ 0 < \alpha, \beta \leq 1. \end{gathered}$$

The existence of solutions for the anti-periodic boundary value problem of a fractional p-Laplacian equation was studied by Chen et al. [11]. They studied the following problem:

$$\begin{gathered} D_C^{\beta}\phi_p(D_C^{\alpha}x(t)) = f(t, x(t)), \\ x(0) = -x(1),\ D_C^{\alpha}x(0) = -D_C^{\alpha}x(1), \\ f \in C([0,T] \times \mathbb{R}),\ 0 < \alpha, \beta \leq 1. \end{gathered}$$

Some anti-periodic problems were also considered in [19,30].

Note that, none of these articles contain singularity. Since we are dealing with singular initial value problems, we would like to give a brief overview of some related results.

In [3], Agarwal et al. studied the existence of positive solutions for the following singular fractional boundary value problem with the Riemann-Liouville fractional derivative:

$$\begin{gathered} D^{\alpha}u(t) + f(t, u(t), D^{\mu}u(t)) = 0,\ 1 < \alpha < 2,\ 0 < \mu \leq \alpha - 1, \\ u(0) = u(1) = 0,\ f \text{ is singular at } 0. \end{gathered}$$

In [4], Agarwal et al. proved the existence of positive solutions to the singular fractional boundary value problem with the Caputo fractional derivative:

$$\begin{gathered} D_C^{\alpha}u(t) + f(t, u(t), u', D_C^{\mu}u(t)) = 0,\ 1 < \alpha < 2,\ 0 < \mu < 1, \\ u'(0) = 0, u(1) = 0,\ f \text{ is singular at } 0. \end{gathered}$$

Webb [36] studied initial value problems for Caputo fractional differential equations with singular nonlinearities in the forms:

$$\begin{gathered} D_C^{m+\alpha}u(t) = t^{-\gamma}f(t, u(t)),\ 0 < \alpha < 1,\ 0 \leq \gamma < \alpha, \\ u'(0) = u_0, ..., u^m(0) = u_m,\ D^m u \in AC[0,T], f \in C([0,T] \times \mathbb{R}). \end{gathered}$$

and

$$\begin{gathered} D_*^{1+\alpha}u(t) = t^{-\gamma}f(t, u(t), D_C^{\beta}u(t)),\ 0 \leq \gamma < \alpha < 1,\ 0 < \beta \leq 1, \\ u'(0) = u_0,\ u'(0) = u_1,\ f \in C([0,T] \times \mathbb{R} \times \mathbb{R}). \end{gathered}$$

The proof of the existence of a solution to these problems is based on the Leray-Schauder fixed point theorem and Gronwall type inequalities. Gronwall inequalities are widely used to obtain a priori bounds ( see [18,29,37–39]).

We note that, for the main concepts used in this article, we generally followed [11,36,37].

This paper consists of three sections. The first section includes the introduction and preliminary information. The second section includes a precise mathematical analysis of the equivalence of the fractional differential equations and Volterra integral equations studied in this paper. The existence of solutions to initial value problems is discussed in Section 3.

## 2. Volterra integral equations

In this section, we establish a relationship between Caputo fractional differential equations and Volterra integral equations.

**Theorem 2.1.** *Let* $f \in C([0,1] \times \mathbb{R})$ *and* $0 \leq \gamma < \alpha, \beta \leq 1$. *If a function* $x(t)$ *satisfies the initial value problem*

$$\begin{cases} D_C^{\beta}\phi_p(D_C^{\alpha}x(t)) = t^{-\gamma}f(t,x(t)), \\ x(0) = x_0,\ D_C^{\alpha}x(0) = x_\alpha, \quad x_0,\ x_\alpha \in \mathbb{R}, \\ x \in AC[0,1],\ D_C^{\alpha}x \in AC[0,1], \end{cases} \tag{2.1}$$

*then* $x(t)$ *satisfies the Volterra integral equation*

$$x(t) = I^{\alpha}\phi_q\big[I^{\beta}(t^{-\gamma}f(t,x(t))) + x_1\big] + x_0, \tag{2.2}$$

*where* $x_1 = \phi_p(x_\alpha)$, $p, q > 1$, *and* $1/p + 1/q = 1$.

*Proof.* The condition $x \in AC[0,1]$ implies $x \in Dom(D_C^{\alpha})$ and $D_C^{\alpha}x \in AC[0,1]$ implies that $\phi_p(D_C^{\alpha}x) \in AC[0,1]$. Consequently, $\phi_p(D_C^{\alpha}x) \in Dom(D_C^{\beta})$. This means that under these conditions the problem (2.1) is well-defined. Here, for the inclusion $\phi_p(D_C^{\alpha}x) \in AC[0,1]$, we have used the composition rule: If the functions $F : [c,d] \to \mathbb{R}$ and $G : [a,b] \to [c,d]$ are absolutely continuous, then $F(G(t))$ is also absolutely continuous.

Let $x(t)$ satisfy (2.1). Since $D_C^{\beta}\phi_p(D_C^{\alpha}x(t)) \in L^1$, we can apply $I^{\beta}$ to (2.1). Then

$$I^{\beta}D_C^{\beta}\phi_p(D_C^{\alpha}x(t)) = I^{\beta}(t^{-\gamma}f(t,x(t)))$$

or the same

$$I^{\beta}I^{1-\beta}D(\phi_p(D_C^{\alpha}x(t))) = I^{\beta}(t^{-\gamma}f(t,x(t))).$$

$D(\phi_p(D_C^{\alpha}x(t)) \in L^1$, then by Proposition 1.1 c) we have

$$ID(\phi_p(D_C^{\alpha}x(t))) = I^{\beta}(t^{-\gamma}f(t,x(t))).$$

Hence, by using $\phi_p(D_C^{\alpha}x) \in AC[0,1]$ we get

$$\phi_p(D_C^{\alpha}x(t)) - \phi_p(D_C^{\alpha}x(0)) = I^{\beta}(t^{-\gamma}f(t,x(t)))$$

and

$$D_C^{\alpha}x(t) = \phi_q[I^{\beta}(t^{-\gamma}f(t,x(t))) + x_1], \tag{2.3}$$

where $x_1 = \phi_p(D_C^{\alpha}x(0)) = \phi_p(x_\alpha)$.

Applying the operator $I^{\alpha}$ to both sides of (2.3) and using Proposition 1.1 f) we get the Volterra integral equation (2.2):

$$x(t) = I^{\alpha}\phi_q\big[I^{\beta}(t^{-\gamma}f(t,x(t))) + x_1\big] + x_0.$$

□

Thus, (2.1) implies (2.2). But in general (2.2) does not imply (2.1). More precisely, if $x \in C[0,1]$ satisfies the Volterra integral equation (2.2) then, in general, it does not follow that $x \in AC[0,1]$ and $D_C^\alpha x \in AC[0,1]$. However, if we use the generalized Caputo derivative $D_*^\alpha$ instead of the Caputo derivative $D_C^\alpha$ then we can show that they are equivalent. The equivalence result is given in Theorem 2.2.

Additionally, we note that $I^\alpha : AC[0,1] \to AC[0,1]$ and consequently

$$AC[0,1] = Dom(D_C^\alpha) \subset Dom(D_*^\alpha) = \{x(t) : I^{1-\alpha}(x - x_0) \in AC[0,1]\}.$$

On the other hand, if $0 < \alpha < 1$ and $x \in Dom(D_C^\alpha)$, then $D_*^\alpha x(t) = D_C^\alpha x(t)$ (Proposition 1.1 f)). This means that the operator $D_*^\alpha$ is an extension of the operator $D_C^\alpha$.

Now we give the equivalence result.

**Theorem 2.2.** *Let* $f \in C([0,1] \times \mathbb{R})$ *and* $0 \le \gamma < \alpha, \beta \le 1$. *A function* $x \in C[0,1]$ *satisfies the Volterra integral equation*

$$x(t) = I^\alpha \phi_q\big[I^\beta(t^{-\gamma} f(t, x(t))) + x_1\big] + x_0, \quad x_0, x_1 \in \mathbb{R} \tag{2.4}$$

*if and only if* $x \in C[0,1]$ *is a solution of the following initial value problem*

$$\begin{cases} D_*^\beta \phi_p(D_*^\alpha x(t)) = t^{-\gamma} f(t, x(t)), \\ x(0) = x_0,\ D_*^\alpha x(0) = x_\alpha^*, \\ x \in Dom(D_*^\alpha),\ \ \phi_p(D_*^\alpha x) \in Dom(D_*^\beta),\ \ D_*^\alpha x \in C[0,1], \end{cases} \tag{2.5}$$

where $x_\alpha^* = \phi_q(x_1)$.

*Proof.* Let $x \in C[0,1]$ satisfy the Volterra integral equation (2.4). By using the substitution $s = \sigma t$ we can write

$$I^\beta(t^{-\gamma} f(t, x(t))) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{-\gamma} f(s, x(s))\, ds =$$

$$\frac{t^{\beta-\gamma}}{\Gamma(\beta)} \int_0^1 (1-\sigma)^{\beta-1} \sigma^{-\gamma} f(t\sigma, x(t\sigma))\, d\sigma.$$

This equality together with the conditions $f \in C([0,1] \times \mathbb{R})$ and $0 \le \gamma < \beta \le 1$ yields

$$\phi_q(I^\beta(t^{-\gamma} f(t, x(t)))) \in C[0,1]. \tag{2.6}$$

It follows from (2.6) that $I^\alpha \phi_q[I^\beta(t^{-\gamma} f(t, x(t))) + x_1]\Big|_{t=0} = 0$ and consequently $x(0) = x_0$. Applying $I^{1-\alpha}$ to (2.4) and using Proposition 1.1. c) we get

$$I^{1-\alpha}(x(t) - x_0) = I\phi_q\big[I^\beta(t^{-\gamma} f(t, x(t))) + x_1\big] \in C^1[0,1] \subset AC[0,1].$$

This means that $x \in Dom(D_*^\alpha)$. By Proposition 1.1 d)

$$\begin{gathered} D_*^\alpha x(t) = D^\alpha(x(t) - x_0) = D^\alpha I^\alpha \phi_q\big[I^\beta(t^{-\gamma} f(t, x(t))) + x_1\big] = \\ \phi_q\big[I^\beta(t^{-\gamma} f(t, x(t))) + x_1\big]. \end{gathered} \tag{2.7}$$

So, $D_*^\alpha x \in C[0,1]$ and $D_*^\alpha x(0) = \phi_q(x_1) = x_\alpha^*$. Applying $\phi_p$ to both sides of (2.7) and using the fact that $t^{-\gamma} f(t, x(t)) \in L^1$ we obtain

$$\begin{gathered}
\phi_p(D_*^\alpha x(t)) = I^\beta(t^{-\gamma} f(t, x(t))) + x_1 \Rightarrow \\
\phi_p(D_*^\alpha x(t)) - \phi_p(D_*^\alpha x(0)) = I^\beta(t^{-\gamma} f(t, x(t))) \Rightarrow \\
I^{1-\beta}[\phi_p(D_*^\alpha x(t)) - \phi_p(D_*^\alpha x(0))] = I(t^{-\gamma} f(t, x(t))) \in AC[0,1] \Rightarrow \\
\phi_p(D_*^\alpha x) \in Dom(D_*^\beta).
\end{gathered} \tag{2.8}$$

It follows from the second equation in (2.8) that

$$D_*^\beta \phi_p(D_*^\alpha x(t)) = t^{-\gamma} f(t, x(t)).$$

Conversely, assume that the conditions $x \in Dom(D_*^\alpha)$, $\phi_p(D_*^\alpha x) \in Dom(D_*^\beta)$, and $D_*^\alpha x \in C[0,1]$ are satisfied and Eq (2.5) holds. By using these conditions, the definition of $D_*^\beta$ and Proposition 1.1, we obtain from Eq (2.5) that

$$\begin{gathered}
DI^{1-\beta}[\phi_p(D_*^\alpha x(t)) - x_1] = t^{-\gamma} f(t, x(t)) \Rightarrow \text{ (by the conditions)} \\
I^{1-\beta}[\phi_p(D_*^\alpha x(t)) - x_1] = I(t^{-\gamma} f(t, x(t))) \Rightarrow \text{(by applying } I^\beta) \\
I[\phi_p(D_*^\alpha x(t)) - x_1] = I^{\beta+1}(t^{-\gamma} f(t, x(t))) \Rightarrow \text{(taking derivative)} \\
\phi_p(D_*^\alpha x(t)) = I^\beta(t^{-\gamma} f(t, x(t))) + x_1 \Rightarrow \text{(by applying } \phi_q) \\
D_*^\alpha x(t) = \phi_q[I^\beta(t^{-\gamma} f(t, x(t))) + x_1].
\end{gathered}$$

Now, using the definition of $D_*^\alpha$ and the condition $x \in Dom(D_*^\alpha)$ we obtain that

$$\begin{gathered}
DI^{1-\alpha}(x(t) - x_0) = \phi_q[I^\beta(t^{-\gamma} f(t, x(t))) + x_1] \Rightarrow \\
I^{1-\alpha}(x(t) - x_0) = I\phi_q[I^\beta(t^{-\gamma} f(t, x(t))) + x_1] \Rightarrow \\
I(x(t) - x_0) = I^{\alpha+1}\phi_q[I^\beta(t^{-\gamma} f(t, x(t))) + x_1].
\end{gathered}$$

Taking the derivative from the last equation we get

$$x(t) = I^\alpha \phi_q[I^\beta(t^{-\gamma} f(t, x(t))) + x_1] + x_0.$$

□

Finally, we compare the problems (2.1) and (2.5).

**Theorem 2.3.** *Let* $0 \le \gamma < \alpha, \beta < 1$. *If a function* $x(t)$ *is a solution of the problem with Caputo derivative:*

$$\begin{cases}
D_C^\beta \phi_p(D_C^\alpha x(t)) = t^{-\gamma} f(t, x(t)), \\
x(0) = x_0,\ D_C^\alpha x(0) = x_\alpha,\ \ x_0,\ x_\alpha \in \mathbb{R}, \\
x \in AC[0,1],\ D_C^\alpha x \in AC[0,1],
\end{cases}$$

*then it is a solution of the problem with generalized Caputo derivative:*

$$\begin{cases}
D_*^\beta \phi_p(D_*^\alpha x(t)) = t^{-\gamma} f(t, x(t)), \\
x(0) = x_0,\ D_*^\alpha x(0) = x_\alpha, \\
x \in Dom(D_*^\alpha),\ \ \phi_p(D_*^\alpha x) \in Dom(D_*^\beta),\ \ D_*^\alpha x \in C[0,1].
\end{cases}$$

*Conversely, if a function* $x(t)$ *is a solution of the problem with the generalized Caputo derivative and additionally* $x \in AC[0,1]$, $D_C^\alpha x \in AC[0,1]$, *then it is a solution of the problem with the Caputo derivative.*

*Proof.* This fact immediately follows from the definition of the domain of the operators $D_C^\alpha$, $D_*^\alpha$, and Proposition 1.1. f), i.e., if $0 < \alpha < 1$ and $x(t) \in AC[0,1]$ then $D_*^\alpha x(t) = D^\alpha(x(t) - x(0)) = D_C^\alpha x(t)$. □

## 3. Solvability of initial and boundary value problems

Our starting initial value problem is:

$$\begin{gathered} D_C^{\beta}\phi_p(D_C^{\alpha}x(t)) = t^{-\gamma}f(t,x(t)),\ 0 \le \gamma < \alpha,\ \beta < 1, \\ x(0) = 0,\ D_C^{\alpha}x(0) = 0, \\ x \in AC[0,1],\ D_C^{\alpha}x \in AC[0,1]. \end{gathered} \tag{3.1}$$

We consider the homogeneous initial value problem because by using the substitution $x(t) = y(t) + \frac{x_\alpha}{\alpha\Gamma(\alpha)}t^{\alpha} + x_0$ one can transform the non-homogeneous initial conditions $x(0) = x_0$, $D_C^{\alpha}x(0) = x_\alpha$ into the homogeneous conditions.

Theorem 2.3 gives us the basis for defining a generalized solution concept as follows.

**Definition 3.1.** *A function $x(t)$ is called a generalized solution to the problem (3.1) if it is a solution to the following problem.*

$$\begin{cases} D_*^{\beta}\phi_p(D_*^{\alpha}x(t)) = t^{-\gamma}f(t,x(t)),\ 0 \le \gamma < \alpha,\ \beta < 1, \\ x(0) = 0,\ D_*^{\alpha}x(0) = 0, \\ x \in Dom(D_*^{\alpha}),\ \ \phi_p(D_*^{\alpha}x) \in Dom(D_*^{\beta}),\ \ D_*^{\alpha}x \in C[0,1]. \end{cases} \tag{3.2}$$

By Theorem 2.2, the problem (3.2) is equivalent to the Volterra integral equation:

$$x(t) = I^{\alpha}\phi_q[I^{\beta}(t^{-\gamma}f(t,x(t)))],\ \ x \in C[0,1]. \tag{3.3}$$

For this reason, we will study the problem (3.3) instead of (3.2).

The main theorem regarding the existence of a solution to the Volterra integral equation is as follows.

**Theorem 3.1.** *Let $f$ be continuous on $[0,1]\times\mathbb{R}$ and there exist nonnegative functions $a, b \in C[0,1]$ such that $|f(t,u)| \le a(t) + b(t)|u|^{p-1}$, for all $t \in [0,1]$ and $u \in \mathbb{R}$. If*

$$\frac{\Gamma(1-\gamma)\|b\|}{\Gamma(\alpha+1)^{p-1}\Gamma(1-\gamma+\beta)} < 1, \tag{3.4}$$

*then the Volterra integral equation (3.3) has a solution in $C[0,1]$. So, the problem (3.1) has a generalized solution in $C[0,1]$.*

*Proof.* We use Schaefer's fixed point theorem to show the existence of a solution to problem (3.3). A version of Schaefer's fixed point theorem is as follows:

Let $X$ be a Banach space. If

i) $T : X \to X$ is a continuous compact operator,

ii) the set $\cup_{0\le\lambda\le1}\{x \in X : x(t) = \lambda Tx(t)\}$ is bounded,

then $T$ has a fixed point in $X$.

We have

$$x(t) = I^{\alpha}\phi_q[I^{\beta}(t^{-\gamma}f(t,x(t)))].$$

Denoting $Tx(t) = I^{\alpha}\phi_q[I^{\beta}(t^{-\gamma}f(t,x(t)))]$ we can write this equation in the following form

$$Tx(t) = x(t).$$

Now, we need to show that $T$ has a fixed point in $C[0,1]$.

$$Tx(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\Big[\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} \tau^{-\gamma} f(\tau, x(\tau))\, d\tau\Big] ds.$$

i) We have to prove that $T : C[0,1] \to C[0,1]$ is a continuous compact operator. First, let us show that $T : C[0,1] \to C[0,1]$ is continuous, i.e.,

$$x_n(t) \to x(t) \,\text{in}\, C[0,1] \Rightarrow Tx_n(t) \to Tx(t) \,\text{in}\, C[0,1].$$

A convergent sequence in a normed space is bounded. Hence,

$$x_n(t) \to x(t) \,\text{in}\, C[0,1] \Rightarrow \|x_n(t)\| \leq M, \text{ for all } t \in C[0,1].$$

Since the function $f$ is continuous on $[0,1] \times [-M, M]$, it is uniformly continuous on this compact set. It means that for $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|f(t, x_n(t)) - f(t, x(t))| < \varepsilon, \text{ for all}\, n \geq n_\varepsilon, \text{ and all}\, t \in [0,1].$$

By the definition of $T$,

$$Tx_n(t) - Tx(t) = I^\alpha[\phi_q(I^\beta(t^{-\gamma} f(t, x_n(t)))) - \phi_q(I^\beta(t^{-\gamma} f(t, x(t))))].$$

Let

$$y_n(t) = I^\beta(t^{-\gamma} f(t, x_n(t))) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{-\gamma} f(s, x_n(s))\, ds,$$

$$y(t) = I^\beta(t^{-\gamma} f(t, x(t))) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{-\gamma} f(s, x(s))\, ds.$$

The function $f$ is bounded on $[0,1] \times [-M, M]$. Hence $|f(t,s)| \leq L$, for all $(t,s) \in [0,1] \times [-M, M]$. Then

$$\|y_n\|,\ \|y\| \leq \frac{L}{\Gamma(\beta)} B(\beta, 1-\gamma)$$

and

$$|y_n(t) - y(t)| \leq \frac{\varepsilon}{\Gamma(\beta)} B(\beta, 1-\gamma) \text{ for all}\, n \geq n_\varepsilon, \text{ and all}\, t \in [0,1],$$

where $B(\nu, \mu) = \int_0^1 (1-s)^{\nu-1} s^{\mu-1}\, ds = \frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}$ for $\nu > 0$, $\mu > 0$.

Finally, it follows from the continuity of the function $\phi_p$ and continuity of the operator $I^\alpha : C[0,1] \to C[0,1]$ that the operator $T$ is continuous.

Now we prove that the operator $Tx(t) = I^\alpha \phi_q[I^\beta(t^{-\gamma} f(t, x(t)))]$ is compact. For this, we need to show that for any bounded set $\Omega \subset C[0,1]$ the set $\overline{T(\Omega)}$ is compact. By the Arzela-Ascoli theorem, $\overline{T(\Omega)}$ is compact if and only if

a) $T(\Omega)$ is bounded in $C[0,1]$ (it is the same that $T(\Omega)$ is uniformly bounded).

b) $T(\Omega)$ is equicontinuous.

We first show that $T(\Omega)$ is bounded in $C[0,1]$. Boundedness of $T(\Omega)$ means that

$$\|Tx(t)\| \leq C,\ \forall x \in \Omega,$$

where $C$ does not depend on $x$. This is a trivial fact. However, we will give a short proof. Since $\Omega$ is bounded, we have $\|x\| \leq C$ for all $x \in \Omega$. From the continuity of $f$ it follows that it is uniformly continuous on $[0,1] \times [-C, C]$. Then $|f(t, x(t)| \leq M$ for all $x \in \Omega$ and all $t \in [0,1]$. Hence,

$$\begin{gathered} |I^{\beta}(t^{-\gamma} f(t, x(t)))| = \\ \Big| \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{-\gamma} f(s, x(s))\, ds \Big| \leq \tfrac{M}{\Gamma(\beta)} B(\beta, 1-\gamma). \end{gathered} \tag{3.5}$$

On the other hand, $I^{\alpha} : C[0,1] \to C[0,1]$ is a bounded operator and

$$\|I^{\alpha} x(t)\| \leq \frac{1}{\Gamma(\alpha+1)} \|x(t\|. \tag{3.6}$$

Then, by using (3.5), (3.6), and the monotonicity of $s^{q-1}$, we obtain that

$$\|Tx(t)\| = \big\| I^{\alpha} \phi_q [I^{\beta}(t^{-\gamma} f(t, x(t)))] \big\| \leq \frac{1}{\Gamma(\alpha+1)} \Big[ \frac{M}{\Gamma(\beta)} B(\beta, 1-\gamma) \Big]^{q-1}.$$

This means that $T(\Omega)$ is uniformly bounded.

b) Now we show that $T(\Omega)$ is equicontinuous. To prove this, we use the same technique as in [11] (see theorem 3.1 in [11]).

Let $0 \leq t_1 < t_2 \leq 1$ and $x \in \Omega$. By (3.5) we have

$$|\phi_q [I^{\beta}(t^{-\gamma} f(t, x(t)))]| \leq \Big( \frac{M}{\Gamma(\beta)} B(\beta, 1-\gamma) \Big)^{q-1}. \tag{3.7}$$

Then, using (3.7) we get

$$\begin{gathered} |Tx(t_2) - Tx(t_1)| = \tfrac{1}{\Gamma(\alpha)} \Big| \int_0^{t_2} (t_2 - s)^{\alpha-1} \phi_q [I^{\beta}(s^{-\gamma} f(s, x(s)))] ds - \\ \int_0^{t_1} (t_1 - s)^{\alpha-1} \phi_q [I^{\beta}(s^{-\gamma} f(s, x(s)))] ds \Big| = \\ \tfrac{1}{\Gamma(\alpha)} \Big| \int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}) \phi_q [I^{\beta}(s^{-\gamma} f(s, x(s)))] ds + \\ \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \phi_q [I^{\beta}(s^{-\gamma} f(s, x(s)))] \Big| \leq \\ \tfrac{1}{\Gamma(\alpha+1)} \Big( \tfrac{M}{\Gamma(\beta)} B(\beta, 1-\gamma) \Big)^{q-1} [t_1^{\alpha} + t_2^{\alpha} + 2(t_2 - t_1)^{\alpha}]. \end{gathered}$$

This inequality together with uniform continuity of $t^{\alpha}$ on $[0,1]$ yields that $T(\Omega)$ is equicontinuous.

ii) Let us prove that the set $\cup_{0 \leq \lambda \leq 1} \{x \in C[0,1] : x(t) = \lambda Tx(t)\}$ is bounded. Let $x(t) = \lambda Tx(t)$, $\lambda \in (0,1]$. By the condition of the theorem $|f(t,u)| \leq a(t) + b(t)|u|^{p-1}$, for all $t \in [0,1]$ and all $u \in \mathbb{R}$. Then

$$|x(t) \leq |Tx(t)| = \frac{1}{\Gamma(\alpha)} \Big| \int_0^t (t-s)^{\alpha-1} \phi_q \Big[ \frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} \tau^{-\gamma} f(\tau, x(\tau))\, d\tau \Big] ds \Big|.$$

Setting $\tau = \sigma s$, we obtain that

$$\begin{gathered} |x(t)| \leq \tfrac{1}{\Gamma(\alpha)} \Big| \int_0^t (t-s)^{\alpha-1} \phi_q \Big[ \tfrac{s^{\beta-\gamma}}{\Gamma(\beta)} \int_0^1 (1-\sigma)^{\beta-1} \sigma^{-\gamma} f(\sigma s, x(\sigma s))\, d\sigma \Big] ds \Big| \leq \\ \tfrac{1}{\Gamma(\alpha)} \Big| \int_0^t (t-s)^{\alpha-1} \big[ \tfrac{B(\beta, 1-\gamma)}{\Gamma(\beta)} (\|a\| + \|b\| \|x\|^{p-1}) \big]^{q-1} ds \Big| = \\ \tfrac{1}{\Gamma(\alpha+1)} \big[ \tfrac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\beta)} (\|a\| + \|b\| \|x\|^{p-1}) \big]^{q-1}, \end{gathered}$$

where we have used $B(\beta, 1-\gamma) = \frac{\Gamma(\beta)\Gamma(1-\gamma)}{\Gamma(1-\gamma+\beta)}$. Since $(p-1)(q-1) = 1$, we obtain from the last inequality that

$$\|x\|^{p-1} \le \frac{1}{\Gamma(\alpha+1)^{p-1}} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\beta)} (\|a\| + \|b\|\|x\|^{p-1})$$

and consequently,

$$\frac{\|b\|\|x\|^{p-1}}{\|a\| + \|b\|\|x\|^{p-1}} \le \frac{\Gamma(1-\gamma)\|b\|}{\Gamma(\alpha+1)^{p-1}\Gamma(1-\gamma+\beta)}. \tag{3.8}$$

If the set $\cup_{0\le\lambda\le1}\{x \in C[0,1] : x(t) = \lambda T x(t)\}$ were unbounded then there would be a sequence in this set that converges to infinity. Then taking limit as $\|x\| \to \infty$ in (3.8) and using the condition of the theorem $\frac{\Gamma(1-\gamma)\|b\|}{\Gamma(\alpha+1)^{p-1}\Gamma(1-\gamma+\beta)} < 1$, we get the following contradiction

$$1 \le \frac{\Gamma(1-\gamma)\|b\|}{\Gamma(\alpha+1)^{p-1}\Gamma(1-\gamma+\beta)} < 1.$$

Hence, the set $\cup_{0\le\lambda\le1}\{x \in C[0,1] : x(t) = \lambda T x(t)\}$ is bounded and, by Schaefer's theorem, the operator $T$ has a fixed point. □

Using the methods applied in Theorem 3.1, we obtain the following result for the smallest eigenvalue of the fractional p-laplacian operator.

**Corollary 3.1.** *Let $\mu_1$ be the smallest eigenvalue of the following eigenvalue problem:*

$$D_*^{\beta}\phi_p(D_*^{\alpha}x(t)) = \mu|x|^{p-2}x, \ \mu > 0, \ p > 1, \ 0 < \alpha, \beta \le 1,$$
$$x(0) = 0, \ D_*^{\alpha}x(0) = 0.$$

*Then*

$$\mu_1 \ge \Gamma(\alpha+1)^{p-1}\Gamma(\beta+1).$$

*Proof.* Let $0 < \mu_1 < \Gamma(\alpha+1)^{p-1}\Gamma(\beta+1)$ be an eigenvalue of the given problem. Then,

$$D_*^{\beta}\phi_p(D_*^{\alpha}x(t)) = \mu_1|x|^{p-2}x,$$
$$x(0) = 0, \ D_*^{\alpha}x(0) = 0.$$

By Theorem 2.2, this problem is equivalent to the Volterra integral equation:

$$x(t) = I^{\alpha}\phi_q[I^{\beta}(\mu_1|x|^{p-2}x)], \ \ x \in C[0,1].$$

According to Theorem 3.1, we have $a = 0, b = \mu_1$ and by the assumption $0 < \mu_1 < \Gamma(\alpha+1)^{p-1}\Gamma(\beta+1)$ the condition

$$\frac{\|b\|}{\Gamma(\alpha+1)^{p-1}\Gamma(1+\beta)} = \frac{\mu_1}{\Gamma(\alpha+1)^{p-1}\Gamma(1+\beta)} < 1$$

is satisfied. Then under these conditions we have proved in Theorem 3.1 that the set $\cup_{0\le\lambda\le1}\{x \in C[0,1] : x(t) = \lambda T x(t)\}$ is bounded, where $Tx(t) = I^{\alpha}\phi_q[I^{\beta}(\mu_1|x|^{p-2}x)]$. But, since $\mu_1$ is an eigenvalue, the set $x(t) = \lambda T x(t)\}$ is unbounded for $\lambda = 1$. This is a contradiction. Consequently, $\mu_1 \ge \Gamma(\alpha+1)^{p-1}\Gamma(\beta+1)$. □

Finally, we give an example as an application of Theorem 3.1.

**Example 3.1.** *If* $0 < b < \frac{1}{2}$ *then the following problem has a solution in* $C[0, 1]$.

$$D_*^{\frac{1}{2}}\phi_3(D_*^{\frac{1}{2}}x(t)) = t^{-1/3}(1 + bx^2(t)),$$
$$x(0) = 0,\ D_*^{\alpha}x(0) = 0.$$

Let us check the conditions of Theorem 3.1. In this case
$f(t, x(t)) = 1 + bx^2(t),\ p = 3,\ \gamma = \frac{1}{3},\ \ \alpha = \beta = \frac{1}{2}$ and $0 < \gamma < \alpha, \beta < 1$. Then

$$\frac{\Gamma(1-\gamma)\|b\|}{\Gamma(\alpha+1)^{p-1}\Gamma(1-\gamma+\beta)} = \frac{\Gamma(\frac{2}{3})b}{\Gamma(\frac{3}{2})^2\Gamma(\frac{7}{6})}.$$

$\Gamma(\frac{2}{3}) = 1,35411..,\ \Gamma\left(\frac{3}{2}\right) = 0,88622..,\ \Gamma\left(\frac{7}{6}\right) = 0,92771..$ and

$$\frac{\Gamma(\frac{2}{3})b}{\Gamma(\frac{3}{2})^2\Gamma(\frac{7}{6})} < \frac{1,4b}{(0,88)^2 0,92} < 2b < 1.$$

Consequently, the condition
$$\frac{\Gamma(1-\gamma)\|b\|}{\Gamma(\alpha+1)^{p-1}\Gamma(1-\gamma+\beta)} < 1$$
is satisfied and the problem has a solution.

## 4. Conclusions

The main subject of this paper was the initial value problems for Caputo fractional differential equations with singular nonlinearities involving the p-Laplacian operator. The most important difference of this paper from other studies on this subject is that the equation

$$D_C^{\beta}\phi_p(D_C^{\alpha}x(t)) = t^{-\gamma}f(t, x(t)),\ 0 \le \gamma < \alpha\beta \le 1$$

contains the singular term $t^{-\gamma}$.

In general, Volterra integral equations are used to solve problems given for fractional differential equations. However, their equivalence is not always shown in the literature. In this article, the equivalence of such problems was discussed in full detail. We would like to emphasize that we benefited from the techniques of Webb [36] in this subject.

For the case $f = \mu|x|^{p-2}x,\ \gamma = 0$, a result related to the lower bound of the eigenvalues is given as Corollary 3.1.

By the methods, applied in the article, similar results can be obtained for higher order equations and different initial and boundary value problems.

## Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The author declares that he has no competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## References


1. M. S. Abdo, S. A. Idris, W. Albalawi, A. Abdel-Aty, M. Zakarya, E. E. Mahmoud, Qualitative study on solutions of piecewise nonlocal implicit fractional differential equations, *J. Funct. Spaces*, **2023** (2023), 2127600. https://doi.org/10.1155/2023/2127600
2. R. P. Agarwal, B. Ahmad, Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions, *Comput. Math. Appl.*, **62** (2011), 1200–1214. https://doi.org/10.1016/j.camwa.2011.03.001
3. R. P. Agarwal, D. O'Regan, S. Stanek, Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations, *J. Math. Anal. Appl.*, **371** (2010), 57–68. https://doi.org/10.1016/j.jmaa.2010.04.034
4. R. P. Agarwal, D. O'Regan, S. Stanek, Positive solutions for mixed problems of singular fractional differential equations, *Math. Nachr.*, **285** (2012), 27–41. https://doi.org/10.1002/mana.201000043
5. B. Ahmad, J. J. Nieto, Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory, *Topol. Method. Nonlinear Anal.*, **35** (2010), 295–304.
6. A. Babakhani, V. Daftardar-Gejji, Existence of positive solutions of nonlinear fractional differential equations, *J. Math. Anal. Appl.*, **278** (2003), 434–442. https://doi.org/10.1016/S0022-247X(02)00716-3
7. L. C. Becker, T. A. Burton, I. K. Purnaras, Complementary equations: A fractional differential equation and a Volterra integral equation, *Electron. J. Qual. Theory Differ. Equ.*, 2015, 12. https://doi.org/10.14232/ejqtde.2015.1.12
8. M. Benchohra, S. Hamani, S. K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonlinear Anal.-Theor.*, **71** (2009), 2391–2396. https://doi.org/10.1016/j.na.2009.01.073
9. T. Chen, W. Liu, C. Yang, Antiperiodic solutions for Lienard-type differential equation with $p$-Laplacian operator, *Bound. Value Probl.*, **210** (2010), 194824. https://doi.org/10.1155/2010/194824
10. T. Chen, W. Liu, J. Liu, Solvability of periodic boundary value problem for fractional p-Laplacian equation, *Appl. Math. Comput*, **244** (2014), 422–431. http://doi.org/10.1016/j.amc.2014.06.105
11. T. Chen, W. Liu, An anti-periodic boundary value problem for the fractional differential equation with a $p$-Laplacian operator, *Appl. Math. Lett.*, **25** (2012), 1671–1675. https://doi.org/10.1016/j.aml.2012.01.035
12. M. A. Darwish, S. K. Ntouyas, On initial and boundary value problems for fractional order mixed type functional differential inclusions, *Comput. Math. Appl.*, **59** (2010), 1253–1265. https://doi.org/10.1016/j.camwa.2009.05.006

13. J. Deng, Z. Deng, Existence of solutions of initial value problems for nonlinear fractional differential equations, *Appl. Math. Lett.*, **32** (2014), 6–12. http://doi.org/10.1016/j.aml.2014.02.001

14. K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Heidelberg: Springer Berlin, 2010. https://doi.org/10.1007/978-3-642-14574-2

15. P. W. Eloe, T. Masthay, Initial value problems for Caputo fractional differential equations, *J. Fract. Calc. Appl.*, **9** (2018), 178–195.

16. M. El-Shahed, J. J. Nieto, Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order, *Comput. Math. Appl.*, **59** (2010), 3438–3443. https://doi.org/10.1016/j.camwa.2010.03.031

17. S. Etemad, M. A. Ragusa, S. Rezapour, A. Zada, Existence property of solutions for multi-order q-difference FBVPs based on condensing operators and end-point technique, *Fixed Point Theor.*, **25** (2024), 115–142. https://doi.org/10.24193/fpt-ro.2024.1.08

18. D. Henry, *Geometric theory of semilinear parabolic equations*, Heidelberg: Springer Berlin, 1981. https://doi.org/10.1007/BFb0089647

19. D. Jiang, W. Gao, Upper and lower solution method and a singular boundary value problem for the one-dimensional $p$-Laplacian, *J. Math. Anal. Appl.*, **252** (2000), 631–648. https://doi.org/10.1006/jmaa.2000.7012

20. E. Kaslik, S. Sivasundaram, Non-existence of periodic solutions in fractional order dynamical systems and a remarkable difference between integer and fractional-order derivatives of periodic functions, *Nonlinear Anal.-Real*, **13** (2012), 1489–1497. https://doi.org/10.1016/j.nonrwa.2011.11.013

21. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006. https://doi.org/10.1016/S0304-0208(06)80011-3

22. N. Kosmatov, A boundary value problem of fractional order at resonance, *Electron. J. Differ. Equ.*, **2010** (2010), 135.

23. N. Kosmatov, Integral equations and initial value problems for nonlinear differential equations of fractional order, *Nonlinear Anal.-Theor.*, **70** (2009), 2521–2529. https://doi.org/10.1016/j.na.2008.03.037

24. S. Krim, A. Salim, M. Benchohra, Nonlinear contractions and Caputo tempered implicit fractional differential equations in b-metric spaces with infinite delay, *Filomat*, **37** (2023), 7491–7503.

25. K. Q. Lan, W. Lin, Positive solutions of systems of Caputo fractional differential equations, *Commun. Appl. Anal.*, **17** (2013), 61–86.

26. M. Marin, A. Hobiny, I. Abbas, The effects of fractional time derivatives in porothermoelastic materials using finite element method, *Mathematics*, **9** (2021), 1606. https://doi.org/10.3390/math9141606

27. M. Marin, A. Seadawy, S. Vlase, A. Chirila, On mixed problem in thermoelasticity of type III for Cosserat media. *J. Taibah Univ. Sci.*, **16** (2022), 1264–1274. https://doi.org/10.1080/16583655.2022.2160290

28. C. Li, S. Sarwar, Existence and continuation of solutions for Caputo type fractional differential equations, *Electron. J. Differ. Equ.*, **2016** (2016), 207. https://ejde.math.txstate.edu/Volumes/2016/207/li.pdf
29. D. S. Mitrinovic, J. E. Pecaric, A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Dordrecht: Springer, 1991. https://doi.org/10.1007/978-94-011-3562-7
30. S. K. Ntouyas, E. Pourhadi, Positive solutions of nonlinear fractional three-point boundary-value problem, *Le Matematiche*, **73** (2018), 139–154. http://doi.org/10.4418/2018.73.1.10
31. I. Podlubny, *Fractional differential equations*, Academic Press, 1999.
32. J. Sabatier, O. P. Agrawal, J. A. T. Machado, *Advances in fractional calculus: Theoretical developments and applications in physics and engineering*, Dordrecht: Springer, 2007. https://doi.org/10.1007/978-1-4020-6042-7
33. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Switzerland; Philadelphia: Gordon and Breach Science Publishers, 1993.
34. G. Wang, B. Ahmad, L. Zhang, Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order, *Nonlinear Anal.-Theor.*, **74** (2011), 792–804. http://doi.org/10.1016/j.na.2010.09.030
35. J. Wang, M. Feckan, Y. Zhou, Nonexistence of periodic solutions and asymptotically periodic solutions for fractional differential equations. *Commun. Nonlinear Sci. Numer. Simulat.*, **18** (2013), 246–256. http://doi.org/10.1016/j.cnsns.2012.07.004
36. J. R. L. Webb, Initial value problems for Caputo fractional equations with singular nonlinerities, *Electron. J. Differ. Equ.*, **2019** (2019), 117.
37. J. R. L. Webb, Weakly singular Gronwall inequalities and applications to fractional differential equations, *J. Math. Anal. Appl.*, **471** (2019), 692–711. https://doi.org/10.1016/j.jmaa.2018.11.004
38. H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.*, **328** (2007), 1075–1081. https://doi.org/10.1016/j.jmaa.2006.05.061
39. T. Zhu, New Henry-Gronwall integral inequalities and their applications to fractional differential equations, *Bull. Braz. Math. Soc.*, **49** (2018), 647–657. https://doi.org/10.1007/s00574-018-0074-z


AIMS Press