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Research article

Tubular surface generated by a curve lying on a regular surface and its characterizations

A. A. Abdel-Salam[1,2,*], **M. I. Elashiry**[3,4], **and M. Khalifa Saad**[5]

[1] Department of Mathematics, Applied College, Northern Border University, Rafha, Saudi Arabia
[2] Department of Mathematics, Faculty of Science, Sohag University, 82524 Sohag, Egypt
[3] Department of Mathematics, Faculty of Science and Arts, Northern Border University, Rafha, KSA
[4] Department of Mathematics, Faculty of Science, Fayoum University, El-Fayoum, Egypt
[5] Department of Mathematics, Faculty of Science, Islamic University of Madinah, KSA

* **Correspondence:** Email: Assem.mahmoud@nbu.edu.sa.

Abstract: In this research, we have constructed and studied special tubular surfaces in Euclidean 3-space $\mathbb{R}^3$. We examined the singularities and geometrical properties of these surfaces. We achieved some significant results for these surfaces via Darboux frame. Also, we have proposed a few geometric invariants that illustrate the geometric characteristics of these surfaces, such as tubular Weingarten surfaces, using the traditional methods of differential geometry. Additionally, taking advantage of the singularity theory, we have given the classification of generic singularities of these surfaces. At last, we have presented some computational examples as an instance of use to validate our theoretical findings.

Keywords: tubular surfaces; Weingarten surfaces; singularities; Darboux frame
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1. Introduction

The envelope of a moving sphere with variable radius is characterized as a canal surface, which is frequently used in computer-aided design (CAD) and computer-aided geometric design (CAGD) for solid and surface modeling. A canal surface is an envelope of a one-parameter set of spheres centered at the center curve $c(s)$ with radius $r(s)$. The spheres that are specified by the radius function $r(s)$ and the center curve $c(s)$ are combined to form a canal surface, which is obtained by the spine curve $c(s)$. These surfaces have a wide range of uses, including form reconstruction, robot movement planning, the creation of blending surfaces, and the easy sight of long and thin objects like pipes, ropes, poles, and live intestines. The term "tubular surface" refers to these canal surfaces if the radius function $r(s)$ is constant (for more details, see [1–8]).

Tubular surfaces are one of the enormous vital subjects of surface theory. In $\mathbb{R}^3$, a tubular surface is a fundamental and well-known device that is used for geometric construction. Due to this place of tubular surfaces, numerous geometers and designers have explored and acquired numerous properties of tubular surfaces, see for instance [9–12].

In this article, we investigate the geometric conditions for the tubular surfaces to have generic singularities as a front (i.e., cuspidal lips, cuspidal beaks, and Swallowtails). Moreover, we study the tubular Weingarten surfaces which fulfill nontrivial connection between components of the set $\{K, K_{II}, H, H_{II}\}$, where (K, H) and (K_{II}, H_{II}) are Gaussian curvatures.

The paper can be organized as follows: We provide a brief review of the geometry of surfaces, particularly Frenet and Darboux frames related to our study of tubular surfaces in Section 2. In Section 3, we investigate the singularities of tubular surfaces with a Darboux frame and provide some findings from these surfaces. Section 4 provides tubular Weingarten and linear Weingarten surfaces (W-and LW-surfaces) in accordance with a nontrivial functional relation between their curvatures. To enhance our findings and provide a practical demonstration, we include some computational examples in Section 5. These examples not only serve to illustrate our primary results but also feature graphical representations for clarity.

2. Preliminaries

In this part, we show a few ideas, equations, and summaries of curves and surfaces in $\mathbb{R}^3$ which can be tracked down in the course readings on differential geometry, see [1–3]. A curve is regular if it admits a tangent line at each point of the curve. In the following , all curves are assumed to be regular. Let $\alpha(s) : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve in $\mathbb{R}^3$; by $\kappa(s)$ and $\tau(s)$ we denote the natural curvature and torsion of α, respectively. The Frenet equations are:

$$\begin{pmatrix} \boldsymbol{T}'(s) \\ \boldsymbol{N}'(s) \\ \boldsymbol{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T}(s) \\ \boldsymbol{N}(s) \\ \boldsymbol{B}(s) \end{pmatrix}. \tag{2.1}$$

The Darboux frame is an alternative approach to defining a new moving frame constructed on a surface. One can exist on a surface in Euclidean or non-Euclidean spaces [13]. The Darboux frame of $\alpha = \alpha(s)$ is expressed as follows:

$$\begin{pmatrix} T'(s) \\ \boldsymbol{g}'(s) \\ \boldsymbol{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g(s) & \kappa_n(s) \\ -\kappa_g(s) & 0 & \tau_g(s) \\ -\kappa_n(s) & -\tau_g(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ \boldsymbol{g}(s) \\ \boldsymbol{n}(s) \end{pmatrix}, \tag{2.2}$$

and the relation matrix between Serret-Frenet and Darboux frames is given by

$$\begin{pmatrix} T(s) \\ \boldsymbol{g}(s) \\ \boldsymbol{n}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \boldsymbol{T}(s) \\ \boldsymbol{N}(s) \\ \boldsymbol{B}(s) \end{pmatrix}, \tag{2.3}$$

where κ_g is the geodesic curvature, κ_n is the normal curvature, and τ_g is the geodesic torsion of $\alpha(s)$.

They are defined as:

$$\kappa_g = \kappa\cos\vartheta, \kappa_n = \kappa\sin\vartheta, \tau_g = \tau + \frac{d\vartheta}{ds}. \tag{2.4}$$

In addition, κ_g and τ_g can be calculated as follows:

$$\kappa_g = \left\langle \frac{d\alpha}{ds}, \frac{d^2\alpha}{ds^2} \times \boldsymbol{n} \right\rangle, \ \tau_g = \left\langle \frac{d\alpha}{ds}, \mathbf{n} \times \frac{d\boldsymbol{n}}{ds} \right\rangle.$$

Let $\boldsymbol{n}(u, v)$ be the standard unit normal vector field on a surface $\Upsilon = \Upsilon(u, v)$ defined by $\mathbf{n} = \frac{\Upsilon_u \times \mathbf{X}_v}{\|\Upsilon_u \times \Upsilon_v\|}$, where $\Upsilon_i = \frac{\partial \Upsilon}{\partial u_i}$ and $i = 1, 2$. Therefore, the metric (I) of Υ is

$$I = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2,$$

where $g_{11} = \langle \Upsilon_u, \Upsilon_u \rangle$, $g_{12} = \langle \Upsilon_u, \Upsilon_v \rangle$, and $g_{22} = \langle \Upsilon_v, \Upsilon_v \rangle$. Also, the 2^{nd} fundamental form (II) of Υ is

$$II = h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2,$$

where $h_{11} = \langle \Upsilon_{uu}, \boldsymbol{n} \rangle$, $h_{12} = \langle \Upsilon_{uv}, \boldsymbol{n} \rangle$, and $h_{22} = \langle \Upsilon_{vv}, \boldsymbol{n} \rangle$. The Gaussian curvature K, and mean curvature H are respectively, expressed as:

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}, \qquad H = \frac{h_{11}g_{22} - 2g_{12}h_{12} + g_{11}h_{22}}{2\left(g_{11}g_{22} - g_{12}^2\right)}. \tag{2.5}$$

From Brioschi's formula [14, 15], the second Gaussian curvature K_{II} is expressed as

$$K_{II} = \frac{1}{h^2} \left(\begin{vmatrix} -\frac{h_{11,22}}{2} + h_{12,12} - \frac{h_{22,11}}{2} & \frac{h_{11,1}}{2} & h_{12,1} - \frac{h_{11,2}}{2} \\ h_{12,2} - \frac{h_{22,1}}{2} & h_{11} & h_{12} \\ \frac{h_{22,2}}{2} & h_{12} & h_{22} \end{vmatrix} - \begin{vmatrix} 0 & \frac{h_{11,2}}{2} & \frac{h_{22,1}}{2} \\ \frac{h_{11,2}}{2} & h_{11} & h_{12} \\ \frac{h_{22,1}}{2} & h_{12} & h_{22} \end{vmatrix} \right), \tag{2.6}$$

where $h = \det(h_{ij})$, $h_{ij,\alpha} = \frac{\partial h_{ij}}{\partial u^\alpha}$, and $h_{ij,\alpha\beta} = \frac{\partial^2 h_{ij}}{\partial u^\alpha \partial u^\beta}$. Furthermore the second mean curvature H_{II} is

$$H_{II} = H - \frac{1}{2}\Delta\left(\ln \sqrt{|K|}\right), \tag{2.7}$$

where

$$\Delta = -\frac{1}{\sqrt{|h|}} \frac{\partial}{\partial u^i} \left[\sqrt{|h|} h^{ij} \frac{\partial}{\partial u^j} \right]; \ \left(h^{ij}\right) = \left(h_{ij}\right)^{-1}. \tag{2.8}$$

2.1. *Criteria for singularities of fronts*

In this subsection, we will utilize a similar strategy on the peculiarity hypothesis for groups of double smooth capabilities. Nitty gritty depictions are viewed as in the books [16, 17]. Let $U \subset \mathbb{R}^2$ be an open set and $f : (U, p) \to (\mathbb{R}^3, \mathbf{0})$ a map germ. Two map germs f_i: $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ $(i = 1, 2)$, are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $g_1 : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$, and $g_2 : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ such that $f_2 \circ g_1 = g_2 \circ f_1$ holds. A map germ $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is called a (wave) front if there exists a unit vector field ν of $\mathbb{R}^3$ along f such that $\mathfrak{L} = \langle f, \nu \rangle$ is a Legendrian immersion. Since $\mathfrak{L} = \langle f, \nu \rangle$ is Legendrian,

$$\langle df, \nu \rangle = 0, \text{ and } \langle \nu, \nu \rangle = 1, \tag{2.9}$$

hold. For a front f , we define a function $\lambda : U \subset \mathbb{R}^2 \to \mathbb{R}$ by $\lambda(u, v) = \det(f_u, f_v, \nu)$. The function λ is called a discriminant function of f.

We call $p \in U$ a singular point of f if $\operatorname{rank}(df_p) \leq 1$. The set of singular points $S(f)$ of f is the zero set of λ. A singular point $p \in U$ of f is said to be non-degenerate if $d\lambda(p) \neq 0$. Let p be a non-degenerate singular point of a front f . Then $S(f)$ is parameterized by a regular curve $\gamma(t) : (-\varepsilon, \varepsilon) \to U$ near p. Moreover, there exists a non-vanishing vector field η along γ such that $df(\eta(t)) = 0$. This vector field η is called a null vector. For further details, see [11, 12]. Under these notations, we present the criterion for the cuspidal edges, Swallowtails, and cuspidal butterfly as follows:

Proposition 2.1. *Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a front and p a non-degenerate singular point of f. Then we have:*
(1) f is $\mathcal{A}$-equivalent to cuspidal edge CE at p if and only if $\eta\lambda(p) \neq 0$, where $\eta\lambda$ means the directional derivative $D_\eta\lambda$;
(2) f is $\mathcal{A}$-equivalent to Swallowtail SW at p if and only if $\eta\lambda(p) = 0$, and $\eta^2\lambda(p) \neq 0$,
(3) f is $\mathcal{A}$-equivalent to cuspidal butterfly CBF at p if and only if $\eta\lambda(p) = \eta^2\lambda(p) = 0$, and $\eta^3\lambda(p) \neq 0$.

Here,

$$
\begin{aligned}
CE &= \left\{(x_1, x_2, x_3) | x_1 = u,\ x_2 = v^2,\ x_3 = v^3\right\}, \\
SW &= \left\{(x_1, x_2, x_3) | x_1 = u,\ x_2 = 3v^2 + uv^2,\ x_3 = 4v^3 + 2uv\right\}, \\
CBF &= \left\{(x_1, x_2, x_3) | x_1 = 4u^5 + u^2 v,\ x_2 = 5u^4 + 2uv,\ x_3 = v\right\}.
\end{aligned}
\tag{2.10}
$$

These surfaces are shown in the following figures (see Figures 1, 2a and 2b, respectively).

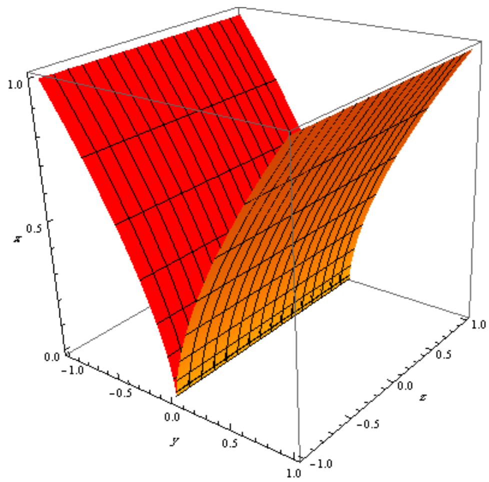
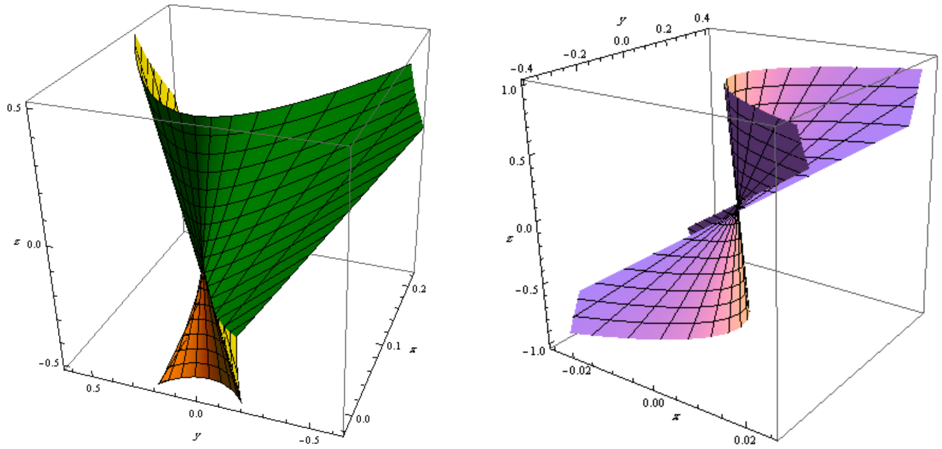


Figure 1. The cuspidal edge (CE) surface.

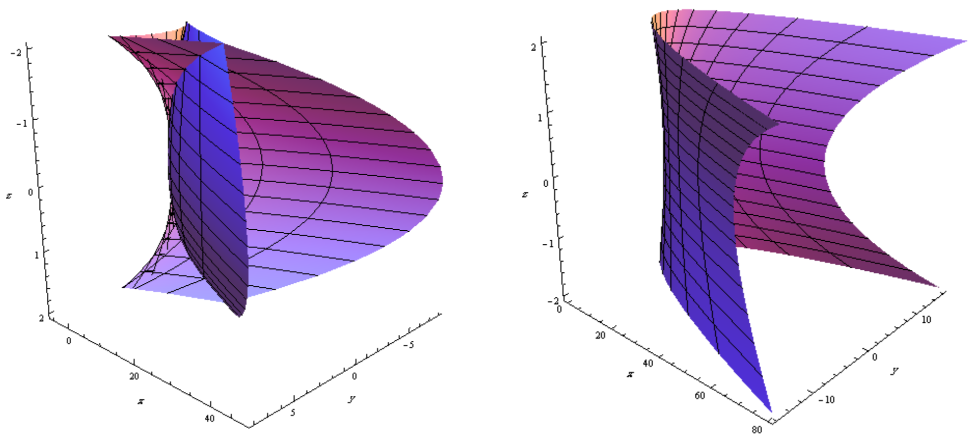


(a) (b)

Figure 2. (a) The Swallowtail (SW) surface; (b) the cuspidal butterfly (CBF) surface.

Now, we turn to degenerate singularities. Let p be a degenerate singular point of the front f. If $\text{rank}(df_p) = 1$, then there exists η near p; if $q \in S(f)$, then $df_q(\eta(t)) = 0$. Criteria for degenerate singularities are as follows:

Proposition 2.2. *Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a front and p a degenerate singular point of f. Then we have the following:*
(1) f is $\mathcal{A}$-equivalent to cuspidal lips CLP if and only if $\text{rank}(df_p) = 1$, and the $\det(\mathcal{H}\lambda(p)) > 0$, where $\det(\mathcal{H}\lambda(p))$ denotes the determinant of the Hessian matrix of λ at p (see Figure 3b);
(2) f is $\mathcal{A}$-equivalent to cuspidal beaks CBK if and only if $\text{rank}(df_p) = 1$, $\det(\mathcal{H}\lambda(p)) < 0$, and $\eta^2\lambda(p) \neq 0$ (see Figure 3a).



(a) (b)

Figure 3. (a) The cuspidal beaks CBK surface; (b) the cuspidal lips CLP surface.

Here, for a function $\lambda : (U, u, v) \to \mathbb{R}$, $\mathcal{H}\lambda$ is the matrix defined by

$$\mathcal{H}\lambda(u, v) = \begin{pmatrix} \lambda_{uu} & \lambda_{uv} \\ \lambda_{uv} & \lambda_{vv} \end{pmatrix}, \tag{2.11}$$

and

$$CLP = \left\{(x_1, x_2, x_3) | x_1 = 3u^4 + 2u^2v^2,\ x_2 = u^3 + uv^2,\ x_3 = v, \right\},$$
$$CBK = \left\{(x_1, x_2, x_3) | x_1 = 3u^4 - 2u^2v^2,\ x_2 = u^3 - uv^2,\ x_3 = v\right\}.$$

3. Singularities of tubular surfaces

Here, we study the singularity of a tubular surface, and give the conditions for this surface to be *CE, SW, CBF, CLP,* and *CBK* singularities in terms of $\kappa_g(s), \kappa_n(s)$, and $\tau_g(s)$. By utilizing the Darboux frame, the tubular surfaces of radius $r > 0$ about the $\beta(s) = \int \boldsymbol{T}(s)\, ds$ is the surface with parametrization

$$\Upsilon : \mathbf{Q}(s, \varphi) = \beta(s) + r(\cos\varphi \boldsymbol{g} + \sin\varphi \boldsymbol{n}). \tag{3.1}$$

3.1. Local singularities

Singularties are essential for understanding the properties of tubular surfaces. So, after simple calculations, we have

$$\begin{cases} \mathbf{Q}_s(s, \varphi) = \left(1 - r\,\kappa_g \cos\varphi - r\,\kappa_n \sin\varphi\right)\boldsymbol{T} - r\,\tau_g \sin\varphi\, \boldsymbol{g} + r\,\tau_g \cos\varphi\, \boldsymbol{n}, \\ \mathbf{Q}_\varphi(s, \varphi) = -r(\sin\varphi\, \boldsymbol{g} - \cos\varphi\, \boldsymbol{n}). \end{cases} \tag{3.2}$$

From Eq (3.2), we can show that Υ has a singularity at $\mathbf{Q}(s, \varphi)$ if and only if

$$\left\|\mathbf{Q}_s \times \mathbf{Q}_\varphi\right\| = 1 - r\kappa_g \cos\varphi - r\kappa_n \sin\varphi = 0.$$

This is equivalent to

$$\lambda(s, \varphi) = r\kappa_g \cos\varphi - r\kappa_n \sin\varphi - 1.$$

Therefore, the Hessian matrix of $\lambda(s, \varphi)$ at a singular point is given by

$$\mathcal{H}\lambda(s, \varphi) = \begin{pmatrix} r\,\kappa_g'' \cos\varphi - r\,\kappa_n'' \sin\varphi & -r\,\kappa_g' \sin\varphi - r\,\kappa_n' \cos\varphi \\ -r\,\kappa_g' \sin\varphi - r\,\kappa_n' \cos\varphi & -r\,\kappa_g \cos\varphi + r\,\kappa_n \sin\varphi \end{pmatrix},$$

and

$$\begin{aligned} \det(\mathcal{H}\lambda(p)) = &- r^2 \cos^2\varphi(\kappa_g\kappa_g'' + \kappa_n'^2) + r^2 \cos\varphi \sin\varphi(\kappa_n\kappa_g'' - 2\kappa_g'\kappa_n' + \kappa_g\kappa_n'') \\ &- r^2 \sin^2\varphi(\kappa_n\kappa_n'' + \kappa_g'^2). \end{aligned}$$

Now it is easy to prove the following lemma.

Lemma 3.1. *The tubular surface Υ parameterized by Eq (3.1) is a front.*

Lemma 3.2. *Let Υ be the tubular surface parameterized by Eq (3.1). Then rank$(dQ_p) \leq 1$ at p if and only if*

$$\kappa_g \cos\varphi + \kappa_n \sin\varphi = \frac{1}{r}, \tag{3.3}$$

is satisfied.

Proof. We suppose that a tubular surface Υ parameterized by Eq (3.1) has singularity at p. Then it satisfies the Eq (3.3). Conversely, if Eq (3.3) holds, in view of Eq (3.2), then

$$\mathbf{Q}_s(s,\varphi) = \tau_g \mathbf{Q}_\varphi(s,\varphi), \tag{3.4}$$

which means that rank$(dQ_p) \leq 1$. This completes the proof.

Proposition 3.1. *Let Υ be a tubular surface parameterized by Eq (3.1). If p is a non-degenerate singular point for Υ, then $d\lambda(p) \neq 0$ if and only if*

$$\frac{\kappa_g^{'}}{\kappa_n^{'}} \neq \tan\varphi, \tag{3.5}$$

or

$$\frac{\kappa_g}{\kappa_n} \neq -\cot\varphi, \tag{3.6}$$

is satisfied.

Proof. Suppose that Υ be the tubular surface parameterized by Eq (3.1). Then, it has singularity at p if and only if Eq (3.3) holds. Conversely, if Eq (3.3) holds, it is clear that:

$$d\lambda(p) = r\left(\left(\kappa_g^{'}\cos\varphi - \kappa_n^{'}\sin\varphi\right)ds + \left(-\kappa_g\sin\varphi - \kappa_n\cos\varphi\right)d\varphi\right), \tag{3.7}$$

which means that p is a non-degenerate singular point $d\lambda(p) \neq 0$ if and only if $\kappa_g^{'}\cos\varphi - \kappa_n^{'}\sin\varphi \neq 0$ or $\kappa_g\sin\varphi + \kappa_n\cos\varphi \neq 0$. Hence, the proof is completed.

So, we find the following result.

Corollary 3.1. *Let p be a degenerate singular point for Υ. Then we have $d\lambda(p) = 0$ if and only if $\kappa_g^{'}/\kappa_n^{'} = \tan\varphi$, and $\kappa_g/\kappa_n = -\cot\varphi$.*

Now we are ready to state our main theorems:

Theorem 3.1. *Let p be a non-degenerate singular point for Υ. Then*
(1) Υ is $\mathcal{A}$-equivalent to the CE at p if and only if $\eta\lambda(p) \neq 0$, that is,

$$\left(\kappa_g\tau_g - \kappa_n^{'}\right)\sin\varphi + \left(\kappa_n\tau_g + \kappa_g^{'}\right)\cos\varphi \neq 0. \tag{3.8}$$

(2) Υ is $\mathcal{A}$-equivalent to the SW at p if and only if $\eta\lambda(p) = 0$, and $\eta^2\lambda(p) \neq 0$, that is,

$$\left(\kappa_g\tau_g - \kappa_n^{'}\right)\sin\varphi + \left(\kappa_n\tau_g + \kappa_g^{'}\right)\cos\varphi = 0,$$

and

$$\left(2\kappa_g^{'}\tau_g + \kappa_g\tau_g^{'} - \kappa_n^{''} + \kappa_n\tau_g^2\right)\sin\varphi + \left(2\kappa_n^{'}\tau_g + \kappa_n\tau_g^{'} + \kappa_g^{''} - \kappa_g\tau_g^2\right)\cos\varphi \neq 0. \tag{3.9}$$

(3) Υ *is $\mathcal{A}$-equivalent to the CBF at p if and only if $\eta\lambda(p) = \eta^2\lambda(p) = 0$, and $\eta^3\lambda(p) \neq 0$, that is,*

$$\left(\kappa_g\tau_g - \kappa_n^{'}\right)\sin\varphi + \left(\kappa_n\tau_g + \kappa_g^{'}\right)\cos\varphi = 0,$$
$$\left(2\kappa_g^{'}\tau_g + \kappa_g\tau_g^{'} - \kappa_n^{''} + \kappa_n\tau_g^2\right)\sin\varphi + \left(2\kappa_n^{'}\tau_g + \kappa_n\tau_g^{'} + \kappa_g^{''} - \kappa_g\tau_g^2\right)\cos\varphi = 0,$$

and

$$\begin{aligned}&\left(3\kappa_g^{''}\tau_g + 3\kappa_g^{'}\tau_g^{'} + \kappa_g\tau_g^{''} - \kappa_n^{'''} + 3\kappa_n^{'}\tau_g^2 + 3\kappa_n\tau_g\tau_g^{'} - \kappa_g\tau_g^3\right)\sin\varphi \\ &+ \left(3\kappa_n^{''}\tau_g + 3\kappa_n^{'}\tau_g^{'} + \kappa_n\tau_g^{''} + \kappa_g^{'''} - 3\kappa_g^{'}\tau_g^2 - 3\kappa_g\tau_g\tau_g^{'} - \kappa_n\tau_g^3\right)\cos\varphi \neq 0.\end{aligned} \tag{3.10}$$

Proof. (1) Since p is a singular point of Υ, we have

$$\lambda(p) = r\kappa_g\cos\varphi - r\kappa_n\sin\varphi - 1.$$

Because p is a non-degenerate singular point, the null vector filed η is defined as

$$\eta = \frac{\partial}{\partial s} - \tau_g\frac{\partial}{\partial\varphi}.$$

Therefore,

$$\eta\lambda(p) = r\left(\left(\kappa_g^{'}\cos\varphi - \kappa_n^{'}\sin\varphi\right) + \left(\kappa_g\tau_g\sin\varphi + \kappa_n\tau_g\cos\varphi\right)\right).$$

So, we get: $\eta\lambda(p) \neq 0$ if and only if

$$\left(\kappa_g\tau_g - \kappa_n^{'}\right)\sin\varphi + \left(\kappa_n\tau_g + \kappa_g^{'}\right)\cos\varphi \neq 0.$$

(2) Similarly, we have:

$$\eta^2\lambda(p) = r\left[\left(2\kappa_g^{'}\tau_g + \kappa_g\tau_g^{'} - \kappa_n^{''} + \kappa_n\tau_g^2\right)\sin\varphi + \left(2\kappa_n^{'}\tau_g + \kappa_n\tau_g^{'} + \kappa_g^{''} - \kappa_g\tau_g^2\right)\cos\varphi\right].$$

By using Case (1), we have: $\eta\lambda(p) = 0$, and $\eta^2\lambda(p) \neq 0$ if and only if

$$\left(\kappa_g\tau_g - \kappa_n^{'}\right)\sin\varphi + \left(\kappa_n\tau_g + \kappa_g^{'}\right)\cos\varphi = 0,$$

and

$$\left(2\kappa_g^{'}\tau_g + \kappa_g\tau_g^{'} - \kappa_n^{''} + \kappa_n\tau_g^2\right)\sin\varphi + \left(2\kappa_n^{'}\tau_g + \kappa_n\tau_g^{'} + \kappa_g^{''} - \kappa_g\tau_g^2\right)\cos\varphi \neq 0.$$

(3) By a similar procedure as in Case (1) and Case (2), we have $\eta\lambda(p) = \eta^2\lambda(p) = 0$, and $\eta^3\lambda(p) \neq 0$, if and only if

$$\left(\kappa_g\tau_g - \kappa_n^{'}\right)\sin\varphi + \left(\kappa_n\tau_g + \kappa_g^{'}\right)\cos\varphi = 0,$$
$$\left(2\kappa_g^{'}\tau_g + \kappa_g\tau_g^{'} - \kappa_n^{''} + \kappa_n\tau_g^2\right)\sin\varphi + \left(2\kappa_n^{'}\tau_g + \kappa_n\tau_g^{'} + \kappa_g^{''} - \kappa_g\tau_g^2\right)\cos\varphi = 0,$$

and

$$\begin{aligned}&\left(3\kappa_g^{''}\tau_g + 3\kappa_g^{'}\tau_g^{'} + \kappa_g\tau_g^{''} - \kappa_n^{'''} + 3\kappa_n^{'}\tau_g^2 + 3\kappa_n\tau_g\tau_g^{'} - \kappa_g\tau_g^3\right)\sin\varphi \\ &+ \left(3\kappa_n^{''}\tau_g + 3\kappa_n^{'}\tau_g^{'} + \kappa_n\tau_g^{''} + \kappa_g^{'''} - 3\kappa_g^{'}\tau_g^2 - 3\kappa_g\tau_g\tau_g^{'} - \kappa_n\tau_g^3\right)\cos\varphi \neq 0.\end{aligned}$$

Therefore, using Proposition 1, the proof is complete.

Theorem 3.2. *Let Υ be a tubular surface parameterized by Eq (3.1), and p is a degenerate singular point. Then, one has the followings:*
(1) Υ is $\mathcal{A}$-equivalent to CLP if and only if rank$(df_p) = 1$, and

$$\cos^2\varphi(\kappa_g\kappa_g'' + \kappa_n'^2) - \cos\varphi\sin\varphi(\kappa_n\kappa_g'' - 2\kappa_g'\kappa_n' + \kappa_g\kappa_n'') + \sin^2\varphi(\kappa_n\kappa_n'' + \kappa_g'^2) < 0. \tag{3.11}$$

(2) Υ is $\mathcal{A}$-equivalent to CBK if and only if rank$(df_p) = 1$,

$$\cos^2\varphi(\kappa_g\kappa_g'' + \kappa_n'^2) - \cos\varphi\sin\varphi(\kappa_n\kappa_g'' - 2\kappa_g'\kappa_n' + \kappa_g\kappa_n'') + \sin^2\varphi(\kappa_n\kappa_n'' + \kappa_g'^2) > 0, \tag{3.12}$$

and

$$\begin{aligned}&\left(3\kappa_g''\tau_g + 3\kappa_g'\tau_g' + \kappa_g\tau_g'' - \kappa_n''' + 3\kappa_n'\tau_g^2 + 3\kappa_n\tau_g\tau_g' - \kappa_g\tau_g^3\right)\sin\varphi\\&+\left(3\kappa_n''\tau_g + 3\kappa_n'\tau_g' + \kappa_n\tau_g'' + \kappa_g''' - 3\kappa_g'\tau_g^2 - 3\kappa_g\tau_g\tau_g' - \kappa_n\tau_g^3\right)\cos\varphi \neq 0.\end{aligned} \tag{3.13}$$

Proof. Let p be a degenerate singular point of Υ, then Eqs (3.7) and (3.8) are hold. Therefore, using Proposition 2, the proof is complete.

3.2. The relation among the curvature functions

Presently, we concentrate on tubular surfaces fulfilling a few conditions concerning their curvatures as follows:

According to Eqs (3.2), we find

$$g_{11} = \left(1 - r\,\kappa_g\cos\varphi - r\,\kappa_n\sin\varphi\right)^2 + r^2\tau_g^2,\ g_{12} = r^2\tau_g,\ \text{and } g_{22} = r^2. \tag{3.14}$$

The normal vector of $\mathbf{Q}$ is

$$\boldsymbol{N}(s,\varphi) = \frac{\mathbf{Q}_s \times \mathbf{Q}_\varphi}{\left\|\mathbf{Q}_s \times \mathbf{Q}_\varphi\right\|} = \cos\varphi\boldsymbol{g} + \sin\varphi\boldsymbol{n}. \tag{3.15}$$

By a straightforward calculation, we get

$$\begin{aligned}\mathbf{Q}_{ss} =& \, r\left(\left(\kappa_g\tau_g - \kappa_n'\right)\sin\varphi - \left(\kappa_n\tau_g + \kappa_g'\right)\cos\varphi\right)\boldsymbol{T}\\&+\left(\kappa_g - r\left(\kappa_g\kappa_n + \tau_g'\right)\sin\varphi - r\left(\kappa_g^2 + \tau_g^2\right)\cos\varphi\right)\boldsymbol{g},\\&+\left(\kappa_n - r\left(\kappa_n^2 + \tau_g^2\right)\sin\varphi - r\left(\kappa_g\kappa_n - \tau_g'\right)\cos\varphi\right)\boldsymbol{n},\\\mathbf{Q}_{s\varphi} =& \, r\left(\kappa_g\sin\varphi - \kappa_n\cos\varphi\right)\,\boldsymbol{T} - r\,\tau_g\cos\varphi\,\boldsymbol{g} - r\,\tau_g\sin\varphi\,\boldsymbol{n},\\\mathbf{Q}_{\varphi\varphi} =& - r\cos\varphi\,\boldsymbol{g} - r\sin\varphi\,\boldsymbol{n}.\end{aligned}$$

This prompts the components of (II); h_{11}, h_{12}, and h_{22} as follows:

$$\begin{aligned}&h_{11} = \kappa_g\cos\varphi + \kappa_n\sin\varphi - 2r\kappa_g\kappa_n\sin\varphi\cos\varphi - r\kappa_g^2\cos^2\varphi - r\kappa_n^2\sin^2\varphi - r\tau_g^2,\\&h_{12} = -\,r\,\tau_g,\ h_{22} = -r.\end{aligned} \tag{3.16}$$

Therefore, we get

$$\begin{cases} K = \dfrac{-\kappa_g \cos\varphi - \kappa_n \sin\varphi + 2r\kappa_g\kappa_n \sin\varphi\cos\varphi - r(\kappa_g^2\cos^2\varphi + \kappa_n^2\sin^2\varphi)}{r\mu}, \\ H = \dfrac{\kappa_g \cos\varphi + \kappa_n \sin\varphi - r\kappa_g\kappa_n \sin\varphi\cos\varphi - r\kappa_g^2\cos^2\varphi - r\kappa_n^2\sin^2\varphi - \frac{\mu}{r}}{2\mu}, \end{cases} \tag{3.17}$$

where $\mu = \left(1 - r\,\kappa_g \cos\varphi - r\,\kappa_n \sin\varphi\right)^2$. And from Eq (3.16), we get

$$\begin{cases} h_{11,2} = -\kappa_g \sin\varphi + \kappa_n \cos\varphi - 2r\kappa_g\kappa_n \cos^2\varphi + 2r\kappa_g\kappa_n \sin^2\varphi + 2r(\kappa_g^2 - \kappa_n^2)\sin\varphi\cos\varphi, \\ h_{11,22} = -\kappa_g \cos\varphi - \kappa_n \sin\varphi + 8r\kappa_g\kappa_n \sin\varphi\cos\varphi + 2r(\kappa_g^2 - \kappa_n^2)(\cos^2\varphi - \sin^2\varphi), \\ h_{11,1} = \kappa_g' \cos\varphi + \kappa_n' \sin\varphi - 2r\left(\kappa_g'\kappa_n + \kappa_g\kappa_n'\right)\sin\varphi\cos\varphi \\ \qquad -2r\left(\kappa_g\kappa_g'\cos^2\varphi + \kappa_n\kappa_n'\sin^2\varphi + \tau_g\tau_g'\right), \\ h_{12,1} = -r\tau_g', \; h_{12,2} = h_{22,1} = h_{22,2} = h_{22,11} = h_{12,12} = 0. \end{cases} \tag{3.18}$$

From Eqs (2.6), (3.16), and (3.18), we find

$$K_{II} = \frac{-1}{4h^2}\left(-2hh_{11,22} - rh_{11,2}^2\right), \tag{3.19}$$

where

$$h = -r\left(\kappa_g \cos\varphi + \kappa_n \sin\varphi - 2r\kappa_g\kappa_n \sin\varphi\cos\varphi - r\kappa_g^2\cos^2\varphi - r\kappa_n^2\sin^2\varphi\right),$$

Also, we have:

$$\begin{aligned} H_{II} = {} & \frac{\kappa_g \cos\varphi + \kappa_n \sin\varphi - r\kappa_g\kappa_n \sin\varphi\cos\varphi - r\kappa_g^2\cos^2\varphi - r\kappa_n^2\sin^2\varphi - \frac{\mu}{r}}{2\mu} \\ & - \frac{1}{2}\Delta\left(\ln\sqrt{\left|\frac{-\kappa_g \cos\varphi - \kappa_n \sin\varphi + 2r\kappa_g\kappa_n \sin\varphi\cos\varphi - r(\kappa_g^2\cos^2\varphi + \kappa_n^2\sin^2\varphi)}{r\mu}\right|}\right). \end{aligned} \tag{3.20}$$

From Eqs (3.17), (3.19), and (3.20), we obtain the result:

Corollary 3.2. *The Gaussian curvatures of* $\boldsymbol{\Upsilon}$ *are*

$$K = \frac{-\kappa_g \cos\varphi - \kappa_n \sin\varphi + r\kappa_g\kappa_n \sin\varphi\cos\varphi + r\kappa_g^2}{r\mu},$$
$$H = \frac{\kappa_g \cos\varphi + \kappa_n \sin\varphi - r\kappa_g\kappa_n \sin\varphi\cos\varphi - r\kappa_g^2 - \frac{\mu}{r}}{2\mu},$$

and

$$K_{II} = \frac{2h\left(\kappa_g \cos\varphi + \kappa_n \sin\varphi - 4r\kappa_g\kappa_n \sin\varphi\cos\varphi\right) - \delta^2}{4h^2},$$

$$\begin{aligned} H_{II} = {} & \frac{\kappa_g \cos\varphi + \kappa_n \sin\varphi - r\kappa_g\kappa_n \sin\varphi\cos\varphi - r\kappa_g^2 - \frac{\mu}{r}}{2\mu} \\ & -\tfrac{1}{2}\Delta\left(\ln\sqrt{\left|\frac{-\kappa_g \cos\varphi - \kappa_n \sin\varphi + r\kappa_g\kappa_n \sin\varphi\cos\varphi + r\kappa_g^2}{r\mu}\right|}\right). \end{aligned}$$

4. Tubular LW-surfaces

Now, for a tubular LW-surfaces Υ, an extension of Eq (3.1) for a nontrivial functional relation between a pair $\{A, B\}$, $A \neq B$, of the curvatures K, K_{II}, H, and H_{II} are studied. Thus, by using Eqs (3.17), (3.19) and (3.20), one can get the differentiation of K, K_{II}, H, and H_{II} concerning s and φ. In any case, the upsides of these estimations are long to such an extent that we can overlook them. In this manner, we have the accompanying cases:

(i) $f(K, H) = (K)_s (H)_\varphi - (K)_\varphi (H)_s = 0$,
(ii) $f(K, K_{II}) = (K)_s (K_{II})_\varphi - (K)_\varphi (K_{II})_s = 0$,
(iii) $f(H, K_{II}) = (H)_s (K_{II})_\varphi - (H)_\varphi (K_{II})_s = 0$,
(iv) $f(H, H_{II}) = (H)_s (H_{II})_\varphi - (H)_\varphi (H_{II})_s = 0$,
(v) $f(K_{II}, H_{II}) = (K_{II})_s (H_{II})_\varphi - (K_{II})_\varphi (H_{II})_s = 0$.

From the primary case, one can see that it has evaporated indistinguishably. Hence we have

Corollary 4.1. *The tubular surface Υ is a W-surface.*

From the second and third cases, one can get the two Jacobian equations, and we conclude that $\kappa_g' = \kappa_n' = 0$, which leads to $\kappa_g = \kappa_n = constant$. Consequently, we obtain the following result.

Theorem 4.1. *The tubular surface Υ is a W-surface generated by a circle.*

Similarly, from the fourth and fifth cases, the two Jacobian equations are split to sixteen conditions and satisfied when $\kappa_g = \tau_g = constant$. Subsequently, we find the following theorem.

Theorem 4.2. *The tubular surface M is a W-surface generated by a circular helix α with non-zero constant curvatures (see Figure 4).*

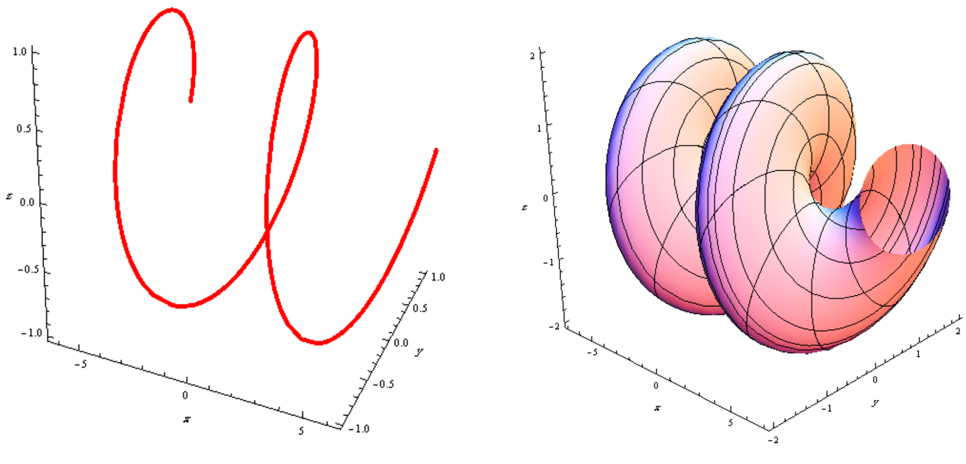


Figure 4. (a) The cylindrical helix $\alpha(s)$; (b) the tubular surface Υ_1 along $\alpha(s)$.

Finally, one can see that the following linear relations hold:

Theorem 4.3. *For a tubular surface, the following hold:*
(i) $aK + bH = c$, *where* $a + c\lambda^2 \neq 0$ *and* $\kappa_g = \kappa_n = 0$,
(ii) $aK + bK_{II} = c$, *where* $b = c = 0$ *and* $\kappa_g = \kappa_n = 0$,
(iii) $aH + bK_{II} = c$, *where* $b = 0$, *and* $\kappa = 0$,
(iv) $aH + bH_{II} = c$, *where* $a + b + c\lambda \neq 0$, $\tau_g \neq 0$, *and* $\kappa_g = \kappa_n = 0$,
(v) $aH_{II} + bK_{II} = c$, *where* $a + b + c\lambda \neq 0$, $\tau_g \neq 0$, *and* $\kappa_g = \kappa_n = 0$.

Here, a, b, and c are non-zero arbitrary constants.
As a result, we give the following corollary:

Corollary 4.2. *The tubular surface M is an open part of a circular cylinder.*

5. Applications

Now, we will introduce two computational examples for constructing tubular surfaces to support our main results.

Example 5.1. *Consider the regular surface parameterized by*

$$S_1(u, v) = (uv, v\cos(u), v\sin(u)).$$

Darboux frame vectors of the curve $\alpha(s) = (s, \cos(s), \sin(s))$, *which lies on the regular surface* S_1, *are*

$$\begin{aligned}
\boldsymbol{T}(s) &= \left(\frac{1}{\sqrt{2}}, -\frac{\sin(s)}{\sqrt{2}}, \frac{\cos(s)}{\sqrt{2}}\right),\\
\boldsymbol{g}(s) &= \left(\frac{s}{\sqrt{2}\sqrt{2+s^2}}, \frac{2\cos(s) + s\sin(s)}{\sqrt{2}\sqrt{2+s^2}}, \frac{-s\cos(s) + 2\sin(s)}{\sqrt{2}\sqrt{2+s^2}}\right),\\
\boldsymbol{n}(s) &= \left(\frac{-\cos^2(s) - \sin^2(s)}{\sqrt{2+s^2}}, \frac{s\cos(s) - \sin(s)}{\sqrt{2+s^2}}, \frac{\cos(s) + s\sin(s)}{\sqrt{2+s^2}}\right),
\end{aligned}$$

Also, we have

$$\begin{aligned}
\kappa_g &= \frac{-2}{\sqrt{2+s^2}},\\
\kappa_n &= \frac{-s}{\sqrt{2+s^2}},\\
\tau_g &= \frac{s^2}{2+s^2}.
\end{aligned}$$

Thus, according to Eq (3.1), the constructed tubular surface $\boldsymbol{\Upsilon}_1(s, v)$ *associated with the Darboux* *frame of radius $r > 0$ along $\alpha(s)$ is parametrized by (see Figure 4b):*

$$\Upsilon_1 = \left\{ \begin{array}{c} s + \frac{\cos(v)(2\cos(s) - \cos(v)\sin(s))}{\sqrt{9+\cos(2v)}}, \\ \cos(s) - \frac{\cos(v)^2}{\sqrt{9+\cos(2v)}} + \frac{2\sin(v)}{\sqrt{4+\cos(v)^2}}, \\ -\frac{2\cos(v)}{\sqrt{9+\cos(2v)}} + \sin(s) - \frac{\cos(v)\sin(v)}{\sqrt{4+\cos(v)^2}} \end{array} \right\}.$$

For Υ_1, we obtain

$$g_{11} = \frac{1}{4(2+s^2)^2} \left(\begin{array}{c} 36 + 36s^2 + 11s^4 + 8\sqrt{2}\left(2+s^2\right)^{3/2}\cos(v) + 4\cos(2v) \\ -s^4\cos(2v) + 8s\left(2+s^2\right)^{3/2}\sin(v) \\ +4\sqrt{2}s\sin(2v) + 2\sqrt{2}s^3\sin(2v) \end{array} \right),$$

$$g_{12} = \frac{s^2}{\sqrt{2}(2+s^2)}, \ g_{22} = 1,$$

$$h_{11} = \frac{1}{\eta_1} \left(\begin{array}{c} 64\sqrt{2} + 96\sqrt{2}s^2 + 64\sqrt{2}s^4 + 16\sqrt{2}s^6 + 2\sqrt{2+s^2}\left(76 + 76s^2 + 23s^4\right)\cos(v) \\ -8\sqrt{2}\left(-2+s^2\right)\left(2+s^2\right)^2\cos(2v) + 8\sqrt{2+s^2}\cos(3v) \\ -8s^2\sqrt{2+s^2}\cos(3v) - 6s^4\sqrt{2+s^2}\cos(3v) \\ +76\sqrt{2}s\sqrt{2+s^2}\sin(v) + 76\sqrt{2}s^3\sqrt{2+s^2}\sin(v) \\ +23\sqrt{2}s^5\sqrt{2+s^2}\sin(v) + 128s\sin(2v) + 128s^3\sin(2v) \\ +32s^5\sin(2v) + 12\sqrt{2}s\sqrt{2+s^2}\sin(3v) \\ +4\sqrt{2}s^3\sqrt{2+s^2}\sin(3v) - \sqrt{2}s^5\sqrt{2+s^2}\sin(3v) \end{array} \right);$$

$$\eta_1 = \left(8\left(2+s^2\right)^{5/2} \sqrt{ \begin{array}{c} 18 + 9s^2 + 8\sqrt{2}\sqrt{2+s^2}\cos(v) \\ -\left(-2+s^2\right)\cos(2v) + 8s\sqrt{2+s^2}\sin(v) + 2\sqrt{2}s\sin(2v) \end{array} } \right),$$

$$h_{12} = \frac{s^2\left(4 + 2s^2 + \sqrt{2}\sqrt{2+s^2}\cos(v) + s\sqrt{2+s^2}\sin(v)\right)}{(2+s^2)^{3/2} \sqrt{ \begin{array}{c} 18 + 9s^2 + 8\sqrt{2}\sqrt{2+s^2}\cos(v) \\ -\left(-2+s^2\right)\cos(2v) + 8s\sqrt{2+s^2}\sin(v) + 2\sqrt{2}s\sin(2v) \end{array} }},$$

$$h_{22} = \frac{2\sqrt{2+s^2}\cos(v) + \sqrt{2}\left(4 + 2s^2 + s\sqrt{2+s^2}\sin(v)\right)}{\sqrt{2+s^2} \sqrt{ \begin{array}{c} 18 + 9s^2 + 8\sqrt{2}\sqrt{2+s^2}\cos(v) \\ -\left(-2+s^2\right)\cos(2v) + 8s\sqrt{2+s^2}\sin(v) + 2\sqrt{2}s\sin(2v) \end{array} }}.$$

Moreover, we obtain

$$K^{\Upsilon_1}(s,1) = \frac{-1}{\eta_2}\begin{pmatrix} -51\left(2+s^2\right)^2 - 100\sqrt{2}\left(2+s^2\right)^{3/2} + 52\left(-4+s^4\right) \\ -24\sqrt{2}\sqrt{2+s^2} + 36\sqrt{2}s^2\sqrt{2+s^2} \\ -\left(4-12s^2+s^4\right) - 200s\sqrt{2+s^2} \\ -100s^3\sqrt{2+s^2} - 104\sqrt{2}s\left(2+s^2\right) \\ -72s\sqrt{2+s^2} + 12s^3\sqrt{2+s^2} \\ +4\sqrt{2}s\left(-2+s^2\right) \end{pmatrix};$$

$$\eta_2 = \begin{pmatrix} 227\left(2+s^2\right)^2 + 304\sqrt{2}\left(2+s^2\right)^{3/2} \\ -100\left(-4+s^4\right) + 32\sqrt{2}\sqrt{2+s^2} + 4\ ] \\ +s\begin{pmatrix} -48\sqrt{2}s\sqrt{2+s^2} + s\left(-12+s^2\right) \\ +4\begin{pmatrix} 76\left(2+s^2\right)^{3/2} + 50\sqrt{2}\left(2+s^2\right) \\ +24\sqrt{2+s^2} - 4s^2\sqrt{2+s^2} \\ -\sqrt{2}\left(-2+s^2\right) \end{pmatrix} \end{pmatrix} \end{pmatrix},$$

$$H^{\Upsilon_1}(s,1) = \frac{1}{\eta_3}\begin{pmatrix} 26\sqrt{2}\left(2+s^2\right)^2 + 70\left(2+s^2\right)^{3/2} \\ -10\sqrt{2}\left(-4+s^4\right) + 4\sqrt{2+s^2} \\ +s\begin{pmatrix} -6s\sqrt{2+s^2} + 35\sqrt{2}\left(2+s^2\right)^{3/2} \\ +40\left(2+s^2\right) + 6\sqrt{2}\sqrt{2+s^2} \\ -\sqrt{2}s^2\sqrt{2+s^2} \end{pmatrix} \end{pmatrix};$$

$$\eta_3 = 2\sqrt{2+s^2}\begin{pmatrix} 18+9s^2-\left(-2+s^2\right) \\ +8s\sqrt{2+s^2} \\ +4\sqrt{2}\left(2\sqrt{2+s^2}+s\right) \end{pmatrix}^{3/2}.$$

Example 5.2. *Let us consider the regular surface parameterized by*

$$S_2(u,v) = \left(1+v\cos(u), v\sin(u), 2v\sin\left(\frac{u}{2}\right)\right).$$

Darboux frame vectors of the curve $\beta(s) = \left(1+\cos(s), \sin(s), 2\sin\left(\frac{s}{2}\right)\right)$*, which lies on the regular surface* S_2 *are:*

$$\boldsymbol{T}(s) = \left(-\frac{\sqrt{2}\sin(s)}{\sqrt{3+\cos(s)}}, \frac{\sqrt{2}\cos(s)}{\sqrt{3+\cos(s)}}, \frac{\sqrt{2}\cos\left(\frac{s}{2}\right)}{\sqrt{3+\cos(s)}}\right),$$

$$\boldsymbol{g}(s) = \begin{pmatrix} \frac{(3+6\cos(s)-\cos(2s))}{2\sqrt{(7-3\cos(s))}\sqrt{3+\cos(s)}}, \\ -\frac{(-3+\cos(s))\sin(s)}{\sqrt{(7-3\cos(s))}\sqrt{3+\cos(s)}}, \\ \frac{4v\sin\left(\frac{s}{2}\right)}{\sqrt{(7-3\cos(s))}\sqrt{3+\cos(s)}} \end{pmatrix},$$

$$\boldsymbol{n}(s) = \begin{pmatrix} \frac{\sqrt{2}\left(2\cos(s)\sin\left(\frac{s}{2}\right)-\cos\left(\frac{s}{2}\right)\sin(s)\right)}{\sqrt{(7-3\cos(s))}}, \\ \frac{\sqrt{2}\left(\cos\left(\frac{s}{2}\right)\cos(s)+2\sin\left(\frac{s}{2}\right)\sin(s)\right)}{\sqrt{(7-3\cos(s))}}, \\ \frac{-\sqrt{2}}{\sqrt{(7-3\cos(s))}} \end{pmatrix},$$

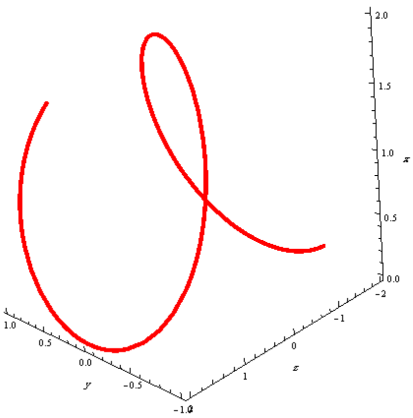
Also, we have

$$\begin{aligned} \kappa_g &= -\frac{2\sqrt{2}}{\sqrt{(7-3\cos(s))}}, \\ \kappa_n &= -\frac{3\sin\left(\frac{s}{2}\right)}{\sqrt{2}\sqrt{(7-3\cos(s))}}, \\ \tau_g &= \frac{3\sin\left(\frac{s}{2}\right)\sin(s)}{7-3\cos(s)}. \end{aligned}$$

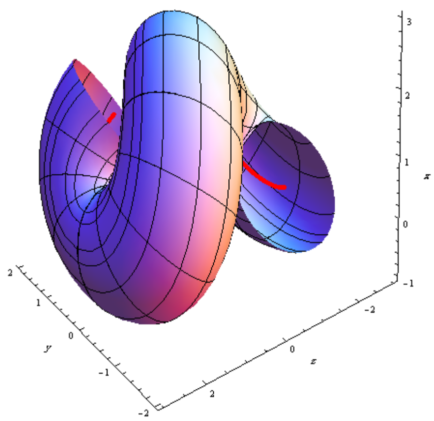
Thus, according to Eq (3.1), the tubular surface $\Upsilon_2(s, v)$ associated with the Darboux frame of radius $r > 0$ along $\beta(s)$ is the surface with the parametrization (see Figure 5b):

$$\Upsilon_2 = \left\{ \begin{array}{c} 1+\cos(s)+\frac{(3+6\cos(s)-\cos(2s))}{2\sqrt{(7-3\cos(s))}\sqrt{3+\cos(s)}}+\frac{\sqrt{2}\left(2\cos(s)\sin\left(\frac{s}{2}\right)-\cos\left(\frac{s}{2}\right)\sin(s)\right)}{\sqrt{(7-3\cos(s))}}, \\ \sin(s)-\frac{(-3+\cos(s))\sin(s)}{\sqrt{(7-3\cos(s))}\sqrt{3+\cos(s)}}+\frac{\sqrt{2}\left(\cos\left(\frac{s}{2}\right)\cos(s)+2\sin\left(\frac{s}{2}\right)\sin(s)\right)}{\sqrt{(7-3\cos(s))}}, \\ 2\sin\left(\frac{s}{2}\right)+\frac{4\sin\left(\frac{s}{2}\right)}{\sqrt{(7-3\cos(s))}\sqrt{3+\cos(s)}}-\frac{\sqrt{2}}{\sqrt{v^2(7-3\cos(s))}} \end{array} \right\}.$$

***Remark**. It should be noted that the calculations of the tubular surface Υ_2 can be calculated using Mathematica.*



(a)



(b)

Figure 5. (a) The regular space curve $\beta(s)$; (b) the tubular surface Υ_2 along $\beta(s)$.

Example 5.3. *Let $\gamma = \gamma(u)$ be a space curve which lies on a regular surface and has a cusp at $u_0 = 0$ (see Figure 6a),*

$$\gamma(u) = \{\cos u + u\sin u, 0, u\cos u - \sin u\}. \tag{5.1}$$

Darboux frame vectors of γ are calculated as follows:

$$\begin{aligned} T &= \left\{\frac{u\cos u}{\sqrt{u^2}}, 0, -\frac{u\sin u}{\sqrt{u^2}}\right\}, \\ g &= \left\{\frac{u\sqrt{u^2}\sin u}{\sqrt{u^2+u^4}}, -\frac{\sqrt{u^2}}{\sqrt{u^2+u^4}}, \frac{u\sqrt{u^2}\cos u}{\sqrt{u^2+u^4}}\right\}, \\ n &= \left\{-\frac{u\sin u}{\sqrt{u^2+u^4}}, \frac{-u^2\cos^2 u - u^2\sin^2 u}{\sqrt{u^2+u^4}}, -\frac{u\cos u}{\sqrt{u^2+u^4}}\right\}. \end{aligned} \tag{5.2}$$

Therefore, the tubular surface associated with these Darboux vectors along $\gamma(u)$ is given by

$$\mathbf{\Upsilon}_3 = \left\{\begin{array}{c} \cos u + u\sin u + \frac{u\sqrt{u^2}\cos v\sin u}{\sqrt{u^2+u^4}} - \frac{u\sin u\sin v}{\sqrt{u^2+u^4}}, \\ -\frac{\sqrt{u^2}\cos v}{\sqrt{u^2+u^4}} + \frac{\left(-u^2 v\cos^2 u - u^2 v\sin^2 u\right)\sin v}{\sqrt{u^2+u^4}v}, \\ u\cos u + \frac{u\sqrt{u^2}\cos u\cos v}{\sqrt{u^2+u^4}} - \sin u - \frac{u\cos u\sin v}{\sqrt{u^2+u^4}} \end{array}\right\}, \tag{5.3}$$

then

$$\begin{aligned} \mathbf{\Upsilon}_{3u} &= \left\{\begin{array}{c} \frac{u\left(1+u^2\right)\cos u\left(u^2\cos v + \sqrt{u^2}\left(\sqrt{u^2+u^4}-\sin v\right)\right)+\sin u\left(u^2\cos v+\left(u^2\right)^{3/2}\sin v\right)}{\sqrt{u^2}\left(1+u^2\right)\sqrt{u^2+u^4}}, \\ \frac{\sqrt{u^2+u^4}\left(\sqrt{u^2}\cos v - \sin v\right)}{u\left(1+u^2\right)^2}, \\ \frac{-u\sin u\left(\left(u^2+u^4\right)\cos v+\sqrt{u^2}\left(1+u^2\right)\left(\sqrt{u^2+u^4}-\sin v\right)\right)+\cos u\left(u^2\cos v+\left(u^2\right)^{3/2}\sin v\right)}{\sqrt{u^2}\left(1+u^2\right)\sqrt{u^2+u^4}} \end{array}\right\}, \\ \mathbf{\Upsilon}_{3v} &= \left\{\begin{array}{c} -\frac{u\sin u\left(\cos v+\sqrt{u^2}\sin v\right)}{\sqrt{u^2+u^4}}, \\ \frac{-u^2\cos v+\sqrt{u^2}\sin v}{\sqrt{u^2+u^4}}, \\ -\frac{u\cos u\left(\cos v+\sqrt{u^2}\sin v\right)}{\sqrt{u^2+u^4}} \end{array}\right\}. \end{aligned} \tag{5.4}$$

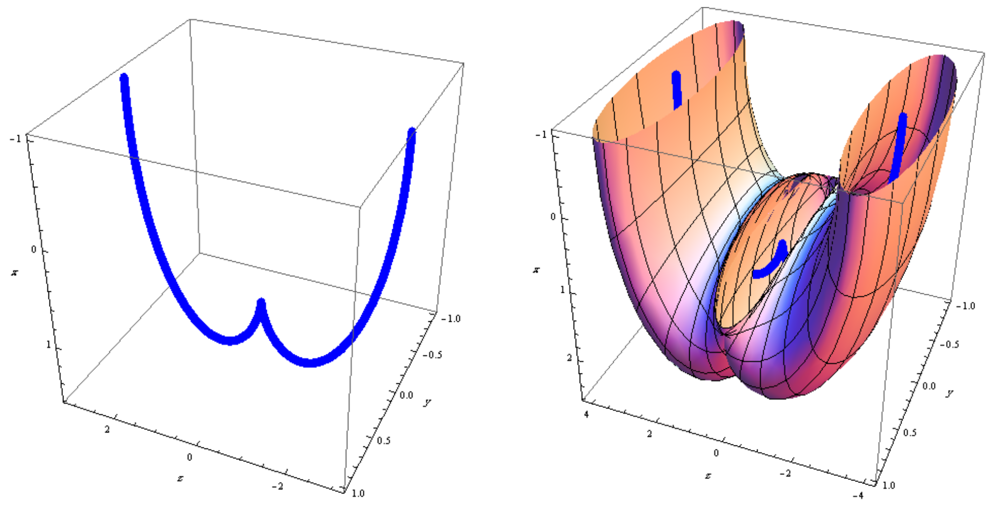
From Eq (5.4), we get

$$\|\mathbf{\Upsilon}_{3u} \times \mathbf{\Upsilon}_{3v}\| = \frac{1}{\sqrt{2}}\sqrt{\frac{1+3u^2+2u^4+(-1+u^2)\cos 2v + 4\sqrt{u^2}\cos v\left(\sqrt{u^2+u^4}-\sin v\right) - 4\sqrt{u^2+u^4}\sin v}{1+u^2}},$$

which means $\mathbf{\Upsilon}_3$ has a singularity if and only if

$$\frac{1}{\sqrt{2}}\sqrt{\frac{1+3u^2+2u^4+(-1+u^2)\cos 2v + 4\sqrt{u^2}\cos v\left(\sqrt{u^2+u^4}-\sin v\right) - 4\sqrt{u^2+u^4}\sin v}{1+u^2}} = 0.$$

Consequently, $\mathbf{\Upsilon}_3$ represents a front surface, and among its singular points are those denoted as $(0, n\pi)$, i.e., $n = 0, 1, 2, ...$ (see Figure 6b).



(a) **(b)**

Figure 6. (a) $\gamma(s)$ has a cusp at $(0,0)$; (b) Υ_3 has singularities at $(0, n\pi)$.

6. Conclusions

In this work, we studied the geometric properties and singularities of tubular surfaces with a Darboux frame in $\mathbb{R}^3$. Also, the local singularities of tubular Weingarten surfaces and relations among their curvature functions were studied. This study was intended to clear away to conduct the geometric analysis of tubular surfaces through the geometric conditions for these surfaces to have generic singularities as a front (i.e., cuspidal lips, cuspidal beaks, and Swallowtails).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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