Mathematics

Research article

Analytical solutions to time-space fractional Kuramoto-Sivashinsky Model using the integrated Bäcklund transformation and Riccati-Bernoulli sub-ODE method

M. Mossa Al-Sawalha[1], **Safyan Mukhtar**[2,3,*], **Albandari W. Alrowaily**[4], **Saleh Alshammari**[1], **Sherif. M. E. Ismaeel**[5,6] **and S. A. El-Tantawy**[7,8]

[1] Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia

[2] Department of Basic Sciences, Preparatory Year, King Faisal University, Al-Ahsa 31982, Saudi Arabia

[3] Department of Mathematics and Statistics, College of Science, King Faisal University, Al-Ahsa 31982, Saudi Arabia

[4] Department of Physics, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

[5] Department of Physics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia

[6] Department of Physics, Faculty of Science, Ain Shams University, Cairo, Egypt

[7] Department of Physics, Faculty of Science, Port Said University, Port Said 42521, Egypt

[8] Research Center for Physics (RCP), Department of Physics, Faculty of Science and Arts, Al-Mikhwah, Al-Baha University, Al-Baha 1988, Saudi Arabia

* **Correspondence:** Email: smahmad@kfu.edu.sa.

Abstract: This paper solves an example of a time-space fractional Kuramoto-Sivashinsky (KS) equation using the integrated Bäcklund transformation and the Riccati-Bernoulli sub-ODE method. A specific version of the KS equation with power nonlinearity of a given degree is examined. Using symbolic computation, we find new analytical solutions to the current problem for modeling many nonlinear phenomena that are described by this equation, like how the flame front moves back and forth, how fluids move down a vertical wall, or how chemical reactions happen in a uniform medium while they oscillate uniformly across space. In the field of mathematical physics, the Riccati-Bernoulli sub-ODE approach is shown to be a valuable tool for producing a variety of single solutions.

Keywords: fractional Kuramoto-Sivashinsky (KS) equation; Bäcklund transformation; Riccati-Bernoulli sub-ODE method; shock and periodic waves; families of traveling wave solutions

1. Introduction

In recent times, fractional calculus (FC) has become an indispensable instrument in numerous scientific domains, serving as a basis for developing numerous innovative and cutting-edge models. Fractional derivative operators are significant because they can increase model accuracy even in situations where crucial real parameters are unknown with high precision. The scientific literature [1–3] covers fractional differential operators of several types, such as Caputo-Fabrizio, Caputo, Riemann-Liouville, and Atangana-Baleanu. We extend fractional constitutive models, which have proven effective in characterizing viscoelastic materials, to study the mechanics of electroosmosis slip flows in viscoelastic fluids that are forced by both pressure gradients and electroosmosis. This work is important for optimizing the design of micro-fluidic devices, and advances our knowledge of flow behaviors in microchannels for viscoelastic fluids [4, 5]. Drawing inspiration from the Glover-Dumm equation (GDE), a practical mathematical model widely utilized for simulating water table profiles between parallel drainpipes in the context of unsteady flow conditions, the GDE is derived through the analytical solution of the Boussinesq equation (BE). Notably, incorporating fractional derivatives, characterized by their non-local properties, mitigates scale effects on parameters. Consequently, this facilitates a more accurate simulation of hydro-geological processes [6, 7]. Traditional advection-diffusion equations may not be able to describe the transport process adequately, and one of the possible ways to model anomalous transport is to consider fractional dispersion equations, which are more general and can overcome some of the limitations of traditional advection-diffusion equations. Indeed, there has been growing interest in using fractional reaction-diffusion equations to describe the complex behavior of transport processes in the presence of anomalous transport of reactive pollutants [8, 9]. Therefore, it is essential to propose and construct fractional reaction-diffusion equations that capture the essential characteristics of the governing system well.

Finding analytical solutions to fractional partial differential equations (FPDEs) is a challenging task, which has led to the invention of a myriad of mathematical techniques to circumvent it. The inherent ability of analytical solutions to reveal the exact behavior of the modeled system is often superior to numerical methods, which has attracted the attention of many researchers. Thus, the search for analytical solutions in the domain of FPDEs has been an essential and continuously evolving domain [10–15]. Many different mathematical schemes have been brought to bear in the scientific literature to compute the solutions of FPDEs [16–22]. Generally, nonlinear partial differential equations (PDEs) are central to model many problems in physics, engineering, biology, and finance. They can be used to model a vast range of complex phenomena, such as turbulent flows, nonlinear waves in plasmas, the dynamics of the heart, traffic flow, population dynamics, and the pricing of financial derivatives [23–25]. The realm of observer design methods for nonlinear generalized systems with nonlinear algebraic constraints have found practical applications in various fields [26, 27]. The development of an observer design method tailored to address the complexities posed by nonlinear algebraic constraints within generalized systems was investigated in [28, 29]. Many valuable approaches were used to model and analyze complex dynamical engineering and physical issues [30–33]. These methods [30–33] have effectively assessed numerous previously challenging problems to study and model. Previous studies [34–36] have demonstrated the potential of the methods used to enhance the understanding and control of nonlinear systems with some nonlinear algebraic constraints through rigorous analysis and

simulation. This has paved the way for progress in the search for new strategies for accurately modeling and simulating the problems researchers address. Unlike their linear counterparts, FPDEs capture the complex interactions and nontrivial dynamics of real-world systems, leading to rich mathematical theories associated with them. Thus, solving these types of nonlinear differential equations is a crucial step to understanding the underlying mechanisms and producing forecasts that linear models frequently miss. Recent years have seen remarkable progress in this direction, thanks to advances in theory and computational methods [37–41].

Moreover, from the viewpoint of the Riccati-Bernoulli sub-ODE method, an efficient algebraic manipulation into various physical phenomena related to reaction-diffusion flows, such as flame front oscillations, dynamic behaviors, oscillations in homogeneous medium in the flow of a fluid through the vertical walls, etc., can be done. In particular, applying the Riccati-Bernoulli sub-ODE method to PDEs transforms those equations into algebraic equations that are easy to solve. Using this technique, many analytical solutions describing the complex dynamics of different physical models can be derived [42–44]. Therefore, the basic physical processes leading to these complex dynamics can be further understood. The most attractive feature of this method is that it generates many solitary wave solutions. The Bäcklund transformation [45] has made this possible, which produces an infinite sequence of solutions. Additionally, the current study aims to use this analytical approach to clarify the complex dynamics of the Kuramoto-Sivashinsky (KS) equation, which will significantly advance our understanding of real-world physical problems, such as precisely describing spatiotemporal chaos, forecasting turbulent transitions in fluid dynamics, plasma physics, and simulating complex patterns in chemical reactions. Through this methodology, the research seeks to expand the understanding of these difficulties, promote progress in analyzing complex phenomena, and improve the predictive power of the KS equation in various scientific contexts.

In this investigation, we derive some exact solutions to the KS equation for fractional dynamics, with particular attention to the case ($n = 1$)

$$D_t^\alpha(F) + F^n D_x^\beta(F) + p D_x^{2\beta}(F) + q D_x^{3\beta}(F) + r D_x^{4\beta}(F) = 0, \tag{1.1}$$

where α and β indicate the time and space fractional parameters, whereas $0 < \alpha, \beta \le 1$.

Here, F indicates the dependent variable/wave function, in which the equation incorporates two independent variables: (x) and (t). Coefficients (p), (q), and (r) are defined as constants, whereas the exponent (n) is an integer. Many studies have extensively examined Eq (1.1) with ($n = 1$) from various perspectives. The equation's attraction lies in its one-dimensional nonlinear nature, incorporating dissipation, instability, and dispersion elements. Its significance stems from its utility in addressing scientific and engineering challenges. Equation (1.1) was used to clarify the source of persistent wave propagation in reaction-diffusion environments [46]. Another study examined Eq (1.1) with (**n = 1**) to clarify the nonlinear development of disrupted flame fronts and gain an understanding of the intricate dynamics of combustion processes [47]. The applicability of this equation to the analysis of viscous incompressible fluid motion moving down an inclined plane highlights its adaptability and offers important insights into the behavior of fluid dynamics in appropriate environments [48–50]. Moreover, Eq (1.1) was used to model dissipative waves in plasma physics, which helped us learn more about how waves behave in complex media [51]. Furthermore, Michelson [52] examined elementary particles as solutions of the KS equation, offering a new perspective on the fundamental physical laws that govern particle behavior in this nonlinear setting.

The operator representing derivatives of order α follows the definition in [53]

$$D_\theta^\alpha q(\theta) = \lim_{m \to 0} \frac{q(m(\theta)^{1-\alpha} - q(\theta))}{m}, 0 < \alpha \le 1. \tag{1.2}$$

This inquiry utilizes the following characteristics of this derivative:

$$\begin{cases} D_\theta^\alpha \theta^j = j\theta^{j-\alpha}, \\ D_\theta^\alpha \left(j_1 \eta(\theta) \pm j_2 m(\theta)\right) = j_1 D_\theta^\alpha(\eta(\theta)) \pm j_2 D_\theta^\alpha(m(\theta)), \\ D_\theta^\alpha \chi\left[\xi^\tau(\theta)\right] = \chi'_\xi(\xi(\theta)) D_\theta^\alpha \xi(\theta). \end{cases} \tag{1.3}$$

2. Methodology

Consider the FPDE

$$P_1\left(f, D_t^\alpha(f), D_{\zeta_1}^\beta(f), D_{\zeta_2}^{2\beta}(f), f D_{\zeta_1}^\beta(f), \ldots\right) = 0, \tag{2.1}$$

where $0 < \alpha, \beta, \le 1$, and the polynomial P_1 is a function of $f(\zeta_1, \zeta_2, \zeta_3, \ldots, t)$. This polynomial includes the fractional order derivatives as well as the nonlinear terms. The primary stages of this method are then thoroughly covered. The wave transformations that follow are our suggestions for looking into possible solutions for Eq (1.1):

$$F(x, t) = f(\psi), \tag{2.2}$$

with

$$\psi(x, t) = \left(\frac{x^\beta}{\beta}\right) - \omega\left(\frac{t^\alpha}{\alpha}\right), \quad \omega \in R, \tag{2.3}$$

where $\psi \equiv \psi(x, t)$ represents the phase function. Equation (2.3) is subjected to a modification resulting in the emergence of a nonlinear ordinary differential equation (NODE), hence assuming a modified mathematical expression

$$P_2\left(f, f'(\psi), f''(\psi), f f'(\psi), \ldots\right) = 0. \tag{2.4}$$

Consider the following formal solution for Eq (2.2)

$$f(\psi) = \sum_{i=-m}^{m} b_i \vartheta(\psi)^i. \tag{2.5}$$

Under the restriction that both $b_m \ne 0$ and $b_{-m} \ne 0$ simultaneously, the b_i constants must be determined. Concurrently, the function is generated via the subsequent Bäcklund transformation

$$\vartheta(\psi) = \frac{-\tau Y + X\phi(\psi)}{X + Y\phi(\psi)}, \tag{2.6}$$

with the requirement that $Y \ne 0$, which we consider (τ), (X), and (Y) to be constants.

Moreover, let $\phi(\psi)$ be a function defined as follows:

$$\frac{d\phi}{d\psi} = \tau + \phi(\psi)^2, \tag{2.7}$$

The solutions to Eq (2.7) are usually recognized to be the following [54]:

(i) If $\tau < 0$,   then $\phi(\psi) = -\sqrt{-\tau}\tanh(\sqrt{-\tau}\psi)$,   or $\phi(\psi) = -\sqrt{-\tau}coth(\sqrt{-\tau}\psi)$.   (2.8)

(ii) If $\tau > 0$,   then $\phi(\psi) = \sqrt{\tau}\tan(\sqrt{\tau}\psi)$,   or $\phi(\psi) = -\sqrt{\tau}cot(\sqrt{\tau}\psi)$.   (2.9)

(iii) If $\tau = 0$,   then $\phi(\psi) = \dfrac{-1}{\psi}$.   (2.10)

Under the framework of Eq (2.2), the positive integer (N) can be found by using homogeneous balancing principles, which entail finding equilibrium between the highest-order derivatives and highest-nonlinearity in Eq (2.5). The degree $f(\psi)$ can be expressed more precisely as $D[f(\psi)] = N$. Therefore, this enables us to perform the following computation of the degree of linked expressions:

$$D\left[\frac{d^k f}{d\psi^k}\right] = N + k,$$

$$D\left[f^J \frac{d^k f}{d\psi^k}\right]^s = NJ + s(k + N). \tag{2.11}$$

Combining Eq (2.4) with Eqs (2.5) and (2.7), grouping terms with the same powers of $f(\psi)$, and then equating them to zero are the steps in creating algebraic equations. Using the Maple software to deduce the pertinent values for various parameters will result in an efficient resolution of this system. Thus, this makes it easier to compute the soliton wave-propagating solutions to Eq (2.1) with accuracy by computational analysis.

3. Execution of the problem

Using the approach described in Section 2, we precisely solve the fractional KS equation (1.1) for solitary waves. In the framework of Eq (1.1), we especially study certain cases, namely when (n=1). We simplify the equations by calculating the wave transformation described in Eq (2.3) to make the final equation easier to calculate. Here, we provide the equation introduced after the transformative step, resulting in the nonlinear ODE derived from the initial fractional PDE

$$r f''''(\psi) + q f'''(\psi) + p f''(\psi) + f'(\psi)(f - \omega) = 0. \tag{3.1}$$

We incorporate the replacement given in Eq (2.5) into Eqs (2.6), (2.7), and (3.1). Through the systematic collection of coefficients associated with $\phi^i(\psi)$, we develop an algebraic system of equations that is ultimately equal to zero. By utilizing Maple's computational tool, we can solve the previously given system of algebraic equations and obtain the following results:

- Case I:

$$b_0 = b_0, b_{1,2,3} = 0, b_{-1} = -\frac{\sqrt{-30\,b_{-3}^3}}{b_{-3}}, b_{-2} = -i\sqrt[4]{-30\,b_{-3}^3}, b_{-3} = b_{-3}, p = p,$$

$$q = \frac{2\,i\sqrt[4]{-30\,b_{-3}^3}}{b_{-3}}, r = -\frac{1}{4}\frac{\sqrt{-30\,b_{-3}^3}}{b_{-3}^2}, \omega = -\frac{1}{15}\frac{11\,i\left(-30\,b_{-3}^3\right)^{3/4} - 15\,b_{-3}^2 b_0}{b_{-3}^2},$$

$$\tau = \frac{1}{30} \frac{\sqrt{-30 b_{-3}^3}}{b_{-3}}, B = B. \tag{3.2}$$

- Case II:

$$b_0 = b_0, b_1 = 15/2, b_{-1} = -\frac{225}{4} b_3^{-1}, b_{-2} = \frac{225}{4} \frac{\sqrt{30}}{b_3^{3/2}}, b_2 = \sqrt{30} \sqrt{b_3},$$

$$b_3 = b_3, b_{-3} = -\frac{3375}{8} b_3^{-2}, p = p, q = 1/15 \sqrt{30} \sqrt{b_3}, r = \frac{1}{120} b_3,$$

$$\omega = -\frac{7 \sqrt{30} \sqrt{b_3} - b_3 b_0}{b_3}, \tau = 15/2 b_3^{-1}, B = B. \tag{3.3}$$

- Case III:

$$b_0 = b_0, b_1 = 30, b_{-1,-2,-3} = 0, b_2 = \sqrt{30} \sqrt{b_3}, b_3 = b_3,$$

$$r = \frac{1}{120} b_3, \omega = -\frac{22 \sqrt{30} \sqrt{b_3} - b_3 b_0}{b_3},$$

$$p = p, q = 1/15 \sqrt{30} \sqrt{b_3}, \tau = 30 b_3^{-1}, B = B. \tag{3.4}$$

According to case I, the solution set for the following values of (ψ) and (τ) are obtained

$$\psi = \frac{x^\beta}{\beta} + \frac{1}{15} \frac{\left(11 i \left(-30 b_{-3}^3\right)^{\frac{3}{4}} - 15 b_{-3}^2 b_0\right) t^\alpha}{b_{-3}^2 \alpha}, \tag{3.5}$$

$$\tau = \frac{1}{30} \frac{\sqrt{-30 b_{-3}^3}}{b_{-3}}. \tag{3.6}$$

Solution Space. 1: For $\tau < 0$, Eq (1.1) brings about the resulting single-wave solutions:

$$F_1(x,t) = b_{-3} \Theta_1^3 \Pi_1^{-3} - i \sqrt[4]{-30 b_{-3}^3} \Theta_1^2 \Pi_1^{-2} - \sqrt{-30 b_{-3}^3} \Theta_1 \Pi_1^{-1} b_{-3}^{-1} + b_0, \tag{3.7}$$

or

$$F_2(x,t) = b_{-3} \Theta_2^3 \Pi_2^{-3} - i \sqrt[4]{-30 b_{-3}^3} \Theta_2^2 \Pi_2^{-2} - \sqrt{-30 b_{-3}^3} \Theta_2 b_{-3}^{-1} \Pi_2^{-1} + b_0, \tag{3.8}$$

with

$$\Pi_1 = \left(-\frac{1}{30} \frac{\sqrt{-30 b_{-3}^3}}{b_{-3}} Y - X \sqrt{-\tau} \tanh\left(\sqrt{-\tau} \psi\right)\right),$$

$$\Pi_2 = \left(-\frac{1}{30} \frac{\sqrt{-30 b_{-3}^3}}{b_{-3}} Y - X \sqrt{-\tau} \coth\left(\sqrt{-\tau} \psi\right)\right),$$

$$\Theta_1 = \left(X - Y\sqrt{-\tau}\tanh\left(\sqrt{-\tau}\psi\right)\right),$$
$$\Theta_2 = \left(X - Y\sqrt{-\tau}\coth\left(\sqrt{-\tau}\psi\right)\right).$$

Solution Space. 2: For $\tau > 0$, Eq (1.1) brings about the resulting single-wave solutions:

$$F_3(x,t) = b_{-3}\Theta_3^3\Pi_3^{-3} - i\sqrt[4]{-30\,b_{-3}^3}\,\Theta_3^2\Pi_3^{-2} - \sqrt{-30\,b_{-3}^3}\,\Theta_3 b_{-3}^{-1}\Pi_3^{-1} + b_0, \tag{3.9}$$

or

$$F_4(x,t) = b_{-3}\Theta_4^3\Pi_4^{-3} - i\sqrt[4]{-30\,b_{-3}^3}\,\Theta_4^2\Pi_4^{-2} - \sqrt{-30\,b_{-3}^3}\,\Theta_4 b_{-3}^{-1}\Pi_4^{-1} + b_0, \tag{3.10}$$

with

$$\Pi_3 = \left(-\frac{1}{30}\frac{\sqrt{-30\,b_{-3}^3}}{b_{-3}}Y + X\sqrt{\tau}\tan\left(\sqrt{\tau}\psi\right)\right),$$

$$\Pi_4 = \left(-\frac{1}{30}\frac{\sqrt{-30\,b_{-3}^3}}{b_{-3}}Y - X\sqrt{\tau}\cot\left(\sqrt{\tau}\psi\right)\right),$$

$$\Theta_3 = \left(X + Y\sqrt{\tau}\tan\left(\sqrt{\tau}\psi\right)\right),$$

$$\Theta_4 = \left(X - Y\sqrt{\tau}\cot\left(\sqrt{\tau}\psi\right)\right).$$

Solution Space. 3: For $\tau = 0$, Eq (1.1) brings about the resulting single-wave solutions:

$$F_5(x,t) = b_{-3}\Theta_5^3\Pi_5^{-3} - i\sqrt[4]{-30\,b_{-3}^3}\,\Theta_5^2\Pi_5^{-2} - \sqrt{-30\,b_{-3}^3}\,b_{-3}^{-1}\Theta_5\Pi_5^{-1} + b_0, \tag{3.11}$$

with

$$\Pi_5 = \left(-\frac{1}{30}\frac{\sqrt{-30\,b_{-3}^3}\,Y}{b_{-3}} - \frac{X}{\psi}\right),$$

$$\Theta_5 = \left(X - \frac{Y}{\psi}\right).$$

According to case II, the solution set for the following values of (ψ) and (τ) are obtained

$$\psi = \frac{x^\beta}{\beta} + \frac{\left(7\sqrt{30}\sqrt{b_3} - b_3 b_0\right)t^\alpha}{b_3 \alpha}, \tag{3.12}$$

$$\tau = \frac{15}{2\,b_3}. \tag{3.13}$$

Solution Space. 1: For $\tau < 0$, Eq (1.1) brings about the resulting single-wave solutions:

$$F_6(x,t) = -\frac{3375}{8}b_3^{-2}\Theta_6^3\Pi_6^{-3} + \frac{225}{4}b_3^{\frac{-3}{2}}\Theta_6^2\Pi_6^{-2} - \frac{225}{4}b_3^{-1}\Theta_6\Pi_6^{-1} + b_0$$
$$+ \frac{15}{2}\Theta_6^{-1}\Pi_6 + \sqrt{30}\sqrt{b_3}\Theta_6^{-2}\Pi_6^2 + b_3\Theta_6^{-3}\Pi_6^3, \tag{3.14}$$

or

$$F_7(x,t) = -\frac{3375}{8}\,b_3^{-2}\Theta_7^3\Pi_7^{-3} + \frac{225}{4}\,\sqrt{30}b_3^{\frac{-3}{2}}\,\Theta_7^2\Pi_7^{-2} - \frac{225}{4}\,b_3^{-1}\Theta_7\Pi_7^{-1} + b_0$$
$$+ \frac{15}{2}\,\Theta_7^{-1}\Pi_7 + \sqrt{30}\,\sqrt{b_3}\Theta_7^{-2}\Pi_7^2 + b_3\Theta_7^{-3}\Pi_7^3,$$

(3.15)

with

$$\Pi_6 = \left(-\frac{15}{2}\frac{Y}{b_3} - X\sqrt{-\tau}\tanh\left(\sqrt{-\tau}\psi\right)\right),$$
$$\Pi_7 = \left(-\frac{15}{2}\frac{Y}{b_3} - X\sqrt{-\tau}\coth\left(\sqrt{-\tau}\psi\right)\right),$$
$$\Theta_6 = \left(X - Y\sqrt{-\tau}\tanh\left(\sqrt{-\tau}\psi\right)\right),$$
$$\Theta_7 = \left(X - Y\sqrt{-\tau}\coth\left(\sqrt{-\tau}\psi\right)\right).$$

Solution Space. 2: For $\tau > 0$, Eq (1.1) brings about the resulting single-wave solutions:

$$F_8(x,t) = -\frac{3375}{8}\,b_3^{-2}\Theta_8^3\Pi_8^{-3} + \frac{225}{4}\,\sqrt{30}b_3^{-3/2}\Theta_8^2\Pi_8^{-2} - \frac{225}{4}b_3^{-1}\Theta_8\Pi_8^{-1} + b_0$$
$$+ \frac{15}{2}\,\Theta_8^{-1}\Pi_8 + \sqrt{30}\,\sqrt{b_3}\Theta_8^{-2}\Pi_8^2 + b_3\Theta_8^{-3}\Pi_8^3,$$

(3.16)

or

$$F_9(x,t) = -\frac{3375}{8}\,b_3^{-2}\Theta_9^3\Pi_9^{-3} + \frac{225}{4}\,\sqrt{30}b_3^{\frac{-3}{2}}\,\Theta_9^2\Pi_9^{-2} - \frac{225}{4}\,b_3^{-1}\Theta_9\Pi_9^{-1} + b_0$$
$$+ \frac{15}{2}\,\Theta_9^{-1}\Pi_9 + \sqrt{30}\,\sqrt{b_3}\Theta_9^{-2}\Pi_9^2 + b_3\Theta_9^{-3}\Pi_9^3,$$

(3.17)

with

$$\Pi_8 = \left(-\frac{15}{2}\frac{Y}{b_3} + X\sqrt{\tau}\tan\left(\sqrt{\tau}\psi\right)\right),$$
$$\Pi_9 = \left(-\frac{15}{2}\frac{Y}{b_3} - X\sqrt{\tau}\cot\left(\sqrt{\tau}\psi\right)\right),$$
$$\Theta_8 = \left(X + Y\sqrt{\tau}\tan\left(\sqrt{\tau}\psi\right)\right),$$
$$\Theta_9 = \left(X - Y\sqrt{\tau}\cot\left(\sqrt{\tau}\psi\right)\right).$$

Solution Space. 3: For $\tau = 0$, Eq (1.1) brings about the resulting single-wave solutions:

$$F_{10}(x,t) = -\frac{3375}{8}\,b_3^{-2}\Theta_{10}^3\Pi_{10}^{-3} + \frac{225}{4}\,\sqrt{30}b_3^{\frac{-3}{2}}\,\Theta_{10}^2\Pi_{10}^{-2} - \frac{225}{4}\,b_3^{-1}\Theta_{10}\Pi_{10}^{-1}$$
$$+ b_0 + \frac{15}{2}\,\Theta_{10}^{-1}\Pi_{10} + \sqrt{30}\,\sqrt{b_3}\Theta_{10}^{-2}\Pi_{10}^2 + b_3\Theta_{10}^{-3}\Pi_{10}^3,$$

(3.18)

with

$$\Pi_{10} = \left(-\frac{15}{2} \frac{Y}{b_3} - \frac{X}{\psi} \right),$$

$$\Theta_{10} = \left(X - \frac{Y}{\psi} \right).$$

According to case III, the solution set for the following values of (ψ) and (τ) are obtained:

$$\psi = \frac{x^\beta}{\beta} + \frac{\left(22 \sqrt{30} \sqrt{b_3} - b_3 b_0 \right) t^\alpha}{b_3 \alpha}, \tag{3.19}$$

$$\tau = 30 \, b_3^{-1}. \tag{3.20}$$

Solution Space. 1: For $\tau < 0$, Eq (1.1) brings about the resulting single-wave solutions:

$$F_{11}(x,t) = b_0 + 30 \, \Theta_{11}^{-1} \Pi_{11} + \sqrt{30} \sqrt{b_3} \Theta_{11}^{-2} \Pi_{11}^2 + b_3 \Theta_{11}^{-3} \Pi_{11}^3, \tag{3.21}$$

or

$$F_{12}(x,t) = b_0 + 30 \, \Theta_{12}^{-1} \Pi_{12} + \sqrt{30} \sqrt{b_3} \Theta_{12}^{-2} \Pi_{12}^2 + b_3 \Theta_{12}^{-3} \Pi_{12}^3, \tag{3.22}$$

with

$$\Pi_{11} = \left(-30 \frac{Y}{b_3} - X \sqrt{-\tau} \tanh\left(\sqrt{-\tau} \psi \right) \right),$$

$$\Pi_{12} = \left(-30 \frac{Y}{b_3} - X \sqrt{-\tau} \coth\left(\sqrt{-\tau} \psi \right) \right),$$

$$\Theta_{11} = \left(X - Y \sqrt{-\tau} \tanh\left(\sqrt{-\tau} \psi \right) \right),$$

$$\Theta_{12} = \left(X - Y \sqrt{-\tau} \coth\left(\sqrt{-\tau} \psi \right) \right).$$

Solution Space. 2: For $\tau > 0$, Eq (1.1) brings about the resulting single-wave solutions:

$$F_{13}(x,t) = b_0 + 30 \, \Theta_{13}^{-1} \Pi_{13} + \sqrt{30} \sqrt{b_3} \Theta_{13}^{-2} \Pi_{13}^2 + b_3 \Theta_{13}^{-3} \Pi_{13}^3, \tag{3.23}$$

or

$$F_{14}(x,t) = b_0 + 30 \, \Theta_{14}^{-1} \Pi_{14} + \sqrt{30} \sqrt{b_3} \Theta_{14}^{-2} \Pi_{14}^2 + b_3 \Theta_{14}^{-3} \Pi_{14}^3, \tag{3.24}$$

with

$$\Pi_{13} = \left(-30 \frac{Y}{b_3} + X \sqrt{\tau} \tan\left(\sqrt{\tau} \psi \right) \right),$$

$$\Pi_{14} = \left(-30 \frac{Y}{b_3} - X \sqrt{\tau} \cot\left(\sqrt{\tau} \psi \right) \right),$$

$$\Theta_{13} = \left(X + Y \sqrt{\tau} \tan\left(\sqrt{\tau} \psi \right) \right),$$

$$\Theta_{14} = \left(X - Y\sqrt{\tau}\cot\left(\sqrt{\tau}\psi\right)\right).$$

Solution Space. 3: For $\tau = 0$, Eq (1.1) bring about the resulting single-wave solutions:

$$F_{15}(x,t) = b_0 + 30\,\Theta_{15}^{-1}\Pi_{15} + \sqrt{30}\,\sqrt{b_3}\Theta_{15}^{-2}\Pi_{15}^2 + b_3\Theta_{15}^{-3}\Pi_{15}^3, \tag{3.25}$$

with

$$\Pi_{15} = \left(-30\,\frac{Y}{b_3} - \frac{X}{\psi}\right),$$

$$\Theta_{15} = \left(X - \frac{Y}{\psi}\right).$$

4. Results and discussion

The Riccati-Bernoulli sub-ODE method is an effective tool in finding exact analytical solutions for dynamical systems characterized by flame front oscillations, fluid flow down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium. This method is known for being easily applied to various physical systems. It can create a large family of periodic and solitary traveling wave solutions, each with its unique property, without using the usual discretization and linearization methods that are commonly used to solve problems.

In addition to providing exact solutions for systems displaying the physical phenomena described above, the obtained results allow for a deeper understanding of the complex dynamics underlying these processes. The versatility of our approach is critical to capturing highly intricate behaviors, hence many solutions with different parameters; any resulting analytical solutions are standard, thus securing a crucial assessment of accuracy, augmenting the reliability of the stability analysis of the relevant solutions manifold in the establishment of the dependability of computational models. A striking feature of our method is that it can engender solitons in conjunction with self-reinforcing solitary waves as it circumvents energy dissipation. Soliton dynamics becomes indispensable when phenomena such as flame front oscillations or fluid descent are explained, where waves propagate unperturbed, thereby unifying propagations before and after interactions in the soliton. The genesis of solitons from a delicate balance between linear and nonlinear processes affords a new vantage point from which to study the dynamics of complex physical systems.

The applied method generates three families of single-wave solutions. For, $\tau = 0$, these solutions show rational qualities, and for $\tau < 0$, they show hyperbolic traits, whereas for $\tau > 0$, they show periodic characteristics. Figures 1–5 show visual representations of the wave behaviors corresponding to some obtained solutions, offering a thorough understanding of their dynamic features. The graphical representations are a valuable tool for clarifying and understanding the complex dynamics present in these single-wave solutions. In these figures, the general shape of the waveforms described by the derived solutions becomes clear, in addition to the effect of the fractional parameters on the behavior of these waves. Figure 1 clearly shows that the complex solution represents the shock wave profile. In Figure 1 (a), the real part of solution (15) is considered, whereas the imaginary part of the solution (3.7) is introduced in Figure 1(b) in the (x,t)−plane. Figures 2 (a) and 3(b) show the impact of time fractional parameter α on the shock wave profile for the real and imaginary parts, respectively. Additionally, the solution (3.11) is investigated, as demonstrated in Figure 4. Moreover, the impact of

the time-fractional parameter α on the solution profile is examined as illustrated in Figures 4 (c) and 4(d) for the real and imaginary parts, respectively. It becomes clear here that the effect of the time-fractional parameter on the real part is opposite to the imaginary part. Also, solution (3.16) is analyzed numerically, as illustrated in Figure 5. In this figure, we examine the effect of both time and space fractional parameters on the real part of the first solution. Furthermore, solution (3.24) is numerically investigated, as evident in Figures 6(a) and 6(b) for the real and imaginary parts, respectively.
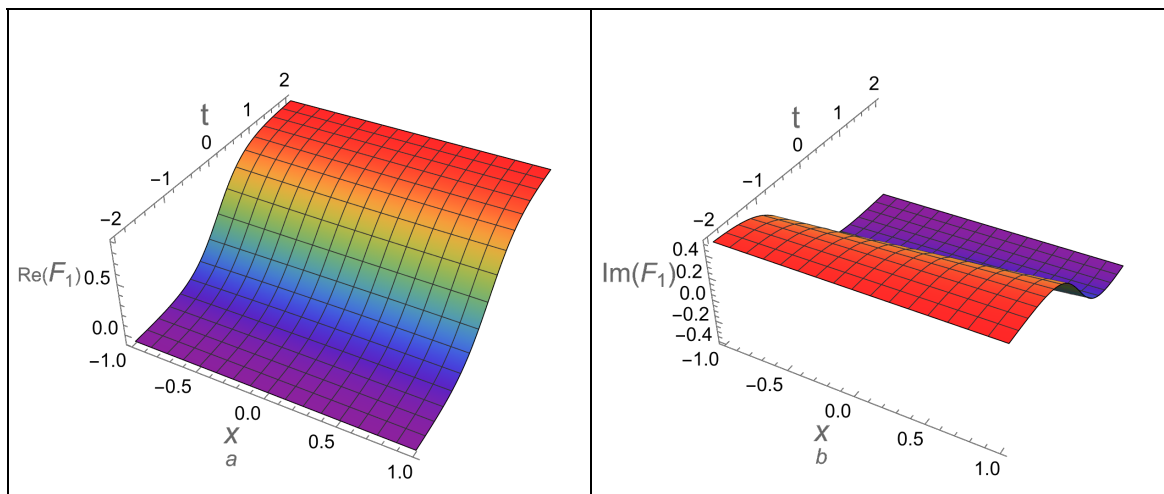
Figure 1. Solution (3.7) is plotted in the (x, t)-plane: (a) The real part of solution (3.7), and (b) the imaginary part of solution (3.7). Here, $(\alpha, \beta, b_0, b_{-3}, X, Y) = (1, 1, 0.1, 0.1, 0.01, 0.1)$.
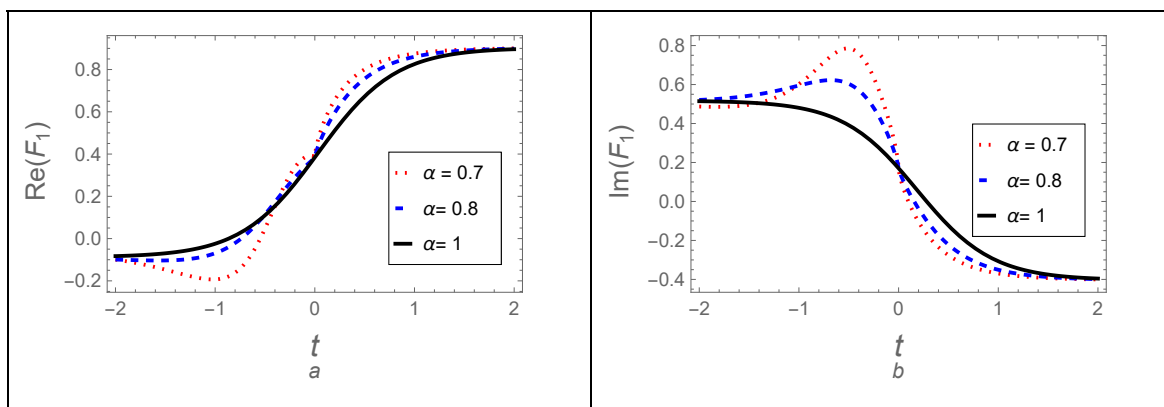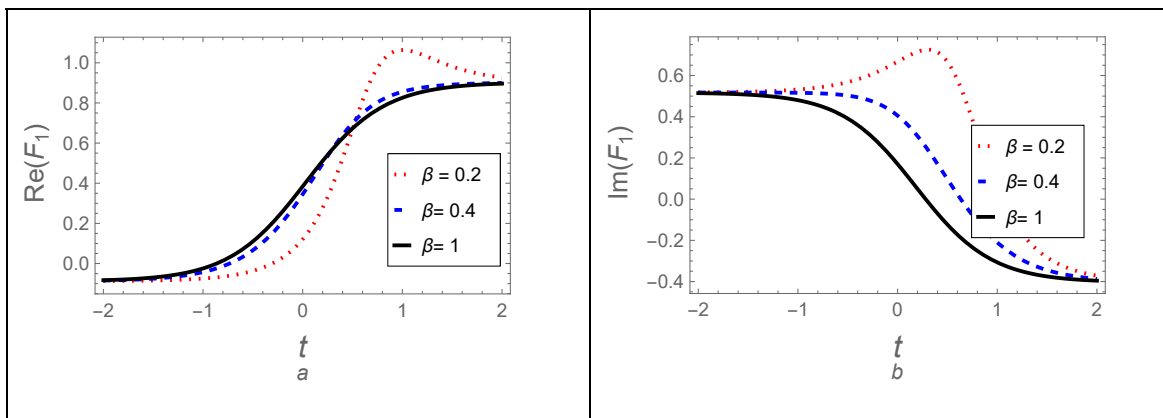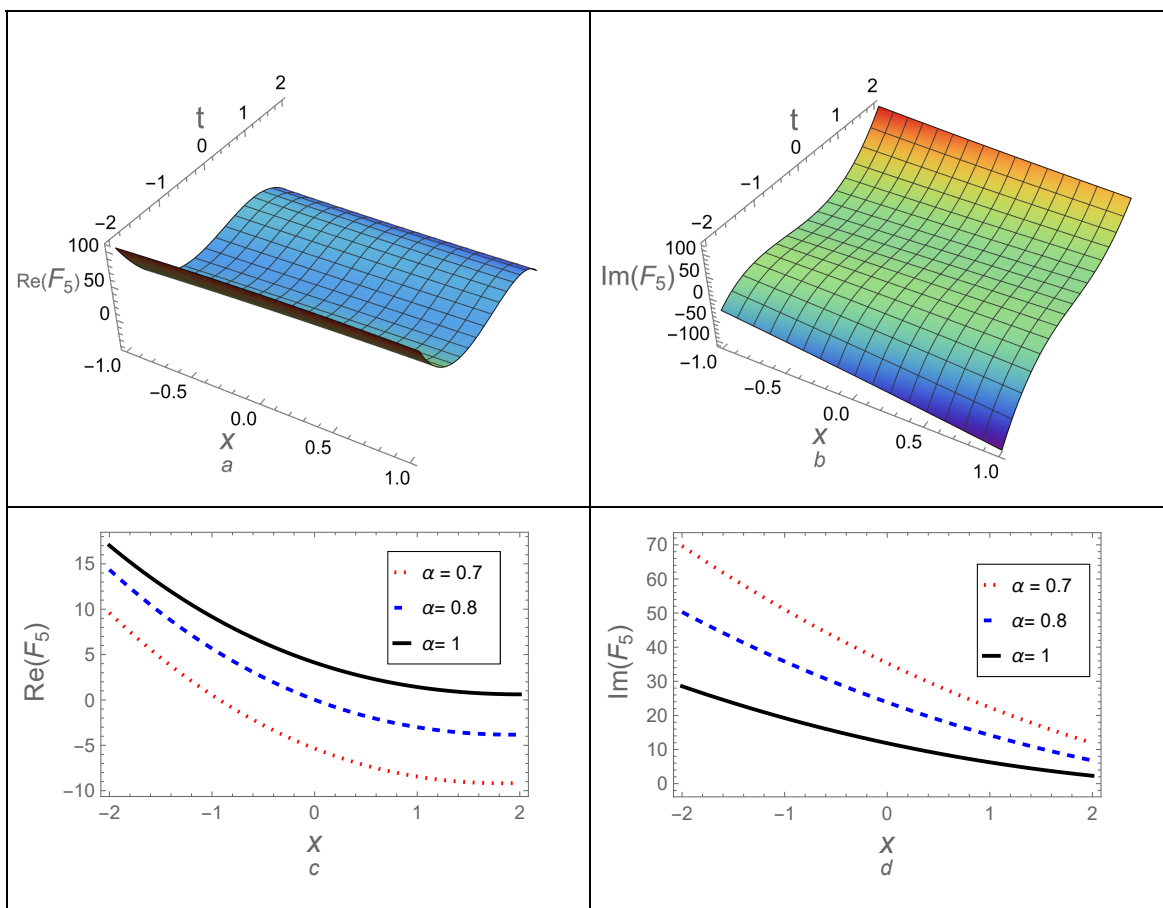

Figure 2. The impact of time fractional parameter α. Solution (3.7) is plotted against t for $x = 1$: (a) The real part of solution (3.7) and (b) The imaginary part of solution (3.7). Here, $(\alpha, \beta, b_0, b_{-3}, X, Y) = (1, 1, 0.1, 0.1, 0.01, 0.1)$.

Figure 3. The impact of space fractional parameter β. Solution (3.7) is plotted against t for $x = 1$: (a) The real part of solution (3.7), and (b) the imaginary part of solution (3.7). Here, $(\alpha, \beta, b_0, b_{-3}, X, Y) = (1, 1, 0.1, 0.1, 0.01, 0.1)$.

Figure 4. Numerical examination of solution (3.11): (a) The real part of solution (3.9) in the (x, t)-plane for $(\alpha, \beta) = (1, 1)$, (b) the imaginary part of solution (3.9) in the (x, t)-plane for $(\alpha, \beta) = (1, 1)$, (c) the real part of solution (3.9) against the time fractional parameter α, and (d) the imaginary part of solution (3.9) against the time fractional parameter α. Here, $(b_0, b_{-3}, X, Y) = (0.1, 0.1, 1, 0.01)$.
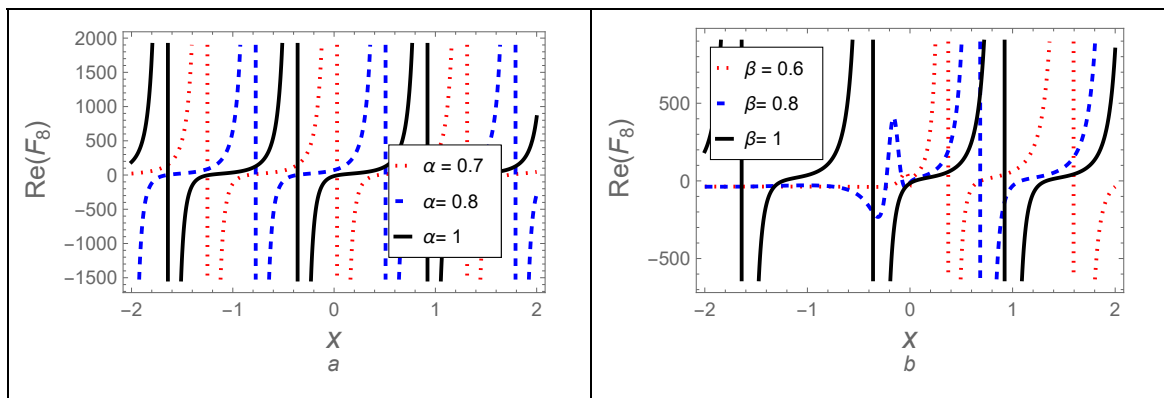
Figure 5. The impact of time and space fractional parameters (α, β) on the profile of solution (3.16). Solution (3.16) is plotted against x for $t = 1$: (a) The real part of against α, and (b) the imaginary part against β. Here, $(b_0, b_3, X, Y) = (0.1, 5, 0.7, 0.01)$.
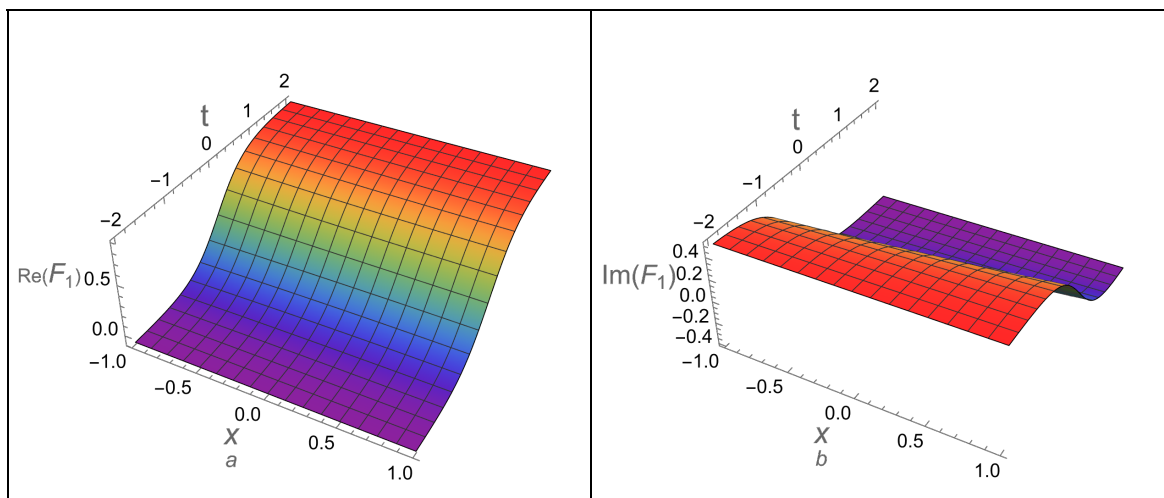

Figure 6. Solution (3.24) is plotted in the (x, t)-plane: (a) The real part of solution (3.24), and (b) the imaginary part of solution (3.24). Here, $(\alpha, \beta, b_0, b_3, X, Y) = (1, 1, 0.01, 0.01, 0.1, 0.09)$.

5. Conclusions

In this investigation, the Bäcklund transformation was employed with the Riccati-Bernoulli sub-ODE technique to derive some analytical solutions for the fractional Kuramoto-Sivashinsky model. The approach outlined in this work is efficient for solving and analyzing both time and space fractional partial differential equations (PDEs), showcasing simplicity and efficiency. Our approach, known for its simplicity and low computational requirements, can be easily expanded to tackle various typical fractional-order PDEs encountered in science and engineering. The approach presented shows potential for future applications in practical research, particularly in tackling large-scale, highly nonlinear fractional-order PDE systems. Three families of traveling wave solutions for the current problem have been derived in this context. Based on the related solution parameter (τ),

we found that for $\tau = 0$, these solutions show rational qualities and for $\tau < 0$, they show hyperbolic traits whereas for $\tau > 0$, they show periodic characteristics. Some derived solutions have been numerically examined, and the impact of both time and space fractional parameters on the profiles of these solutions has been investigated.

Our results provide significant insights into intricate processes, such as the vibrations of flame boundaries, the descent of fluid on a vertical plane, and the consistent vibrations of chemical reactions in a uniform environment. These discoveries enhance our comprehension of chemical kinetics, fluid dynamics, and combustion behavior, offering crucial knowledge for examining dynamic systems in various fields like engineering, biology, physics, chemistry, and economics. Hence, our proposed approach proves crucial in thoroughly grasping and articulating fractional-order evaluations within significant physical models.

Future work: It is also expected that the current approach will succeed in analyzing and solving many nonlinear evolution equations related to many nonlinear phenomena that arise and propagate in various plasma systems. Therefore, this approach can be applied to derive analytical solutions to some different time and space fractional wave equations, such as the family of fractional KdV-type equations of third, fifth, and seventh-order dispersion [55–57], and the fractional nonlinear Schrödinger-type (NLS-type) equations [58–60], which are used to describe solitons, shocks, cnoidal waves, and rogue waves that arise and propagate in plasma physics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Authors contributions

All authors contributed equally and approved the final version of the current manuscript.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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