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*Research article*

# On graded weakly $J_{gr}$-semiprime submodules

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**Abstract:** Let $\Gamma$ be a group, $\mathcal{A}$ be a $\Gamma$-graded commutative ring with unity 1, and $\mathcal{D}$ a graded $\mathcal{A}$-module. In this paper, we introduce the concept of graded weakly $J_{gr}$-semiprime submodules as a generalization of graded weakly semiprime submodules. We study several results concerning of graded weakly $J_{gr}$-semiprime submodules. For example, we give a characterization of graded weakly $J_{gr}$-semiprime submodules. Also, we find some relations between graded weakly $J_{gr}$-semiprime submodules and graded weakly semiprime submodules. In addition, the necessary and sufficient condition for graded submodules to be graded weakly $J_{gr}$-semiprime submodules are investigated. A proper graded submodule $U$ of $\mathcal{D}$ is said to be a graded weakly $J_{gr}$-semiprime submodule of $\mathcal{D}$ if whenever $r_g \in h(\mathcal{A})$, $m_h \in h(\mathcal{D})$ and $n \in \mathbb{Z}^+$ with $0 \neq r_g^n m_h \in U$, then $r_g m_h \in U + J_{gr}(\mathcal{D})$, where $J_{gr}(\mathcal{D})$ is the graded Jacobson radical of $\mathcal{D}$.

**Keywords:** graded weakly $J_{gr}$-semiprime submodule; graded $J_{gr}$-semiprime submodule; graded weakly semiprime submodule
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## 1. Introduction

Throughout this work, we assume that $\mathcal{A}$ is a commutative $\Gamma$-graded ring with identity and $\mathcal{D}$ is a unitary graded $\mathcal{A}$-module.

The study of graded rings and modules has attracted the attentions of many researchers for a long time due to their important applications in many fields in such as geometry and physics. For example, graded Lie algebra plays a significant role in differential geometry, such as with Frolicher-Nijenhuis, as well as the Nijenhuis-Richardson bracket (see [12]). In addition, they solve many physical problems related to supermanifolds, supersymmetries and quantizations of systems with symmetry (see [6, 17]). Recently, some classical notions and definitions have been extended and generalized. For instance: the concepts of graded weakly semiprime ideals have been extended to the concepts of graded weakly semiprime submodules (see [2, 9, 10, 13, 18]). The main goal of this paper is to study the theory of graded modules over graded commutative rings. In particular, we introduce graded weakly $J_{gr}$-semiprime submodules, which are a generalization of graded weakly semiprime submodules. Also, several results concerning graded weakly $J_{gr}$-semiprime submodules will be given.

Let $\Gamma$ be a group. A ring $\mathcal{A}$ is said to be a $\Gamma$-graded ring if there exist additive subgroups $\mathcal{A}_g$ of $\mathcal{A}$ indexed by the elements $g \in \Gamma$ with $\mathcal{A} = \bigoplus_{g\in\Gamma} \mathcal{A}_g$ and $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in \Gamma$. We set $h(\mathcal{A}) := \cup_{g\in\Gamma}\mathcal{A}_g$. If $t \in \mathcal{A}$, then $t$ can be written uniquely as $\sum_{g\in\Gamma} t_g$, where $t_g$ is called a homogeneous component of $t$ in $\mathcal{A}_g$. Let $\mathcal{A} = \bigoplus_{g\in\Gamma} \mathcal{A}_g$ be a $\Gamma$-graded ring. An ideal $L$ of $\mathcal{A}$ is said to be a graded ideal if $L = \bigoplus_{g\in\Gamma}(L \cap \mathcal{A}_g) := \bigoplus_{g\in\Gamma} L_g$. By $L \leq^{id}_{\Gamma} \mathcal{A}$, we mean that $L$ is a graded ideal of $\mathcal{A}$. Also, by $L <^{id}_{\Gamma} \mathcal{A}$, we mean that $L$ is a proper graded ideal of $\mathcal{A}$. Let $\mathcal{A}$ be a $\Gamma$-graded ring, and $\mathcal{D}$ an $\mathcal{A}$-module. Then, $\mathcal{D}$ is a $\Gamma$-graded $\mathcal{A}$-module if there exists a family of additive subgroups $\{\mathcal{D}_g\}_{g\in\Gamma}$ of $\mathcal{D}$ with $\mathcal{D} = \bigoplus_{g\in\Gamma} \mathcal{D}_g$ and $\mathcal{A}_g\mathcal{D}_h \subseteq \mathcal{D}_{gh}$ for all $g, h \in \Gamma$. We set $h(\mathcal{D}) := \cup_{g\in\Gamma} \mathcal{D}_g$. Let $\mathcal{D} = \bigoplus_{g\in\Gamma} \mathcal{D}_g$ be a graded $\mathcal{A}$-module. A submodule $U$ of $\mathcal{D}$ is said to be *a graded submodule of $M$* if $U = \bigoplus_{g\in\Gamma}(U \cap \mathcal{D}_g) := \bigoplus_{g\in\Gamma} U_g$. By $U \leq^{sub}_{\Gamma} \mathcal{D}$, we mean that $U$ is a $\Gamma$-graded submodule of $\mathcal{D}$. Also, by $U <^{sub}_{\Gamma} \mathcal{D}$, we mean that $U$ is a proper $\Gamma$-graded submodule of $\mathcal{D}$. These basic properties and more information on graded rings and modules can be found in [11, 14–16]. A $<^{sub}_{\Gamma} \mathcal{D}$ is said to be *a Gr-maximal* if there is a $L \leq^{sub}_{\Gamma} \mathcal{D}$ with $U \subseteq L \subseteq \mathcal{D}$, and then $U = L$ or $L = \mathcal{D}$ (see [16]). The graded Jacobson radical of a graded module $\mathcal{D}$, denoted by $J_{gr}(\mathcal{D})$, is defined to be the intersection of all $Gr$-maximal submodules of $\mathcal{D}$ (if $\mathcal{D}$ has no $Gr$-maximal submodule then we shall take, by definition, $J_{gr}(\mathcal{D}) = \mathcal{D}$), (see [16]). A $U <^{sub}_{\Gamma} \mathcal{D}$ is called *a graded semiprime* (briefly, $Gr$-semiprime) *submodule* if, whenever $t_g \in h(\mathcal{A})$, $m_h \in h(\mathcal{D})$ and $n \in \mathbb{Z}^+$ with $t^n_g m_h \in U$, then $t_g m_h \in U$ (see [10]). A $U <^{sub}_{\Gamma} \mathcal{D}$ is called *a graded weakly semiprime (briefly, Gr-W-semiprime) submodule* if whenever $t_g \in h(\mathcal{A})$, $m_h \in h(\mathcal{D})$ and $n \in \mathbb{Z}^+$ with $0 \neq t^n_g m_h \in U$, then $t_g m_h \in U$ (see [18]). It is shown in [4, Lemma 2.11] that if $U \leq^{sub}_{\Gamma} \mathcal{D}$, then $(U :_{\mathcal{A}} \mathcal{D}) = \{r \in \mathcal{A} : rU \subseteq \mathcal{D}\}$ is a graded ideal of $\mathcal{A}$. Let $N \leq^{sub}_{\Gamma} \mathcal{D}$ and $I \leq^{id}_{\Gamma} \mathcal{A}$. We use the notation $(N :_{\mathcal{D}} I)$ to denote the graded submodule $\{m \in \mathcal{D} : Im \subseteq N\}$ of $\mathcal{D}$.

## 2. Results

**Definition 2.1.** A proper graded submodule $U$ of $\mathcal{D}$ is said to be a graded weakly $J_{gr}$-semiprime (briefly, $Gr$-$W$-$J_{gr}$-semiprime) submodule of $\mathcal{D}$ if, whenever $0 \neq r^n_g m_h \in U$ where $r_g \in h(\mathcal{A})$, $m_h \in h(\mathcal{D})$ and $n \in \mathbb{Z}^+$, then $r_g m_h \in U + J_{gr}(\mathcal{D})$. In particular, a graded ideal $L$ of $\mathcal{A}$ is said to be a graded weakly $J_{gr}$-semiprime ideal of $\mathcal{A}$ if $L$ is a graded weakly $J_{gr}$-semiprime submodule of the graded $\mathcal{A}$-module $\mathcal{A}$.

It is clear that every $Gr$-$W$-semiprime submodule is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$, but the converse is not true in general. This is clear from the following examples.

**Example 2.2.** Let $\Gamma = \mathbb{Z}_2$ and $\mathcal{A} = \mathbb{Z}$ be a $\Gamma$-graded ring with $\mathcal{A}_0 = \mathbb{Z}$, $\mathcal{A}_1 = \{0\}$. Then $\mathcal{D} = \mathbb{Z}_{24}$ is a graded $\mathcal{A}$-module with $\mathcal{D}_0 = \mathbb{Z}_{24}$ and $\mathcal{D}_1 = \{\overline{0}\}$. Let $U = \{\overline{0}, \overline{8}, \overline{16}\} \leq^{sub}_{\Gamma} \mathbb{Z}_{24}$. Since $J_{gr}(\mathbb{Z}_{24}) = \langle\overline{2}\rangle \cap \langle\overline{3}\rangle = \langle\overline{6}\rangle = \{\overline{0}, \overline{6}, \overline{12}, \overline{18}\}$, and whenever $0 \neq r^k m \in U$ for $r \in h(\mathbb{Z})$, $m \in h(\mathbb{Z}_{24})$ and $k \in \mathbb{Z}^+$ implies that $rm \in U + J_{gr}(\mathbb{Z}_{24}) = \{\overline{0}, \overline{8}, \overline{16}\} + \{\overline{0}, \overline{6}, \overline{12}, \overline{18}\} = \langle\overline{2}\rangle$, we have $U$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$. However, $U$ is not a$Gr$-$W$-semiprime submodule of $\mathcal{D}$ since there exist $2 \in h(\mathbb{Z})$, $\overline{2} \in h(\mathbb{Z}_{24})$, and $2 \in \mathbb{Z}^+$ such that $0 \neq 2^2 \cdot \overline{2} = \overline{8} \in U$, but $2 \cdot \overline{2} = \overline{4} \notin U$.

**Example 2.3.** Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a $G$-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_{p^\infty} = \{\frac{a}{p^n} + \mathbb{Z} : a, n \in \mathbb{Z}, n \geqslant 0\}$ be a graded $R$-module with $M_0 = \mathbb{Z}_{p^\infty}$ and $M_1 = \{0_{\mathbb{Z}_{p^\infty}}\} = \{\mathbb{Z}\}$, where $p$ is a fixed prime number. Consider the graded submodule $N = \langle \frac{1}{p^3} + \mathbb{Z} \rangle$ of $M$. Then $N$ is not a *Gr-W*-semiprime submodule of $M$, since $0 \neq p^l(\frac{1}{p^{3+l}} + \mathbb{Z}) = \frac{1}{p^3} + \mathbb{Z} \in N$ but $p(\frac{1}{p^{3+l}} + \mathbb{Z}) \notin N$, where $1 \neq l \in \mathbb{Z}^+$. However, easy computations show that $N$ is a $Gr\text{-}W\text{-}J_{gr}$-semiprime submodule of $M$.

Following are theorems that give some equivalent characterizations of the $Gr\text{-}W\text{-}J_{gr}$-semiprime submodule.

**Theorem 2.4.** *Let $U <^{sub}_{\Gamma} \mathcal{D}$. Then the following statements are equivalent.*

(i) *$U$ is a $Gr\text{-}W\text{-}J_{gr}$-semiprime submodule of $\mathcal{D}$.*

(ii) $(U :_{\mathcal{D}} \langle r_g^n \rangle) \subseteq (\langle 0 \rangle :_{\mathcal{D}} \langle r_g^n \rangle) \cup (U + J_{gr}(\mathcal{D}) :_{\mathcal{D}} \langle r_g \rangle)$*, for each* $r_g \in h(\mathcal{A})$.

(iii) *Either* $(U :_{\mathcal{D}} \langle r_g^n \rangle) \subseteq (\langle 0 \rangle :_{\mathcal{D}} \langle r_g^n \rangle)$ *or* $(U :_{\mathcal{D}} \langle r_g^n \rangle) \subseteq (U + J_{gr}(\mathcal{D}) :_{\mathcal{D}} \langle r_g \rangle)$*, for each* $r_g \in h(\mathcal{A})$.

*Proof.* $(i) \to (ii)$: Let $r_g \in h(\mathcal{A})$ and $m_h \in (U :_{\mathcal{D}} \langle r_g^n \rangle) \cap h(\mathcal{D})$. Then $\langle r_g^n \rangle m_h \subseteq U$, and hence $r_g^n m_h \in U$. If $r_g^n m_h \neq 0$, then $r_g m_h \in U + J_{gr}(\mathcal{D})$ as $U$ is a $Gr\text{-}W\text{-}J_{gr}$-semiprime submodule of $\mathcal{D}$. Hence, $\langle r_g \rangle m_h \subseteq U + J_{gr}(\mathcal{D})$, and it follows that $m_h \in (U + J_{gr}(\mathcal{D}) :_{\mathcal{D}} \langle r_g \rangle)$. Thus, $m_h \in (\langle 0 \rangle :_{\mathcal{D}} \langle r_g^n \rangle) \cup (U + J_{gr}(\mathcal{D}) :_{\mathcal{D}} \langle r_g \rangle)$. If $r_g^n m_h = 0$, then $\langle r_g^n \rangle m_h \subseteq \{0\}$, and so $m_h \in (\langle 0 \rangle :_{\mathcal{D}} \langle r_g^n \rangle)$. Hence, $m_h \in (\langle 0 \rangle :_{\mathcal{D}} \langle r_g^n \rangle) \cup (U + J_{gr}(\mathcal{D}) :_{\mathcal{D}} \langle r_g \rangle)$. Therefore, $(U :_{\mathcal{D}} \langle r_g^n \rangle) \subseteq (\langle 0 \rangle :_{\mathcal{D}} \langle r_g^n \rangle) \cup (U + J_{gr}(\mathcal{D}) :_{\mathcal{D}} \langle r_g \rangle)$.

$(ii) \to (iii)$: It is clear.

$(iii) \to (i)$: Let $r_g \in h(\mathcal{A})$, $m_h \in h(\mathcal{D})$, and $n \in \mathbb{Z}^+$ with $0 \neq r_g^n m_h \in U$. Then $\{0\} \neq \langle r_g^n \rangle m_h \subseteq U$, which implies that $m_h \in (U : \langle r_g^n \rangle)$ and $m_h \notin (\langle 0 \rangle :_{\mathcal{D}} \langle r_g^n \rangle)$. Now, by (iii), we get $m_h \in (U + J_{gr}(\mathcal{D}) :_{\mathcal{D}} \langle r_g \rangle)$ and so $r_g m_h \in U + J_{gr}(\mathcal{D})$. Therefore, $U$ is a $Gr\text{-}W\text{-}J_{gr}$-semiprime submodule of $\mathcal{D}$. □

**Theorem 2.5.** *Let $U <^{sub}_{\Gamma} \mathcal{D}$. Then the following statements are equivalent.*

(i) *$U$ is a $Gr\text{-}W\text{-}J_{gr}$-semiprime submodule of $\mathcal{D}$.*

(ii) *For every* $K \leq^{sub}_{\Gamma} \mathcal{D}$, $r_g \in h(\mathcal{A})$, *and* $n \in \mathbb{Z}^+$ *with* $\{0\} \neq \langle r_g \rangle^n K \subseteq U$, *then* $\langle r_g \rangle K \subseteq U + J_{gr}(\mathcal{D})$.

*Proof.* $(i) \Rightarrow (ii)$ Let $K \leq^{sub}_{\Gamma} \mathcal{D}$, $r_g \in h(\mathcal{A})$, and $n \in \mathbb{Z}^+$ with $\{0\} \neq \langle r_g \rangle^n K \subseteq U$. This implies that, $K \subseteq (U :_{\mathcal{D}} \langle r_g^n \rangle)$ and $K \nsubseteq (\langle 0 \rangle :_{\mathcal{D}} \langle r_g^n \rangle)$. Since $U$ is a $Gr\text{-}W\text{-}J_{gr}$-semiprime submodule of $\mathcal{D}$, by Theorem 2.4 we have $K \subseteq (U :_{\mathcal{D}} \langle r_g^n \rangle) \subseteq (U + J_{gr}(\mathcal{D}) :_{\mathcal{D}} \langle r_g \rangle)$, hence $\langle r_g \rangle K \subseteq U + J_{gr}(\mathcal{D})$.

$(ii) \Rightarrow (i)$ Let $0 \neq r_g^n m_h \in U$ where $r_g \in h(\mathcal{A})$, $m_h \in h(\mathcal{D})$, and $n \in \mathbb{Z}^+$. Then $\{0\} \neq \langle r_g \rangle^n \langle m_h \rangle \subseteq U$. Now, by (ii), we have $\langle r_g \rangle \langle m_h \rangle \subseteq U + J_{gr}(\mathcal{D})$, and it follows that $r_g m_h \in U + J_{gr}(\mathcal{D})$. Therefore, $U$ is a $Gr\text{-}W\text{-}J_{gr}$-semiprime submodule of $\mathcal{D}$. □

The following corollaries follow directly from Theorem 2.5.

**Corollary 2.6.** *Let $U <^{sub}_{\Gamma} \mathcal{D}$. Then $U$ is a $Gr\text{-}W\text{-}J_{gr}$-semiprime submodule of $\mathcal{D}$ if and only if for every* $r_g \in h(\mathcal{A})$, *and* $n \in \mathbb{Z}^+$ *with* $\{0\} \neq \langle r_g^n \rangle \mathcal{D} \subseteq U$, *then* $\langle r_g \rangle \mathcal{D} \subseteq U + J_{gr}(\mathcal{D})$.

**Corollary 2.7.** *Let $U <^{sub}_{\Gamma} \mathcal{D}$. Then $U$ is a $Gr\text{-}W\text{-}J_{gr}$-semiprime submodule of $\mathcal{D}$ if and only if for every* $r_g \in h(\mathcal{A})$, $K \leq^{sub}_{\Gamma} \mathcal{D}$ *and* $n \in \mathbb{Z}^+$ *with* $\{0\} \neq r_g^n K \subseteq U$, *then* $r_g K \subseteq U + J_{gr}(\mathcal{D})$.

**Theorem 2.8.** *Let $U$ be a $Gr\text{-}W\text{-}J_{gr}$-semiprime submodule of $\mathcal{D}$ with $J_{gr}(\mathcal{D}) \subseteq U$. Then $U$ is a Gr-W-semiprime submodule of $\mathcal{D}$.*

*Proof.* Let $r_g \in h(\mathcal{A})$, $m_h \in h(\mathcal{D})$, and $n \in \mathbb{Z}^+$ with $0 \neq r_g^n m_h \in U$. Since $U$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$ and $J_{gr}(\mathcal{D}) \subseteq U$, we have $r_g m_h \in U + J_{gr}(\mathcal{D}) = U$. Therefore, $U$ is a $Gr$-$W$-semiprime submodule of $\mathcal{D}$. □

Let $\mathcal{A}$ be a $\Gamma$-graded ring and $\mathcal{D}$, $\mathcal{D}'$ be two graded $\mathcal{A}$-modules. Let $\varphi : \mathcal{D} \to \mathcal{D}'$ be an $\mathcal{A}$-module homomorphsim. Then, $\varphi$ is said to be a graded homomorphsim if $\varphi(\mathcal{D}_g) \subseteq \mathcal{D}'_g$ for all $g \in \Gamma$, see [16].

**Theorem 2.9.** *Let $U <_{\Gamma}^{sub} \mathcal{D}$. If $J_{gr}(\mathcal{D}/U) = \{U\}$, then $J_{gr}(\mathcal{D}) \subseteq U$.*

*Proof.* Define $\varphi : \mathcal{D} \to \mathcal{D}/U$ as a graded homeomorphism given by $\varphi(x) = x + U$ for all $x \in h(\mathcal{D})$; by [3, Theorem 2.12 (i)], $\varphi(J_{gr}(\mathcal{D})) \subseteq J_{gr}(\mathcal{D}/U)$. Since $J_{gr}(\mathcal{D}/U) = \{U\}$, then $\{U\} \subseteq \varphi(J_{gr}(\mathcal{D})) \subseteq \{U\}$, so we have $\varphi(J_{gr}(\mathcal{D})) = \{U\}$, thus $J_{gr}(\mathcal{D}) \subseteq Ker\varphi = U$. □

**Corollary 2.10.** *Let $U$ be a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$ with $J_{gr}(\frac{\mathcal{D}}{U}) = \{U\}$. Then $U$ is a $Gr$-$W$-semiprime submodule of $\mathcal{D}$.*

*Proof.* This is clear by Theorems 2.9 and 2.8. □

A graded $\mathcal{A}$-module $\mathcal{D}$ is a graded semisimple ($Gr$-semisimple) if and only if every graded submodule $U$ of $\mathcal{D}$ is a direct summand. That is $\mathcal{D}$ is a $Gr$-semisimple if and only if for every graded submodule $U$ of $\mathcal{D}$ there exists $L$, a graded submodule of $\mathcal{D}$ such that $\mathcal{D} = U \oplus L$.

A graded submodule $U$ is called a graded small ($Gr$-small) if $\mathcal{D} = U + V$ for $V \leq_{\Gamma}^{sub} \mathcal{D}$ implies that $V = \mathcal{D}$, see [1].

**Theorem 2.11.** *Let $\mathcal{D}$ be a $Gr$-semisimple $\mathcal{A}$-module and $U$ be a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$. Then $U$ is a $Gr$-$W$-semiprime submodule of $\mathcal{D}$.*

*Proof.* Let $\mathcal{D}$ be a $Gr$-semisimple $\mathcal{A}$-module. Then every graded submodule of $\mathcal{D}$ is a direct summand. Thus, the only $Gr$-small sumodule of $\mathcal{D}$ is $\{0\}$, and it follows that $J_{gr}(\mathcal{D}) = \sum\{S : S$ is a $Gr$-small submodule of $\mathcal{D}\} = \{0\} \subseteq U$ by [3, Theorem 2.10]. Since $U$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$, by Theorem 2.8, then $U$ is a $Gr$-$W$-semiprime submodule of $\mathcal{D}$. □

Recall from [4] that a graded module $\mathcal{D}$ is said to be a graded torsion ($Gr$-torsion) free $\mathcal{A}$-module if, whenever $r_g m_h = 0$ where $r_g \in h(\mathcal{A})$ and $m_h \in$h$(\mathcal{D})$, then either $m_h = 0$ or $r_g = 0$.

**Theorem 2.12.** *Let $\mathcal{D}$ be a $Gr$-torsion free $\mathcal{A}$-module, and $U \leq_{\Gamma}^{sub} \mathcal{D}$ with $J_{gr}(\frac{\mathcal{D}}{U}) = \{U\}$. Then $U$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$ if and only if for any nonzero $L \leq_{\Gamma}^{id} \mathcal{A}$, $(U :_{\mathcal{D}} L)$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$.*

*Proof.* ($\Longrightarrow$) Let $0 \neq L \leq_{\Gamma}^{id} \mathcal{A}$, $m_h \in h(\mathcal{D})$, $r_g \in h(\mathcal{A})$, and $n \in \mathbb{Z}^+$ with $0 \neq r_g^n m_h \in (U :_{\mathcal{D}} L)$. Then $\{0\} \neq \langle r_g^n \rangle m_h \subseteq (U :_{\mathcal{D}} L)$, and hence $\langle r_g^n \rangle (L m_h) \subseteq U$. If $\langle r_g^n \rangle (L m_h) = \{0\}$, so there exists $0 \neq i \in L \cap h(\mathcal{A})$ with $\langle r_g^n \rangle i m_h = \{0\}$, so $i \cdot r_g^n m_h = 0$. Hence, $r_g^n m_h = 0$ as $\mathcal{D}$ is a $Gr$-torsion free $\mathcal{A}$-module, which is a contradiction. So, assume that $\langle r_g \rangle^n (L m_h) = \langle r_g^n \rangle (L m_h) \neq \{0\}$. Since $U$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$, by Theorem 2.5, $\langle r_g \rangle (L m_h) \subseteq U + J_{gr}(\mathcal{D})$. But, $J_{gr}(\frac{\mathcal{D}}{U}) = \{U\}$, and by Theorem 2.9, we have $J_{gr}(\mathcal{D}) \subseteq U$ so $\langle r_g \rangle (L m_h) \subseteq U$. This implies that, $\langle r_g \rangle m_h \subseteq (U :_{\mathcal{D}} L) \subseteq (U :_{\mathcal{D}} L) + J_{gr}(\mathcal{D})$ and hence $r_g m_h \in (U :_{\mathcal{D}} L) \subseteq (U :_{\mathcal{D}} L) + J_{gr}(\mathcal{D}, )$. Therefore, $(U :_{\mathcal{D}} L)$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$.

($\Longleftarrow$) Assume that $(U :_{\mathcal{D}} L)$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$ for any nonzero $L \leq_{\Gamma}^{id} \mathcal{A}$. Put $\mathcal{A} = L$, then $U = (U :_{\mathcal{D}} \mathcal{A})$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$. □

Recall from [7] that a graded $\mathcal{A}$-module $\mathcal{D}$ is called a graded multiplication module ($Gr$-multiplication module) if for every $U \leq_{\Gamma}^{sub} \mathcal{D}$ there exists a $K \leq_{\Gamma}^{id} \mathcal{A}$ such that $U = K\mathcal{D}$. If $\mathcal{D}$ is a $Gr$-multiplication $\mathcal{A}$-module, $U = (U :_{\mathcal{A}} \mathcal{D})\mathcal{D}$ for every $U \leq_{\Gamma}^{sub} \mathcal{D}$.

The set of all homogeneous zero divisors of $\mathcal{A}$ is $G\text{-}Z(\mathcal{A}) = \{r \in h(\mathcal{A}) : rs = 0$ for some $0 \neq s \in h(\mathcal{A})\}$, and the set of all homogeneous regular elements is $G\text{-}C(\mathcal{A}) = \{c \in h(\mathcal{A}) : c \notin G\text{-}Z(\mathcal{A})\} = \{c \in h(\mathcal{A}) : cr \neq 0$ for all $0 \neq r \in h(\mathcal{A})\}$. It is clear that $\mathcal{D}$ is a $Gr$-torsion free if and only if $cm \neq 0$ for all $c \in G\text{-}C(\mathcal{A})$ and $0 \neq m \in h(\mathcal{D})$.

**Theorem 2.13.** *Every faithful Gr-multiplication $\mathcal{A}$-module is a Gr-torsion free.*

*Proof.* Suppose that, $\mathcal{D}$ is not $Gr$-torsion free. Hence, there exist $c \in G\text{-}C(\mathcal{A})$ and $0 \neq m \in h(\mathcal{D})$ with $cm = 0$. Since $\mathcal{D}$ is a faithful $Gr$-multiplication $\mathcal{A}$-module, there exists an $L \leq_{\Gamma}^{id} \mathcal{A}$ with $\mathcal{A}m = L\mathcal{D}$, and so $\mathcal{A}cm = cL\mathcal{D}$. This implies that $(cL)\mathcal{D} = \{0\}$. Since $\mathcal{D}$ is a faithful, $cL = \{0\}$. Hence, $c \in G\text{-}Z(\mathcal{A})$ since $L \neq 0$, and so $c \notin G\text{-}C(\mathcal{A})$, which is a contradiction. Therefore, $\mathcal{D}$ is $Gr$ -torsion free. □

**Corollary 2.14.** *Let $\mathcal{D}$ be a faithful Gr-multiplication $\mathcal{A}$-module and $U \leq_{\Gamma}^{sub} \mathcal{D}$, with $J_{gr}(\frac{\mathcal{D}}{U}) = \{U\}$. Then, U is a Gr-W-$J_{gr}$-semiprime submodule of $\mathcal{D}$ if and only if for any nonzero $L \leq_{\Gamma}^{id} \mathcal{A}$, $(U :_{\mathcal{D}} L)$ is a Gr-W-$J_{gr}$-semiprime submodule of $\mathcal{D}$.*

*Proof.* Follows by Theorems 2.13 and 2.12. □

**Theorem 2.15.** *Let U be Gr-small submodule of $\mathcal{D}$ with $J_{gr}(\mathcal{D})$ a Gr-W-semiprime submodule of $\mathcal{D}$. Then, U is a Gr-W-$J_{gr}$-semiprime submodule of $\mathcal{D}$.*

*Proof.* Let $r_g \in h(\mathcal{A})$, $m_h \in h(\mathcal{D})$, and $n \in \mathbb{Z}^+$ with $0 \neq r_g^n m_h \in U$. Since $U$ is a $Gr$-small submodule of $\mathcal{D}$, then by [3, Theorem 2.10], $U \subseteq J_{gr}(\mathcal{D}) = \sum\{A : A$ is a $Gr$-small submodule of $\mathcal{D}\}$, so $0 \neq r_g^n m_h \in J_{gr}(\mathcal{D})$, since $J_{gr}(\mathcal{D})$ is a $Gr$-$W$-semiprime submodule of $\mathcal{D}$, then $r_g m_h \in J_{gr}(\mathcal{D}) \subseteq U + J_{gr}(\mathcal{D})$. Therefore $U$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$. □

Recall from [16] that graded $\mathcal{A}$-module $\mathcal{D}$ is said to be a graded finitely generated ($Gr$-finitely generated) if $\mathcal{D} = \mathcal{A}a_{g1} + \cdots + \mathcal{A}a_{gn}$ for some $a_{g1}, a_{g2}, ..., a_{gn} \in h(\mathcal{D})$.

**Theorem 2.16.** *Let $\mathcal{D}$ be a Gr-finitely generated Gr-multiplication $\mathcal{A}$-module and L be a Gr-W-$J_{gr}$-semiprime ideal of $\mathcal{A}$ with $ann_{\mathcal{A}}(\mathcal{D}) \subseteq L$. Then $L\mathcal{D}$ is a Gr-W-$J_{gr}$-semiprime submodule of $\mathcal{D}$.*

*Proof.* Let $r_g \in h(\mathcal{A})$, $m_h \in h(\mathcal{D})$, and $k \in \mathbb{Z}^+$ with $0 \neq r_g^k m_h \in L\mathcal{D}$. Then $\{0\} \neq r_g^k\langle m_h\rangle \subseteq L\mathcal{D}$. Since $\mathcal{D}$ is a $Gr$-multiplication, $\langle m_h\rangle = J\mathcal{D}$ for some $J \leq_{\Gamma}^{id} \mathcal{A}$, and hence $\{0\} \neq r_g^k J\mathcal{D} \subseteq L\mathcal{D}$. This implies that $\{0\} \neq r_g^k J \subseteq L + ann_{\mathcal{A}}(\mathcal{D})$ by [5, Lemma 3.9]. Since $ann_{\mathcal{A}}(\mathcal{D}) \subseteq L$, and it follows that $\{0\} \neq r_g^k J \subseteq L$. Hence, $r_g J \subseteq L + J_{gr}(\mathcal{A})$ as $L$ is a $Gr$-$W$-$J_{gr}$-semiprime ideal of $\mathcal{A}$. Thus, $r_g J\mathcal{D} \subseteq L\mathcal{D} + J_{gr}(\mathcal{A})\mathcal{D} \subseteq L\mathcal{D} + J_{gr}(\mathcal{D})$, and so $r_g m_h \in r_g\langle m_h\rangle \subseteq L\mathcal{D} + J_{gr}(\mathcal{D})$. Therefore, $L\mathcal{D}$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$. □

The following example shows that the residual of the $Gr$-$W$-$J_{gr}$-semiprime submodule is not necessarily a $Gr$-$W$-$J_{gr}$-semiprime ideal.

**Example 2.17.** Let $\Gamma = \mathbb{Z}_2$ and $\mathcal{A} = \mathbb{Z}$ be a $\Gamma$-graded ring such that $\mathcal{A}_0 = \mathbb{Z}$ and $\mathcal{A}_1 = \{0\}$. Let $\mathcal{D} = \mathbb{Z}_8$ be a graded $\mathcal{A}$-module such that $\mathcal{D}_0 = \mathbb{Z}_8$ and $\mathcal{D}_1 = \{\bar{0}\}$. Let $U = \{\bar{0}, \bar{4}\} = \langle\bar{4}\rangle \leq_{\Gamma}^{sub} \mathcal{D}$. Thus, it is a $Gr$-$W$- $J_{gr}$-semiprime submodule of $\mathcal{D}$ where $J_{gr}(\mathcal{D}) = \langle\bar{2}\rangle$. However $(U :_{\mathcal{A}} \mathcal{D}) = 4\mathbb{Z}$ is not a $Gr$-$W$-$J_{gr}$-semiprime ideal of $\mathcal{A}$, since $0 \neq 2^2 \cdot 1 \in (U :_{\mathcal{A}} \mathcal{D})$ where $2, 1 \in h(\mathcal{A})$, but $2 \cdot 1 \notin (U :_{\mathcal{A}} \mathcal{D}) + J_{gr}(\mathcal{A}) = 4\mathbb{Z} + (0) = 4\mathbb{Z}$.

The following theorems show that the residual of a $Gr$-$W$-$J_{gr}$-semiprime submodule is a $Gr$-$W$-$J_{gr}$-semiprime ideal with under conditions.

**Theorem 2.18.** *Let $\mathcal{D}$ be a Gr-faithful $\mathcal{A}$-module, and $U \leq_{\Gamma}^{sub} \mathcal{D}$ with $J_{gr}(\mathcal{D}/U) = \{U\}$ and $J_{gr}(\mathcal{A}) \subseteq (U :_{\mathcal{A}} \mathcal{D})$. Then $U$ is a Gr-W-$J_{gr}$-semiprime submodule of $\mathcal{D}$ if and only if $(U :_{\mathcal{A}} \mathcal{D})$ is a Gr-W-$J_{gr}$-semiprime ideal of $\mathcal{A}$.*

*Proof.* ($\Rightarrow$) Let $a_g, b_h \in h(\mathcal{A})$ and $k \in \mathbb{Z}^+$ with $0 \neq a_g^k b_h \in (U :_{\mathcal{A}} \mathcal{D})$. Hence, $\{0\} \neq a_g^k b_h \mathcal{D} \subseteq U$. Since $U$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$, by Corollary 2.7, we have $a_g b_h \mathcal{D} \subseteq U + J_{gr}(\mathcal{D})$. By Theorem 2.9, we have $J_{gr}(\mathcal{D}) \subseteq U$ since $J_{gr}(\mathcal{D}/U) = \{U\}$. This implies that $a_g b_h \mathcal{D} \subseteq U$. Thus, $a_g b_h \in (U :_{\mathcal{A}} \mathcal{D}) \subseteq (U :_{\mathcal{A}} \mathcal{D}) + J_{gr}(\mathcal{A})$. Therefore, $(U :_{\mathcal{A}} \mathcal{D})$ is a $Gr$-$W$-$J_{gr}$-semiprime ideal of $\mathcal{A}$.

($\Leftarrow$) Let $r_g \in h(\mathcal{A})$ and $n \in \mathbb{Z}^+$ with $\{0\} \neq \langle r_g \rangle^n \mathcal{D} \subseteq U$. Hence, $\{0\} \neq \langle r_g \rangle^n \subseteq (U :_{\mathcal{A}} \mathcal{D})$ (if $\langle r_g \rangle^n = \{0\}$, then $\langle r_g \rangle^n \mathcal{D} = \{0\}$ as a contradiction), it follows that $0 \neq r_g^n.1 \in (U :_{\mathcal{A}} \mathcal{D})$. Hence. $r_g \cdot 1 \in (U :_{\mathcal{A}} \mathcal{D}) + J_{gr}(\mathcal{A})$ as $(U :_{\mathcal{A}} \mathcal{D})$ is a $Gr$-$W$-$J_{gr}$-semiprime ideal of $\mathcal{A}$. Since $J_{gr}(\mathcal{A}) \subseteq (U :_{\mathcal{A}} \mathcal{D})$, we have $r_g \in (U :_{\mathcal{A}} \mathcal{D})$, and it follows that $\langle r_g \rangle \subseteq (U :_{\mathcal{A}} \mathcal{D})$. This yields that $\langle r_g \rangle \mathcal{D} \subseteq U \subseteq U + J_{gr}(\mathcal{D})$. Thus¡ $U$ is a $Gr$-$W$ $J_{gr}$-semiprime submodule of $\mathcal{D}$ by Corollary 2.6. □

Recall that a graded $\mathcal{A}$-module $\mathcal{D}$ is said to be a graded cancellation ($Gr$-cancellation) if for any graded ideals $K$ and $L$ of $\mathcal{A}$, $K\mathcal{D} = L\mathcal{D}$, we have $K = L$, see [8].

**Theorem 2.19.** *Let $\mathcal{D}$ be a Gr-finitely generated faithful Gr-multiplication $\mathcal{A}$-module and $U <_{\Gamma}^{sub} \mathcal{D}$. Then $U$ is a Gr-W-$J_{gr}$-semiprime submodule of $\mathcal{D}$ if and only if $(U :_{\mathcal{A}} \mathcal{D})$ is a Gr-W-$J_{gr}$-semiprime ideal of $\mathcal{A}$.*

*Proof.* ($\Rightarrow$) Let $a_g, b_h \in h(\mathcal{A})$ and $k \in \mathbb{Z}^+$ with $0 \neq a_g^k b_h \in (U :_{\mathcal{A}} \mathcal{D})$. Hence, $\{0\} \neq a_g^k b_h \mathcal{D} \subseteq U$. ( if $a_g^k b_h \mathcal{D} = \{0\}$, then $a_g^k b_h = 0$ since $\mathcal{D}$ is a faithful as a contradiction). Since $U$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$, by Corollary 2.7 we get $a_g b_h \mathcal{D} \subseteq U + J_{gr}(\mathcal{D})$. This implies that $a_g b_h \mathcal{D} \subseteq (U :_{\mathcal{A}} \mathcal{D})\mathcal{D} + J_{gr}(\mathcal{D})$ as $\mathcal{D}$ is a $Gr$-multiplication module. Since $J_{gr}(\mathcal{D}) = J_{gr}(\mathcal{A})\mathcal{D}$, we have $a_g b_h \mathcal{D} \subseteq (U :_{\mathcal{A}} \mathcal{D})\mathcal{D} + J_{gr}(\mathcal{A})\mathcal{D} = ((U :_{\mathcal{A}} \mathcal{D}) + J_{gr}(\mathcal{A}))\mathcal{D}$, so $\langle a_g b_h \rangle \mathcal{D} \subseteq ((U :_{\mathcal{A}} \mathcal{D}) + J_{gr}(\mathcal{A}))\mathcal{D}$. Since $\mathcal{D}$ is a $Gr$-finitely generated faithful $Gr$-multiplication by [8, Theorem 2.10], we get $\langle a_g b_h \rangle \subseteq (U :_{\mathcal{A}} \mathcal{D}) + J_{gr}(\mathcal{A})$. Hence, $\langle a_g b_h \rangle \subseteq (U :_{\mathcal{A}} \mathcal{D}) + J_{gr}(\mathcal{A})$, so $a_g b_h \in (U :_{\mathcal{A}} \mathcal{D}) + J_{gr}(\mathcal{A})$. Therefore, $(U :_{\mathcal{A}} \mathcal{D})$ is a $Gr$-$W$-$J_{gr}$-semiprime ideal of $\mathcal{A}$.

($\Leftarrow$) Let $\{0\} \neq r_g^n K \subseteq U$ where $r_g \in h(\mathcal{A})$ and $K \leq_{\Gamma}^{sub} \mathcal{D}$. Since $\mathcal{D}$ is a $Gr$-multiplication $\mathcal{A}$-module, then there exists nonzero $L \leq_{\Gamma}^{id} \mathcal{A}$ with $K = L\mathcal{D}$, and it follows that $\{0\} \neq r_g^n L\mathcal{D} \subseteq U$, hence $\{0\} \neq r_g^n L \subseteq (U :_{\mathcal{A}} \mathcal{D})$. So, $r_g L \subseteq (U :_{\mathcal{A}} \mathcal{D}) + J_{gr}(\mathcal{A})$ as $(U :_{\mathcal{A}} \mathcal{D})$ is a $Gr$-$W$-$J_{gr}$-semiprime ideal of $\mathcal{A}$. Hence, $r_g L\mathcal{D} \subseteq (U :_{\mathcal{A}} \mathcal{D})\mathcal{D} + J_{gr}(\mathcal{A})\mathcal{D} \subseteq (U :_{\mathcal{A}} \mathcal{D})\mathcal{D} + J_{gr}(\mathcal{D})$. This implies that, $r_g K \subseteq U + J_{gr}(\mathcal{D})$ as $\mathcal{D}$ is a $Gr$-multiplication $\mathcal{A}$-module. Thus, $U$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$ by Corollary 2.7. □

**Theorem 2.20.** *Let $\mathcal{D}$ be a Gr-finitely generated faithful Gr-multiplication $\mathcal{A}$-module and $U <_{\Gamma}^{sub} \mathcal{D}$. Then the following statements are equivalent:*

(i) *$U$ is a Gr-W-$J_{gr}$-semiprime submodule of $\mathcal{D}$.*

(ii) *$(U :_{\mathcal{A}} \mathcal{D})$ is a Gr-W-$J_{gr}$-semiprime ideal of $\mathcal{A}$.*

(iii) *$U = L\mathcal{D}$ for some a Gr-W-$J_{gr}$-semiprime ideal $L$ of $\mathcal{A}$.*

*(i)⇒(ii)*. By Theorem 2.19,

[(ii)⇒(iii)] Since $\mathcal{D}$ is a $Gr$-multiplication $\mathcal{A}$-module, $U = (U :_{\mathcal{A}} \mathcal{D})\mathcal{D}$, where $(U :_{\mathcal{A}} \mathcal{D})$ is a $Gr$-$W$-$J_{gr}$-semiprime ideal of $\mathcal{A}$.

[(iii)⇒(i)] Let $U = L\mathcal{D}$ for some $Gr$-$W$-$J_{gr}$-semiprime ideal $L$ of $\mathcal{A}$. Let $\{0\} \neq \langle r_g\rangle^n\mathcal{D} \subseteq U$ where $r_g \in h(\mathcal{A})$ and $n \in \mathbb{Z}^+$, then $\{0\} \neq \langle r_g\rangle^n\mathcal{D} \subseteq L\mathcal{D}$. Since $\mathcal{D}$ is a $Gr$-finitely generated faithful $Gr$-multiplication, by [8, Theorem 2.10], $\mathcal{D}$ is a $Gr$-cancellation. Thus, $\{0\} \neq \langle r_g\rangle^n \subseteq L$. Since $L$ is a $Gr$-$W$-$J_{gr}$-semiprime ideal of $\mathcal{A}$, $\langle r_g\rangle \subseteq L + J_{gr}(\mathcal{A})$, and it follows that $\langle r_g\rangle\mathcal{D} \subseteq L\mathcal{D} + J_{gr}(\mathcal{A})\mathcal{D}$. This yields that $\langle r_g\rangle\mathcal{D} \subseteq U + J_{gr}(\mathcal{D})$ since $J_{gr}(\mathcal{D}) = J_{gr}(\mathcal{A})\mathcal{D}$. Therefore, $U$ is a $Gr$-$W$-$J_{gr}$-semiprime submodule of $\mathcal{D}$. □

## 3. Conclusions

In this paper, we introduced the concept of graded weakly $J_{gr}$-semiprime submodules of a graded module over a commutative graded ring, which is a generalization of graded weakly semiprime submodules. Also, we proved several properties as well as characterizations of graded weakly $J_{gr}$-semiprime submodules. Finally, we established the necessary and sufficient condition for graded submodules to be graded weakly $J_{gr}$-semiprime submodules.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflict of interest.

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