



Research article

Well-posedness and Ulam-Hyers stability results of solutions to pantograph fractional stochastic differential equations in the sense of conformable derivatives

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Abstract: One kind of stochastic delay differential equation in which the delay term is dependent on a proportion of the current time is the pantograph stochastic differential equation. Electric current collection, nonlinear dynamics, quantum mechanics, and electrodynamics are among the phenomena modeled using this equation. A key idea in physics and mathematics is the well-posedness of a differential equation, which guarantees that the solution to the problem exists and is a unique and meaningful solution that relies continuously on the initial condition and the value of the fractional derivative. Ulam-Hyers stability is a property of equations that states that if a function is approximately satisfying the equation, then there exists an exact solution that is close to the function. Inspired by these findings, in this research work, we established the Ulam-Hyers stability and well-posedness of solutions of pantograph fractional stochastic differential equations (PFSDEs) in the framework of conformable derivatives. In addition, we provided examples to analyze the theoretical results.

Keywords: conformable fractional derivative; pantograph fractional stochastic differential equations; well-posedness; Ulam-Hyers stability

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1. Introduction

A generalization of classical calculus that permits the differentiation and integration of any order is known as fractional calculus (FC). FC serves as a suitable tool for modeling systems with memory effects, where past actions impact current behavior. Fractional-order derivatives better capture memory effects in biological systems, diffusion processes, and viscoelastic materials. Among the noteworthy applications are [1–13]:

- Complex physical systems like viscoelastic materials, diffusion processes, electrical circuits, and fluid dynamics are all being modeled and analyzed using FC. Systems that exhibit anomalous diffusion, non-local behaviors, and memory effects are better described using fractional differential equations.
- In control systems, fractional-order controllers, fractional-order observers, and systems with temporal delays are all being modeled using FC. In some applications, fractional-order controllers perform better, are more robust, and are more stable than integer-order controllers.
- In signal processing, FC is used for applications including image processing, time series analysis, noise reduction, and signal denoising. Long-range dependence, fractal features, and handling non-stationary signals are all benefits of fractional-order filters and transforms.
- FC is used to model physiological processes, biological systems, and medical imaging procedures. Applications include neuronal dynamics modeling, biological signal analysis, drug kinetics, and gene expression.
- Fractional-order damping, nonlinear dynamics, and viscoelastic features of mechanical systems are all being modeled using FC. Applications include vibration analysis, control of flexible structures, and structural health monitoring.
- The dynamics of batteries, electrochemical systems, and materials with unusual transport properties are all being modeled using FC. Mass and charge transport processes are more accurately described by fractional diffusion and reaction-diffusion equations.
- Precipitation patterns, pollutant transfer, and groundwater movement are all being modeled using FC. In hydrological systems, long-term memory effects and anomalous transport processes are better described by fractional-order differential equations.
- In robotics, FC is used to model robot dynamics and to create motion and control algorithms. The benefits of fractional-order controllers include robustness against shocks, trajectory tracking, and adaptive control.
- Within its own domain, FC finds use in the analysis of differential equations involving fractional operators, integral transforms, and special functions. Strong tools for deciphering intricate mathematical and physical systems and phenomena are offered by FC.
- FC is employed to analyze economic time series data, which often exhibits long-range dependence and non-stationary behavior. Fractional processes can capture the memory effects and persistence observed in economic variables over time. Economic models incorporating fractional calculus better capture the dynamics of financial markets, including stock price movements, trading volume, and market volatility.

Researchers utilize FC to find applications in various fields, including the study of synchronization phenomena in complex networks [14–17]. Global bipartite synchronization refers to the phenomenon where two distinct groups of nodes in a network synchronize their dynamics while remaining

unsynchronized with nodes within their group. This phenomenon is crucial in understanding the collective behavior of complex systems and has applications in various domains such as neuroscience, biology, and engineering. The combination of fractional calculus and global bipartite synchronization allows researchers to model and analyze complex network dynamics with memory effects and long-range dependencies. Fractional-order differential equations capture the non-local interactions and memory effects present in real-world systems, while the concept of global bipartite synchronization provides insights into the emergent collective behavior of networked systems. By integrating FC with the study of global bipartite synchronization, researchers develop more accurate models and algorithms for understanding and controlling the dynamics of complex networks. This interdisciplinary approach has the potential to advance our understanding of synchronization phenomena in networked systems and facilitates the design of robust and efficient communication and control strategies.

Fractional order derivative (FOD) is defined in different ways; some definitions are conformable, Riemann-Liouville (R-L), Caputo-Fabrizio, Atangana-Baleanu, Caputo-Hadamard, and Grunwald Letnikov [18–22]. There are various definitions for FOD, in contrast to integer-order derivatives. In general, these definitions do not match each other. The conformable fractional derivative (CFD), which provides a unique description of the FOD in contrast to previous versions, is developed by Khalil et al. [23].

Regarding a mapping $\Psi(\tau) : [0, \infty[\rightarrow \mathcal{R}$, the CFD is:

$$\mathcal{T}_\tau^\ell \Psi(\tau) = \lim_{\epsilon \rightarrow 0} \frac{\Psi^{\lceil \ell \rceil - 1}(\tau + \epsilon \tau^{\lceil \ell \rceil - \ell}) - \Psi^{\lceil \ell \rceil - 1}(\tau)}{\epsilon},$$

where \mathcal{T}_τ^ℓ is the CFD with respect to time and $u - 1 < \ell \leq u$, $\tau > 0$, $u \in \mathcal{N}$ and $\lceil \ell \rceil$, the lowest integer that is equal to or larger than ℓ . In a certain instance, if $0 < \ell \leq 1$, we acquire

$$\mathcal{T}_\tau^\ell \Psi(\tau) = \lim_{\epsilon \rightarrow 0} \frac{\Psi(\tau + \epsilon \tau^{1-\ell}) - \Psi(\tau)}{\epsilon}, \quad \tau > 0.$$

If $\Psi(\tau)$ is ℓ -differentiable in some $(0, \Xi)$, $\Xi > 0$ and $\lim_{\tau \rightarrow 0^+} \Psi^\ell(\tau)$ exists, so we have $\Psi^{(\ell)}(0) = \lim_{\tau \rightarrow 0^+} \Psi^{(\ell)}(\tau)$. The fractional integral in the sense of conformable of a function $\Psi(\tau)$ starting from $\tau_0 \geq 0$ is defined as:

$$\mathfrak{I}_\ell^{\tau_0}(\Psi)(\tau) = \int_{\tau_0}^{\tau} \frac{\Psi(\chi)}{(\chi - \tau_0)^{1-\ell}} d\chi, \quad \ell \in (0, 1].$$

Differential equations (DEs) with a random or stochastic component are known as stochastic differential equations (SDEs). In the natural and social sciences, including physics, chemistry, biology, ecology, economics, finance, and engineering, SDEs are extensively employed to model a wide range of phenomena. These are only a few of the several scientific and engineering domains where SDEs are used [24–26].

- **Physics:** Systems that are sensitive to random forces, noise, or temperature changes have their motion described by SDEs. One SDE that simulates the Brownian motion of a particle in a fluid is the Langevin equation.
- **Chemistry:** The dynamics of chemical reactions involving random events, including collisions, transitions, or catalysis, are captured by SDEs. The intermolecular forces and thermodynamic fluctuations are taken into consideration in the molecular dynamics equation, which explains the motion of individual molecules in a system.

- **Biology:** Genetic mutations, environmental variations, and population interactions are examples of random events that can affect the behavior of biological systems, and SDEs simulate their behavior. An example of an SDE that depicts the dynamics of predator-prey relationships in an ecosystem is the Lotka-Volterra equation.
- **Ecology:** SDEs simulate the growth and decline of populations that are affected by stochastic events, such as births, deaths, migrations, or competitions. For example, the logistic equation is a SDE that describes the growth of a population with a carrying capacity.
- **Economics:** Random shock-driven economic variables, such as interest rates, prices, and exchange rates, are analyzed using SDEs. One SDE that establishes the value of a European option in a financial market is the Black-Scholes equation.
- **Finance:** Financial assets like stocks, bonds, and derivatives that are susceptible to uncertainty have their risk and return estimated by SDEs. One SDE that simulates the price of a stock that moves in a random walk with drift and volatility is the geometric Brownian motion.
- **Engineering:** Engineering systems subject to random disturbances, including signal processing, control, or communication systems, have their performance and dependability improved by SDEs. One SDE that approximates a system's state from noisy observations is the Kalman filter.

A DE with a FOD and a stochastic process is known as a fractional stochastic differential equation (FSDE). Systems with long-range dependencies, turbulence, economics, and biology are being modeled by these equations. The idea of FSDEs still faces many questions and challenges that need to be addressed. The existence and uniqueness of solutions, stability and convergence, numerical techniques, and applications to physics, biology, finance, and other domains are a few of the interesting subjects.

Some authors have been actively researching the FSDEs lately. Li and Xu developed exponential stability in the mean square of delay FSDEs [27]. This study endeavored to provide additional conditions for the exponential stability of the mean square of the scenarios it investigated. Sadovskii's fixed-point theory was applied by Li and Peng [28] and worked on the important concept of the controllability of FSDEs. Cui and Yan [29] established results for delay integrodifferential equations with stochastic terms with respect to the existence of solutions in Hilbert spaces with the help of Sadovskii's fixed-point. Specifically, in [30], Niu and Xie investigated the existence, uniqueness, and regularity of FSDEs. With the help of the Schauder theorem, Chen and Li [31] worked on some characteristics of solutions to FSDEs. Under different assumptions, the authors of the articles [32, 33] showed that solutions to FSDEs exist. Karczewska and Lizama reported a number of findings in [34] on the solutions to FSDEs. Schnaubelt and Veraar [35] have worked on path-wise continuous features for solutions to stochastic problems. The stopping time technique was used by Xiao and Wang [36] to investigate the stability of FSDEs of the Caputo type. Saifullah et al. [37] investigated the existence and uniqueness of solutions for FSDEs with R-L. The authors applied the fixed point theory (FPT) for uniqueness and the Móach condition for the existence of a solution. The writers also present the Hyers-Ulam stability of these equations. Moumen et al. [38] subjected FSDEs with nonlocal conditions to an approximate controllability analysis. The authors derived the approximate controllability of nonlinear FSDEs. For the desired results, the FPT for multi-valued operators was employed.

Dynamic systems depend on states other than their present ones from the past; these occurrences are often simulated using pantograph differential equations (PDEs) [39–44]. PDEs, FC, and stochastic calculus are combined in stochastic fractional pantograph differential equations (PFSDs). They are

useful in many domains where systems show memory effects in addition to random fluctuations. Among the noteworthy uses of PFSDEs are [45–50]:

- PFSDEs are used to simulate stochastic volatility and memory effects in financial processes, including option pricing models, asset price dynamics, and risk management techniques. These models aid in the development of successful investing strategies as well as the comprehension of the intricate behavior of financial markets.
- PFSDEs are used in the modeling and analysis of random delay and memory effect communication systems, including packet-switched networks, wireless networks, and data transfer protocols. These models help maximize dependability, reduce latency, and optimize network performance.
- Stochastic dynamical systems, random walks in fractal media, diffusion processes, and other physical systems with random fluctuations and memory effects are among the systems that PFSDEs simulate. These models aid in forecasting the long-term evolution of complex physical systems and in understanding their behavior.
- PFSDEs are used to represent biological systems, including gene expression pathways, neuronal dynamics, and physiological responses to external stimuli, that exhibit both stochastic variability and memory effects. These models aid in the development of strategies for the management and treatment of disease as well as the comprehension of the fundamental mechanisms driving biological processes.
- PFSDEs are useful for simulating stochastic variability and memory effects in environmental processes, including pollution dispersion, ecological systems, and climate models. These models are helpful for forecasting changes in the environment, evaluating the effects of human activity on ecosystems, and formulating conservation plans for the environment.
- PFSDEs are employed in the modeling and analysis of control systems, including time-delay systems, networked control systems, and adaptive control algorithms, that have both stochastic disturbances and memory effects. These models aid in the development of reliable control schemes that successfully manage disturbances and uncertainty.
- PFSDEs are used to create stochastic models for optimization issues, reinforcement learning, and machine learning methods. These models make it possible for learning algorithms to identify long-term dependencies as well as short-term oscillations in the data, resulting in learning algorithms that are more reliable and effective.

The existence and uniqueness of solutions to FSDEs are one of the main subjects covered in mathematics. Knowing whether or not a certain FSDE has a unique solution is essential. One of the most important qualitative ideas in dynamical systems is stability theory. Subsequently, the concept of stability garners increased interest across several fields of study and practical uses. In mathematics, Ulam-Hyers stability refers to the approximations of functional equation solutions. In the 1940s, Ulam and Hyers introduced it, and Rassias and others later generalized it.

Inspired by these findings, we first demonstrated the existence and uniqueness of solutions of PFSDEs in the framework of CFD by using the Banach fixed point theorem; after that, we demonstrated the continuous dependency (Con-D) of solutions on both the initial value (In-V) and the fractional exponent ℓ . The second section was devoted to examining the Ulam-Hyers stability (UHS) of PFSDE using the generalized Gronwall inequalities. The main part of the proof involved the use of the Cauchy-

Schwartz inequality (CS-Ineq), Itô isometry formula (I-IsF), generalized Gronwall's inequality (GG-Ineq), and temporally weighted norm.

Compared with the research results of [39–50], the major contributions of this paper include at least the following three aspects:

- (i) In contrast to [39–44], the system we studied was more generalized because it included the stochastic term.
- (ii) In contrast to [45–50], in our research work, we established results regarding the existence, uniqueness, continuous dependency on both the initial condition and fractional component, and Ulam-Hyers stability in the sense of CFD.
- (iii) In contrast to [45–51], in which continuous dependency was not discussed, we have evaluated this factor in our research work. We presented continuous dependency of solutions of PFSDEs on the initial value and fractional exponent in the sense of CFD.

We examined the following PFSDEs of order $\frac{1}{2} < \ell < 1$:

$$\mathcal{T}_\tau^\ell \mathbb{Z}(\tau) = \mathcal{A}_1(\tau, \mathbb{Z}(\tau), \mathbb{Z}(\eta\tau)) + \mathcal{A}_2(\tau, \mathbb{Z}(\tau), \mathbb{Z}(\eta\tau)) \frac{d\mathcal{W}_\tau}{d\tau}, \quad (1.1)$$

where $\eta\tau$ represents the past state and $\eta \in (0, 1)$ and ℓ represent the CFD, $\mathcal{A}_1 : [\tau_0, \mathfrak{J}] \times \mathcal{R}^m \times \mathcal{R}^m \rightarrow \mathcal{R}^m$, $\mathcal{A}_2 : [\tau_0, \mathfrak{J}] \times \mathcal{R}^m \times \mathcal{R}^m \rightarrow \mathcal{R}^m$ are measurable and on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathfrak{P})$, and $(\mathcal{W}_\tau)_{\tau \in [0, \infty)}$ is a scalar Brownian motion, with the filtration $\mathfrak{F}_\tau = (\mathcal{F}_\tau)_{\tau_0 \leq \tau \leq \mathfrak{J}}$.

Fractional orders in the interval $(\frac{1}{2}, 1)$ are a popular choice [52–55]; the precise range may differ based on the situation, the nature of the issue, and the intended trade-off between correctly capturing the dynamics of the system and maintaining mathematical tractability. For instance, the fractional order in viscoelastic materials may indicate the ratio of elasticity to viscosity. Values between $\frac{1}{2}$ and 1 more closely represent the behavior of the system in such circumstances. When compared to fractional-order systems with orders below $\frac{1}{2}$, those with orders in the interval $(\frac{1}{2}, 1)$ frequently show superior stability and regularity features. Systems having orders smaller than $\frac{1}{2}$ can result in non-unique solutions or a deficiency of solutions in mathematics.

Extending the fractional order range from $(\frac{1}{2}, 1)$ to $(0, 1)$ poses challenges due to the inclusion of the point 0. However, we have provided unified criteria for selecting fractional orders within the extended range $(0, 1)$ by considering the following factors:

- (i) We established criteria based on stability and regularity properties of solutions. Fractional orders within the range $(0, 1)$ yield stable and well-behaved solutions for the systems under consideration. Analyzing stability and regularity conditions guided the selection of suitable fractional orders.
- (ii) Unified criteria ensure the existence and uniqueness of solutions for fractional differential equations within the extended range. We developed mathematical conditions that guarantee the existence and uniqueness of solutions, considering the fractional order's impact on these properties.
- (iii) Criteria were based on the physical interpretation of fractional orders within the range $(0, 1)$. We ensured that selected fractional orders aligned with the underlying dynamics of the systems

being modeled. This involved considering the physical meaning of fractional orders in specific applications, such as viscoelasticity or diffusion processes.

- (iv) Unified criteria address the numerical stability and efficiency of methods used to solve fractional differential equations with fractional orders in the range $(0, 1)$. We established guidelines for selecting numerical methods that accurately and efficiently handle fractional order equations while maintaining stability and convergence properties.
- (v) We conducted comparative analyses between fractional orders within the range $(0, 1)$ and those within the range $(\frac{1}{2}, 1)$. By comparing the behavior and performance of systems modeled with different fractional orders, the authors identified criteria that effectively differentiated between suitable fractional orders.

By considering these factors and establishing unified criteria, we provided guidance for selecting fractional orders within the extended range $(0, 1)$ that ensured the stability, regularity, and physical relevance of solutions in various applications.

The study is structured as follows: We use some fundamental ideas in the next section to provide the framework for the PFSDE conclusions. We first demonstrate the well-posedness of the PFSDE solution in the first subsection of Section 3, and we demonstrate the UHS and offer two instances to support our findings in the second section. Section 4 then presents the conclusion.

2. Preliminaries

In this section, we will go over some basic concepts, assumptions, characteristics, and definitions that will be useful in this paper.

First, we present some useful characteristics of CFD that make it distinguishable from other fractional derivatives. The primary advantages of CFD are outlined in the following order [56–67]:

- (i) In contrast to the other fractional formulations, it satisfies every need and regulation of an ordinary derivative, such as the product, quotient, and chain rules; Rolle's theorem; and mean-value theorem.
- (ii) It reduces the complexity of well-known transforms, such as the Sumudu and Laplace transforms, which are used as instruments to solve some DEs.
- (iii) It may be quickly and simply modified to solve systems, analytical, and numerical DEs.
- (iv) Fuzzy generalized CFD, Katugampola fractional derivatives, M-CFD, deformable fractional derivatives, and class CFD are just a few of the new concepts that may be created and expanded upon thanks to it.
- (v) In many different applications, it produces new comparisons between CFD and previous fractional definitions.
- (vi) CFD draws a lot of interest from researchers since it can be used to depict a wide range of phenomena and applications that need to be addressed.

The following are some significant aspects of CFD that are relevant to differentiation processes.

Theorem 2.1. [68] Let $0 < \ell \leq 1$, $\Psi_1(\tau)$ and $\Psi_2(\tau)$ be ℓ -differentiable at a point $\tau > 0$. Then

- (i) $\mathcal{T}_\tau^\ell(\hbar_1\Psi_1(\tau) + \hbar_2\Psi_2(\tau)) = \hbar_1\mathcal{T}_\tau^\ell\Psi_1(\tau) + \hbar_2\mathcal{T}_\tau^\ell\Psi_2(\tau)$, $\forall \hbar_1, \hbar_2 \in \mathcal{R}$.
- (ii) $\mathcal{T}_\tau^\ell(\tau^\hbar) = \hbar\tau^{\hbar-\ell}$, $\forall \hbar \in \mathcal{R}$.
- (iii) $\mathcal{T}_\tau^\ell(\Upsilon) = 0$, $\Upsilon \in \mathcal{R}$.
- (iv) $\mathcal{T}_\tau^\ell\Psi_1(\tau)\Psi_2(\tau) = \Psi_1(\tau)\mathcal{T}_\tau^\ell\Psi_2(\tau) + \Psi_2(\tau)\mathcal{T}_\tau^\ell\Psi_1(\tau)$.
- (v) $\mathcal{T}_\tau^\ell \frac{\Psi_1(\tau)}{\Psi_2(\tau)} = \frac{\Psi_2(\tau)\mathcal{T}_\tau^\ell\Psi_1(\tau) - \Psi_1(\tau)\mathcal{T}_\tau^\ell\Psi_2(\tau)}{\Psi_2^2(\tau)}$.

Denote by $(\Omega, \mathcal{F}, (\mathcal{F}_\tau)_{\tau_0 \leq \tau \leq \mathfrak{J}}, \mathfrak{P})$ the complete probability space and $\mathcal{W}(\tau)$ is the standard Brownian motion. Suppose $\mathbb{Z}_\tau = \mathcal{L}^2(\Omega, \mathcal{F}_\tau, \mathfrak{P})$ be the space of all \mathcal{F}_τ -measurable, integrable, and mean square functions $\mathbb{Z} = (\mathbb{Z}_1, \mathbb{Z}_2, \dots, \mathbb{Z}_m)^T: \Omega \rightarrow \mathcal{R}^m$ with

$$\|\mathbb{Z}\|_{\mathcal{M}, \mathcal{S}} = \left(\sum_{j=1}^m \mathcal{E}[|\mathbb{Z}_j|^2] \right)^{\frac{1}{2}} = \left(\mathcal{E}[\|\mathbb{Z}\|^2] \right)^{\frac{1}{2}},$$

making use of the usual Euclidian norm, $\|\cdot\|$.

Definition 2.2. To prove the desired results, we assume that coefficients \mathcal{A}_1 and \mathcal{A}_2 meet the following assumptions:

(\mathcal{H}_1) : There is $\mathcal{L} > 0$ such that $\forall (\tau, \nu_1, \nu_2, \tilde{\nu}_1, \tilde{\nu}_2) \in [\tau_0, \mathfrak{J}] \times \mathcal{R}^m \times \mathcal{R}^m \times \mathcal{R}^m \times \mathcal{R}^m$:

$$\|\mathcal{A}_1(\tau, \nu_1, \nu_2) - \mathcal{A}_1(\tau, \tilde{\nu}_1, \tilde{\nu}_2)\|^2 \leq \mathcal{L}^2(\|\nu_1 - \tilde{\nu}_1\|^2 + \|\nu_2 - \tilde{\nu}_2\|^2),$$

$$\|\mathcal{A}_2(\tau, \nu_1, \nu_2) - \mathcal{A}_2(\tau, \tilde{\nu}_1, \tilde{\nu}_2)\|^2 \leq \mathcal{L}^2(\|\nu_1 - \tilde{\nu}_1\|^2 + \|\nu_2 - \tilde{\nu}_2\|^2).$$

(\mathcal{H}_2) : $\mathcal{A}_1(\cdot, 0, 0)$ and $\mathcal{A}_2(\cdot, 0, 0)$ are bounded, i.e,

$$\|\mathcal{A}_1(\cdot, 0, 0)\|_\infty = \text{ess sup}_{\tau \in [\tau_0, \mathfrak{J}]} \|\mathcal{A}_1(\tau, 0, 0)\| < \infty,$$

$$\|\mathcal{A}_2(\cdot, 0, 0)\|_\infty = \text{ess sup}_{\tau \in [\tau_0, \mathfrak{J}]} \|\mathcal{A}_2(\tau, 0, 0)\| < \infty,$$

$$\int_{\tau_0}^{\mathfrak{J}} \|\mathcal{A}_1(\chi, 0, 0)\|^2 d\chi < \infty,$$

$$\int_{\tau_0}^{\mathfrak{J}} \|\mathcal{A}_2(\chi, 0, 0)\|^2 d\chi < \infty.$$

We give now the definition of UHS.

Definition 2.3. The Eq (1.1) is UHS with respect to ϵ if there is a constant $\mathfrak{N} > 0$ such that for each $\epsilon > 0$ and $\mathcal{N}(\tau) \in \mathcal{H}^2([\eta\tau_0, \mathfrak{J}])$ of the following inequality: $\forall \tau \in [\tau_0, \mathfrak{J}]$

$$\mathcal{E} \left[\left\| \mathcal{N}(\tau) - \mathcal{N}(\tau_0) - \left(\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \left(\mathcal{A}_1(\chi, \mathcal{N}(\chi), \mathcal{N}(\eta\chi)) d\chi + \mathcal{A}_2(\chi, \mathcal{N}(\chi), \mathcal{N}(\eta\chi)) d\mathcal{W}(\chi) \right) \right) \right\|^2 \right] \leq \epsilon, \quad (2.1)$$

there exists a solution $\mathbb{Z}(\tau) \in \mathcal{H}^2([\eta\tau_0, \mathfrak{J}])$ of Eq (1.1), with $\mathbb{Z}(\tau) = \mathcal{N}(\tau)$ for $\tau \in [\eta\tau_0, \tau_0]$, satisfies $\mathcal{E}[\|\mathcal{N}(\tau) - \mathbb{Z}(\tau)\|^2] \leq \mathfrak{N}\epsilon$, $\forall \tau \in [\tau_0, \mathfrak{J}]$.

3. The main results

In this part of the paper, we present our results for the well-posedness solutions of PFSDE.

3.1. Well-posedness of PFSDEs under the standard Lipschitz condition of coefficients

Assume $\widetilde{\mathcal{H}}^2(\eta\tau_0, \mathfrak{J})$ is the space where all processes $\mathbb{Z}(\tau)$ which are measurable $\mathcal{F}_{\mathfrak{J}}$ -adapted, with $\mathcal{F}_{\mathfrak{J}} = (\mathfrak{F}_{\tau})_{\tau \in [\eta\tau_0, \mathfrak{J}]}$ and satisfy the following:

$$\|\mathbb{Z}\|_{\widetilde{\mathcal{H}}^2} = \sup_{\eta\tau_0 \leq \tau \leq \mathfrak{J}} \|\mathbb{Z}(\tau)\|_{\mathcal{M}_{\mathcal{S}}} < \infty.$$

It is easy to show that $(\widetilde{\mathcal{H}}^2([\eta\tau_0, \mathfrak{J}]), \|\cdot\|_{\widetilde{\mathcal{H}}^2})$ is a Banach space. For $\varpi \in \mathcal{H}^2([\eta\tau_0, \tau_0])$, we consider the operator $\Lambda_{\varpi} : \widetilde{\mathcal{H}}^2([\eta\tau_0, \mathfrak{J}]) \rightarrow \widetilde{\mathcal{H}}^2([\eta\tau_0, \mathfrak{J}])$ defined by

$$\begin{aligned} \Lambda_{\varpi}(\mathbb{Z}(\tau)) = & \varpi(\tau_0) + \int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) d\chi \\ & + \int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_2(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) d\mathcal{W}(\chi), \end{aligned} \quad (3.1)$$

for $\tau \in [\tau_0, \mathfrak{J}]$ and $\Lambda_{\varpi}(\mathbb{Z}(\tau)) = \varpi(\tau)$ for $\tau \in [\eta\tau_0, \tau_0]$.

The subsequent lemma shows this operator's well-defined property. It is essential to demonstrate the well-definedness of an operator in the context of DEs because a well-defined operator in DEs guarantees a clear, consistent mapping from the domain to the range that is independent of the representation of the domain's elements. In the study of DEs, this ensures the accuracy and consistency of mathematical operations and solutions.

The elementary inequality below is employed in the proof of this result as well as multiple others that follow:

$$\|\mathbb{Z}_1 + \mathbb{Z}_2 + \cdots + \mathbb{Z}_j\|^2 \leq j(\|\mathbb{Z}_1\|^2 + \|\mathbb{Z}_2\|^2 + \cdots + \|\mathbb{Z}_j\|^2), \quad \forall \mathbb{Z}_1, \mathbb{Z}_2, \dots, \mathbb{Z}_j \in \mathcal{R}^m. \quad (3.2)$$

Lemma 3.1. Assume that assumptions \mathcal{A}_1 and \mathcal{A}_2 are true, then operator Λ_{ϖ} is well-defined for all $\varpi \in \widetilde{\mathcal{H}}^2([\eta\tau_0, \tau_0])$.

Proof. Let $\mathbb{Z} \in \widetilde{\mathcal{H}}^2([\eta\tau_0, \mathfrak{J}])$. By utilizing Eqs (3.1) and (3.2), we get the following result:

$$\begin{aligned} \|\Lambda_{\varpi}(\mathbb{Z}(\tau))\|_{\mathcal{M}_{\mathcal{S}}}^2 \leq & 3\|\varpi(\tau_0)\|_{\mathcal{M}_{\mathcal{S}}}^2 + 3\mathcal{E}\left[\left\|\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) d\chi\right\|^2\right] \\ & + 3\mathcal{E}\left[\left\|\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_2(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) d\mathcal{W}(\chi)\right\|^2\right]. \end{aligned} \quad (3.3)$$

With CS-ineq, we derive

$$\begin{aligned} & \mathcal{E}\left[\left\|\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) d\chi\right\|^2\right] \\ & \leq \left(\int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} d\chi\right) \mathcal{E}\left[\int_{\tau_0}^{\tau} \|\mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi))\|^2 d\chi\right]. \end{aligned} \quad (3.4)$$

Via the use of (\mathcal{A}_1) , we conclude that

$$\|\mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi))\|^2 \leq 2\mathcal{L}^2\left(\|\mathbb{Z}(\chi)\| + \|\mathbb{Z}(\eta\chi)\|\right)^2 + 2\|\mathcal{A}_1(\chi, 0, 0)\|_\infty^2. \quad (3.5)$$

Then, from Eq (3.5), we get

$$\begin{aligned} \mathcal{E}\left[\int_{\tau_0}^{\tau} \|\mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi))\|^2 d\chi\right] &\leq 4\mathcal{L}^2(\mathfrak{J} - \tau_0) \sup_{\tau \in [\tau_0, \mathfrak{J}]} \mathcal{E}[\|\mathbb{Z}(\tau)\|^2 \\ &\quad + \|\mathbb{Z}(\eta\tau)\|^2] + 2 \int_{\tau_0}^{\mathfrak{J}} \|\mathcal{A}_1(\chi, 0, 0)\|^2 d\chi \\ &\leq 8\mathcal{L}^2(\mathfrak{J} - \tau_0) \sup_{\tau \in [\eta\tau_0, \mathfrak{J}]} \mathcal{E}[\|\mathbb{Z}(\tau)\|^2] + 2 \int_{\tau_0}^{\mathfrak{J}} \|\mathcal{A}_1(\chi, 0, 0)\|_\infty^2 d\chi. \end{aligned} \quad (3.6)$$

By utilizing Eq (3.6) in Eq (3.4), we acquire the following outcomes:

$$\begin{aligned} &\mathcal{E}\left[\left\|\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) d\chi\right\|^2\right] \\ &\leq \frac{1}{2\ell-1} (\mathfrak{J} - \tau_0)^{2\ell-1} \left(8\mathcal{L}^2(\mathfrak{J} - \tau_0) \sup_{\tau \in [\eta\tau_0, \mathfrak{J}]} \mathcal{E}[\|\mathbb{Z}(\tau)\|^2] + 2 \int_{\tau_0}^{\mathfrak{J}} \|\mathcal{A}_1(\chi, 0, 0)\|^2 d\chi\right). \end{aligned} \quad (3.7)$$

Considering the use of I-IsF, we can determine that

$$\mathcal{E}\left[\left\|\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_2(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) d\mathcal{W}(\chi)\right\|^2\right] = \mathcal{E}\left[\int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} \|\mathcal{A}_2(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi))\|^2 d\chi\right]. \quad (3.8)$$

Applying (\mathcal{A}_1) , we may deduce that

$$\|\mathcal{A}_2(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi))\|^2 \leq 4\mathcal{L}^2\left(\|\mathbb{Z}(\chi)\|^2 + \|\mathbb{Z}(\eta\chi)\|^2\right) + 2\|\mathcal{A}_2(\cdot, 0, 0)\|_\infty^2. \quad (3.9)$$

Thus, using Eq (3.9), we derive the next outcomes of our findings from Eq (3.8).

$$\begin{aligned} &\mathcal{E}\left[\left\|\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_2(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) d\mathcal{W}(\chi)\right\|^2\right] \\ &\leq \mathcal{E}\left[\int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} \left(4\mathcal{L}^2(\|\mathbb{Z}(\chi)\|^2 + \|\mathbb{Z}(\eta\chi)\|^2) + 2\|\mathcal{A}_2(\cdot, 0, 0)\|^2\right) d\chi\right] \\ &\leq \frac{8\mathcal{L}^2}{2\ell-1} (\mathfrak{J} - \tau_0)^{2\ell-1} \|\mathbb{Z}(\tau)\|_{\mathcal{H}_2}^2 + \frac{2}{2\ell-1} (\mathfrak{J} - \tau_0)^{2\ell-1} \|\mathcal{A}_2(\cdot, 0, 0)\|_\infty^2. \end{aligned} \quad (3.10)$$

From Eqs (3.7) and (3.10) consequently, we extract our intended conclusions.

To establish the EU, we have to prove the following lemma.

Lemma 3.2. For any $\ell > \frac{1}{2}$ and $\tau > 0$, the inequality that follows is valid:

$$\frac{\mathcal{X}}{\Gamma(2\ell-1)} \int_0^{\tau} \chi^{2\ell-2} \mathbb{E}_{2\ell-1}(\mathcal{X} \chi^{2\ell-1}) d\chi \leq \mathbb{E}_{2\ell-1}(\mathcal{X} \tau^{2\ell-1}), \quad (3.11)$$

where $\mathbb{E}_{2\ell-1}(\cdot)$ is a Mittag-Leffler function which is defined as

$$\mathbb{E}_{(2\ell-1)}(\tau) = \sum_{j=0}^{\infty} \frac{\tau^j}{\Gamma((2\ell-1)j+1)}. \quad (3.12)$$

Proof. Let $\mathcal{X} > 0$ be arbitrary. First, we interchange integral and sum, and then we apply the procedures that follow identity.

$$\int_0^{\tau} \chi^{2\ell-2} \chi^{j(2\ell-1)} d\chi = \tau^{(j+1)(2\ell-1)} \mathbb{B}(2\ell-1, j(2\ell-1)+1), \quad j = 0, 1, 2, \dots$$

So, we get

$$\begin{aligned} \frac{\mathcal{X}}{\Gamma(2\ell-1)} \int_0^{\tau} \chi^{2\ell-2} \mathbb{E}_{2\ell-1}(\mathcal{X} \chi^{2\ell-1}) d\chi &= \mathcal{X} \sum_{j=0}^{\infty} \frac{\mathcal{X}^j}{\Gamma(j(2\ell-1)+1)} \int_0^{\tau} \chi^{2\ell-2} \chi^{j(2\ell-1)} d\chi \\ &= \sum_{j=0}^{\infty} \frac{\mathcal{X}^{j+1} \tau^{(j+1)(2\ell-1)}}{\Gamma(2\ell-1) \Gamma(j(2\ell-1)+1)} \\ &= \sum_{j=1}^{\infty} \frac{\mathcal{X}^j \tau^{j(2\ell-1)}}{\Gamma(j(2\ell-1)+1)} \\ &= \mathbb{E}_{2\ell-1}(\mathcal{X} \tau^{2\ell-1}) - 1 \\ &\leq \mathbb{E}_{2\ell-1}(\mathcal{X} \tau^{2\ell-1}). \end{aligned}$$

In this case, the beta function is \mathbb{B} . This brings the proof to an end.

Theorem 3.3. Assume that the assumptions (\mathcal{A}_1) and (\mathcal{A}_2) are fulfilled, then the problem Eq (1.1) with In.C ϖ has unique solution.

Proof. In the beginning, proceed to select a fixed positive constant \mathcal{X} :

$$\mathcal{X} > 8\mathcal{L}^2(\mathfrak{J} - \tau_0 + 1)\Gamma(2\tau - 1). \quad (3.13)$$

We establish a weighted norm $\|\cdot\|_{\mathcal{X}}$ over the space $\widetilde{\mathcal{H}}^2([\eta\tau_0, \mathfrak{J}])$ as:

$$\|\mathbb{Z}(\tau)\|_{\mathcal{X}} = \sup_{\tau \in [\eta\tau_0, \mathfrak{J}]} \left(\frac{\mathcal{E}[\|\mathbb{Z}(\tau)\|^2]}{\mathcal{Y}(\tau)} \right)^{\frac{1}{2}}, \quad \forall \mathbb{Z}(\tau) \in \widetilde{\mathcal{H}}^2([\eta\tau_0, \mathfrak{J}]), \quad (3.14)$$

$\mathcal{Y}(\tau) = \mathbb{E}_{2\ell-1}(\mathcal{X}(\tau - \tau_0)^{2\ell-1})$ when $\tau \in [\tau_0, \mathfrak{J}]$ and $\mathcal{Y}(\tau) = 1$ for $\tau \in [\eta\tau_0, \tau_0]$. $\|\cdot\|_{\widetilde{\mathcal{H}}^2}$ and $\|\cdot\|_{\mathcal{X}}$, two norms, are equivalent. For this reason, $(\widetilde{\mathcal{H}}^2([\eta\tau_0, \mathfrak{J}]), \|\cdot\|_{\mathcal{X}})$ is a Banach space. Let $\mathbb{Z}_1, \mathbb{Z}_2 \in \widetilde{\mathcal{H}}^2([\eta\tau_0, \mathfrak{J}])$, we have $\forall \tau \in [\eta\tau_0, \tau_0]$, $\Lambda_{\varpi}(\mathbb{Z}_1(\tau)) - \Lambda_{\varpi}(\mathbb{Z}_2(\tau)) = 0$. For $\tau \in [\tau_0, \mathfrak{J}]$, we get

$$\begin{aligned} \mathcal{E} \left[\left\| \Lambda_{\varpi}(\mathbb{Z}_1(\tau)) - \Lambda_{\varpi}(\mathbb{Z}_2(\tau)) \right\|^2 \right] &\leq 2\mathcal{E} \left[\left\| \int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \left(\mathcal{A}_1(\chi, \mathbb{Z}_1(\chi), \mathbb{Z}_1(\eta\chi)) \right. \right. \right. \\ &\quad \left. \left. \left. - \mathcal{A}_1(\chi, \mathbb{Z}_2(\chi), \mathbb{Z}_2(\eta\chi)) \right) d\chi \right\|^2 \right] \\ &\quad + 2\mathcal{E} \left[\left\| \int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \left(\mathcal{A}_2(\chi, \mathbb{Z}_1(\chi), \mathbb{Z}_1(\eta\chi)) \right. \right. \right. \\ &\quad \left. \left. \left. - \mathcal{A}_2(\chi, \mathbb{Z}_2(\chi), \mathbb{Z}_2(\eta\chi)) \right) d\mathcal{W}(\chi) \right\|^2 \right]. \quad (3.15) \end{aligned}$$

Employing the CS-ineq and (\mathcal{A}_1) , we acquire

$$\begin{aligned} & \mathcal{E} \left[\left\| \int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \left(\mathcal{A}_1(\chi, \mathbb{Z}_1(\chi), \mathbb{Z}_1(\eta\chi)) - \mathcal{A}_1(\chi, \mathbb{Z}_2(\chi), \mathbb{Z}_2(\eta\chi)) \right) d\chi \right\|^2 \right] \\ & \leq 2\mathcal{L}^2(\mathfrak{J} - \tau_0) \int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} \left(\mathcal{E} \left[\|\mathbb{Z}_1(\chi) - \mathbb{Z}_2(\chi)\|^2 \right] + \mathcal{E} \left[\|\mathbb{Z}_1(\eta\chi) - \mathbb{Z}_2(\eta\chi)\|^2 \right] \right) d\chi. \end{aligned} \quad (3.16)$$

By applying I-IsF and (\mathcal{A}_1) , we reach the subsequent outcome:

$$\begin{aligned} & \mathcal{E} \left[\left\| \int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \left(\mathcal{A}_2(\chi, \mathbb{Z}_1(\chi), \mathbb{Z}_1(\eta\chi)) - \mathcal{A}_2(\chi, \mathbb{Z}_2(\chi), \mathbb{Z}_2(\eta\chi)) \right) d\mathcal{W}(\chi) \right\|^2 \right] \\ & = \mathcal{E} \left[\int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} \left\| \mathcal{A}_2(\chi, \mathbb{Z}_1(\chi), \mathbb{Z}_1(\eta\chi)) - \mathcal{A}_2(\chi, \mathbb{Z}_2(\chi), \mathbb{Z}_2(\eta\chi)) \right\|^2 d\chi \right] \\ & \leq 2\mathcal{L}^2 \int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} \left(\mathcal{E} \left[\|\mathbb{Z}_1(\chi) - \mathbb{Z}_2(\chi)\|^2 \right] + \mathcal{E} \left[\|\mathbb{Z}_1(\eta\chi) - \mathbb{Z}_2(\eta\chi)\|^2 \right] \right) d\chi. \end{aligned} \quad (3.17)$$

Then,

$$\begin{aligned} \mathcal{E} \left[\|\Lambda_{\varpi}(\mathbb{Z}_1(\tau)) - \Lambda_{\varpi}(\mathbb{Z}_2(\tau))\|^2 \right] & \leq \delta \int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} \left(\mathcal{E} \left[\|\mathbb{Z}_1(\chi) - \mathbb{Z}_2(\chi)\|^2 \right] \right. \\ & \quad \left. + \mathcal{E} \left[\|\mathbb{Z}_1(\eta\chi) - \mathbb{Z}_2(\eta\chi)\|^2 \right] \right) d\chi, \end{aligned} \quad (3.18)$$

where $\delta = \frac{4\mathcal{L}^2(\mathfrak{J}-\tau_0+1)}{\Gamma(\ell)^2}$. Then,

$$\begin{aligned} \mathcal{E} \left[\|\Lambda_{\varpi}(\mathbb{Z}_1(\tau)) - \Lambda_{\varpi}(\mathbb{Z}_2(\tau))\|^2 \right] & \leq \delta \int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} \left(\frac{\mathcal{Y}(\chi) \mathcal{E} \left[\|\mathbb{Z}_1(\chi) - \mathbb{Z}_2(\chi)\|^2 \right]}{\mathcal{Y}(\chi)} \right. \\ & \quad \left. + \frac{\mathcal{Y}(\eta\chi) \mathcal{E} \left[\|\mathbb{Z}_1(\eta\chi) - \mathbb{Z}_2(\eta\chi)\|^2 \right]}{\mathcal{Y}(\eta\chi)} \right) d\chi \\ & \leq \delta \|\mathbb{Z}_1(\tau) - \mathbb{Z}_2(\tau)\|_{\mathcal{X}}^2 \int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} (\mathcal{Y}(\chi) + \mathcal{Y}(\eta\chi)) d\chi \\ & \leq 2\delta \|\mathbb{Z}_1(\tau) - \mathbb{Z}_2(\tau)\|_{\mathcal{X}}^2 \int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} \mathcal{Y}(\chi) d\chi \\ & \leq \frac{2\delta\Gamma(2\ell-1)}{\mathcal{X}} E_{2\ell-1} \left(\mathcal{X} (\tau - \tau_0)^{2\ell-1} \right) \|\mathbb{Z}_1(\tau) - \mathbb{Z}_2(\tau)\|_{\mathcal{X}}^2. \end{aligned}$$

Therefore,

$$\|\Lambda_{\varpi}(\mathbb{Z}_1(\tau)) - \Lambda_{\varpi}(\mathbb{Z}_2(\tau))\|_{\mathcal{X}} \leq \sqrt{\frac{2\delta\Gamma(2\ell-1)}{\mathcal{X}}} \|\mathbb{Z}_1(\tau) - \mathbb{Z}_2(\tau)\|_{\mathcal{X}}.$$

Then, there is a unique solution to Eq (1.1).

We will demonstrate the Con-D of solutions on In-V that are ϖ and Θ to PFSDEs in the following theorem.

Theorem 3.4. (\mathcal{A}_1) and (\mathcal{A}_2) are assumed to be valid. The solution $\mathcal{Z}_\ell(\cdot, \varpi)$ of Eq (1.1) is always dependent on ϖ , i.e.,

$$\lim_{\Theta \rightarrow \varpi} \sup_{\tau \in [\eta\tau_0, \mathfrak{S}]} \|\mathcal{Z}_\ell(\tau, \varpi) - \mathcal{Z}_\ell(\tau, \Theta)\|_{\mathcal{M}\mathcal{S}} = 0. \quad (3.19)$$

Proof. Because $\mathcal{Z}_\ell(\cdot, \varpi)$ and $\mathcal{Z}_\ell(\cdot, \Theta)$ are solutions of Eq (1.1), employing Eq (3.2), the CS-Ineq, I-ISF, (\mathcal{A}_1) , (\mathcal{A}_2) , and Lemma 3.2, we yield

$$\begin{aligned} \sup_{\tau \in [\eta\tau_0, \mathfrak{S}]} \frac{\mathcal{E}[\|\mathcal{Z}_\ell(\tau, \varpi) - \mathcal{Z}_\ell(\tau, \Theta)\|^2]}{\mathbb{E}_{2\ell-1}(\mathcal{X}(\tau - \tau_0)^{2\ell-1})} &\leq 3\mathcal{E}[\|\varpi - \Theta\|^2] + \frac{\Gamma(2\ell - 1)}{\mathcal{X}} 6\mathcal{L}^2(\tau + 1) \\ &\times \sup_{\tau \in [\eta\tau_0, \mathfrak{S}]} \frac{\mathbb{E}[\|\mathcal{Z}_\ell(\tau, \varpi) - \mathcal{Z}_\ell(\tau, \Theta)\|^2]}{\mathbb{E}_{2\ell-1}(\mathcal{X}(\tau - \tau_0)^{2\ell-1})}. \end{aligned} \quad (3.20)$$

By virtue of definition of $\|\cdot\|_{\mathcal{X}}$, we have

$$\left(1 - \frac{\Gamma(2\ell - 1)}{\mathcal{X}} 6\mathcal{L}^2(\tau + 1)\right) \|\mathcal{Z}_\ell(\tau, \varpi) - \mathcal{Z}_\ell(\tau, \Theta)\|_{\mathcal{X}}^2 \leq 3\|\varpi - \Theta\|_{\mathcal{M}\mathcal{S}}^2, \quad (3.21)$$

which, together with Eq (3.13), we derive

$$\lim_{\varpi \rightarrow \Theta} \sup_{\tau \in [\eta\tau_0, \tau]} \|\mathcal{Z}_\ell(\tau, \varpi) - \mathcal{Z}_\ell(\tau, \Theta)\|_{\mathcal{M}\mathcal{S}}^2 = 0. \quad (3.22)$$

We will demonstrate the Con-D of solutions on fractional exponent ℓ to PFSDEs in the following theorem.

Theorem 3.5. Let us suppose that both (\mathcal{A}_1) and (\mathcal{A}_2) are valid. After that, the solution $\mathcal{Z}_\ell(\cdot, \varpi)$ to Eq (1.1) is Con-D on ℓ , i.e.,

$$\lim_{\tilde{\ell} \rightarrow \ell} \sup_{\tau \in [\eta\tau_0, \mathfrak{S}]} \|\mathcal{Z}_\ell(\tau, \varpi) - \mathcal{Z}_{\tilde{\ell}}(\tau, \varpi)\|_{\mathcal{M}\mathcal{S}} = 0. \quad (3.23)$$

Proof. Let $\ell, \tilde{\ell}$ be arbitrary but fixed. As $\mathcal{Z}_\ell(\cdot, \varpi)$ and $\mathcal{Z}_{\tilde{\ell}}(\cdot, \varpi)$ are solutions of Eq (1.1) and we implement Eq (3.2), the CS-Ineq, I-IsF, (\mathcal{A}_1) , (\mathcal{A}_2) , and Lemma 3.2, we achieve

$$\begin{aligned} \frac{\mathcal{E}[\|\mathcal{Z}_\ell(\tau, \varpi) - \mathcal{Z}_{\tilde{\ell}}(\tau, \varpi)\|^2]}{\mathbb{E}_{2\ell-1}(\mathcal{X}(\tau - \tau_0)^{2\ell-1})} &\leq \sup_{\tau \in [\eta\tau_0, \mathfrak{S}]} \frac{\mathcal{E}[\|\mathcal{Z}_\ell(\tau, \varpi) - \mathcal{Z}_{\tilde{\ell}}(\tau, \varpi)\|^2]}{\mathbb{E}_{2\ell-1}(\mathcal{X}(\tau - \tau_0)^{2\ell-1})} 8(\tau + 1)\mathcal{L}^2 \\ &\frac{\int_{\tau_0}^{\tau} \mathcal{X}^{2\ell-2} \mathbb{E}_{2\ell-1}(\mathcal{X}(\chi - \tau_0)^{2\ell-1}) d\chi}{\mathbb{E}_{2\ell-1}(\mathcal{X}(\chi - \tau_0)^{2\ell-1})} \\ &+ 4(\tau + 1) \left(4\mathcal{L}^2 \sup_{\tau \in [\eta\tau_0, \mathfrak{S}]} \|\mathcal{Z}_{\tilde{\ell}}(\tau, \varpi)\|^2 + 2\mathcal{L}^2 \|\varpi\|_{\mathcal{M}\mathcal{S}}^2 \right) \end{aligned} \quad (3.24)$$

$$\begin{aligned} &+ \|\mathcal{A}_1(\cdot, 0, 0)\|_{\infty}^2 + \|\mathcal{A}_2(\cdot, 0, 0)\|_{\infty}^2 \\ &\times \int_{\tau_0}^{\tau} (\lambda(\chi, \tau_0, \ell, \tilde{\ell}))^2 d\chi, \end{aligned} \quad (3.25)$$

where

$$\lambda(\chi, \tau_0, \ell, \tilde{\ell}) = |(\chi - \tau_0)^{\ell-1} - (\chi - \tau_0)^{\tilde{\ell}-1}|. \quad (3.26)$$

By virtue of Lemma 3.2, we attain

$$\begin{aligned} & \left(1 - \frac{8(\mathfrak{Y} + 1)\mathcal{L}^2\Gamma(2\ell - 1)}{\mathcal{X}}\right) \|\mathcal{Z}_\ell(\tau, \varpi) - \mathcal{Z}_{\tilde{\ell}}(\tau, \varpi)\|_{\mathcal{X}}^2 \\ & \leq 4(\mathfrak{Y} + 1) \left(4\mathcal{L}^2 \sup_{\tau \in [\eta\tau_0, \mathfrak{Y}]} \|\mathcal{Z}_\ell(\tau, \varpi)\|^2 + 2\mathcal{L}^2 \|\varpi\|_{\mathcal{M}\mathcal{F}}^2 + \|\mathcal{A}_1(\cdot, 0, 0)\|_{\infty}^2 + \|\mathcal{A}_2(\cdot, 0, 0)\|_{\infty}^2\right) \\ & \quad \times \int_{\tau_0}^{\ell} (\lambda(\chi, \tau_0, \ell, \tilde{\ell}))^2 d\chi. \end{aligned} \quad (3.27)$$

However, we yield

$$\begin{aligned} \int_{\tau_0}^{\tau} (\lambda(\chi, \tau_0, \ell, \tilde{\ell}))^2 d\chi &= \int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} d\chi + \int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\tilde{\ell}-2} d\chi + \int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell+\tilde{\ell}-2} d\chi \\ &= \frac{(\tau - \tau_0)^{2\ell-1}}{2\ell - 1} + \frac{(\tau - \tau_0)^{2\tilde{\ell}-1}}{2\tilde{\ell} - 1} + \frac{(\tau - \tau_0)^{\ell+\tilde{\ell}-1}}{\ell + \tilde{\ell} - 1}. \end{aligned} \quad (3.28)$$

As a result,

$$\lim_{\tilde{\ell} \rightarrow \ell} \sup_{\chi \in [\eta\tau_0, \mathfrak{Y}]} \int_{\tau_0}^{\tau} (\lambda(\chi, \tau_0, \ell, \tilde{\ell}))^2 d\chi = 0.$$

This suggests the completion of the proof when combined with Eq (1.1).

In the next subsection, we will prove that Eq (1.1) is Ulam-Hyers stable.

3.2. The Ulam-Hyers stability of PFSDEs

Theorem 3.6. Under (\mathcal{A}_1) and (\mathcal{A}_2) , the PFSDE Eq (1.1) is UHS on $[\tau_0, \mathfrak{Y}]$.

Proof. Set $\epsilon > 0$ and $\mathcal{N}(\tau) \in \mathcal{H}^2([\eta\tau_0, \mathfrak{Y}])$ satisfies Eq (2.1). Let $\mathbb{Z}(\tau) \in \mathcal{H}^2([\eta\tau_0, \mathfrak{Y}])$ be the unique solution of Eq (1.1) with $\text{In.V } \mathbb{Z}(\tau) = \mathcal{N}(\tau)$ for $\tau \in [\eta\tau_0, \tau_0]$, then

$$\begin{aligned} \mathbb{Z}(\tau) &= \mathcal{N}(\tau_0) + \left(\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) d\chi \right. \\ & \quad \left. + \int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_2(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) d\mathcal{W}(\chi) \right). \end{aligned} \quad (3.29)$$

Thus,

$$\begin{aligned} \mathcal{E}[\|\mathcal{N}(\tau) - \mathbb{Z}(\tau)\|^2] &\leq 2\mathcal{E} \left[\left\| \mathcal{N}(\tau) - \mathcal{N}(\tau_0) - \left(\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_1(\chi, \mathcal{N}(\chi), \mathcal{N}(\eta\chi)) d\chi \right. \right. \right. \\ & \quad \left. \left. + \int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \mathcal{A}_2(\chi, \mathcal{N}(\chi), \mathcal{N}(\eta\chi)) d\mathcal{W}(\chi) \right) \right\|^2 \right] \\ &+ 2\mathcal{E} \left[\left\| \left(\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \left(\mathcal{A}_1(\chi, \mathcal{N}(\chi), \mathcal{N}(\eta\chi)) - \mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) \right) d\chi \right. \right. \right. \\ & \quad \left. \left. + \int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \left(\mathcal{A}_2(\chi, \mathcal{N}(\chi), \mathcal{N}(\eta\chi)) \right. \right. \right. \\ & \quad \left. \left. \left. - \mathcal{A}_2(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi)) \right) d\mathcal{W}(\chi) \right) \right\|^2 \right]. \end{aligned} \quad (3.30)$$

Thus, using CS-Ineq and (\mathcal{A}_1) , (\mathcal{A}_2) , we can derive that

$$\begin{aligned} \mathcal{E}[\|\mathcal{N}(\tau) - \mathbb{Z}(\tau)\|^2] &\leq 2\epsilon + 4\mathcal{E}\left[\left\|\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \left(\mathcal{A}_1(\chi, \mathcal{N}(\chi), \mathcal{N}(\eta\chi)) - \mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi))\right) d\chi\right\|^2\right] \\ &\quad + 4\mathcal{E}\left[\left\|\int_{\tau_0}^{\tau} (\chi - \tau_0)^{\ell-1} \left(\mathcal{A}_2(\chi, \mathcal{N}(\chi), \mathcal{N}(\eta\chi)) - \mathcal{A}_2(\chi, \mathbb{Z}(\chi), \mathbb{Z}(\eta\chi))\right) d\mathcal{W}(\chi)\right\|^2\right] \\ &\leq 2\epsilon + \frac{8\mathcal{L}^2(\mathfrak{J} - \tau_0)^{2\ell-1}}{(2\ell - 1)} \mathcal{E}\left[\int_{\tau_0}^{\tau} \left(\|\mathcal{N}(\chi) - \mathbb{Z}(\chi)\|^2 + \|\mathcal{N}(\eta\chi) - \mathbb{Z}(\eta\chi)\|^2\right) d\chi\right] \\ &\quad + 8\mathcal{L}^2 \mathcal{E}\left[\int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} \left(\|\mathcal{N}(\chi) - \mathbb{Z}(\chi)\|^2 + \|\mathcal{N}(\eta\chi) - \mathbb{Z}(\eta\chi)\|^2\right) d\chi\right]. \end{aligned}$$

Let $F(\tau) = \sup_{\chi \in [\eta\tau_0, \tau]} \mathcal{E}[\|\mathcal{N}(\chi) - \mathbb{Z}(\chi)\|^2]$, for $\tau \in [\tau_0, \mathfrak{J}]$.

We have

$$\mathcal{E}[\|\mathcal{N}(\chi) - \mathbb{Z}(\chi)\|^2] \leq F(\chi), \quad (3.31)$$

and

$$\mathcal{E}[\|\mathcal{N}(\eta\chi) - \mathbb{Z}(\eta\chi)\|^2] \leq F(\chi), \quad \forall \chi \in [\tau_0, \mathfrak{J}]. \quad (3.32)$$

Then, for $\tau \in [\tau_0, \mathfrak{J}]$, we obtain

$$\mathcal{E}[\|\mathcal{N}(\tau) - \mathbb{Z}(\tau)\|^2] \leq 2\epsilon + \frac{16\mathcal{L}^2(\mathfrak{J} - \tau_0)^{2\ell-1}}{(2\ell - 1)} \int_{\tau_0}^{\tau} F(\chi) d\chi + 16\mathcal{L}^2 \int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} F(\chi) d\chi.$$

Hence, for all $\tau \in [\tau_0, \tau]$,

$$\mathcal{E}[\|\mathcal{N}(\chi) - \mathbb{Z}(\chi)\|^2] \leq 2\epsilon + \frac{16\mathcal{L}^2(\mathfrak{J} - \tau_0)^{2\ell-1}}{(2\ell - 1)} \int_{\tau_0}^{\tau} F(\chi) d\chi + 16\mathcal{L}^2 \int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} F(\chi) d\chi.$$

Then,

$$F(\tau) \leq 2\epsilon + \delta_1 \int_{\tau_0}^{\tau} F(\chi) d\chi + \delta_2 \int_{\tau_0}^{\tau} (\chi - \tau_0)^{2\ell-2} F(\chi) d\chi, \quad (3.33)$$

for all $\tau \in [\tau_0, \mathfrak{J}]$, where $\delta_1 = \frac{16\mathcal{L}^2(\mathfrak{J} - \tau_0)^{2\ell-1}}{(2\ell-1)}$ and $\delta_2 = 16\mathcal{L}^2$.

Using GG-Ineq, we get

$$\begin{aligned} F(\tau) &\leq \left(2\epsilon + \delta_1 \int_{\tau_0}^{\tau} F(\chi) d\chi\right) \mathbb{E}_{2\ell-1} \left(\delta_2 \Gamma(2\ell - 1) (\tau - \tau_0)^{2\ell-1}\right) \\ &\leq \delta_3 \epsilon + \delta_4 \int_{\tau_0}^{\tau} F(\chi) d\chi, \end{aligned}$$

where $\delta_3 = 2\mathbb{E}_{2\ell-1} \left(\delta_2 \Gamma(2\ell - 1) (\mathfrak{J} - \tau_0)^{2\ell-1}\right)$ and $\delta_4 = \delta_1 \mathbb{E}_{2\ell-1} \left(\delta_2 \Gamma(2\ell - 1) (\mathfrak{J} - \tau_0)^{2\ell-1}\right)$.

Using GG-Ineq, we get

$$F(\tau) \leq \delta_3 \epsilon e^{\delta_4(\tau - \tau_0)}. \quad (3.34)$$

Hence,

$$\mathcal{E}[\|\mathcal{N}(\tau) - \mathbb{Z}(\tau)\|^2] \leq \mathfrak{N}\epsilon, \quad \forall \tau \in [\tau_0, \mathfrak{J}], \quad (3.35)$$

where $\mathfrak{N} = \delta_3 e^{\delta_4(\mathfrak{J} - \tau_0)}$. Thus, Eq (1.1) is UHS.

4. Examples

The purpose of this section is to understand the results that we established in the above section. For this purpose, we present two numerical examples of PFSDEs.

The PFSDEs find applications in various fields where systems exhibit random behavior and are influenced by both deterministic and stochastic factors. One physical application of PFSDEs is in modeling the dynamics of certain types of mechanical systems subject to random perturbations. In such cases, PFSDEs can be used to model the dynamics of the system. The pantograph term arises when there are delays in the system's response, for instance, due to the time it takes for vibrations to propagate through the structure. The stochastic component captures the randomness inherent in the external forcing or system parameters.

A suspension bridge subjected to random wind loads can be modeled using PFSDEs. The deterministic part of the equation accounts for the bridge's structural properties, such as stiffness and damping, while the stochastic part represents the random nature of wind loads. The pantograph term may account for delays in the bridge's response to wind gusts due to factors like the propagation speed of vibrations along its structure.

Example 4.1. Let $\tau \in [1, 2]$ and the PFSDE for each $\epsilon > 0$ be provided by

$$\begin{cases} \mathcal{T}_\tau^\ell \mathbb{Z}(\tau) = \mathcal{A}_1(\tau, \mathbb{Z}(\tau), \mathbb{Z}(0.5\tau)) + \mathcal{A}_2(\tau, \mathbb{Z}(\tau), \mathbb{Z}(0.5\tau)) \frac{d\mathcal{W}(\tau)}{d\tau}, \\ \mathcal{E} \left[\left\| \mathbb{Z}(\tau) - \mathbb{Z}(1) - \left(\int_1^\tau (\chi - \tau_0)^{\ell-1} \left(\mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(0.5\chi)) d\chi \right. \right. \right. \right. \\ \left. \left. \left. + \mathcal{A}_2(\chi, \mathbb{Z}(\chi), \mathbb{Z}(0.5\chi)) d\mathcal{W}(\chi) \right) \right\|^2 \right] \leq \epsilon, \\ \mathbb{Z}(\tau) = \varpi(\tau), \tau \in [0.5, 1], \end{cases} \quad (4.1)$$

where

$$\begin{aligned} \mathbb{Z}(\tau) &\in \mathcal{H}^2([0.5, 2], \mathcal{R}), \\ \mathcal{A}_1(\tau, \mathbb{Z}(\tau), \mathbb{Z}(0.5\tau)) &= e^{0.5\tau} \cos(\mathbb{Z}(\tau)), \\ \mathcal{A}_2(\tau, \mathbb{Z}(\tau), \mathbb{Z}(0.5\tau)) &= \frac{\cos(\mathbb{Z}(0.5\tau))}{\sqrt{1+0.5\tau}}. \end{aligned}$$

Where, the deterministic and stochastic parts of Eq (4.1) are $e^{0.5\tau} \cos(\mathbb{Z}(\tau))$ and $\frac{\cos(\mathbb{Z}(0.5\tau))}{\sqrt{1+0.5\tau}}$ respectively. Now we demonstrate that, with relation to ϵ , Eq (4.1) is UHS. Let $(\tau, \mathbb{Z}_1, \mathbb{Z}_2) \in [1, 2] \times \mathcal{R} \times \mathcal{R}$, therefore,

$$\begin{aligned} &\left\| \mathcal{A}_1(\tau, \mathbb{Z}_1, \widetilde{\mathbb{Z}}_1) - \mathcal{A}_1(\tau, \mathbb{Z}_2, \widetilde{\mathbb{Z}}_2) \right\| + \left\| \mathcal{A}_2(\tau, \mathbb{Z}_1, \widetilde{\mathbb{Z}}_1) - \mathcal{A}_2(\tau, \mathbb{Z}_2, \widetilde{\mathbb{Z}}_2) \right\| \\ &\leq e^1 \left(\left\| \mathbb{Z}_1 - \mathbb{Z}_2 \right\| + \left\| \widetilde{\mathbb{Z}}_1 - \widetilde{\mathbb{Z}}_2 \right\| \right). \end{aligned}$$

This means that (\mathcal{A}_1) is satisfied. Furthermore,

$$\left\| \mathcal{A}_2(\cdot, 0, 0) \right\|_\infty \leq 1,$$

and

$$\int_1^2 \|\mathcal{A}_1(\tau, 0, 0)\|^2 d\tau \leq e^2.$$

The assumptions (\mathcal{A}_1) and (\mathcal{A}_2) are therefore met. Thus, with respect to ϵ on $[1, 2]$, Eq (4.1) is UHS utilizing Theorem 3.6. For Systems Eqs (4.1) and (4.2), we conduct a simulation based on the Euler-Maruyama scheme with a step size of 10^{-3} . We can see from Figure 1 that the distance between $\mathcal{N}(\tau)$ and $\mathbb{Z}(\tau)$ is less than a constant, and according to Definition 2.3, the solution of Eq (4.1) is UHS.

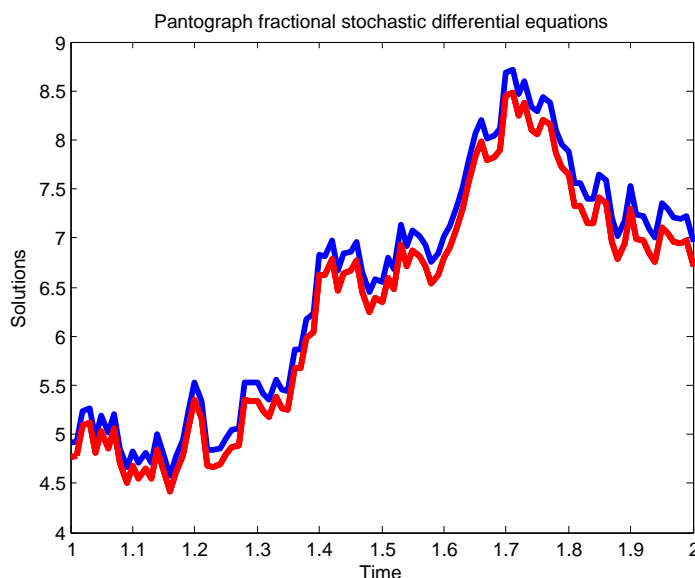


Figure 1. The behavior of the solutions $\mathcal{N}(\tau)$ and $\mathbb{Z}(\tau)$ when $\epsilon = 0.01$. Red color: $\mathcal{N}(\tau)$; blue color: $\mathbb{Z}(\tau)$.

Example 4.2. Let $\tau \in [1, 4]$ and the PFSDE for each $\epsilon > 0$ be provided by

$$\left\{ \begin{array}{l} \mathcal{T}_\tau^\ell \mathbb{Z}(\tau) = \mathcal{A}_1(\tau, \mathbb{Z}(\tau), \mathbb{Z}(0.25\tau)) + \mathcal{A}_2(\tau, \mathbb{Z}(\tau), \mathbb{Z}(0.25\tau)) \frac{d\mathcal{W}(\tau)}{d\tau}, \\ \mathcal{E} \left[\left\| \mathbb{Z}(\tau) - \mathbb{Z}(1) - \left(\int_1^\tau (\chi - 1)^{\ell-1} \left(\mathcal{A}_1(\chi, \mathbb{Z}(\chi), \mathbb{Z}(0.25\chi)) d\chi \right. \right. \right. \right. \\ \left. \left. \left. + \mathcal{A}_2(\chi, \mathbb{Z}(\chi), \mathbb{Z}(0.25\chi)) d\mathcal{W}(\chi) \right) \right\|^2 \right] \leq \epsilon, \\ \mathbb{Z}(\tau) = \varpi(\tau), \tau \in [0.25, 1], \end{array} \right. \quad (4.2)$$

where

$$\begin{aligned} \mathbb{Z}(\tau) &\in \mathcal{H}^2([0.25, 4], \mathcal{R}), \\ \mathcal{A}_1(\tau, \mathbb{Z}(\tau), \mathbb{Z}(0.25\tau)) &= \frac{\cos \mathbb{Z}(\tau)}{\sqrt{1 + (0.25\tau)^2}}, \\ \mathcal{A}_2(\tau, \mathbb{Z}(\tau), \mathbb{Z}(0.25\tau)) &= \frac{\mathbb{Z}(0.25\tau)}{1 + \tau^2}. \end{aligned}$$

Where, the deterministic and stochastic parts of the Eq (4.2) are $\frac{\cos \mathbb{Z}(\tau)}{\sqrt{1+(0.25\tau)^2}}$ and $\frac{\mathbb{Z}(0.25\tau)}{1+\tau^2}$ respectively. Now, with regard to ϵ , we demonstrate that Eq (4.2) is UHS. Assume that $(\tau, \mathbb{Z}_1, \mathbb{Z}_2) \in [1, 4] \times \mathcal{R} \times \mathcal{R}$, hence

$$\|\mathcal{A}_1(\tau, \mathbb{Z}_1, \tilde{\mathbb{Z}}_1) - \mathcal{A}_1(\tau, \mathbb{Z}_2, \tilde{\mathbb{Z}}_2)\| + \|\mathcal{A}_2(\tau, \mathbb{Z}_1, \tilde{\mathbb{Z}}_1) - \mathcal{A}_2(\tau, \mathbb{Z}_2, \tilde{\mathbb{Z}}_2)\| \leq \|\mathbb{Z}_1 - \mathbb{Z}_2\| + \|\tilde{\mathbb{Z}}_1 - \tilde{\mathbb{Z}}_2\|.$$

Consequently, (\mathcal{A}_1) is met. Likewise

$$|\mathcal{A}_2(\cdot, 0)|_\infty = 0,$$

and

$$\int_1^4 |\mathcal{A}_1(\tau, 0)|^2 d\tau \leq \pi.$$

The assumptions (\mathcal{A}_1) and (\mathcal{A}_2) are therefore met. Thus, with respect to ϵ on $[1, 4]$, Eq (4.2) is UHS employing Theorem 3.6. We can see from Figure 2 that the distance between $\mathcal{N}(\tau)$ and $\mathbb{Z}(\tau)$ is less than a constant, and according to Definition 2.3, the solution of Eq (4.2) is UHS.

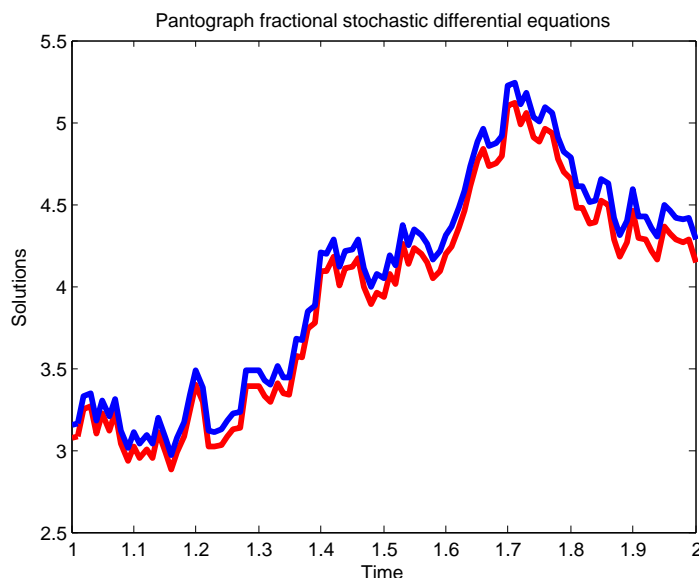


Figure 2. The behavior of the solutions $\mathcal{N}(\tau)$ and $\mathbb{Z}(\tau)$ when $\epsilon = 0.01$. Red color: $\mathcal{N}(\tau)$; blue color: $\mathbb{Z}(\tau)$.

Continuous dependency is a key property that contributes to the well-posedness of systems of differential equations, ensuring stability in the solutions by guaranteeing that small changes in parameters or initial conditions lead to correspondingly small changes in the solutions. Therefore, continuous dependency is closely tied to the overall stability and reliability of mathematical models described by differential equations. As both models presented in Examples 4.1 and 4.2 satisfy the criteria of continuous dependency, we conclude that both models are stable.

5. Conclusions

In the present paper, we examined the existence and uniqueness of the pantograph fractional stochastic differential equations using the Banach fixed point theorem. Second, we demonstrated that solutions continuously depend on the fractional order ℓ and initial value. The fractional-order derivative is considered in the conformable sense. Using the generalized Gronwall inequalities and stochastic analytic methods, we examined the Ulam-Hyers stability of the considered problem. We demonstrated our findings with two examples in the final section.

We will use numerical techniques in our future work to solve various kinds of real-world challenges modeled with pantograph fractional stochastic differential equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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