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Research article

Existence of solutions of fractal fractional partial differential equations through different contractions

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Abstract: In the past, the existence and uniqueness of the solutions of fractional differential equations have been investigated by many researchers theoretically in various approaches in the literature. In this paper, there is no discussion of the existence of solutions for the nonlinear differential equations with fractal fractional operators. The objective of this work is to present novel contraction approaches, notably the $\propto$-ψ-contraction $\propto$-type of the $\tilde{\mathrm{F}}$-contraction, within the context of $\hat{F}$-metric and orbital metric spaces. The aim of this study is to illustrate certain fixed point theorems that offer a new and direct approach to establish the existence and uniqueness of the solution to the general partial differential equations by employing the fractal fractional operators.

Keywords: partial differential equation; fractal fractional operator; metric spaces; fixed point
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

Fractional order problems are receiving considerable attention in different scientific fields because they can model complicated processes more precisely than conventional integer-order equations. Artificial neural networks (ANNs) have been one of the notable applications used in recent studies to solve fractional higher-order linear integro-differential equations [1]. By utilizing the natural parallel processing skills of ANNs, these equations can be effectively solved, leading to progress in areas such as physics, engineering, and finance. The stability study of pandemics like the COVID-19 outbreak has been improved by using Caputo-Fabrizio fractional differential equations (*FDEs*) [2, 3]. This method allows for a more detailed comprehension of epidemic dynamics, which helps create successful containment tactics. Using spectral methods with fractional basis functions provides a robust framework for solving integral equations in various scientific fields, namely, fractional

Fredholm integro-differential equations. Researching stability and the existence of solutions in complex structures, such as the triple problem of fractional hybrid delay differential equations, along with modern mathematical modeling techniques, provides valuable insights into the behavior of intricate dynamic systems [4–6].

Nonlinear differential equations are widely recognized for describing a variety of physical events. Partial differential equations, specifically belonging to the category of the Cauchy problem, have long been recognized as effective mathematical tools for modeling real-world problems in various areas of engineering and science. There are numerous differential operators in use today in the literature, the most popular of which is related to the rate of change [7]. Because of a newly introduced parameter known as fractal dimension, it has now been demonstrated in several exceptional studies that the fractal derivative, a differential operator introduced by Wenfeng Chen, predicts aspects of nature more precisely than ordinary differentiation. The fractal derivative is a term used in applied mathematics to describe a variable scaled according to t^{α}. New avenues of research for science, engineering, and technological progress have been opened because of these new mathematical tools. The fractal derivative was developed to describe physical phenomena that are beyond the scope of classical physical rules. Media having non-integral fractal dimensions do not conform to these supposedly Euclidean geometrical considerations. Fractal features are frequently seen in practical situations such as porous materials, aquifers turbulence, etc. [8, 9].

As a result, utilizing diverse numerical and analytical approaches for solving the nonlinear differential equations is essential for scientific problem identification [10, 11]. The majority of researchers (see [12, 13]) have looked at theoretical conclusions in different ways that prove the existence results for the *FDEs*. Afshari and Baleanu [14] have recently investigated the theoretical solution (existence and uniqueness) for some Atangana-Baleanu *FDEs* in the sense of Caputo. Also, Karapinar et al. [15] demonstrate the existence of solutions to ordinary and fractional boundary value problems (BVPs) with integral type boundary conditions (BC) in the context of some Caputo-type fractional operators.

Therefore, fixed point theory has attracted much attention in recent decades; it is a beautiful technique for determining the solution of existence to differential/integral equations. For this reason, one of the efficient results proposed by Wardowski [16] guarantees the existence and uniqueness of a fixed point in the context of the usual metric space. Samet et al. [17] introduce the idea of $\propto$-admissibility of mappings, which was subsequently expanded upon by Karapinar and Samet [18]. Gopal introduced a novel idea of $\propto$-type $\tilde{\mathsf{F}}$-contraction mapping in their paper [19]. Recently, Jleli with Samet [20] has proposed the concept of $\mathbb{F}$-metric space as an approach to extend the applicability of the Banach contraction principle.

This manuscript employs various contractions, including the $\propto$-ψ-contraction and $\propto$-type of $\tilde{\mathsf{F}}$-contraction, along with outcomes from $\hat{F}$-metric and orbital metric spaces. These are utilized to obtain the theoretical solution for a general partial differential equation involving a fractal fractional differential operator

$$\begin{aligned} {}_{0}^{FF}D_{\ni}^{\flat}\hbar(\mathsf{q},\ni) &= \mathsf{F}(\mathsf{q},\ni,\hbar(\mathsf{q},\ni)), \qquad 0<\flat<1, \\ \hbar(\mathsf{q},0) &= \hbar_{\circ}, \end{aligned} \tag{1.1}$$

where

$$(\mathsf{q},\ni)\in[0,\pounds]\times[0,H], \quad V=[0,\pounds]\times[0,H]\hbar(\mathsf{q},\ni)\in \mathsf{C}(V,\mathcal{R}),$$

and F is a function that is continuous and non-linear, i.e., $\mathsf{F}(0,0,\hbar(0,0))=0$.

Definition 1.1. *[10] For a non-empty set $\mho$, define m_d: $\mho\times\mho\to[0,+\infty)$, which is termed as a b-metric if these conditions hold:*

(1) If $m_d(w,s)=0$, then $w=s$ for all $w,s\in\mho$.

(2) $m_d(w,s)=m_d(s,w)$ for all $w,s\in\mho$.

(3) $m_d(w,s)\leq b[m_d(w,q)+m_d(q,s)]$.

Then, we call $(\mho,m_d)$ a b-metric space.

Consider $(\mho,m_d)$ to represent a complete b-metric space, and $\mathcal{P}$ is the set of functions ψ: $[0,+\infty)\to[0,+\infty)$ possessing the following two properties:

(1) ψ being increasing and continuous.

(2) $\psi(\ell\mathsf{q})\leq\ell\psi(\mathsf{q})\leq\ell\mathsf{q}$, provided $\ell>0$.

Further, we assume that $\mathcal{Q}$ consist of elements in the form as non-decreasing mapping Φ, which are defined by Φ: $[0,+\infty)\to[0,\frac{1}{a})$ with $a\geq1$.

Definition 1.2. *[17] Let $\mho$ be a non-empty set, Υ: $\mho\to\mho$, and $\propto$: $\mho\times\mho\to\mathcal{R}$ such that*

$$\propto(g,s)\ \geq\ 1\implies\ \propto(\Upsilon g,\Upsilon s)\geq1\quad\textit{for all}\quad g,s\in\mho.$$

Then Υ is termed as $\propto$-admissible

Definition 1.3. *[11] Let $(\mho,m_d)$ represent a complete b-metric space, and Υ: $\mho\to\mho$, and $\propto$: $\mho\times\mho\to[0,+\infty)$ such that,*

$$\propto(g,s)\psi\left(a^3m_d(\Upsilon g,\Upsilon s)\right)\ \leq\ \Phi\left(\psi\left(m_d(g,s)\right)\right)\psi\left(m_d(g,s)\right),$$

where $g,s\in\mho$, $\Phi\in\mathcal{Q}$, $a\geq1$, and $\psi\in\mathcal{P}$. Then Υ is termed as an $\propto$-ψ-contraction function.

The upcoming result illustrates that $\propto$-ψ-contractive mapping have a fixed point.

Corollary 1.4. *[17] Assume $(\mho,m_d)$ represents a complete b-metric space, and Υ: $\mho\to\mho$ is an $\propto$-ψ-contraction in a way that:*

($\wp$**1**) *There exists $\wp\in\mho$ in a manner that $\propto(\wp,\psi\wp)\geq1$.*

($\wp$**2**) $\{\wp_n\}\subseteq\mho$, $\lim\limits_{n\to+\infty}\wp_n=\wp$, *where $\wp\in\mho$ and $\propto(\wp_n,\wp_{n+1})\geq1$ implies $\propto(\wp_n,\wp)\geq1$.*

Then Υ possesses a fixed point.

2. Existence results

For the proof of the following theorem consider

$$\mho=\mathsf{C}(\varsigma,\mathcal{R})\quad\text{and}\quad m_d(\hbar,\omega)=\sup_{\mathsf{q},\ni\in\varsigma}\|\hbar(\mathsf{q})-\omega(\ni)\|^2.$$

Theorem 2.1. *Let* $\mathsf{J}: \mathcal{R}^2 \to \mathcal{R}$ *in a way that*

(**i**)

$$\|\mathsf{F}(q,\ni,\hbar(q,\ni)) - \mathsf{F}(q,\ni,\omega(q,\ni))\| \leq \frac{\varsigma^{\flat}}{2\sqrt{2}\flat\top^{2\flat-1}\mathcal{B}(\flat,\flat)},$$

then, $\sqrt{\Phi(\psi(\|\hbar(q,\ni)-\omega(q,\ni)\|^2))\psi(\|\hbar(q,\ni)-\omega(q,\ni)\|^2)}$ *for* $(q,\ni) \in \varsigma$ *and* $\hbar(q,\ni), \omega(q,\ni) \in \mathsf{C}(\varsigma,\mathcal{R})$ *with* $\mathsf{J}(\hbar,\omega) \geq 0$.

(**ii**) *There exists* $\hbar_1 \in \mathsf{C}(\varsigma,\mathcal{R})$ *with* $\mathsf{J}(\hbar,\Upsilon\hbar_1) \geq 0$*, where* $\Upsilon: \mathsf{C} \to \mathsf{C}$ *is defined as*

$$\Upsilon(\hbar) = \hbar_\circ + \frac{1}{\varsigma(\flat)}\int_0^t \flat \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1}\mathsf{F}(q,\curlyvee,\hbar(q,\curlyvee))d\curlyvee.$$

(**iii**) $(q,\ni) \in \varsigma$ *and* $\hbar,\omega \in \mathsf{C}, \mathsf{J}(\hbar,\omega) \geq 0$ *emphasize the fact that* $\mathsf{J}(\Upsilon\hbar,\Upsilon\omega) \geq 0$.

(**iv**) $\{\hbar_n\} \subseteq \mathsf{C}, \hbar_n \to \hbar$*, where* $\hbar \in \mathsf{C}$ *and* $\mathsf{J}(\hbar_n,\hbar_{n+1}) \geq 0$*, for* $n \in N$; *then there exists at least one solution of the problem (1.1).*

Proof. In problem (1.1) F a is nonlinear mapping, and

$$ {}_0^{FF}D_{\ni}^{\flat}\hbar(q,\ni) = \frac{1}{\varsigma(1-\flat)}\frac{d}{d\ni^{\flat}}\int_0^{\ni}\hbar(q,\curlyvee)(\ni-\curlyvee)^{-\flat}d\curlyvee. \tag{2.1}$$

Since $\int_0^{\ni}\hbar(q,\curlyvee)(\ni-\curlyvee)^{-\flat}d\curlyvee$ is differentiable, Eq (2.1) can be converted into

$$\frac{1}{\flat\ni^{\flat-1}}\frac{1}{\varsigma(1-\flat)}\frac{d}{d\ni}\int_0^{\ni}\hbar(q,\curlyvee)(\ni-\curlyvee)^{-\flat}d\curlyvee.$$

Consequently, Eq (1.1) could be transformed into

$$\hbar(q,\ni) - \hbar(q,0) = \curlyvee^{\flat-1}(\ni-\curlyvee)^{\flat-1}\mathsf{F}(\ell,\curlyvee,\hbar)d\curlyvee, \curlyvee^{\flat-1}(\ni-\curlyvee)^{\flat-1}\mathsf{F}(q,\curlyvee,\hbar)d\curlyvee.$$

Consequently,

$$\hbar(q,\ni) = \hbar_\circ + \frac{1}{\varsigma(\flat)}\int_0^{\ni}\flat\curlyvee^{\flat-1}(\ni-\curlyvee)^{\flat-1}\mathsf{F}(q,\curlyvee,\hbar) = \Upsilon\hbar. \tag{2.2}$$

Here, we show that Υ has a fixed point

$$\begin{aligned}
\|\Upsilon\hbar - \Upsilon\omega\|^2 &= \|\frac{1}{\varsigma(\flat)}\int_0^{\ni}\flat\curlyvee^{\flat-1}(\ni-\curlyvee)^{\flat-1}(\mathsf{F}(q,\curlyvee,\hbar)-\mathsf{F}(q,\curlyvee,\omega))d\curlyvee\|^2\\
&\leq \{\frac{1}{\varsigma(\flat)}\int_0^{t}\flat\curlyvee^{\flat-1}(\ni-\curlyvee)^{\flat-1}\|\mathsf{F}(q,\curlyvee,\hbar)-\mathsf{F}(q,\curlyvee,\omega)\|d\curlyvee\}^2\\
&\leq \{\frac{1}{2\sqrt{2}\top^{2\flat-1}\mathcal{B}(\flat,\flat)}\int_0^{\ni}\curlyvee^{\flat-1}(\ni-\curlyvee)^{\flat-1}\sqrt{\Phi(\psi(\|\hbar(q,\ni)-\omega(q,\ni)\|^2))\psi(\|\hbar(q,\ni)-\omega(q,\ni)\|^2)}d\curlyvee\}^2\\
&= \{\frac{1}{2\sqrt{2}\top^{2\flat-1}\mathcal{B}(\flat,\flat)}\sqrt{\Phi(\psi(\|\hbar(q,\ni)-\omega(q,\ni)\|^2))\psi(\|\hbar(q,\ni)-\omega(q,\ni)\|^2)}\int_0^{\ni}\curlyvee^{\flat-1}(\ni-\curlyvee)^{\flat-1}d\curlyvee\}^2\\
&\leq \{\frac{1}{2\sqrt{2}\top^{2\flat-1}\mathcal{B}(\flat,\flat)}\sqrt{\Phi(\psi(m_d(\hbar-\omega)))\psi(m_d(\hbar-\omega))}\top^{2\flat-1}\mathcal{B}(\flat,\flat)\}^2\\
&= \{\frac{1}{2\sqrt{2}}\sqrt{\Phi(\psi(m_d(\hbar-\omega)))\psi(m_d(\hbar-\omega))}\}^2\\
&= \frac{1}{8}\Phi(\psi(m_d(\hbar-\omega)))\psi(m_d(\hbar-\omega)),
\end{aligned}$$

for $\hbar, \omega \in \mathsf{C}(\varsigma, \mathcal{R})$ with $\mathsf{J}(\hbar, \omega) \geq 0$, we have

$$8\|\Upsilon\hbar - \Upsilon\omega\|^2 \leq \Phi(\psi(m_d(\hbar, \omega)))\psi(m_d(\hbar, \omega)).$$

Now defining

$$\propto : \mathsf{C}(\varsigma, \mathcal{R}) \times \mathsf{C}(\varsigma, \mathcal{R}) \to [0, +\infty)$$

by

$$\propto (\hbar, \omega) = \begin{cases} 1, & \text{if } \mathsf{J}(\hbar, \omega) \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} \propto (\hbar, \omega)\psi(8m_d(\Upsilon\hbar, \Upsilon\omega)) &\leq 8m_d(\Upsilon\hbar, \Upsilon\omega) \\ &= \Phi(\psi(m_d(\hbar, \omega)))\psi(m_d(\hbar, \omega)). \end{aligned}$$

To justify Υ is $\propto$-admissible, we have from (**iii**)

$$\begin{aligned} \propto (\hbar, \omega) \geq 1 \Rightarrow \mathsf{J}(\hbar, \omega) \geq 0 \Rightarrow \mathsf{J}(\Upsilon\hbar, \Upsilon\omega) \geq 0 \\ \Rightarrow \propto (\Upsilon\hbar, \Upsilon\omega) \geq 1, \end{aligned}$$

$\hbar, \omega \in \mathsf{C}(\varsigma, \mathcal{R})$. By (**ii**), it is obvious that $\hbar_\circ \in \mathsf{C}(\varsigma, \mathcal{R})$ in a way that $\propto (\hbar_\circ, \Upsilon\hbar_\circ) \geq 1$, from (**iv**) and the Corollary 1.4 there exist $\hbar^* \in \mathsf{C}(\varsigma, \mathcal{R})$ that ensure $\hbar^* = \Upsilon\hbar^*$. □

Next, the definition of $\propto$-type $\tilde{F}$-contraction is to be presented. For this, we need certain assumptions. Suppose that F represents the mappings of the form, $\tilde{F}: \mathcal{R}_+ \to \mathcal{R}$ in the sense that:

(k1) $\tilde{F}$ needs to be increasing strictly;

(k2) $\lim_{g \to 0^+} g^\sigma \tilde{F}(g) = 0$ for $\sigma \in (0, 1)$;

(k3) $\lim_{n \to +\infty} \tilde{F}(g_n) = -\infty$ if and only if $\lim_{n \to +\infty} g_n = 0$ for every $\{g_n\}_{n \in N}$.

Definition 2.2. *[19] Let $\Upsilon: \mho \to \mho$, $\propto: \mho \times \mho \to \{+\infty\} \cup (0, +\infty)$ and $\tilde{F} \in F$; and there exists $\varepsilon > 0$ such that*

$$\varepsilon + \propto (\hbar, \omega)\tilde{F}(m_d(\Upsilon\hbar, \Upsilon\omega)) \leq \tilde{F}(m_d(\hbar, \omega)),$$

for each $\hbar, \omega \in \mho$, with $m_d(\Upsilon\hbar, \Upsilon\omega) > 0$, and then Υ is called an $\propto$-type $\tilde{F}$-contraction on $\mho$.

Theorem 2.3. *[19] Assume $(\mho, m_d)$ is a metric space, and $\Upsilon: \mho \to \mho$ such that*

(λ**1**) *Υ is an $\propto$-type $\tilde{F}$-contraction;*

(λ**2**) *There exist $\hbar_\circ \in \mho$ with $\propto (\hbar_\circ, \Upsilon\hbar_\circ) \geq 1$;*

(λ**3**) *Υ is $\propto$-admissible;*

(λ**4**) *If $\{\hbar_n\} \subseteq \mho$ with $\propto (\hbar_n, \hbar_{n+1}) \geq 1$ and $\hbar_n \to \hbar$, then $\propto (\hbar_n, \hbar) \geq 1$;*

(λ**5**) *$\tilde{F}$ is continuous. Then there exists $\hbar^* \in \mho$ such that $\Upsilon(\hbar^*) = \hbar^*$, and $\{\Upsilon^n\hbar_\circ\}_{n \in N}$ converge to $\hbar^*$.*

For the proof of the next theorem the metric

$$m_d(\hbar, \omega) = \sup_{(\mathsf{q},\ni)\in\varsigma} |\hbar(\mathsf{q}, \ni) - \omega(\mathsf{q}, \ni)| = \|\hbar - \omega\|$$

will be taken under consideration.

Theorem 2.4. *Let* $\mathsf{J}\colon \mathcal{R}^2 \to \mathcal{R}$ *in a way that*

(**p1**)

$$\|\mathsf{F}\hbar - \mathsf{F}\omega\| \leq \frac{e^{-\varepsilon}\varsigma(\flat)}{\flat\top^{2\flat-1}\mathcal{B}(\flat,\flat)}\|\hbar - \omega\|$$

for $(q, \ni) \in \varsigma$ *and* $\hbar(q, \ni), \omega(q, \ni) \in \mathsf{C}(\varsigma, \mathcal{R})$ *with* $\mathsf{J}(\hbar, \omega) \geq 0$*;*

(**p2**) *There exists* $\hbar_1 \in \mathsf{C}(\varsigma, \mathcal{R})$ *with* $\mathsf{J}(\hbar, \Upsilon\hbar_1) \geq 0$*, where* $\Upsilon : \mathsf{C} \to \mathsf{C}$*, defined by*

$$\Upsilon(\hbar) \;=\; \hbar_\circ + \frac{1}{\varsigma(\flat)}\int_0^{\ni} \flat\, \curlyvee^{\flat-1}\, (\ni - \curlyvee)^{\flat-1}\mathsf{F}(q, \curlyvee, \hbar(q, \curlyvee))d\curlyvee;$$

(**p3**) $(q, \ni) \in \varsigma$ *and* $\hbar, \omega \in \mathsf{C}, \mathsf{J}(\hbar, \omega) \geq 0$ *imply that* $\mathsf{J}(\Upsilon\hbar, \Upsilon\omega) \geq 0$*;*

(**p4**) $\{\hbar_n\} \subseteq \mathsf{C}, \hbar_n \to \hbar$*, where* $\hbar \in \mathsf{C}$ *and* $\mathsf{J}(\hbar_n, \hbar_{n+1}) \geq 0$*, for* $n \in N$*; Then there exists at least one solution of the problem (1.1).*

Proof. The following integral equation can be formed from Eq (1.1):

$$\hbar(\mathsf{q}, \ni) = \hbar_\circ + \int_0^{\ni} \flat\, \curlyvee^{\flat-1}\, (\ni - \curlyvee)^{\flat-1}\mathsf{F}(\mathsf{q}, \curlyvee, \hbar)d\curlyvee = \Upsilon\hbar.$$

To verify the fixed point of Υ, we have

$$\begin{aligned}
\|\Upsilon\hbar - \Upsilon\omega\| &= \|\frac{1}{\varsigma(\flat)}\int_0^{\ni} \flat\, \curlyvee^{\flat-1}\, (\ni - \curlyvee)^{\flat-1}(\mathsf{F}(\mathsf{q}, \curlyvee, \hbar) - \mathsf{F}(\mathsf{q}, \curlyvee, \omega))d\curlyvee\,\| \\
&\leq \frac{\flat}{\varsigma(\flat)}\int_0^{\ni} \curlyvee^{\flat-1}(\ni - \curlyvee)^{\flat-1}\|\mathsf{F}(\mathsf{q}, \curlyvee, \hbar - \mathsf{F}(\mathsf{q}, \curlyvee, \omega))\|d\curlyvee \\
&\leq \frac{e^{-\varepsilon}}{\top^{2\flat-1}\mathcal{B}(\flat,\flat)}\|\hbar - \omega\|\int_0^{\ni} \curlyvee^{\flat-1}(\ni - \curlyvee)^{\flat-1}d\curlyvee \\
&\leq \frac{e^{-\varepsilon}}{\top^{2\flat-1}\mathcal{B}(\flat,\flat)}\|\hbar - \omega\|\top^{2\flat-1}\mathcal{B}(\flat,\flat).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|\Upsilon\hbar - \Upsilon\omega\| &\leq e^{-\varepsilon}\|\hbar - \omega\|, \\
\varepsilon + \ln(\|\Upsilon\hbar - \Upsilon\omega\|) &\leq \ln(\|\hbar - \omega\|).
\end{aligned}$$

Or,

$$\varepsilon + \ln(m_d(\Upsilon\hbar - \Upsilon\omega)) \leq \ln(m_d(\hbar - \omega)).$$

Setting $\mathsf{F}(\hbar) = \ln\hbar$, then quite smoothly it can be shown that $\mathsf{F} \in \tilde{F}$.

Next, defining $\propto$ as, $\propto$: $\mathsf{C} \times \mathsf{C} \to \{-\infty\} \cup [0, +\infty)$ such that,

$$\propto (\hbar, \omega) = \begin{cases} 1, & \text{if } \mathsf{J}(\hbar, \omega) \geq 0, \\ -\infty, & \text{else,} \end{cases}$$

then we deduce

$$\varepsilon + \propto (\hbar, \omega)\mathsf{F}(m_d(\Upsilon\hbar, \Upsilon\omega)) \leq \mathsf{F}(m_d(\hbar, \omega))$$

for $\hbar, \omega \in \mathsf{C}$, and $m_d(\Upsilon\hbar, \Upsilon\omega) > 0$, and with utilisation of (**p3**), we have

$$\begin{aligned} \propto (\hbar, \omega) \geq 1 \Rightarrow \mathsf{J}(\hbar, \omega) \geq 0 \\ \Rightarrow \mathsf{J}(\Upsilon\hbar, \Upsilon\omega) \geq 0 \\ \Rightarrow \propto (\Upsilon\hbar, \Upsilon\omega) \geq 1 \end{aligned}$$

for $\hbar, \omega \in \mathsf{C}$. Hence, Υ is $\propto$-admissible by (**p2**), and we have $\hbar_\circ \in \mathsf{C}$ such that $\propto (\hbar_\circ, \Upsilon\hbar_\circ) \geq 1$. From the condition (**p4**) and Theorem 2.3, we get $\hbar^* = \Upsilon\hbar^*$, where $\hbar^* \in \mathsf{C}$, and then there must be at least one solution for (1.1). □

Let $\hat{F}$ be the family of functions ϑ: $(0, +\infty) \to \mathcal{R}$ in a way that:

(ζ**1**) $0 < u < v$ $\vartheta(u) \leq \vartheta(v)$;

(ζ**2**) $\ni_n \to 0$ if and only if $\vartheta(\ni_n) \to -\infty$, where $\{\ni_n\} \subset (0, +\infty)$.

Definition 2.5. *[20] Let $\mho$ be a non empty set, m_d: $\mho \times \mho \to [0, +\infty)$, $\vartheta \in \mathbb{F}$, and $\xi \in [0, +\infty)$ in a way that, when $x, v \in \mho$, the below conditions hold true:*

(υ**1**) *$m_d(x, t) = 0 \Leftrightarrow x = t$;*

(υ**2**) *$m_d(x, t) = m_d(t, x)$;*

(υ**3**) *If $\{x_i\}_{i=1}^n \subset \mho$ in the sense that $(x_1, x_n) = (x, t)$, $n \geq 2$, we have*

$$m_d(x, t) > 0 \quad \Rightarrow \quad \vartheta(m_d(x, t)) \;\leq\; \vartheta(\sum_{i=1}^{n-1} m_d(x_i, x_{i+1})) + \xi.$$

Then $(\mho, m_d)$ is termed as an $\hat{F}$-metric space with $\hat{F}$-metric m_d.

Convergence, Cauchyness, and sequence completeness are all defined in $\hat{F}$-metric space as similar as defined in standard metric space.

Let η be the mappings ψ: $[0, +\infty) \to [0, +\infty)$ in a way that:

(η**1**) ψ is non-decreasing;

(η**2**) $\sum_{n=1}^{+\infty} \psi^n(e) < +\infty$, for $e \in \mathcal{R}^+$.

Definition 2.6. *[21] If a mapping Υ: $\mho \to \mho$ is such that*

$$\propto (\ni, \Upsilon \ni) \geq 1 \quad \Rightarrow \quad \propto (\Upsilon \ni, \Upsilon^2 \ni) \geq 1,$$

for $\ni \in \mho$ and $\propto$: $\mho \times \mho \to [0, +\infty)$, then Υ is called $\propto$-orbital admissible.

Corollary 2.7. *[21] Let $(\mho, m_d)$ be a complete $\hat{F}$-metric space, and $\Upsilon\colon \mho \to \mho$ and $\psi\colon [0,+\infty) \to [0,+\infty)$ in the sense that:*

(α**1**)

$$\propto (\hbar,\omega) m_d(\Upsilon\hbar, \Upsilon\omega) \le \psi(M_d(\hbar,\omega)),$$

where $M_d(\hbar,\omega) = \max m_d(\hbar,\omega), m_d(\hbar,\Upsilon\hbar), m_d(\omega,\Upsilon\omega), \hbar,\omega \in \mho$;

(α**2**) Υ *is $\propto$-orbital admissible;*

(α**3**) $\propto (s_\circ, \Upsilon s_\circ) \ge 1$ *for* $s_\circ \in \mho$*;*

(α**4**) $\Upsilon \in \mathbb{F}$ *is continuous and verify the condition* (υ_3)*, and ψ is continuous and satisfying* $\Upsilon(s) > \Upsilon(\psi(s)) + \xi$, $0 < s < +\infty$, *where* $\xi \in [0,+\infty)$. *Then Υ must have a fixed point.*

Assume $\mho = \mathsf{C}(\varsigma, \mathcal{R})$ and $m_d\colon \mho \times \mho \to [0,+\infty)$ defined by

$$m_d(\hbar,\omega) = \begin{cases} e^{\|\hbar-\omega\|}, & \text{if } \hbar \ne \omega, \\ 0, & \text{if } \hbar = \omega, \end{cases}$$

where

$$\|\hbar(\mathsf{q},\ni) - \omega(\mathsf{q},\ni)\| = \sup_{(\mathsf{q},\ni)\in\varsigma} |\hbar(\mathsf{q},\ni) - \omega(\mathsf{q},\ni)|,$$

and then m_d is an $\hat{F}$-metric on $\mho$. We have $\vartheta \in \mathbb{F}$ defined by

$$\vartheta(\ni) = -\frac{1}{\mathtt{t}}$$

for $\mathtt{t} > 0$ as well. So, it is obvious that $\vartheta(\mu) > \vartheta(\psi(\mu)) + \xi$, $\mu > 0$, such that, ψ possess the properties

$$\psi < \frac{\mu}{1+\mu}, \quad e^{\psi(n_\circ)} \le \psi(e^{n_\circ}),$$

where $n_\circ \in \{0,1,2,3,\ldots\}$.

The problem (1.1) has a solution in $\hat{F}$-metric space, which can be observed in the following theorem.

Theorem 2.8. *Suppose there is* $\mathsf{J}\colon \mathcal{R}^2 \to \mathcal{R}$ *in a way that*

($\curlyvee$**1**)

$$\|\mathsf{F}(q,\ni,\hbar(q,\ni)) - \mathsf{F}(q,\ni,\omega(q,\ni))\| \le \frac{\varsigma(\flat)}{\flat\top^{2\flat-1}\mathcal{B}(\flat,\flat)}\psi(\|\hbar-\omega\|),$$

where $(q,\ni) \in \varsigma$, *and* $\hbar,\omega \in \mho$ *with* $\mathsf{J}(\hbar,\omega) \ge 0$;

($\curlyvee$**2**) *There exists* $\hbar_1 \in \mho$ *with* $\mathsf{J}(\hbar_1, \Upsilon\hbar_1) \ge 0$ *where* $\Upsilon : \mho \to \mho$ *is defined by*

$$\Upsilon(\hbar) = \hbar_\circ + \frac{1}{\varsigma(\flat)} \int_0^{\ni} \flat \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} \mathsf{F}(q,\curlyvee,\hbar(q,\curlyvee)) d\curlyvee;$$

($\curlyvee$**3**) $\hbar \in \mho$ *with* $\mathsf{J}(\hbar,\Upsilon\hbar) \ge 0$ *implies that* $\mathsf{J}(\Upsilon\hbar, \Upsilon^2\hbar) \ge 0$. *Then, there exist a fixed point of* Υ.

Proof. We can write Eq (1.1) as

$$\hbar(\mathsf{q},\ni) = \hbar_\circ + \int_0^{\ni} \flat \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} \mathsf{F}(\mathsf{q},\curlyvee,\hbar) d\curlyvee = \Upsilon\hbar.$$

To derive the fixed point of Υ, we have

$$\begin{aligned}
\|\Upsilon\hbar - \Upsilon\omega\| &= \|\frac{1}{\varsigma(\flat)} \int_0^{\ni} \flat \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} (\mathsf{F}(\mathsf{q},\curlyvee,\hbar) - \mathsf{F}(\mathsf{q},\curlyvee,\omega)) d\curlyvee \| \\
&\leq \frac{\flat}{\varsigma(\flat)} \int_0^{\ni} \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} \|\mathsf{F}(\mathsf{q},\curlyvee,\hbar) - \mathsf{F}(\mathsf{q},\curlyvee,\omega)\| d\curlyvee \\
&\leq \frac{1}{\mathrm{T}^{2\flat-1}\mathcal{B}(\flat,\flat)} \int_0^{\ni} \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} \psi(\|\hbar - \omega\|) d\curlyvee \\
&= \frac{\psi(\|\hbar - \omega\|)}{\mathrm{T}^{2\flat-1}\mathcal{B}(\flat,\flat)} \int_0^{\ni} \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} d\curlyvee \\
&\leq \psi(\|\hbar - \omega\|).
\end{aligned}$$

Thus, for $\hbar, \omega \in \mho$ having $\mathsf{J}(\hbar,\omega) \geq 0$, we have

$$\begin{aligned}
m_d(\Upsilon\hbar, \Upsilon\omega) = e^{\|\Upsilon\hbar - \Upsilon\omega\|} &\leq e^{\psi(\|\hbar-\omega\|)} \leq \psi(e^{\|\hbar-\omega\|}) \\
&= \psi(m_d(\hbar,\omega)) \leq \psi(M_d(\hbar,\omega)).
\end{aligned}$$

Define $\propto: \mho \times \mho \to [0,+\infty)$ by

$$\propto(\hbar,\omega) = \begin{cases} 1, & \text{if } \mathsf{J}(\hbar,\omega), \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\propto(\hbar,\omega) m_d(\Upsilon\hbar,\Upsilon\omega) \leq m_d(\Upsilon\hbar,\Upsilon\omega) \leq \psi(M_d(\hbar,\omega)),$$

for $\hbar, \omega \in \mho$ with $m_d(\Upsilon\hbar,\Upsilon\omega) \geq 0$. By ($\curlyvee$**3**), we have

$$\begin{aligned}
\propto(\hbar,\Upsilon\hbar) \geq 1 \quad &\Rightarrow \quad \mathsf{J}(\hbar,\Upsilon\hbar) \geq 0 \quad \Rightarrow \mathsf{J}(\Upsilon\hbar, \mathfrak{T}^y\hbar) \geq 0 \\
&\Rightarrow \quad \propto(\Upsilon\hbar, \Upsilon^2\hbar) \geq 1.
\end{aligned}$$

Hence, Υ is an orbital $\propto$-admissible, and from the condition ($\curlyvee$**2**), there exist $\hbar_1 \in \mho$ in a way that $\propto(\hbar_1, \Upsilon\hbar_1) \geq 1$. Additionally, by ($\curlyvee$**3**) and the Corollary 2.7, we obtain $\hbar^* \in \mho$ in a way that $\hbar^* = \Upsilon\hbar^*$. Hence there is a solution $\hbar^*$ of the problem (1.1). □

Now for defining an orbitally complete metric space, suppose $(\mho, m_d)$ is a metric space and Υ: $\mho \to \mho$. If $\ell_\circ \in \mho$, then $\ell_\circ$ has the orbit in the set form as

$$O(\mathsf{q}_\circ) = \{\Upsilon^n \mathsf{q}_\circ : n = 0, 1, 2, 3 \ldots\},$$

where Υ^n is the nth iteration of Υ and $\mathcal{D}(\mathsf{q}_\circ)$ is the diameter of $O(x_\circ)$. $\mho$ is characterised as Υ-orbitally complete metric space if all the Cauchy sequences from $O(x)$ converge in $\mho$ for some $\mathsf{q} \in \mho$.

Theorem 2.9. *[22] Let $(\mho, m_d)$ represent Υ-orbitally complete metric space, Υ: $\mho \to \mho$ and θ: $\mho \to \mathcal{N}$. Then Υ has a unique fixed point, if there exists $\nu > 0$ and $\mathsf{q}_\circ \in \mho$ with $0 < \mathcal{D} < +\infty$ such as*

$$m_d(\Upsilon^{\theta(q)}(\hbar), \Upsilon^{\theta(q)}(\omega)) \leq e^{\nu} m_d(\hbar, \omega).$$

Let

$$\mho = \mathsf{C}(\varsigma, \mathcal{R}) \quad \text{and} \quad m_d(\hbar, \omega) = \sup_{(\mathsf{q},\ni)\in\varsigma} |\hbar - \omega|.$$

The below theorem explores the existence and uniqueness of the problem (1.1) in Υ-orbitally complete metric space.

Theorem 2.10. *Let Υ: $\mho \to \mho$ define by*

$$\Upsilon(\hbar) = \hbar_\circ + \frac{1}{\varsigma(\flat)} \int_0^{\ni} \flat\, \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} \mathsf{F}(q, \curlyvee, \hbar(q, \curlyvee)) d\curlyvee$$

and

$$|\mathsf{F}(q, \ni, \hbar(q, \ni) - \mathsf{F}\omega(q, \ni))| \leq \frac{\varsigma(\flat)}{\flat \top^{2\flat-1} \mathcal{B}(\flat, \flat)} e^{-\nu} |\sqrt{|\hbar|} - \sqrt{|\omega|}|,$$

$$|\mathsf{F}(q, \ni, \hbar)| + |\mathsf{F}(q, \ni, \omega)| \leq \frac{\varsigma(\flat)}{\flat \top^{2\flat-1} \mathcal{B}(\flat, \flat)} e^{-\nu} |\sqrt{|\hbar|} + \sqrt{|\omega|}|.$$

Then the problem (1.1) must have a unique solution.

Proof. Equation (1.1) can be written as

$$\hbar(\mathsf{q}, \ni) = \hbar_\circ + \int_0^{\ni} \flat\, \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} \mathsf{F}(\mathsf{q}, \curlyvee, \hbar) d\curlyvee = \Upsilon\hbar.$$

To prove a unique solution of Υ, we have

$$\begin{aligned} |\Upsilon\hbar - \Upsilon\omega| &= |\frac{1}{\varsigma(\flat)} \int_0^{\ni} \flat\, \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} (\mathsf{F}(\mathsf{q}, \curlyvee, \hbar) - \mathsf{F}(\mathsf{q}, \curlyvee, \omega)) d\curlyvee| \\ &\leq \frac{1}{\varsigma(\flat)} \int_0^{\ni} \flat\, \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} |\mathsf{F}(\mathsf{q}, \curlyvee, \hbar - \mathsf{F}(\mathsf{q}, \curlyvee, \omega))| d\curlyvee \\ &\leq \frac{e^{-\nu}}{\flat \top^{2\flat-1} \mathcal{B}(\flat, \flat)} \int_0^{\ni} \flat\, \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} |\sqrt{|\hbar|} - \sqrt{|\omega|}| d\curlyvee \\ &= \frac{e^{-\nu}}{\flat \top^{2\flat-1} \mathcal{B}(\flat, \flat)} \sup |\sqrt{|\hbar|} - \sqrt{|\omega|}| \int_0^{\ni} \flat\, \curlyvee^{\flat-1} (t - \curlyvee)^{\flat-1} d\curlyvee \\ &\leq e^{-\nu} \sup |\sqrt{|\hbar|} - \sqrt{|\omega|}|. \end{aligned}$$

Also,

$$\begin{aligned} |\Upsilon\hbar| + |\Upsilon\omega| &= |\frac{1}{\varsigma(\flat)} \int_0^{\ni} \flat\, \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} \mathsf{F}(\mathsf{q}, \curlyvee, \hbar) d\curlyvee| + |\frac{1}{\varsigma(\flat)} \int_0^{\ni} \flat\, \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} \mathsf{F}(\mathsf{q}, \curlyvee, \omega) d\curlyvee| \\ &\leq \frac{1}{\varsigma(\flat)} \int_0^{\ni} \flat\, \curlyvee^{\flat-1} (\ni - \curlyvee)^{\flat-1} (|\mathsf{F}(\mathsf{q}, \curlyvee, \hbar)| + |\mathsf{F}(\mathsf{q}, \curlyvee, \omega)|) d\curlyvee \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{e^{-\nu}}{\top^{2\flat-1}\mathcal{B}(\flat,\flat)} \int_0^{\ni} \curlyvee^{\flat-1}(\ni - \curlyvee)^{\flat-1} |\sqrt{|\hbar|} + \sqrt{|\omega|}| d\curlyvee \\
&= \frac{e^{-\nu}}{\top^{2\flat-1}\mathcal{B}(\flat,\flat)} \sup |\sqrt{|\hbar|} + \sqrt{|\omega|}| \int_0^{\ni} \curlyvee^{\flat-1}(\ni - \curlyvee)^{\flat-1} d\curlyvee \\
&\leq e^{-\nu} \sup |\sqrt{|\hbar|} + \sqrt{|\omega|}| \leq \sup |\sqrt{|\hbar|} + \sqrt{|\omega|}|.
\end{aligned}
$$

Now,

$$
\begin{aligned}
m_d(\Upsilon^2\hbar, \Upsilon^2\omega) &= \sup |\Upsilon^2\hbar - \Upsilon^2\omega| \\
&= \sup |\Upsilon\hbar - \Upsilon\omega| \times \sup |\Upsilon\hbar + \Upsilon\omega| \\
&\leq \sup |\Upsilon\hbar - \Upsilon\omega| \times \sup(|\Upsilon\hbar| + |\Upsilon\omega|) \\
&\leq e^{-\nu} \sup |\sqrt{|\hbar|} - \sqrt{|\omega|}| \times \sup \sup |\sqrt{|\hbar|} + \sqrt{|\omega|}| \\
&= e^{-\nu} \sup ||\hbar| - |\omega|| \\
&\leq e^{-\nu} \sup |\hbar - \omega| \\
&= e^{-\nu} m_d(\hbar, \omega).
\end{aligned}
$$

If we consider θ: $\mho \to \mathcal{N}$ in a way that $\theta(\hbar) = 2$ for every $\hbar \in \mho$, then all necessities of Theorem 2.9 are true. As a result, the problem (1.1) ensures a unique solution in $\mho$. □

3. Conclusions

Exploring the solutions for fractional differential equations has been a central focus of this research. There still needs to be more research regarding solutions for nonlinear differential equations that involve fractal fractional operators. This study has focused on introducing new contraction methods, specifically the $\propto$-ψ-contraction and $\propto$-type of $\tilde{\text{F}}$-contraction, in the framework of $\hat{F}$-metric and orbitally metric spaces. This paper has developed particular fixed point theorems that provide a novel and direct approach for investigating the existence and uniqueness of solutions for general partial differential equations using fractal fractional operators. This discovery helps further the knowledge and use of fractional calculus in dealing with complicated nonlinear events and opens the door for future development in this field.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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