*Mathematics*

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 12128–12143.
DOI: 10.3934/math.2024592
Received: 30 December 2023
Revised: 13 March 2024
Accepted: 18 March 2024
Published: 27 March 2024

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*Research article*

# Global behavior of a discrete population model

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**Abstract:** In this work, the global behavior of a discrete population model

$$\begin{cases} x_{n+1} &= \alpha x_n e^{-y_n} + \beta, \\ y_{n+1} &= \alpha x_n (1 - e^{-y_n}), \end{cases} \quad n = 0, 1, 2, \ldots,$$

is considered, where $\alpha \in (0, 1), \beta \in (0, +\infty)$, and the initial value $(x_0, y_0) \in [0, \infty) \times [0, \infty)$. To illustrate the dynamics behavior of this model, the boundedness, periodic character, local stability, bifurcation, and the global asymptotic stability of the solutions are investigated.

**Keywords:** discrete population model; boundedness; prime period-two solution; global asymptotic stablility; transcritical bifurcation; center manifold theorem
**Mathematics Subject Classification:** 39A10

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## 1. Introduction

The model

$$\begin{cases} x_{n+1} &= \lambda x_n e^{-a y_n}, \\ y_{n+1} &= c x_n (1 - e^{-a y_n}), \end{cases} \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

is used to describe the Nicholson-Bailey host-parasitoid system, where $x_n$ and $y_n$ represent the densities of host and parasitoid at the $n$th generation, respectively, $a$ is the searching efficiency of the parasitoid, $\lambda$ is the host reproductive rate, and $c$ is the average number of viable eggs laid by a parasitoid on a single host. System (1.1) is simple and its positive equilibrium is unstable [4, 5, 19], which indicates that the parasitoid populations, or both the parasitoid and host populations, will go extinct. Therefore this simple model is unrealistic for any practical applications. Up to now, the model has been developed to describe the population dynamic behavior of a coupled host-parasitoid (or predator-prey) system. The improved models display more various dynamic behaviors such as stability, bifurcation, and chaotic phenomenon, see [1, 7–10, 16, 17, 20, 21, 23]. For more detailed information, refer to [13–15, 22, 25].

As mentioned in [18], in many populations it is reasonable to believe that either a refuge exists which isolates some small fraction of the population from density-dependent effects, or that there is a small amount of immigration from outside the system each generation. Therefore, in this paper we consider the system

$$\begin{cases} x_{n+1} &= \alpha x_n e^{-y_n} + \beta, \\ y_{n+1} &= \alpha x_n (1 - e^{-y_n}), \end{cases} \quad n = 0, 1, 2, \dots, \tag{1.2}$$

where

$$\alpha \in (0, 1), \;\; \beta \in (0, +\infty), \tag{1.3}$$

and the initial value $(x_0, y_0) \in [0, +\infty) \times [0, +\infty)$. The parameter $\alpha$ is the host reproductive rate at per generation (in the absence of a parasitoid), and the term $\beta$ represents a refuge or a constant amount of immigration of hosts from outside the system per generation.

In [12], Kulenović and Ladas proposed an open problem (Open Problem 6.10.16) asking for investigating the global character of all solutions of system (1.2) with parameters $\alpha \in (0, 1)$ and $\beta \in (1, +\infty)$.

Inspired by the aforementioned open problem, in this paper the boundedness, periodic character, transcritical bifurcation, local asymptotic stability, and global asymptotic stability of system (1.2) are discussed under condition (1.3). Our result partially solves the above open problem.

The paper is organized as follows:

Section 1 is the introduction, and Section 2 involves the preliminaries, where some necessary lemmas are presented. Section 3 deals with the boundedness and periodic character of system (1.2). The linearized stability and bifurcation analysis are discussed in Section 4. Section 5 focuses on the global asymptotic stability of the equilibria of system (1.2). Section 6 is the conclusion.

## 2. Preliminaries

Prior to commencing the discussion, we present some essential lemmas.

**Lemma 2.1.** *(A Comparison Result [11]) Assume that $\alpha \in (0, +\infty)$ and $\beta \in \mathbb{R}$. Let $\{x_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ be sequences of real numbers such that $x_0 \le z_0$ and*

$$\begin{cases} x_{n+1} \le \alpha x_n + \beta, \\ z_{n+1} = \alpha z_n + \beta, \end{cases} \quad n = 0, 1, 2, \dots.$$

*Then $x_n \le z_n$ for $n \ge 0$.*

The following lemma is proved in [6], which will be applied in analyzing the global attractivity of Eq (2.1). Additionally, one can refer to [3, 11, 12] for further information.

**Lemma 2.2.** *Consider the difference equation*

$$u_{n+1} = g(u_n), \;\; n = 0, 1, 2, \dots. \tag{2.1}$$

*Let $I \subseteq [0, +\infty)$ be some interval and assume that $g \in C[I, (0, +\infty)]$ satisfies the following conditions:*

**(i)** *$g(u)$ is non-decreasing in $u$.*

**(ii)** *Equation* (2.1) *has a unique positive equilibrium $\bar{u} \in I$ and the function $g(u)$ satisfies the negative feedback condition:*

$$(u - \bar{u})(g(u) - u) < 0 \quad \text{for every} \;\; u \in I \setminus \{\bar{u}\}.$$

*Then, every positive solution of Eq* (2.1) *with initial conditions in $I$ converges to $\bar{u}$.*

Consider the difference equation

$$y_{n+1} = G(y_n, y_{n-1}), \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

The following strategy for obtaining global attractivity results of Eq (2.2) is derived from [12], which is also referenced in [2].

**Lemma 2.3.** *Let $[a, b]$ be an interval of real numbers and assume that $G : [a, b] \times [a, b] \to [a, b]$ is a continuous function satisfying the following properties:*

**(i)** *$G(x, y)$ is non-decreasing in $x \in [a, b]$ for each $y \in [a, b]$, and $G(x, y)$ is non-increasing in $y \in [a, b]$ for each $x \in [a, b]$.*

**(ii)** *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$G(m, M) = m, \quad \text{and} \quad G(M, m) = M,$$

*then $m = M$.*

*Then, Eq* (2.2) *has a unique equilibrium $\bar{x}$, and every solution of Eq* (2.2) *converges to $\bar{x}$.*

## 3. Boundedness and periodic character

**Theorem 3.1.** *Assume that* (1.3) *holds. Then every nonnegative solution of system* (1.2) *is bounded and eventually enters an invariant rectangle $[\beta, \frac{\beta}{1-\alpha}] \times [0, \frac{\alpha\beta}{1-\alpha}]$.*

*Proof.* Using (1.2) and noting that $0 < e^{-y_n} \leq 1$ for $y_n \geq 0$, we get

$$\beta < x_{n+1} = \alpha x_n e^{-y_n} + \beta < \alpha x_n + \beta, \quad n = 0, 1, 2, \ldots.$$

Consider the initial value problem

$$z_{n+1} = \alpha z_n + \beta, \quad n = 0, 1, 2, \ldots, \tag{3.1}$$

with initial value $z_0 = x_0$. It follows by Lemma 2.1 that

$$x_n \leq z_n, \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

The solution of Eq (3.1) is given by

$$z_n = \alpha^n (z_0 - \frac{\beta}{1-\alpha}) + \frac{\beta}{1-\alpha}, \quad n \geq 1,$$

and for $n > 1$,

$$z_{n+1} - z_n = \alpha^n(1-\alpha)(\frac{\beta}{1-\alpha} - z_0).$$

Therefore, the sequence $\{z_n\}$ is decreasing and bounded below by $\frac{\beta}{1-\alpha}$ with the initial value $z_0 > \frac{\beta}{1-\alpha}$, and it is increasing and bounded above by $\frac{\beta}{1-\alpha}$ with the initial value $z_0 < \frac{\beta}{1-\alpha}$, and $z_n = \frac{\beta}{1-\alpha}$ for $n \geq 1$ with the initial value $z_0 = \frac{\beta}{1-\alpha}$. Thus, $\lim_{n\to\infty} z_n = \frac{\beta}{1-\alpha}$. Hence, for every $\epsilon > 0$, there is an integer $N$ such that, for $n > N$,

$$x_n \leq z_n < \frac{\beta}{1-\alpha} + \varepsilon,$$

and so $x_n \leq \frac{\beta}{1-\alpha}$ for $n > N$. Furthermore, when $n > N$,

$$0 \leq y_{n+1} = \alpha x_n(1 - e^{-y_n}) \leq \alpha x_n \leq \frac{\alpha\beta}{1-\alpha},$$

holds.

Set

$$M = \max\{x_0, x_1, \ldots, x_N, \frac{\beta}{1-\alpha}\}, \quad L = \max\{y_0, y_1, \ldots, y_{N+1}, \frac{\alpha\beta}{1-\alpha}\}.$$

Then

$$\beta \leq x_n \leq M, \quad 0 \leq y_n \leq L, \quad \text{for} \quad n \geq 0.$$

Moreover, if $(x_0, y_0) \in [\beta, \frac{\beta}{1-\alpha}] \times [0, \frac{\alpha\beta}{1-\alpha}]$, then

$$\beta \leq x_1 = \alpha x_0 e^{-y_0} + \beta \leq \alpha x_0 + \beta \leq \frac{\alpha\beta}{1-\alpha} + \beta = \frac{\beta}{1-\alpha},$$

$$0 \leq y_1 = \alpha x_0(1 - e^{-y_0}) \leq \alpha x_0 \leq \frac{\alpha\beta}{1-\alpha},$$

and by using induction, we obtain

$$(x_n, y_n) \in [\beta, \frac{\beta}{1-\alpha}] \times [0, \frac{\alpha\beta}{1-\alpha}] \quad \text{for } n \geq 0.$$

So, the rectangle $[\beta, \frac{\beta}{1-\alpha}] \times [0, \frac{\alpha\beta}{1-\alpha}]$ is invariant, which completes the proof. □

**Theorem 3.2.** *Assume that* (1.3) *holds. Then system* (1.2) *has no positive prime period-two solution.*

*Proof.* Assume for the sake of contradiction that

$$\cdots, (\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_1, \eta_1), (\xi_2, \eta_2), \cdots$$

is a positive prime period-two solution of system (1.2). Then it should satisfy

$$\xi_2 = \alpha\xi_1 e^{-\eta_1} + \beta, \qquad \eta_2 = \alpha\xi_1(1 - e^{-\eta_1}), \tag{3.2}$$

and

$$\xi_1 = \alpha\xi_2 e^{-\eta_2} + \beta, \qquad \eta_1 = \alpha\xi_2(1 - e^{-\eta_2}). \tag{3.3}$$

Clearly, $\xi_1, \xi_2 \geq \beta$.

From (3.2) and (3.3), we derive

$$\xi_2 - \beta = \alpha\xi_1 - \eta_2, \qquad \xi_1 - \beta = \alpha\xi_2 - \eta_1,$$

which are equivalent to

$$\eta_2 - \eta_1 = (1+\alpha)(\xi_1 - \xi_2). \tag{3.4}$$

Thus, $\xi_1 = \xi_2 \Longleftrightarrow \eta_2 = \eta_1$.

Moreover, (3.2) and (3.3) imply that $\xi_1, \xi_2 > \beta$. This is because, if $\xi_1 = \beta$, then $\xi_2 = 0$ and $\eta_1 = \eta_2 = 0$, which is a contradiction. Similarly, if $\xi_2 = \beta$, then $\xi_1 = 0$ and $\eta_2 = \eta_1 = 0$, which leads to a contradiction as well.

Additionally, combining (3.2), (3.3), and (3.4), we can obtain

$$\frac{\xi_2 - \beta}{\xi_1 - \beta} = \frac{\xi_1 e^{-\eta_1}}{\xi_2 e^{-\eta_2}} = \frac{\xi_1}{\xi_2} e^{\eta_2 - \eta_1} = \frac{\xi_1}{\xi_2} e^{(1+\alpha)(\xi_1 - \xi_2)},$$

and thus

$$e^{(1+\alpha)(\xi_1 - \xi_2)} = \frac{\xi_2(\xi_2 - \beta)}{\xi_1(\xi_1 - \beta)},$$

which means that

$$\xi_1(\xi_1 - \beta)e^{(1+\alpha)\xi_1} = \xi_2(\xi_2 - \beta)e^{(1+\alpha)\xi_2}. \tag{3.5}$$

Set

$$A(t) = t(t-\beta)e^{(1+\alpha)t}.$$

Then

$$A'(t) = e^{(1+\alpha)t}[(t-\beta)(t+\alpha t+1)+t],$$

from which it follows that $A'(t) > 0$ for $t \geq \beta > 0$, and thus $A(t)$ is strictly increasing in $t$ for $t \geq \beta > 0$. So, (3.5) implies that $\xi_1 = \xi_2$. Therefore, $\eta_1 = \eta_2$, a contradiction.

The proof is complete. □

## 4. Linearized stability and bifurcation analysis

### 4.1. Linearized stability

**Theorem 4.1.** **(i)** *Assume that* (1.3) *holds and* $\beta \leq \frac{1-\alpha}{\alpha}$. *Then system* (1.2) *possesses a unique nonnegative equilibrium* $\bar{E}_x = (\frac{\beta}{1-\alpha}, 0)$.

**(ii)** *Assume that* (1.3) *holds and* $\beta > \frac{1-\alpha}{\alpha}$. *Then system* (1.2) *possesses two equilibria:* $\bar{E}_x = (\frac{\beta}{1-\alpha}, 0)$ *and* $\bar{E} = (\bar{x}, \bar{y}) \in [\beta, \frac{\beta}{1-\alpha}] \times [0, \frac{\alpha\beta}{1-\alpha}]$.

*Proof.* The equilibria of system (1.2) can be obtained by solving the following equations:

$$\begin{cases} x &= \alpha x e^{-y} + \beta, \\ y &= \alpha x(1-e^{-y}). \end{cases} \tag{4.1}$$

Clearly, $y = 0$ is always the solution of the second equation of (4.1), and thus $\bar{E}_x = (\frac{\beta}{1-\alpha}, 0)$ is always the equilibrium of system (1.2).

From the first equation of (4.1), we get

$$x = \frac{\beta}{1 - \alpha e^{-y}},$$

and thus

$$y = \frac{\alpha\beta(1 - e^{-y})}{1 - \alpha e^{-y}},$$

or, equivalently,

$$y - \alpha y e^{-y} - \alpha\beta(1 - e^{-y}) = 0. \tag{4.2}$$

Let

$$\phi(y) = y - \alpha y e^{-y} - \alpha\beta(1 - e^{-y}). \tag{4.3}$$

Then, $\phi(0) = 0$, and $\phi(y) \sim y$ as $y \to +\infty$. Moreover, we have

$$\phi'(y) = 1 - \alpha e^{-y} + \alpha y e^{-y} - \alpha\beta e^{-y} = \frac{1}{e^y}(e^y + \alpha y - \alpha - \alpha\beta).$$

Let

$$\psi(y) = e^y + \alpha y - \alpha - \alpha\beta. \tag{4.4}$$

Then, $\psi'(y) = e^y + \alpha > 0$, from which it follows that the function $\psi(y)$ is strictly increasing in $[0, +\infty)$.

(i) When $\beta \leq \frac{1-\alpha}{\alpha}$, $\psi(y) > \psi(0) = 1 - \alpha - \alpha\beta \geq 0$ with $y > 0$. Consequently, $\phi'(y) = \frac{1}{e^y}\psi(y) > 0$ for $y > 0$, and system (1.2) has no other equilibrium, which implies that conclusion (i) is valid.

(ii) When $\beta > \frac{1-\alpha}{\alpha}$, $\psi(0) = 1 - \alpha - \alpha\beta < 0$, and $\psi(+\infty) = +\infty$. By the continuity of the function $\psi(y)$, there exists a unique root $y^* \in (0, +\infty)$ such that

$$\psi(y^*) = 0. \tag{4.5}$$

Hence, $\psi(y) < 0$ with $0 < y < y^*$, and $\psi(y) > 0$ with $y > y^*$. Moreover, $\phi'(y) < 0$ with $0 < y < y^*$, and $\phi'(y) > 0$ with $y > y^*$. It follows that $\phi(y)$ is decreasing in $(0, y^*)$, and $\phi(y)$ is increasing in $(y^*, +\infty)$. Thus, the function $\phi(y)$ attains its minimum at $y^*$, $\phi(y^*) < \phi(0) = 0$, and by the continuity of the function $\phi(y)$, equation $\phi(y) = 0$ has a unique positive root $\bar{y}$ such that $\bar{y} > y^*$.

Adding the two equations of system (4.1) yields

$$\bar{x} + \bar{y} = \alpha\bar{x} + \beta, \tag{4.6}$$

hence

$$\bar{x} = (\beta - \bar{y})/(1 - \alpha). \tag{4.7}$$

By (4.1) and (4.7), it is easy to obtain that $\bar{x} \in [\beta, \frac{\beta}{1-\alpha}]$ and $\bar{y} \in [0, \frac{\alpha\beta}{1-\alpha}]$. Thus, in this case, system (1.2) possesses an additional equilibrium $\bar{E} = (\bar{x}, \bar{y})$, and conclusion (ii) follows.

The proof is complete. □

**Theorem 4.2.** **(i)** *Assume that* (1.3) *holds. Then the equilibrium* $\bar{E}_x = (\frac{\beta}{1-\alpha}, 0)$ *is locally asymptotically stable when* $\beta < \frac{1-\alpha}{\alpha}$, *is nonhyperbolic when* $\beta = \frac{1-\alpha}{\alpha}$, *and is unstable (a saddle point) when* $\beta > \frac{1-\alpha}{\alpha}$.

**(ii)** *Assume that* (1.3) *holds and* $\beta > \frac{1-\alpha}{\alpha}$. *Then the unique positive equilibrium* $\bar{E}$ *is locally asymptotically stable (a sink).*

*Proof.* Let

$$F(x,y)=\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}=\begin{pmatrix} \alpha x e^{-y}+\beta \\ \alpha x(1-e^{-y}) \end{pmatrix}.$$

By simple calculation, we have

$$\begin{aligned} \frac{\partial f}{\partial x} &= \alpha e^{-y}, & \frac{\partial f}{\partial y} &= -\alpha x e^{-y}, \\ \frac{\partial g}{\partial x} &= \alpha(1-e^{-y}), & \frac{\partial g}{\partial y} &= \alpha x e^{-y}. \end{aligned}$$

(i) The Jacobian matrix of $F$ evaluated at $\bar{E}_x$ is given by

$$J_F(\bar{E}_x)=\begin{pmatrix} \alpha & -\frac{\alpha\beta}{1-\alpha} \\ 0 & \frac{\alpha\beta}{1-\alpha} \end{pmatrix},$$

and its eigenvalues are

$$\lambda_1=\alpha, \quad \lambda_2=\frac{\alpha\beta}{1-\alpha}.$$

Notice that $\alpha\in(0,1)$, so $0<\lambda_1<1$, and $0<\lambda_2<1$ with $\beta<\frac{1-\alpha}{\alpha}$, $\lambda_2=1$ with $\beta=\frac{1-\alpha}{\alpha}$, and $\lambda_2>1$ with $\beta>\frac{1-\alpha}{\alpha}$, which means that result (i) follows.

(ii) The Jacobian matrix of $F$ evaluated at $\bar{E}$ is given by

$$J_F(\bar{E})=\begin{pmatrix} \alpha e^{-\bar{y}} & -\alpha\bar{x}e^{-\bar{y}} \\ \alpha(1-e^{-\bar{y}}) & \alpha\bar{x}e^{-\bar{y}} \end{pmatrix},$$

and its characteristic equation is

$$\lambda^2-p\lambda+q=0,$$

where $p=\alpha e^{-\bar{y}}(1+\bar{x})$, $q=\alpha^2\bar{x}e^{-\bar{y}}$.

Since the second equation of (4.1) implies that $\bar{x}=\bar{y}/(\alpha(1-e^{-\bar{y}}))$, it can be concluded that

$$0<q=\alpha^2\bar{x}e^{-\bar{y}}=\alpha\bar{x}\cdot\alpha e^{-\bar{y}}=\frac{\bar{y}}{1-e^{-\bar{y}}}\cdot\alpha e^{-\bar{y}}=\frac{\alpha\bar{y}}{e^{\bar{y}}-1}<\frac{\alpha\bar{y}}{\bar{y}}=\alpha<1.$$

Moreover, noticing that the function $\psi(y)$ defined by (4.4) is strictly increasing in $(0,+\infty)$ and that $\bar{y}>y^*$, we can utilize (4.5) to derive

$$\psi(\bar{y})=e^{\bar{y}}+\alpha\bar{y}-\alpha-\alpha\beta>\psi(y^*)=0,$$

where $y^*$ is the minimum point of $\phi(y)$. Thus,

$$1+\alpha\bar{y}e^{-\bar{y}}-\alpha e^{-\bar{y}}-\alpha\beta e^{-\bar{y}}>0. \tag{4.8}$$

In addition, the fact that $\bar{y}$ is the root of the function $\phi(y)$ given by (4.3) implies that $\phi(\bar{y})=0$, namely,

$$\bar{y}-\alpha\bar{y}e^{-\bar{y}}+\alpha\beta e^{-\bar{y}}-\alpha\beta=0. \tag{4.9}$$

Adding (4.8) and (4.9) yields

$$1+\bar{y}-\alpha e^{-\bar{y}}-\alpha\beta>0. \tag{4.10}$$

From (4.6), we have

$$\bar{y} = \beta - (1-\alpha)\bar{x}, \tag{4.11}$$

and from the first equation of system (4.1), we have

$$e^{-\bar{y}} = \frac{\bar{x}-\beta}{\alpha\bar{x}}. \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.10) yields

$$1 + \beta - (1-\alpha)\bar{x} - \frac{\bar{x}-\beta}{\bar{x}} - \alpha\beta > 0,$$

from which it follows that

$$\alpha(\bar{x}-\beta) > \bar{x} - \beta - \frac{\beta}{\bar{x}}.$$

Applying (4.12), we have $q = \alpha^2\bar{x}e^{-\bar{y}} = \alpha(\bar{x}-\beta)$ and

$$|p| = \alpha e^{-\bar{y}}(1+\bar{x}) = \frac{\bar{x}-\beta}{\bar{x}}(1+\bar{x}) = 1 + \bar{x} - \beta - \frac{\beta}{\bar{x}} < 1 + \alpha(\bar{x}-\beta) = 1 + q < 2.$$

By the Schur-Cohn criterion, we obtain that $\bar{E} = (\bar{x}, \bar{y})$ is locally asymptotically stable.

The proof is complete. □

*4.2. Bifurcation analysis*

When parameters $\alpha$ and $\beta$ satisfy the condition $\beta = \frac{1-\alpha}{\alpha}$, the equilibrium $\bar{E}_x = (\frac{\beta}{1-\alpha}, 0)$ is non-hyperbolic with eigenvalue $\lambda_2 = 1$. This indicates a bifurcation probably occurs as the parameter $\beta$ varies and goes through the critical value $\frac{1-\alpha}{\alpha}$. In fact, in this case, a transcritical bifurcation takes place at $\bar{E}_x$.

**Theorem 4.3.** *Assume that* (1.3) *holds and let* $\beta^* = \frac{1-\alpha}{\alpha}$. *Then system* (1.2) *undergoes a transcritical bifurcation at* $\bar{E}_x$ *when the parameter* $\beta$ *passes through the critical value* $\beta^*$.

*Proof.* Letting $u_n = x_n - \frac{\beta}{1-\alpha}$, $v_n = y_n$ shifts the equilibrium $\bar{E}_x$ to the origin, and tranforms the system (1.2) into

$$\begin{cases} u_{n+1} &= \alpha u_n e^{-v_n} + \frac{\alpha\beta}{1-\alpha}e^{-v_n} - \frac{\alpha\beta}{1-\alpha}, \\ v_{n+1} &= \alpha u_n(1-e^{-v_n}) - \frac{\alpha\beta}{1-\alpha}e^{-v_n} + \frac{\alpha\beta}{1-\alpha}, \end{cases} \quad n = 0, 1, 2, \ldots. \tag{4.13}$$

Define $\tau = \beta - \beta^*$ as a small perturbation around $\beta^*$ with $0 < |\tau| \ll 1$. Then, the map of system (4.13) can be expressed as:

$$\begin{pmatrix} u \\ v \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} \alpha u e^{-v} + e^{-v} + \frac{\alpha\tau}{1-\alpha}e^{-v} - \frac{\alpha\tau}{1-\alpha} - 1 \\ \alpha u(1-e^{-v}) - e^{-v} - \frac{\alpha\tau}{1-\alpha}e^{-v} + \frac{\alpha\tau}{1-\alpha} + 1 \\ \tau \end{pmatrix}. \tag{4.14}$$

Expanding (4.14) in a Taylor series at $(u, v, \tau) = (0, 0, 0)$ gives

$$\begin{pmatrix} u \\ v \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} \alpha & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ \tau \end{pmatrix} + \begin{pmatrix} F_1(u, v, \tau) \\ G_1(u, v, \tau) \\ 0 \end{pmatrix}, \tag{4.15}$$

where

$$F_1(u,v,\tau) = \frac{1}{2}v^2 - \alpha uv - \frac{\alpha}{1-\alpha}v\tau - \frac{1}{6}v^3 + \frac{1}{2}\alpha uv^2 + \frac{\alpha}{2(1-\alpha)}v^2\tau + O(3),$$

$$G_1(u,v,\tau) = -\frac{1}{2}v^2 + \alpha uv + \frac{\alpha}{1-\alpha}v\tau + \frac{1}{6}v^3 - \frac{1}{2}\alpha uv^2 - \frac{\alpha}{2(1-\alpha)}v^2\tau + O(3),$$

and $O(3)$ is the sum of all remainder terms with a frequency greater than 3.

Let

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \alpha-1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be an invertible matrix. Through the variable transformation

$$\begin{pmatrix} u \\ v \\ \tau \end{pmatrix} = T \begin{pmatrix} X \\ Y \\ \omega \end{pmatrix},$$

the map (4.15) is transformed into the form

$$\begin{pmatrix} X \\ Y \\ \omega \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ \omega \end{pmatrix} + \begin{pmatrix} F_2(X,Y,\omega) \\ G_2(X,Y,\omega) \\ 0 \end{pmatrix}, \tag{4.16}$$

where

$$\begin{aligned} F_2(X,Y,\omega) =& \frac{1}{2}(1-\alpha^2)Y^2 - \alpha(\alpha-1)XY + \alpha Y\omega + \frac{1}{6}(\alpha-1)^2(2\alpha+1)Y^3 \\ &+ \frac{1}{2}\alpha(\alpha-1)^2XY^2 - \frac{1}{2}\alpha(\alpha-1)Y^2\omega + O(3), \\ G_2(X,Y,\omega) =& \frac{1}{2}(\alpha^2-1)Y^2 + \alpha(\alpha-1)XY - \alpha Y\omega - \frac{1}{6}(\alpha-1)^2(2\alpha+1)Y^3 \\ &- \frac{1}{2}\alpha(\alpha-1)^2XY^2 + \frac{1}{2}\alpha(\alpha-1)Y^2\omega + O(3). \end{aligned}$$

By the center manifold Theorem 2.1.4 in [24], for the map (4.16), there exists a center manifold that can be locally represented in the form:

$$W^c(0,0) = \{(X,Y,\omega) \in R^3 | X = h(Y,\omega),\ |Y| < \delta,\ |\omega| < \delta,\ h(0,0) = 0,\ Dh(0,0) = 0\},$$

for $\delta$ sufficiently small. Suppose that the center manifold has the representation

$$X = h(Y,\omega) = m_1Y^2 + m_2Y\omega + m_3\omega^2 + O(2).$$

Then, it satisfies

$$N(h(Y,\omega)) = h(Y + G_2(h(Y,\omega),Y,\omega),\omega) - [\alpha h(Y,\omega) + F_2(h(Y,\omega),Y,\omega)] = 0,$$

where $O(2)$ represents the sum of all remainder terms with a frequency greater than 2. Hence,

$$m_1Y^2 + m_2Y\omega + m_3\omega^2 = \alpha m_1Y^2 + \alpha m_2Y\omega + \alpha m_3\omega^2 + O(2). \tag{4.17}$$

Comparing the corresponding coefficients of terms in Eq (4.17), we have

$$m_1 = 0, \quad m_2 = 0, \quad m_3 = 0,$$

so the map (4.16) on the center manifold can be written as

$$F^* : Y \mapsto Y + \frac{1}{2}(\alpha^2 - 1)Y^2 - \alpha Y\omega + O(2).$$

Since

$$F^*(0,0) = 0, \ \ \frac{\partial F^*}{\partial Y}|_{(0,0)} = 1,$$

$$\frac{\partial F^*}{\partial \omega}|_{(0,0)} = 0, \ \ \frac{\partial^2 F^*}{\partial Y \partial \omega}|_{(0,0)} = -\alpha \neq 0, \ \ \frac{\partial^2 F^*}{\partial Y^2}|_{(0,0)} = \alpha^2 - 1 \neq 0,$$

therefore a transcritical bifurcation takes place at the equilibrium $(Y, \omega) = (0, 0)$ of the map (4.16), implying that, as the parameter $\beta$ changes and passes through the critical value $\beta^*$, system (1.2) undergoes a transcritical bifurcation at $\bar{E}_x$.

The proof is complete. □

## 5. Global asymptotic stability

In view of Lemma 4.2, to deal with the global asymptotic stability of $E_x$ and $\bar{E}$, it is sufficient to solve its global attractivity.

Consider the difference equation

$$u_{n+1} = A(1 - e^{-u_n}), \quad n = 0, 1, 2, \dots, \tag{5.1}$$

with $A \in (0, +\infty)$ and the initial value $u_0 \in [0, +\infty)$.

**Lemma 5.1.** *When $A \leq 1$, Eq (5.1) possesses a unique equilibrium zero, and when $A > 1$, an additional positive equilibrium $\bar{u}$ emerges satisfying $\bar{u} > \ln A$.*

*Proof.* Clearly, zero is always an equilibrium of Eq (5.1). The positive equilibrium can be obtained by solving the equation

$$u = A(1 - e^{-u}) \quad u \in (0, +\infty).$$

Let

$$h(u) = A(1 - e^{-u}) - u. \tag{5.2}$$

Then, $h(0) = 0$, $h(+\infty) = -\infty$, and $h'(u) = Ae^{-u} - 1$, $h''(u) = -Ae^{-u} < 0$.

When $A \leq 1$, $h'(u) < h'(0) = A - 1 \leq 0$ for $u > 0$, and thus Eq (5.1) has a unique equilibrium, namely zero.

When $A > 1$, the function $h(u)$ attains its maximum at $u = \ln A$. Hence, by the continuity of $h(u)$, there exists a unique $\bar{u}$ such that $h(\bar{u}) = 0$, namely Eq (5.1) has a unique positive equilibrium $\bar{u}$ which satisfies $\bar{u} > \ln A$ and $h(u) > 0$ for $0 < u < \bar{u}$, and $h(u) < 0$ for $u > \bar{u}$. □

**Lemma 5.2.** **(i)** *Assume that $A \leq 1$. Then every nonnegative solution of Eq (5.1) converges to the zero equilibrium.*

**(ii)** *Assume that* $A > 1$. *Then every positive solution of Eq* (5.1) *converges to the unique positive equilibrium* $\bar{u}$.

*Proof.* (i) Clearly, $u_n = 0$ with $u_0 = 0$ for $n \geq 0$, and the result follows. Given $u_0 > 0$, then $u_n > 0$ for $n \geq 1$, and

$$u_{n+1} = A(1 - e^{-u_n}) < Au_n \leq u_n,$$

from which it follows by induction that the sequence $\{u_n\}$ is strictly decreasing and bounded below by zero, so it is convergent. Since, in this case Eq (5.1) has a unique equilibrium zero, hence $\lim_{n\to\infty} u_n = 0$.

(ii) Let $g(u) = A(1 - e^{-u})$. Observing that the function $g(u)$ is increasing for $u > 0$, and using the properties of the function $h(u)$ defined by (5.2), we obtain

$$g(u) = h(u) + u > u \quad \text{with} \quad 0 < u < \bar{u}$$

and

$$g(u) = h(u) + u < u \quad \text{with} \quad u > \bar{u}.$$

Hence,

$$(u - \bar{u})(g(u) - u) < 0 \;\; \text{for} \;\; u \in (0, \infty) \setminus \{\bar{u}\},$$

and condition (ii) in Lemma 2.2 is satisfied. It follows that $\lim_{n\to\infty} u_n = \bar{u}$ with $u_0 > 0$. □

We now start the discussion of our main results.

**Theorem 5.3.** *Every solution* $\{(x_n, y_n)\}$ *of system* (1.2) *with* $x_0 y_0 = 0$ *converges to* $\bar{E}_x$.

*Proof.* Notice that $y_1 = 0$ with $x_0 = 0$, so it is sufficient to discuss the case that $y_0 = 0$. Obviously, in this case, $y_n = 0$ for $n \geq 1$, and thus system (1.2) becomes

$$x_{n+1} = \alpha x_n + \beta, \;\; n = 1, 2, \ldots,$$

and

$$x_n = \alpha^n x_0 + \frac{\beta}{1-\alpha}(1 - \alpha^n) \to \frac{\beta}{1-\alpha}, \quad \text{as } n \to \infty,$$

since $\alpha \in (0, 1)$, finishing the proof. □

**Theorem 5.4.** *Assume that* (1.3) *holds and* $\beta \leq \frac{1-\alpha}{\alpha}$. *Then the unique equilibrium* $\bar{E}_x$ *of system* (1.2) *is a global attractor of all nonnegative solutions.*

*Proof.* Let $\{(x_n, y_n)\}$ be a nonnegative solution of system (1.2). Then, from Theorem 3.1, the subsequence $\{x_n\}$ is eventually bounded and thus there exists an integer $N$ such that $x_n \leq \frac{\beta}{1-\alpha}$ for $n > N$. Using the second equation of system (1.2), we get

$$y_{n+1} = \alpha x_n (1 - e^{-y_n}) \leq \frac{\alpha\beta}{1-\alpha}(1 - e^{-y_n}). \tag{5.3}$$

Noticing that, in this case $\frac{\alpha\beta}{1-\alpha} \leq 1$ and applying 5.2 (i), we obtain that every nonnegative solution of the difference equation

$$\tilde{y}_{n+1} = \frac{\alpha\beta}{1-\alpha}(1 - e^{-\tilde{y}_n}), \;\; n = 0, 1, 2, \ldots,$$

converges to zero. Using the boundedness of the subsequence $\{y_n\}$, (5.3) yields

$$0 \le \liminf_{n\to\infty} y_{n+1} \le \limsup_{n\to\infty} y_{n+1} \le \lim_{n\to\infty} \tilde{y}_{n+1} = 0,$$

from which it follows that $\lim_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} x_n = \frac{\beta}{1-\alpha}$. Thus, $\lim_{n\to\infty}(x_n, y_n) = \bar{E}_x$.

The proof is complete. □

In view of Theorems 5.4 and 4.2 (i), we have the following result:

**Theorem 5.5.** *Assume that* (1.3) *holds and* $\beta < \frac{1-\alpha}{\alpha}$. *Then the unique equilibrium* $\bar{E}_x$ *of system* (1.2) *is globally asymptotically stable.*

Next, we deal with the global asymptotic stability of the unique positive equilibrium $\bar{E}$. We will provide a sufficient condition for $\bar{E}$ to be globally asymptotically stable with respect to all positive solutions $\{(x_n, y_n)\}$ of system (1.2). The positive solution we talk about here means a solution of system (1.2) satisfying $x_n, y_n > 0$ for $n \ge 0$.

**Theorem 5.6.** *Assume that* (1.3) *holds and* $\beta \ge \frac{1+\alpha}{\alpha}$. *Then the unique positive equilibrium* $\bar{E}$ *of system* (1.2) *is a global attractor of all positive solutions.*

*Proof.* Let $\{(x_n, y_n)\}$ be a solution of system (1.2) with $x_0 y_0 \ne 0$, then $y_n > 0$ for $n \ge 1$.

From the second equation of system (1.2), we get

$$x_n = \frac{y_{n+1}}{\alpha(1-e^{-y_n})}, \quad n = 0, 1, 2, \ldots,$$

then

$$\frac{y_{n+2}}{\alpha(1-e^{-y_{n+1}})} = \frac{y_{n+1}}{1-e^{-y_n}} e^{-y_n} + \beta,$$

or, equivalently,

$$y_{n+2} = \alpha y_{n+1}(1-e^{-y_{n+1}}) \frac{e^{-y_n}}{1-e^{-y_n}} + \alpha\beta(1-e^{-y_{n+1}}), \quad n = 0, 1, 2, \ldots, \tag{5.4}$$

which is a second-order difference equation with initial values $y_1 = \alpha x_0(1-e^{-y_0})$, $y_0 > 0$.

Clearly, the equilibrium of Eq (5.4) is not equal to zero and it must satisfy the equation

$$y - \alpha y e^{-y} - \alpha\beta(1-e^{-y}) = 0,$$

which is the equation defined by (4.2). Hence, Eq (5.4) has a unique positive equilibrium, namely $\bar{y}$.

Equation (5.4) implies that

$$y_{n+1} \ge \alpha\beta(1-e^{-y_n}), \quad n = 1, 2, \ldots. \tag{5.5}$$

If $\beta \ge \frac{1+\alpha}{\alpha}$, then $\alpha\beta > 1$. By utilizing Lemma 5.2 (ii), it can be concluded that every positive solution of the difference equation

$$\tilde{y}_{n+1} = \alpha\beta(1-e^{-\tilde{y}_n}), \quad n = 1, 2, \ldots,$$

converges to its positive equilibrium, denoted by $\tilde{y}$, and by Lemma 5.1, $\tilde{y} > \ln \alpha\beta$. Hence, for $\epsilon = \tilde{y} - \ln \alpha\beta > 0$, there exists an integer $N$ such that $\tilde{y}_n > \tilde{y} - \epsilon = \ln \alpha\beta$ for $n > N$. Further, $y_n \geq \tilde{y}_n > \ln \alpha\beta > 0$ for $n > N$. Therefore,

$$\liminf_{n\to\infty} y_{n+1} \geq \ln \alpha\beta > 0.$$

In view of Theorem 3.1, it follows that every positive solution of Eq (5.4) eventually enters an invariant interval $[\ln \alpha\beta, \frac{\alpha\beta}{1-\alpha}] \subset [0, \frac{\alpha\beta}{1-\alpha}]$, and $\bar{y} \in [\ln \alpha\beta, \frac{\alpha\beta}{1-\alpha}]$ is unique.

Set

$$G(u, v) = \alpha u(1 - e^{-u})\frac{e^{-v}}{1 - e^{-v}} + \alpha\beta(1 - e^{-u}),$$

then $G$ is increasing in $u$ for $v > 0$, and is decreasing in $v$ for $u > 0$.

Let $(m, M) \in [\ln \alpha\beta, \frac{\alpha\beta}{1-\alpha}] \times [\ln \alpha\beta, \frac{\alpha\beta}{1-\alpha}]$ be a solution of the following system:

$$\begin{cases} m &= \alpha m(1 - e^{-m})\frac{e^{-M}}{1-e^{-M}} + \alpha\beta(1 - e^{-m}), \\ M &= \alpha M(1 - e^{-M})\frac{e^{-m}}{1-e^{-m}} + \alpha\beta(1 - e^{-M}). \end{cases}$$

Then we have

$$\frac{1}{1 - e^{-m}} - \frac{\alpha\beta}{m} = \frac{\alpha e^{-M}}{1 - e^{-M}}, \tag{5.6}$$

$$\frac{1}{1 - e^{-M}} - \frac{\alpha\beta}{M} = \frac{\alpha e^{-m}}{1 - e^{-m}}. \tag{5.7}$$

Adding (5.6) and (5.7) yields

$$\frac{1}{1 - e^{-m}} - \frac{\alpha\beta}{m} + \frac{\alpha e^{-m}}{1 - e^{-m}} = \frac{1}{1 - e^{-M}} - \frac{\alpha\beta}{M} + \frac{\alpha e^{-M}}{1 - e^{-M}},$$

which is equivalent to

$$\frac{e^m + \alpha}{e^m - 1} - \frac{\alpha\beta}{m} = \frac{e^M + \alpha}{e^M - 1} - \frac{\alpha\beta}{M}. \tag{5.8}$$

Consider the function

$$I(t) = \frac{e^t + \alpha}{e^t - 1} - \frac{\alpha\beta}{t}, \quad t \in [\ln \alpha\beta, \frac{\alpha\beta}{1 - \alpha}].$$

To prove that $m = M$, it is sufficient to show that the function $I(t)$ is injective on the interval $[\ln \alpha\beta, \frac{\alpha\beta}{1-\alpha}]$ under the condition that $\beta \geq \frac{1+\alpha}{\alpha}$. Simple computation shows that

$$\begin{aligned} I'(t) =& -\frac{(1 + \alpha)e^t}{(e^t - 1)^2} + \frac{\alpha\beta}{t^2} = \frac{1}{t^2(e^t - 1)^2}[\alpha\beta(e^t - 1)^2 - (1 + \alpha)t^2 e^t] \\ \geq& \frac{1 + \alpha}{t^2(e^t - 1)^2}[(e^t - 1)^2 - t^2 e^t]. \end{aligned}$$

Let

$$J(t) = (e^t - 1)^2 - t^2 e^t,$$

then

$$J'(t) = 2e^t(e^t - 1 - t - \frac{1}{2}t^2) > 0 \quad \text{for} \quad t > 0,$$

and so, for $t > 0$,

$$J(t) > J(0) = 0.$$

Therefore, $I'(t) > 0$ for $t > 0$, which implies that the function $I(t)$ is strictly increasing on the interval $[\ln \alpha\beta, \frac{\alpha\beta}{1-\alpha}]$. Thus, equality (5.8) yields $m = M$. By applying Lemma 2.3, we get that every positive solution of Eq (5.4) converges to $\bar{y}$.

Consequently, every positive solution of system (1.2) satisfies $\lim_{n\to\infty} y_n = \bar{y}$ and $\lim_{n\to\infty} x_n = \bar{x}$, and so $\lim_{n\to\infty}(x_n, y_n) = \bar{E}$.

The proof is complete. □

In view of Theorems 5.6 and 4.2 (ii), we have the following result:

**Theorem 5.7.** *Assume that* (1.3) *holds and* $\beta \geq \frac{1+\alpha}{\alpha}$. *Then the unique positive equilibrium* $\bar{E}$ *of system* (1.2) *is globally asymptotically stable.*

## 6. Conclusions

In this work, the global behavior of a discrete population model (1.2) is considered with the conditions $\alpha \in (0, 1), \beta \in (0, +\infty)$. It is shown that, for all $\alpha \in (0, 1)$ and $\beta \in (0, +\infty)$, every nonnegative solution of this system is bounded and there is no positive prime period-two solution. However, the existence of equilibria, the local stability, bifurcation, and the global asymptotic stability depend upon the parameters $\alpha, \beta$. Specifically, if $\beta \leq \frac{1-\alpha}{\alpha}$, then this system possesses a unique equilibrium $\bar{E}_x$. It is globally asymptotically stable for $\beta < \frac{1-\alpha}{\alpha}$, and as parameter $\beta$ varies and passes through the critical value $\frac{1-\alpha}{\alpha}$, this system experiences a transcritical bifurcation at $\bar{E}_x$. If $\beta > \frac{1-\alpha}{\alpha}$, then this system possesses two equilibria, $\bar{E}_x$ and $\bar{E}$, where $\bar{E}_x$ is unstable and $\bar{E}$ is locally asymptotically stable. Finally, a sufficient condition $\beta \geq \frac{1+\alpha}{\alpha}$ is established, under which $\bar{E}$ is globally asymptotically stable.

The research result indicates that the use of refuge or external immigration of hosts can contribute to stabilizing the system. If the level of the use of refuge or external immigration of hosts per generation remains at or below the threshold $\frac{1-\alpha}{\alpha}$, the parasitoids will go extinction for all initial populations. Once this threshold is surpassed, the extinct equilibrium $\bar{E}_x$ loses its stability and the stable coexistence equilibrium $\bar{E} = (\bar{x}, \bar{y})$ emerges. Specifically, maintaining the level at or above $\frac{1+\alpha}{\alpha}$ guarantees the hosts and the parasitoids will eventually coexist at a steady density $(\bar{x}, \bar{y})$ for all initial populations. Therefore, it is essential to keep enough of a level of refuge or external immigration of hosts for the long-term survival and stability of this system.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

This work is supported by the National Natural Science Foundation of China (No. 11701425, 12261078). The second author acknowledges the support of the Longyuan Youth Talent Project of Gansu Province.

## Conflict of interest

All authors declare that there are no conflicts of interests regarding the publication of this paper.

## References


1. J. R. Beddington, C. A. Free, J. H. Lawton, Dynamic complexity in predator-prey models framed in difference equations, *Nature*, **255** (1975), 58–60. http://dx.doi.org/10.1038/255058a0
2. E. Camouzis, G. Ladas, *Dynamics of third-Order rational difference equations with open problems and conjectures*, Boca Raton: Chapman & Hall/CRC, 2008.
3. E. A. Grove, G. M. Kent, R. Levins, G. Ladas, S. Valicenti, *Global stability in some population models*, Poznan: Gordon and Breach, 2000, 149–176.
4. M. P. Hassell, R. M. May, Stability in insect host-parasite models, *J. Anim. Ecol.*, **42** (1973), 693–726. http://dx.doi.org/10.2307/3133
5. M. P. Hassell, *The dynamics of arthropod predator-prey systems*, Princeton University Press, 1978.
6. M. L. J. Hautus, A. Emerson, T. S. Bolis, Solution to problem E2721, *Amer. Math. Monthly*, **86** (1979), 865–866.
7. S. B. Hsu, M. C. Li, W. S. Liu, M. Malkin, Heteroclinic foliation, global oscillations for the Nicholson-Bailey model and delay of stability loss, *Discrete Cont. Dyn. S.*, **9** (2003), 1465–1492. http://dx.doi.org/10.3934/dcds.2003.9.1465
8. W. T. Jamieson, J. Reis, Global behavior for the classical Nicholson-Bailey model, *J. Math. Anal. Appl.*, **461** (2018), 492–499. https://doi.org/10.1016/j.jmaa.2017.12.071
9. S. R. J. Jang, J. L. Yu, Discrete-time host-parasitoid models with pest control, *J. Biol. Dynam.*, **6** (2012), 718–739. http://dx.doi.org/10.1080/17513758.2012.700074
10. S. Kapçak, S. Elaydi, Ü. Ufuktepe, Stability of a predator-prey model with refuge effect, *J. Differ. Equ. Appl.*, **22** (2016), 989–1004. http://dx.doi.org/10.1080/10236198.2016.1170823
11. V. L. Kocic, G. Ladas, *Global behavior of nonlinear difference equations of higher order with application*, Kluwer Academic Publishers, 1993.
12. M. R. S. Kulenović, G. Ladas, *Dynamics of second order rational difference equations with open problems and conjectures*, Boca Raton: Chapman & Hall/CRC, 2001.
13. T. H. Li, X. T. Zhang, C. R. Zhang, Pattern dynamics analysis of a space-time discrete spruce budworm model, *Chaos, Soliton. Fract.*, **179** (2024), 114423. http://dx.doi.org/10.1016/j.chaos.2023.114423
14. W. T. Li, H. R. Sun, X. X. Yan, The asymptotic behavior of a higher order delay nonlinear-difference-equations, *Indian J. Pure Appl. Math.*, **34** (2003), 1431–1441.
15. W. T. Li, H. R. Sun, Dynamics of a rational difference equation, *Appl. Math. Comput.*, **163** (2005), 577–591. http://dx.doi.org/10.1016/j.amc.2004.04.002
16. X. Y. Li, Y. Q. Liu, Transcritical bifurcation and flip bifurcation of a new discrete ratio-dependent predator-prey system, *Qual. Theory Dyn. Syst.*, **21** (2022). http://dx.doi.org/10.1007/s12346-022-00646-2

17. X. Y. Li, X. M. Shao, Flip bifurcation and Neimark-Sacker bifurcation in a discrete predator-prey model with Michaelis-Menten functional response, *Electron. Res. Arch.*, **31** (2023), 37–57. http://dx.doi.org/10.3934/era.2023003

18. H. I. McCallum, Effects of immigration on chaotic popution dynamics, *J. Theor. Biol.*, **154** (1992), 277–284. http://dx.doi.org/10.1016/S0022-5193(05)80170-5

19. J. D. Murray, *Mathematical Biology II*, New York: Springer-Verlag, 1989. https://doi.org/10.1007/b98869

20. A. J. Nicholson, An outline of the dynamics of animal populations, *Aust. J. Zool.*, **2** (1954), 9–65. http://dx.doi.org/10.1071/ZO9540009

21. S. Sinha, S. Parthasarathy, Unusual dynamics of extinction in a simple ecological model, *Proc. Natl. Acad. Sci. USA*, **93** (1996), 1504–1508. http://dx.doi.org/10.1073/pnas.93.4.1504

22. S. Stević, Solvability and representations of the general solutions to some nonlinear difference equations of second order, *AIMS Math.*, **8** (2023), 15148–15165. http://dx.doi.org/10.3934/math.2023773

23. V. Weide, M. C. Varriale, F. M. Hilker, Hydra effect and paradox of enrichment in discrete-time predator-prey models, *Math. Biosci.*, **310** (2019), 120–127. https://doi.org/10.1016/j.mbs.2018.12.010

24. S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*, New York: Springer-Verlag, 1990. https://doi.org/10.1007/b97481

25. X. T. Zhang, C. R. Zhang, Y. Z. Zhou, Pattern dynamics analysis of a time-space discrete FitzHugh-Nagumo (FHN) model based on coupled map lattices, *Comput. Math. Appl.*, **157** (2024), 92–123. http://dx.doi.org/10.1016/j.camwa.2023.12.030


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