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Research article

Analytic properties and numerical representations for constructing the extended beta function using logarithmic mean

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Abstract: This paper aimed to obtain generalizations of both the logarithmic mean (L_{mean}) and the Euler's beta function (EBF), which we call the extended logarithmic mean (EL_{mean}) and the extended Euler's beta-logarithmic function (EEBLF), respectively. Also, we discussed various properties, including functional relations, inequalities, infinite sums, finite sums, integral formulas, and partial derivative representations, along with the Mellin transform for the EEBLF. Furthermore, we gave numerical comparisons between these generalizations and the previous studies using MATLAB R2018a in the form of tables and graphs. Additionally, we presented a new version of the beta distribution and acquired some of its characteristics as an application in statistics. The outcomes produced here are generic and can give known and novel results.

Keywords: extended beta function; logarithmic mean; extended beta logarithmic distribution
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1. Introduction

The logarithmic mean (L_{mean}) of $\theta, \phi \in \mathbb{R}^+$, which is of interest in many fields such as engineering, statistics, geometry, and thermodynamics (for more details, see [1–3]), is defined in the following integral formula

$$L_{mean}(\theta, \phi) = \int_0^1 \theta^{1-x}\, \phi^x dx = \begin{cases} \frac{\theta-\phi}{\ln(\theta)-\ln(\phi)}, & \theta \neq \phi, \\ \theta, & \text{otherwise}. \end{cases} \tag{1.1}$$

We can rewrite Eq (1.1) as

$$\begin{aligned}\mathrm{L}_{mean}(\theta,\phi) =& \left(\int_0^1 \frac{du}{(1-u)\theta+u\phi}\right)^{-1}\\ =& \left(\int_0^\infty \frac{du}{(\theta+u)(u+\phi)}\right)^{-1}, \qquad (\theta,\phi\in\mathbb{R}^+).\end{aligned}$$

Following [1], the $\mathrm{L}_{mean}(\theta,\phi)$ is continuous when $\theta=\phi$ and homogeneous and symmetric in θ and ϕ. Further, for $\theta,\phi\in\mathbb{R}^+$, we have

$$\sqrt{\theta\,\phi} < \mathrm{L}_{mean}(\theta,\phi) < \frac{\theta+\phi}{2}. \tag{1.2}$$

Recently, various studies and generalizations for the L_{mean} have been presented by several researchers (see e.g., [1–6]).

The Euler's beta function and the gamma function are defined by (see [7] and [8, Chapter 5, p.215])

$$\mathrm{B}(u,v) = \int_0^1 w^{u-1}(1-w)^{v-1}dw,\ u,\ v>0, \tag{1.3}$$

and

$$\Gamma(\sigma) = \int_0^\infty e^{-w}\,w^{\sigma-1}\,\mathrm{d}w \quad (\mathrm{Re}(\sigma)>0), \tag{1.4}$$

respectively.

An essential characteristic of the beta function is its tight relationship to the gamma function in the form

$$\mathrm{B}(u,v) = \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)}. \tag{1.5}$$

In recent years, the Euler's beta function has been a hotly disputed research area. Many authors have examined various extensions of the beta function (see, for example, [9–14]). The factor $\exp\left(\frac{-\ell}{w(1-w)}\right)$ has been used to extend the domain of the beta function to the entire complex plane. This function is called the extended beta function, which is defined by Choudhary et al. [9] and (see also [8, Chapter 5, p.244]) in the form

$$\mathrm{EB}\,(u,v;\ell) = \int_0^1 w^{u-1}(1-w)^{v-1}\exp\left(\frac{-\ell}{w(1-w)}\right)dw, \tag{1.6}$$

$$\Big(\mathrm{Re}(u)>0,\ \mathrm{Re}(v)>0,\ \text{and}\ \mathrm{Re}(\ell)>0\Big). \tag{1.7}$$

Moreover, this extension yields an exciting connection with several special functions such as Meijer G-function, Bessel function, Macdonald function, generalized hypergeometric function, Whittaker function, and Laguerre polynomial (see [8, Chapter 5, p.241–253]). Furthermore, this extension has applications in several fields, such as mathematics, engineering, or physics, (see, for instance, [15–17]).

In mathematics, it has mainly been used to derive certain probability distributions archived by Good [18] and other applications in fractional operators [8].

Recently, Raïssouli and Chergui [19] used the concept of a logarithmic mean in presenting the beta-logarithmic function (BL_{mean}) as follows

$$\begin{gathered}\mathrm{BL}_{mean}\left(\theta,\phi;\delta_1,\delta_2\right)=\int_0^1 \theta^{1-u}\ \phi^{u}\ u^{\delta_1-1}(1-u)^{\delta_2-1}du,\\ \Big(\mathrm{Re}(\delta_1)>0,\ \mathrm{Re}(\delta_2)>0,\ \theta,\phi\in\mathbb{R}^{+}\text{ such that }\theta\neq\phi\Big).\end{gathered}\tag{1.8}$$

Moreover, they proved that this integral satisfies several properties. It is clear that $\mathrm{BL}_{mean}\left(\theta,\phi;\delta_1,\delta_2\right)$ is an extension of both the logarithmic mean (1.1) and the beta function (1.3).

Motivated by the preceding literature, we offer new generalizations of the logarithmic mean and the extended beta function, along with supplying some of its features and applications. The article's organization is as follows: Section 2 introduces the extended logarithmic mean and the extended Euler's beta-logarithmic function based on the logarithmic mean in (1.1). Also, we provide the Mellin transform for the latter, besides proving several properties such as functional relations, inequalities, infinite sums, finite sums, integral formulas, and partial derivative representations. In Section 3, the numerical values of the comparison results and their graphical explanations are interpreted to study the behavior of the new generalizations using MATLAB. In Section 4, we give a new extension of the conventional beta distribution by using the extended Euler's beta-logarithmic function as an application in statistics. Eventually, we exhibit some concluding remarks in Section 5.

2. The extended Euler's beta-logarithmic function

In this section, we propose the following novel generalization of the standard Euler's beta function.

$$\begin{gathered}\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]=\int_0^1 \alpha^{1-w}\ \beta^{w}\ w^{u-1}(1-w)^{v-1}\ \exp\left(-\frac{\ell}{w(1-w)}\right)dw,\\ \Big(\alpha,\beta\in\mathbb{R}^{+}\text{ with }\alpha\neq\beta,\mathrm{Re}(u)>0,\mathrm{Re}(v)>0,\text{ and }\mathrm{Re}(\ell)>0\Big),\end{gathered}\tag{2.1}$$

which we call the extended Euler's beta-logarithmic function (EEBLF).

Remark 2.1. *Since*

$$\begin{aligned}0&\leq \alpha^{1-w}\ \beta^{w}\ w^{u-1}(1-w)^{v-1}\ \exp\left(-\frac{\ell}{w(1-w)}\right)\\ &\leq\ \kappa\ w^{u-1}(1-w)^{v-1}\ \exp\left(-\frac{\ell}{w(1-w)}\right)\ \textit{ for all } w\in(0,1),\end{aligned}$$

where $\kappa>0$ *for any fixed* $\alpha,\beta\in\mathbb{R}^{+}$*, the function* $w\to\alpha^{1-w}\ \beta^{w}$ *is continuous and bounded on* $[0,1]$*, and* $\left|\exp\left(-\frac{\ell}{w(1-w)}\right)\right|\leq\exp\left(\left|-\frac{\ell}{w(1-w)}\right|\right)\leq 1$ *for all* $\ell\in\mathbb{R}^{+}$*. Therefore, the integral* (2.1) *exists and is convergent for all* $w\in(0,1)$.

Remark 2.2. *The following relationships can be acquired straightway from* (2.1)*:*

$$\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]=\mathrm{EBL}\Big[\beta,\alpha;v,u;\ell\Big],\tag{2.2}$$

$$\mathrm{EBL}\Big[\alpha, \alpha; u, v; \ell\Big] = \alpha\ \mathrm{EB}\Big(u, v; \ell\Big), \tag{2.3}$$

$$\mathrm{EBL}\Big[c\alpha, c\beta; u, v; \ell\Big] = c\ \mathrm{EBL}\Big[\alpha, \beta; u, v; \ell\Big], \quad c > 0. \tag{2.4}$$

Remark 2.3. *We see certain particular cases of the* $\mathrm{EBL}\Big[\alpha, \beta; u, v; \ell\Big]$ *as follows:*

(i) *If* $\alpha = \beta = 1$, *then Eq* (2.1) *reduces to the extended beta function (EBF) defined in* (1.6).

(ii) *When* $\ell = 0$ *in* (2.1)*, we obtain the beta-logarithmic function (BLF) given in* (1.8).

(iii) *If* $u = v = 1$, *then Eq* (2.1) *reduces to a new extension of* L_{mean}, *which is called the extended logarithmic mean* (EL_{mean}) *as*

$$\begin{gathered}\mathrm{EL}_{mean}\Big[\alpha, \beta; \ell\Big] = \int_0^1 \alpha^{1-w}\, \beta^{w}\, \exp\left(-\frac{\ell}{w(1-w)}\right)dw, \\ \Big(\alpha, \beta \in \mathbb{R}^{+}\ \textit{with}\ \alpha \neq \beta,\ \textit{and}\ \mathrm{Re}(\ell) > 0\Big).\end{gathered} \tag{2.5}$$

(iv) *If we choose* $\ell = 0$ *in* (2.5)*, then we get the* L_{mean} *defined in* (1.1).

(v) *Taking* $\alpha = \beta = 1$ *and* $\ell = 0$ *in* (2.1)*, we have the classical beta function defined in* (2.1).

2.1. Some properties of the EEBLF

In this section, we establish essential characteristics of the EEBLF.

Theorem 2.1. *The* $\mathrm{EBL}\Big[\alpha, \beta; u, v; \ell\Big]$ *satisfies the following functional relation:*

$$\mathrm{EBL}\Big[\alpha, \beta; u+1, v; \ell\Big] + \mathrm{EBL}\Big[\alpha, \beta; u, v+1; \ell\Big] = \mathrm{EBL}\Big[\alpha, \beta; u, v; \ell\Big]. \tag{2.6}$$

Proof. From (2.1) into the LHS of (2.6), we obtain

$$LHS = \int_0^1 \alpha^{1-w}\, \beta^{w} \Big[w^{u}\,(1-w)^{v-1} + w^{u-1}\,(1-w)^{v}\Big] \exp\left(-\frac{\ell}{w(1-w)}\right)dw. \tag{2.7}$$

After simple calculations, we get the RHS of (2.6).

We state the following corollaries as direct results from (2.6), which were proved in previous works [8, 9, 19].

Corollary 2.1. *In case* $\ell = 0$ *in* (2.6)*, we have*

$$BL_{mean}\Big[\alpha, \beta; u+1, v\Big] + BL_{mean}\Big[\alpha, \beta; u, v+1\Big] = BL_{mean}\Big[\alpha, \beta; u, v\Big]. \tag{2.8}$$

Corollary 2.2. *Choosing* $\alpha = \beta = 1$ *in* (2.6)*, yields*

$$EB\,(u+1, v; \ell) + EB\,(u, v+1; \ell) = EB\,(u, v; \ell)\,. \tag{2.9}$$

Corollary 2.3. *When* $\ell = 0$ *and* $\alpha = \beta = 1$ *in* (2.6)*, we get*

$$B(u, v) = B(u+1, v) + B(u, v+1). \tag{2.10}$$

Theorem 2.2. *The following inequality holds for the* $\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]$*:*

$$\begin{aligned}&\min(\alpha,\beta)\ \leq \mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big] \leq \max(\alpha,\beta),\\&\quad\Big(\alpha,\beta\in\mathbb{R}^{+}\ \textit{with}\ \alpha\neq\beta, \mathrm{Re}(u)>0, \mathrm{Re}(v)>0,\ \textit{and}\ \mathrm{Re}(\ell)>0\Big).\end{aligned}\tag{2.11}$$

Proof. From (1.2) and (1.6), we observe that

$$\min(\alpha,\beta)\leq\ \sqrt{\alpha\beta}\leq \mathrm{L}_{mean}(\alpha,\beta)\leq\left(\frac{\alpha+\beta}{2}\right)\leq\max(\alpha,\beta),$$

and

$$\mathrm{EB}\Big[u,v;\ell\Big]>0.$$

Thus, we obtain

$$\min(\alpha,\beta)\leq \mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big].\tag{2.12}$$

According to Young's inequality

$$\alpha^{1-\tau}\,\beta^{\tau}\ \leq\alpha(1-\tau)+\beta\tau,\ \text{ for all }\tau\in[0,1],$$

and Eq (2.9), we find that

$$\begin{aligned}\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]&\leq\alpha\mathrm{EB}\Big[u,v+1;\ell\Big]+\beta\mathrm{EB}\Big[u+1,v;\ell\Big]\\&\leq\max(\alpha,\beta)\Bigg[\mathrm{EB}\Big[u,v+1;\ell\Big]+\mathrm{EB}\Big[u+1,v;\ell\Big]\Bigg]\\&\leq\max(\alpha,\beta).\end{aligned}\tag{2.13}$$

From (2.12) and (2.13), we arrive at the desired assertion in (2.6).

Corollary 2.4. *Let* $\alpha=\beta=1$ *in Theorem* (2.2)*, the following inequality holds:*

$$\mathrm{EB}\Big[u,v;\ell\Big]\leq\ \exp(-4\ell)\,B(u,v),\quad\Big(u>0,\ v>0,\ \textit{and}\ \ell\geq0\Big).\tag{2.14}$$

Proof. [8, Theorem 5.5, p.224] is available to view as proof.

Corollary 2.5. *For* $\ell=0$ *in Theorem* (2.2)*, we have*

$$\min(\alpha,\beta)\,B(u,v)\leq BL_{mean}\Big[\alpha,\beta;u,v\Big]\leq\max(\alpha,\beta)\,B(u,v),\quad\Big(u>0,\ v>0,\ \textit{and}\ \alpha,\beta>0\Big).\tag{2.15}$$

Proof. The proof can be viewed in [19, Proposition 2.2, p.133].

Theorem 2.3. *For* $\alpha,\beta\in\mathbb{R}^{+}$ *such that* $\alpha\neq\beta$, $\mathrm{Re}(u)>0$, $\mathrm{Re}(v)>0$, *and* $\mathrm{Re}(\ell)>0$, *the following finite sum holds:*

$$\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]=\sum_{\jmath=0}^{m}\binom{m}{\jmath}\mathrm{EBL}\Big[\alpha,\beta;u+\jmath,v+m-\jmath;\ell\Big].\tag{2.16}$$

Proof. From (2.1), we consider

$$\begin{aligned} \mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big] &= \int_0^1 \alpha^{1-w}\,\beta^{w}\,[w+(1-w)]\,w^{u-1}(1-w)^{v-1}\,\exp\left(-\frac{\ell}{w(1-w)}\right)dw \\ &= \mathrm{EBL}\Big[\alpha,\beta;u+1,v;\ell\Big] + \mathrm{EBL}\Big[\alpha,\beta;u,v+1;\ell\Big]. \end{aligned} \tag{2.17}$$

Similarly, we arrive at

$$\begin{aligned} \mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big] &= \int_0^1 \alpha^{1-w}\,\beta^{w}\,[w+(1-w)]\,w^{u-1}(1-w)^{v-1}\,\exp\left(-\frac{\ell}{w(1-w)}\right)dw \\ &= \mathrm{EBL}\Big[\alpha,\beta;u+2,v;\ell\Big] + \mathrm{EBL}\Big[\alpha,\beta;u,v+2;\ell\Big] + 2\,\mathrm{EBL}\Big[\alpha,\beta;u+1,v+1;\ell\Big]. \end{aligned} \tag{2.18}$$

Using mathematical induction, we attain the desired result (2.16).

Corollary 2.6. *The following finite sum holds:*

$$\mathrm{EB}\Big[u,v;\ell\Big] = \sum_{\jmath=0}^{m} \binom{m}{\jmath}\,\mathrm{EBL}\Big[u+\jmath,v+m-\jmath;\ell\Big]. \tag{2.19}$$

Proof. This directly results from (2.16) when $\alpha=\beta=1$.

Theorem 2.4. *For* $\alpha,\beta\in\mathbb{R}^+$ *such that* $\alpha\neq\beta$, $\mathrm{Re}(u)>0$, $\mathrm{Re}(v)>0$, *and* $\mathrm{Re}(\ell)>0$, *the* $\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]$ *satisfies the following infinite sums:*

(I)

$$\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big] = \sum_{\jmath=0}^{\infty}\mathrm{EBL}\Big[\alpha,\beta;u+\jmath,v+1;\ell\Big]. \tag{2.20}$$

(II)

$$\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big] = \sum_{\imath,\jmath=0}^{\infty}\mathrm{EBL}\Big[\alpha,\beta;u+\imath,v+\jmath;\ell\Big]\frac{(\ln(\alpha))^{\imath}(\ln(\beta))^{\jmath}}{\jmath!\,\imath!}. \tag{2.21}$$

(III)

$$\mathrm{EBL}\Big[\alpha,\beta;u,1-v;\ell\Big] = \sum_{\jmath=0}^{\infty}\frac{(v)_\jmath}{\jmath!}\mathrm{EBL}\Big[\alpha,\beta;u+\jmath,1;\ell\Big], \tag{2.22}$$

where $(v)_\jmath$ *denotes the Pochhammer symbol, which is defined in terms of the gamma function* $\Gamma(v)$ *as*

$$(v)_\jmath = \frac{\Gamma(v+\jmath)}{\Gamma(v)} = v(v+1)(v+2)...(v+(\jmath-1)),\quad \jmath\geq 1,\ (v)_0=1,\ \mathrm{Re}(v)>0.$$

Proof. To prove (2.20), we use the relation

$$(1-w)^{v-1}=(1-w)^{v}\sum_{r=0}^{\infty}w^{r}\quad |w|<1,$$

in the Definition (2.1). Thus, we get

$$\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]=\sum_{r=0}^{\infty}\int_{0}^{1}\alpha^{1-w}\,\beta^{w}\,w^{u+r-1}(1-w)^{v}\,\exp\left(-\frac{\ell}{w(1-w)}\right)dw,$$

which, given (2.1), we obtain the result (2.20).

To prove (2.21), substitute

$$\alpha^{1-w}=\sum_{r=0}^{\infty}\frac{(\ln\alpha)^{r}}{r!}\,(1-w)^{r},\quad |1-w|<\infty,$$

and

$$\beta^{w}=\sum_{s=0}^{\infty}\frac{(\ln\beta)^{s}}{s!}\,w^{r},\quad |w|<\infty,$$

in (2.1) yields

$$\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]=\int_{0}^{1}\sum_{r,s=0}^{\infty}\frac{(\ln\alpha)^{r}\,(\ln\beta)^{s}}{r!\,s!}\,w^{u+s-1}(1-w)^{v+r-1}\,\exp\left(-\frac{\ell}{w(1-w)}\right)dw.$$

Thus, in light of (2.1) and after simplification, we get the infinite sum in (2.21).

To demonstrate (2.22), we observe that

$$\mathrm{EBL}\Big[\alpha,\beta;u,1-v;\ell\Big]=\int_{0}^{1}\alpha^{1-w}\,\beta^{w}\,w^{u-1}(1-w)^{-v}\,\exp\left(-\frac{\ell}{w(1-w)}\right)dw.$$

Using the Binomial relation

$$(1-w)^{-v}=\sum_{r=0}^{\infty}\frac{(v)_{r}}{r!}\,w^{r},\qquad |w|<1,$$

we thus achieve

$$\mathrm{EBL}\Big[\alpha,\beta;u,1-v;\ell\Big]=\sum_{r=0}^{\infty}\frac{(v)_{r}}{r!}\int_{0}^{1}\alpha^{1-w}\,\beta^{w}\,w^{u+r-1}\,\exp\left(-\frac{\ell}{w(1-w)}\right)dw.$$

After simple calculation, we get the infinite sum in (2.22).

Corollary 2.7. *For $\alpha=\beta=1$ in Theorem 2.4, the following infinite sums hold:*

(i)

$$\mathrm{EB}\Big[u,v;\ell\Big] = \sum_{\jmath=0}^{\infty} \mathrm{EB}\Big[u+\jmath, v+1;\ell\Big]. \tag{2.23}$$

(ii)

$$\mathrm{EB}\Big[u,v;\ell\Big] = \sum_{\imath,\jmath=0}^{\infty} \mathrm{EB}\Big[u+\imath, v+\jmath;\ell\Big]. \tag{2.24}$$

(iii)

$$\mathrm{EB}\Big[u,1-v;\ell\Big] = \sum_{\jmath=0}^{\infty} \frac{(v)_\jmath}{\jmath!}\mathrm{EB}\Big[u+\jmath, 1;\ell\Big]. \tag{2.25}$$

Proof. [8, 9] are accessible for inspection as proof.

Corollary 2.8. *For $\ell = 0$ in Theorem 2.4, the following infinite sums hold:*

- $$BL_{mean}\Big[\alpha,\beta;u,v\Big] = \sum_{\jmath=0}^{\infty} BL_{mean}\Big[\alpha,\beta;u+\jmath, v+1\Big]. \tag{2.26}$$

- $$BL_{mean}\Big[\alpha,\beta;u,v\Big] = \sum_{\imath,\jmath=0}^{\infty} BL_{mean}\Big[\alpha,\beta;u+\imath, v+\jmath\Big]\frac{(\ln(\alpha))^\imath(\ln(\beta))^\jmath}{\jmath!\imath!}. \tag{2.27}$$

- $$BL_{mean}\Big[\alpha,\beta;u,1-v\Big] = \sum_{\jmath=0}^{\infty} \frac{(v)_\jmath}{\jmath!}BL_{mean}\Big[\alpha,\beta;u+\jmath, 1\Big]. \tag{2.28}$$

Proof. [19] is available for inspection as evidence.

Theorem 2.5. *For $\alpha,\beta \in \mathbb{R}^+$ such that $\alpha \neq \beta$, $\mathrm{Re}(u) > 0$, $\mathrm{Re}(v) > 0$, and $\mathrm{Re}(\ell) > 0$, the $\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]$ satisfies the following integral representations:*

(I)

$$\begin{aligned}\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big] =& 2\alpha \int_0^{\frac{\pi}{2}} \left(\frac{\beta}{\alpha}\right)^{\cos^2(\varphi)} \cos^{2u-1}(\varphi)\sin^{2v-1}(\varphi) \\ &\times \exp\Big(-\ell\sec^2(\varphi)\csc^2(\varphi)\Big)\,d\varphi.\end{aligned} \tag{2.29}$$

(II)

$$\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big] = e^{-2\ell} \int_0^{\infty} (\alpha)^{\frac{\tau}{\tau+1}} (\beta)^{\frac{1}{\tau+1}} \frac{\tau^{v-1}}{(1+\tau)^{u+v}} \exp\Big(-\ell(\tau+\tau^{-1})\Big)\,d\tau. \tag{2.30}$$

(III)

$$\text{EBL}\Big[\alpha,\beta;u,v;\ell\Big] = \sqrt{\alpha\beta}\,2^{1-u-v}\int_0^{\infty}\left(\frac{\beta}{\alpha}\right)^{\frac{\tau}{2}}(1+\tau)^{u-1}(1-\tau)^{v-1} \times \exp\Big(-4\ell/(1-\tau^2)\Big)\,d\tau. \tag{2.31}$$

Proof. Every situation is straightforward. Taking $w = \cos(\varphi)$ in (2.1) yields (2.29) after calculations. Also, replacing $w = \frac{\tau}{\tau+1}$ in (2.1) provides (2.30). Similarly, substituting $w = \frac{1+\tau}{2}$ in (2.1) gives the result (2.31).

Corollary 2.9. *When $\ell = 0$ in Theorem 2.5, the following integral formulas hold:*

- $$BL_{mean}\Big[\alpha,\beta;u,v\Big] = 2\alpha\int_0^{\frac{\pi}{2}}\left(\frac{\beta}{\alpha}\right)^{\cos^2(\varphi)}\cos^{2u-1}(\varphi)\sin^{2v-1}(\varphi)\,d\varphi.$$
- $$\text{EBL}\Big[\alpha,\beta;u,v\Big] = \int_0^{\infty}(\alpha)^{\frac{\tau}{\tau+1}}\,(\beta)^{\frac{1}{\tau+1}}\,\frac{\tau^{v-1}}{(1+\tau)^{u+v}}\,d\tau.$$
- $$\text{EBL}\Big[\alpha,\beta;u,v\Big] = \sqrt{\alpha\beta}\,2^{1-u-v}\int_0^{\infty}\left(\frac{\beta}{\alpha}\right)^{\frac{\tau}{2}}(1+\tau)^{u-1}(1-\tau)^{v-1}d\tau.$$

Proof. The reference [19] is available to view as evidence.

Remark 2.4. *Applying the results in Remark 2.3 for Theorems 2.3–2.5 generates other correspondent results in [8, 9, 19].*

Theorem 2.6. *For $\alpha,\ \beta \in \mathbb{R}^+$ such that $\alpha \neq \beta$, $\text{Re}(u) > 0$, $\text{Re}(v) > 0$, and $\text{Re}(\ell) > 0$, the Mellin transform of EEBLF is*

$$\mathcal{M}\Big\{\text{EBL}\Big[\alpha,\beta;u,v;\ell\Big];\sigma\Big\} = \Gamma(\sigma)\,BL_{mean}\Big[\alpha,\beta;u+\sigma,v+\sigma\Big], \tag{2.32}$$

where the Mellin transform is defined in [8] by

$$\mathcal{M}\{f(x);\sigma\} = \int_0^{\infty}x^{\sigma-1}f(x)\,dx,\ \text{Re}(\sigma) > 0, \tag{2.33}$$

provided that the integral (2.33) exists.

Proof. Applying (2.33) to (2.1), we observe that

$$\begin{aligned}\mathcal{M}\Big\{\text{EBL}\Big[\alpha,\beta;u,v;\ell\Big];\sigma\Big\} &= \int_0^{\infty}\ell^{\sigma-1}\text{EBL}\Big[\alpha,\beta;u,v;\ell\Big]\,d\ell \\ &= \int_0^{\infty}\ell^{\sigma-1}\left(\int_0^{1}\alpha^{1-w}\,\beta^{w}\,w^{u-1}(1-w)^{v-1}\ \exp\left(-\frac{\ell}{w(1-w)}\right)dw\right)d\ell\end{aligned}$$

$$= \int_0^1 \alpha^{1-w} \beta^w w^{u-1}(1-w)^{v-1} \left(\int_0^\infty \ell^{\sigma-1} \exp\left(-\frac{\ell}{w(1-w)}\right) d\ell\right) dw.$$

Replacing the inner integral by $z = \frac{\ell}{w(1-w)}$ and after simplification, we have

$$\mathcal{M}\Big\{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big];\sigma\Big\} = \Gamma(\sigma) \int_0^1 \alpha^{1-w} \beta^w w^{u+\sigma-1}(1-w)^{v+\sigma-1}\, dw. \tag{2.34}$$

Inserting (1.8) into (2.34), we arrive at (2.32).

Setting $\alpha = \beta = 1$ in (2.32), we get a desired Mellin transform involving the extended beta function (cf., [8, 9]), which is archived in the following corollary.

Corollary 2.10. *Let* $\ell \in \mathbb{R}^+$, $\mathrm{Re}(u) > 0$, *and* $\mathrm{Re}(v) > 0$. *Then*

$$\mathcal{M}\Big\{\mathrm{EB}\Big[u,v;\ell\Big];\sigma\Big\} = \Gamma(\sigma)\, B\Big[u+\sigma, v+\sigma\Big],\ \mathrm{Re}(\sigma) > 0. \tag{2.35}$$

Theorem 2.7. *For* $k, h \in \mathbb{N}_0$, *the following higher-order derivatives are valid for the* $\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]$*:*

(I)

$$\frac{\partial^k}{\partial \ell^k}\left\{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]\right\} = (-1)^k\, \mathrm{EBL}\Big[\alpha,\beta;u-k,v-k;\ell\Big], \quad \mathrm{Re}(\ell) > 0. \tag{2.36}$$

(II)

$$\begin{aligned} &\frac{\partial^k}{\partial \alpha^k}\left\{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]\right\} \\ &= \sum_{r=0}^{\infty} \frac{1}{\alpha^k (r-k)!} (\ln(\alpha))^{r-k} \mathrm{EBL}\Big[1,\beta;u,v+r;\ell\Big], \quad \mathrm{Re}(\alpha) > 0,\ r > k. \end{aligned} \tag{2.37}$$

(III)

$$\begin{aligned} &\frac{\partial^k}{\partial \beta^k}\left\{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]\right\} \\ &= \sum_{r=0}^{\infty} \frac{1}{\beta^k (r-k)!} (\ln(\beta))^{r-k}\, \mathrm{EBL}\Big[\alpha,1;u+r,v;\ell\Big], \quad \mathrm{Re}(\beta) > 0,\ r > k. \end{aligned} \tag{2.38}$$

(IV)

$$\begin{aligned} \frac{\partial^{k+h}}{\partial v^h\, \partial u^k}\left\{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]\right\} &= \int_0^1 \alpha^{1-w} \beta^w w^{u-1}(1-w)^{v-1} \\ &\quad \times \ln^k(w)\ \ln^h(1-w) \exp\left(-\frac{\ell}{w(1-w)}\right) dw, \end{aligned} \tag{2.39}$$

$$\Big(\alpha,\ \beta \in \mathbb{R}^+ \text{ with } \alpha \neq \beta,\ \mathrm{Re}(u) > 0,\ \mathrm{Re}(v) > 0,\ \text{and}\ \mathrm{Re}(\ell) > 0\Big).$$

Proof. The proof is direct by the differentiation of (2.1) for the parameters ℓ, α, β, u, and v, respectively. Thus, we obtain the results in (2.36)–(2.39) applying mathematical induction.

Remark 2.5. *We can attain several outcomes in the literature using the results considered in Remark 2.3 for Theorem 2.7 (e.g., [1, 7–9, 17, 19–21]).*

3. Numerical representations and graphs

The numerical representations of the values of the new generalizations of the logarithmic mean and Euler's beta logarithmic function, besides some of its exceptional cases, are given in the form of tabulated data and graphical outcomes utilizing the MATLAB program in this section.

First, Table 1 shows tabular expressions of the EL_{mean} for various values of the parameters $\alpha,\ \beta$, and ℓ. For different ranges of values of these parameters, one can note the increase in the EL_{mean} values as the value of ℓ decreases. As evident, the last column corresponds to L_{mean} since $\ell = 0$. That explains the agreement of the values in the last two columns in this table when ℓ is close to zero. Figure 1 depicts the plots of the difference in the infinity norm, $\|\mathbf{z}\|_\infty := \max\limits_{J} \left|z_J\right|$, between the EL_{mean} and L_{mean} where the fixed values of the parameters α and β are chosen to be $\{0.1,\ 0.825,\ 2.275,\ 3\}$ and $\{0.25,\ 0.5625,\ 1.1875,\ 1.5\}$, respectively, against the values of the parameter $\ell \in \{10^{-3},\ 8.9\times 10^{-4},\ 7.7\times 10^{-4},\ 6.7\times 10^{-4},\ 5.5\times 10^{-4},\ 4.4\times 10^{-4},\ 3.3\times 10^{-4},\ 2.2\times 10^{-4},\ 1.1\times 10^{-4},\ 10^{-8}\}$. The graph assures that the infinity norm $\|\mathrm{EL}_{mean} - \mathrm{L}_{mean}\|_\infty$ tends to zero when ℓ approaches 0 and increases otherwise.

Second, Table 2 shows tabular representations of the EEBLF for the different values of α, β, u, v, and ℓ. Here, we chose the thier values as $u \in \{0.1,\ 0.1,\ 0.25,\ 0.4,\ 0.55,\ 0.7,\ 0.85,\ 1\}$, $v \in \{0.25,\ 0.5,\ 0.75,\ 1\}$, $\ell \in \{0,\ 0.4,\ 0.80\}$, and $\alpha = \beta \in \{1,\ 3,\ 6,\ 9\}$. It should be mentioned that the values of EEBLF increase as ℓ decreases while fixing the values of other parameters. Choosing $\alpha = \beta = 1$ and $v = 0.25$ allows the reader to compare the values of EEBLF in this table with those presented for extended beta function in [8, Chapter 5, p.268–278].

By selecting fixed values of $\alpha = 0.1, 0.825, 2.275, 3$ and $\beta = 0.25, 0.5625, 1.1875, 1.5$ versus equal values of $u = v \in [0.1, 0.99]$, Figure 2 illustrates the behavior of the $\|\mathrm{EBL} - \mathrm{EL}_{mean}\|_\infty$ for distinct values of $\ell = 0, 10^{-3}, 7.5\times 10^{-4}, 5\times 10^{-4}, 2.5\times 10^{-4},\ 10^{-8}$. The concluded results can be interpreted as the difference in the infinity norm decreasing to zero as both u and v approach one for any chosen value of ℓ.

Finally, as is seen in Tables 1 and 2 and Figures 1 and 2, one can conclude that EEBLF is a more generalized form than those presented in the previous studies for the extensions of the beta function [8, 10, 22].

Table 1. Comparison of numerical values of EL_{mean} in (2.5) for different values for all α, β, and ℓ.

N	α	β	EL_{mean}					
			$\ell = 10^{-3}$	$\ell = 7.5 \times 10^{-4}$	$\ell = 5 \times 10^{-4}$	$\ell = 2.5 \times 10^{-4}$	$\ell = 10^{-8}$	$\ell = 0$
1	0.1	0.25	0.16121	0.16176	0.16233	0.16296	0.1637	0.1637
2	0.1	0.5625	0.2633	0.26427	0.26531	0.26642	0.26777	0.26777
3	0.1	0.875	0.35094	0.35232	0.35378	0.35537	0.3573	0.3573
4	0.1	1.1875	0.4313	0.43307	0.43495	0.437	0.43949	0.43949
5	0.1	1.5	0.50698	0.50913	0.51142	0.51392	0.51698	0.51698
6	0.825	0.25	0.47407	0.47572	0.47746	0.47935	0.48161	0.48161
7	0.825	0.5625	0.67528	0.67751	0.67985	0.68238	0.68539	0.68539
8	0.825	0.875	0.83731	0.84005	0.84294	0.84605	0.84975	0.84975
9	0.825	1.1875	0.9806	0.98383	0.98723	0.9909	0.99527	0.99527
10	0.825	1.5	1.1122	1.1159	1.1198	1.1241	1.1291	1.1291
11	1.55	0.25	0.70045	0.70307	0.70585	0.70887	0.7125	0.7125
12	1.55	0.5625	0.95925	0.96254	0.96601	0.96975	0.97423	0.97423
13	1.55	0.875	1.1629	1.1668	1.1709	1.1753	1.1805	1.1805
14	1.55	1.1875	1.3407	1.3451	1.3498	1.3548	1.3607	1.3607
15	1.55	1.5	1.5025	1.5074	1.5126	1.5182	1.5249	1.5249
16	2.275	0.25	0.9006	0.90415	0.90793	0.91203	0.91701	0.91701
17	2.275	0.5625	1.2059	1.2102	1.2147	1.2196	1.2255	1.2255
18	2.275	0.875	1.4428	1.4477	1.4529	1.4585	1.4652	1.4652
19	2.275	1.1875	1.6477	1.6532	1.659	1.6653	1.6727	1.6727
20	2.275	1.5	1.8332	1.8392	1.8456	1.8525	1.8607	1.8607
21	3	0.25	1.086	1.0905	1.0952	1.1004	1.1067	1.1067
22	3	0.5625	1.432	1.4372	1.4428	1.4488	1.4561	1.4561
23	3	0.875	1.6975	1.7035	1.7097	1.7165	1.7246	1.7246
24	3	1.1875	1.9259	1.9324	1.9394	1.9468	1.9557	1.9557
25	3	1.5	2.1316	2.1387	2.1463	2.1544	2.164	2.164

Table 2. Comparison of numerical values of $EEBLF$ in (2.1) for different values for all α, β, u, v, and ℓ.

N	$\alpha=\beta$	u	v	EEBLF		
				$\ell=0$	$\ell=0.40$	$\ell=0.80$
1	1	0.1	0.25	13.547	0.3861	0.058419
2	1	0.25	0.25	7.4163	0.34229	0.05213
3	1	0.4	0.25	5.8075	0.30491	0.046648
4	1	0.55	0.25	5.0329	0.27285	0.041855
5	1	0.7	0.25	4.5627	0.2452	0.037651
6	1	0.85	0.25	4.2397	0.22123	0.033953
7	1	1	0.25	4	0.20035	0.030691
8	3	0.1	0.5	33.969	0.96446	0.14647
9	3	0.25	0.5	15.732	0.84903	0.13015
10	3	0.4	0.5	11.037	0.75113	0.11597
11	3	0.55	0.5	8.8274	0.66763	0.10363
12	3	0.7	0.5	7.5174	0.59602	0.092838
13	3	0.85	0.5	6.638	0.53431	0.083383
14	3	1	0.5	6	0.48085	0.075074
15	6	0.1	0.75	62.876	1.6251	0.24658
16	6	0.25	0.75	26.657	1.421	0.21819
17	6	0.4	0.75	17.479	1.2488	0.19363
18	6	0.55	0.75	13.24	1.1028	0.17232
19	6	0.7	0.75	10.776	0.97823	0.15376
20	6	0.85	0.75	9.1543	0.87145	0.13756
21	6	1	0.75	8	0.77946	0.12337
22	9	0.1	1	90	2.0755	0.31341
23	9	0.25	1	36	1.8032	0.27622
24	9	0.4	1	22.5	1.5747	0.24415
25	9	0.55	1	16.364	1.3818	0.21642
26	9	0.7	1	12.857	1.2182	0.19236
27	9	0.85	1	10.588	1.0786	0.17142
28	9	1	1	9	0.95902	0.15315

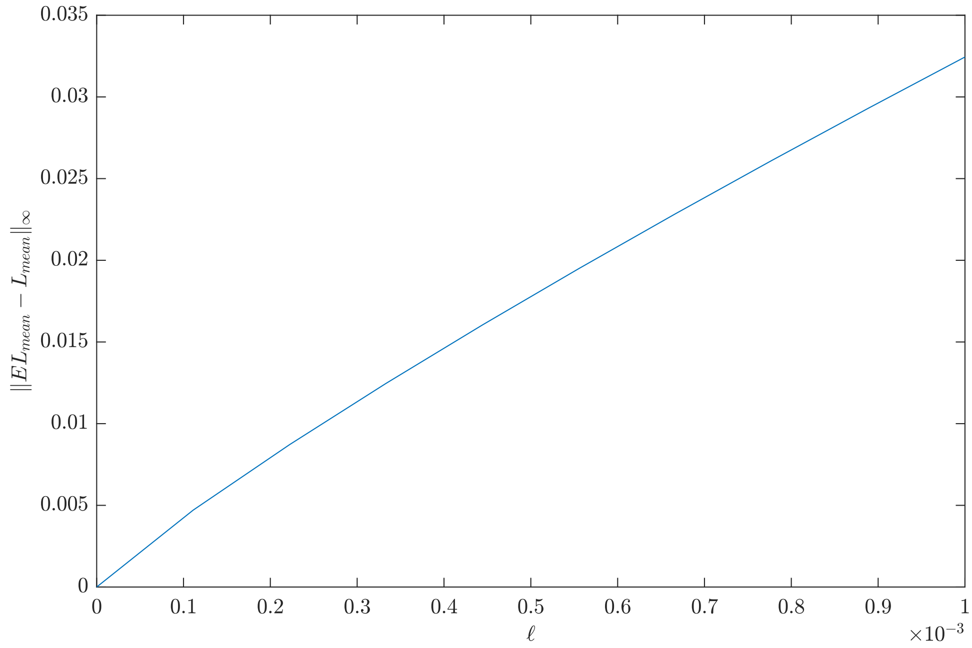

Figure 1. Graphical representation of $||\mathrm{EL}_{mean} - \mathrm{L}_{mean}||_\infty$ various values of ℓ.

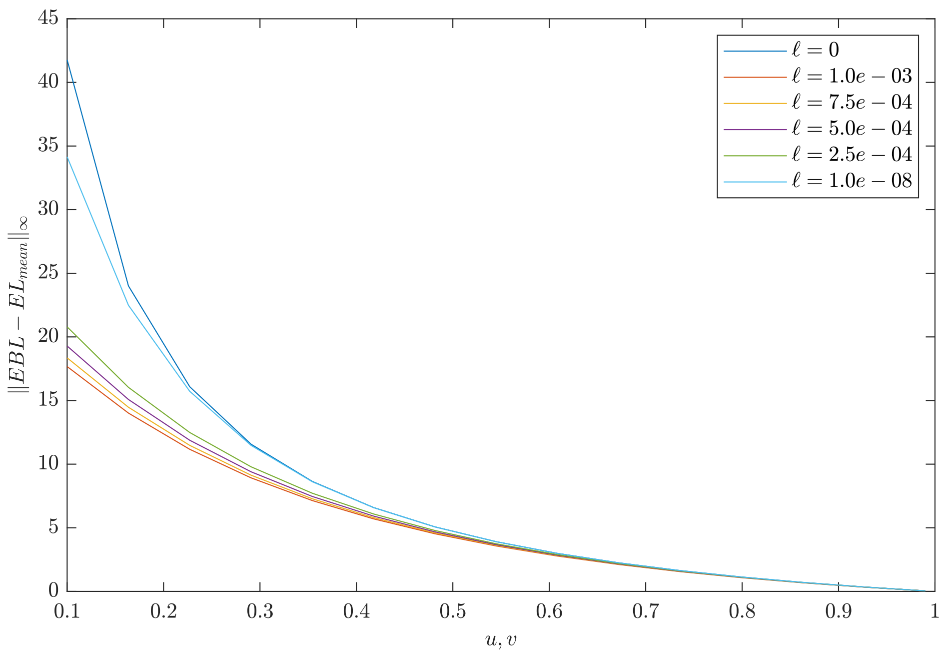

Figure 2. Plots of $||\mathrm{EBL} - \mathrm{EL}_{mean}||_\infty$ with equal values of u, v, and various values of ℓ.

4. An application

The beta distribution is a type of continuous probability distribution defined on the interval $[0, 1]$ by any two positive parameters, which appear as exponents of the random variable and control the shape of the distribution (see [23–26]). As a helpful distribution, it can be rescaled and shifted to produce distributions with diverse shapes over any finite range. The beta function can take on various shapes depending on the values of the two parameters. Later, typical beta distributions were introduced in [8, 10, 13, 14, 16] using expanded beta functions. They suggested that these distributions could be useful for analyzing and reviewing techniques employed in specific circumstances during project evaluation and review.

In this section, we define a random variable associated with $\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]$ and discuss some of their properties.

Definition 4.1. *For* $\alpha,\ \beta \in \mathbb{R}^+$ *such that* $\alpha \neq \beta$, $\mathrm{Re}(u) > 0$, $\mathrm{Re}(v) > 0$*, and* $\mathrm{Re}(\ell) > 0$*, the extended Euler's beta logarithmic distribution (EEBLD) is defined as*

$$f(w) = \begin{cases} \frac{1}{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]}\alpha^{1-w}\beta^{w}w^{u-1}(1-w)^{v-1}\ \exp\left(-\frac{\ell}{w(1-w)}\right), & (0 < w < 1), \\ \\ 0, & \textit{otherwise.} \end{cases} \tag{4.1}$$

We have the ϖ^{th} moment of a random variable χ as for any real number ϖ.

$$\begin{aligned} &\mathcal{E}\left(\chi^{\varpi}\right) = \frac{\mathrm{EBL}\Big[\alpha,\beta;u+\varpi,v;\ell\Big]}{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]}, \\ &\left(\alpha,\beta \in \mathbb{R}^+ \text{such that } \alpha \neq \beta, \mathrm{Re}(u) > 0, \mathrm{Re}(v) > 0, \text{and } \mathrm{Re}(\ell) > 0\right). \end{aligned} \tag{4.2}$$

When $\varpi = 1$, the mean is obtained as a special case of (4.2) given by

$$\mu = \mathcal{E}(\chi) = \frac{\mathrm{EBL}\Big[\alpha,\beta;u+1,v;\ell\Big]}{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]}. \tag{4.3}$$

The variance of a distribution is discussed as follows:

$$\begin{aligned} \sigma^2 &= \mathcal{E}\left(\chi^2\right) - \{\mathcal{E}(\chi)\}^2 \\ &= \frac{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]\mathrm{EBL}\Big[\alpha,\beta;u+2,v;\ell\Big] - \Big\{\mathrm{EBL}\Big[\alpha,\beta;u+1,v;\ell\Big]\Big\}^2}{\Big\{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]\Big\}^2}. \end{aligned} \tag{4.4}$$

The moment generating function (MGF) of the distribution is defined as

$$\mathcal{M}(w) = \sum_{m=0}^{\infty}\frac{w^m}{m!}\mathcal{E}\left(\chi^m\right) = \frac{1}{\mathrm{EBL}\Big[\alpha,\beta;u,v;\ell\Big]}\sum_{m=0}^{\infty}\mathrm{EBL}\Big[\alpha,\beta;u+m,v;\ell\Big]\frac{w^m}{m!}. \tag{4.5}$$

Proposition 4.1. *Let χ represent the extended Euler's beta logarithmic random variable with parameters $(\alpha, \beta\,; u, v)$. Then, for any $\varpi, \varepsilon > 0$, the following suppositions are true:*

$$\left|\mathcal{P}\Big(\chi \leq \varepsilon\Big) - \frac{\mathrm{EBL}\Big[\alpha, \beta; u, v+1; \ell\Big]}{\mathrm{EBL}\Big[\alpha, \beta; u, v; \ell\Big]}\right| \leq \frac{1}{2} + \left|\varepsilon - \frac{1}{2}\right|, \tag{4.6}$$

and

$$\mathcal{P}\left(\chi^{\varpi} \geq \varepsilon\right) \leq \frac{\mathrm{EBL}\Big[\alpha, \beta; u+\varpi, v; \ell\Big]}{\varepsilon \mathrm{EBL}\Big[\alpha, \beta; u, v; \ell\Big]}. \tag{4.7}$$

Proof. By invoking (2.6) and (4.3), we have

$$\mathcal{E}(\chi) = 1 - \frac{\mathrm{EBL}\Big[\alpha, \beta; u, v+1; \ell\Big]}{\mathrm{EBL}\Big[\alpha, \beta; u, v; \ell\Big]}, \tag{4.8}$$

using Lemma 3.1 in [19, 24], we arrive at the required result (4.6).

The second inequality (4.7) can be easily obtained by an application of Markov's inequality, namely

$$\mathcal{P}\left(\chi^{\varpi} \geq \varepsilon\right) \leq \frac{\mathcal{E}\left(\chi^{\varpi}\right)}{\varepsilon}, \tag{4.9}$$

and the definition of $\mathcal{E}\left(\chi^{\varpi}\right)$, we obtain the coveted result (4.7).

Remark 4.1. *As special cases of the results in this part,*

- *we get Proposition 3.2 in [19, pp.137] from* (4.1) *when $\ell = 0$;*
- *we can reduce the symmetric results in [8, Chapter 5, p.258] and [16] when $\alpha = \beta = 1$ in* (4.1)*;*
- *we can obtain the previous results in [23–26] when $\alpha = \beta = 1$ and $\ell = 0$ in* (4.1)–(4.5).

5. Conclusions

Recently, a great variety of extensions of the beta function have been broadly and usefully employed in describing and solving several vital problems of statistics, physics, probability theory, and astrophysics [10–14, 22–26]. In particular, it should be mentioned that further generalizations of classical beta function and logarithmic mean have been introduced and investigated in [19], and their properties and applications have been archived. The works above will motivate the new studies, where the authors can introduce a more general definition of the beta function and logarithmic mean defined in (2.1) of this manuscript. This newly defined function, which is called EEBLF, will have many prospects and applications in different fields.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

This work does not have any conflict of interest.

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