Mathematics

Research article

Mild solutions for fractional non-instantaneous impulses integro-differential equations with nonlocal conditions

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Abstract: In this paper, we investigated Caputo fractional integro-differential equations with non-instantaneous impulses and nonlocal conditions. By employing the solution operator, the Mönch fixed point theorem, and the stepwise estimation method, we eliminated the Lipschitz condition of the nonlinear term, while also dispensing with the requirement for the compressibility coefficient condition $0 < k < 1$. The main results presented represented a generalization and enhancement of previous findings. Furthermore, an example was provided to verify the application of our main results.

1. Introduction

Fractional differential equations have developed rapidly due to their applications, see [1–7]. Compared to classical integer-order equations, fractional differential equations offer a more accurate representation of memory effects in diverse materials and processes. Many scholars have devoted their efforts to exploring this topic and have achieved significant advancements. By using semigroup theory, the noncompact measures conditions, the references [8] studied local and global existence and uniqueness of mild solutions for fractional impulsive evolution differential equations of order $0 < \alpha \leq 1$. By using semigroup theory, the properties of noncompact measures, the references [10–12] studied local and global existence of solutions for a class of fractional evolution equations of order $0 < \alpha \leq 1$. By the generalized Darbo fixed point theorem and Schauder fixed point theorem, the reference [14] investigated the existence and uniqueness of mild solutions for fractional impulsive integro-differential evolution equations of order $1 < \beta \leq 2$ with nonlocal conditions. By using the cosine family theory, the Mönch fixed point theorem, the reference [15] studied the existence of mild

solutions for fractional impulsive integro-differential evolution equations of order $1 < \beta \le 2$ with nonlocal conditions. By using the two-step Adomian decomposition method, the reference [16, 17] found an analytical solution of for fractional order integro-differential equation, meanwhile, established results for the existence and uniqueness of a solution with the help of some fixed point theorems and the Banach contraction principle. Many evolution processes are characterized by abrupt state changes, which can be effectively modeled using impulsive differential equations. There are two types of impulses: instantaneous impulses and non-instantaneous impulses. The theory of instantaneous impulses often describes some processes which have a sudden change in their states at certain times, particularly in the fields of biology, dynamics, physics, and engineering. However, in the management and optimal control of life phenomena, there exist numerous non-instantaneous impulsive phenomena. The classical instantaneous impulsive differential system is not suitable to describe such phenomena, whereas differential equations with the non-instantaneous impulse offer a more suitable framework. Differential equations with non-instantaneous impulses have found extensive applications in areas such as pesticide dynamics, pest control, integrated environmental management, and fishery resources. Therefore, it is worthwhile to explore the non-instantaneous impulsive differential system. Many scholars have dedicated their efforts to investigating this subject and have achieved notable advancements. For further details, please refer to [18–23]. In comparison to the results of differential equations with instantaneous impulses, there are fewer results available for differential equations with non-instantaneous impulses.

The non-instantaneous impulses were first proposed by Hernandez and Regan [18]. In [18], the authors studied the following integer order equations:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), t \in (s_i, t_{i+1}], i = 1, 2, ..., n, \\ u(t) = g_i(t, u(t)), t \in (t_i, s_i], i = 1, 2, ..., n, \\ u(0) = u_0, \end{cases}$$

where $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup of bounded linear operators $(T(t)), t \ge 0$ is defined on a Banach space X, and the impulses start abruptly at the points t_i and their action continues on the interval $[t_i, s_i]$. Utilizing the semigroup theory and fixed point theory, they established the existence of mild solutions.

Pierri et al. [20] studied the following differential equation:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), t \in (s_i, t_{i+1}], \\ x(t) = g_i(t, N_i(t)(x)), t \in (t_i, s_i], i \in \mathbb{N}, \\ x(0) = x_0, \end{cases}$$

where they gave the existence of mild solutions on $[0, + \propto)$.

The literature [21] considered the following fractional nonautonomous evolution equations with delay:

$$\begin{cases} {}^C D_t^\beta x(t) = A(t)x(t) + f(t, x(\tau_1(t)), x(\tau_2(t)), ..., x(\tau_l(t))), t \in J = [0, T], \\ x(0) = x_0 \in E, \end{cases}$$

where $A(t)_{t \in J}$ is a family of closed and linear operators defined on a dense domain $D(A)$ in Banach space E into E and the operator $A(t)$ depends on t.

In [18, 20], the authors studied the integer order equation with non-instantaneous impulse, where the operator A did not depend on t. In the process of proving the theorem, the nonlinear function f required the Lipschitz condition; meanwhile, the compression coefficient was less than 1. In [21], the authors derived mild solutions using the Banach contraction mapping principle without employing the contraction coefficient $0 < k < 1$.

Inspired by the literature mentioned earlier, we explore the following nonlinear fractional integro-differential equations with nonlocal conditions and non-instantaneous impulses:

$$\begin{cases} {}^{C}_{0}D^{\beta}_{t}u(t) = A(t)u(t) + f(t, u(t), (H_1u)(t), (H_2u)(t)), \ t \in (s_i, t_{i+1}], i = 1, 2, ..., n, \\ u(t) = l_i(t, u(t)), t \in (t_i, s_i], i = 1, 2, ..., n, \\ u(0) + g_1(u) = u_0 \in X, \end{cases} \tag{1.1}$$

where ${}^{C}_{0}D^{\beta}_{t}$ represents the Caputo fractional derivatives of order $\beta \in (0, 1]$, $u(\cdot) \in X$, $A(t) \subset X$ is a closed linear operator, and $A(t)$ depends on t. $J = [0, T]$, $0 = s_0 < t_1 \leq s_1 < t_2 \leq ... < t_n \leq s_n < t_{n+1} = T < +\infty$. $f \in C[J \times X^3, X]$, $l_i : J \times X \to X$, and $g_1 : X \to X$ are also continuous. H_1 and H_2 are defined by

$$H_1u(t) = \int_0^t h_1(t, s, u(s))ds, \ \ H_2u(t) = \int_0^T h_2(t, s, u(s))ds,$$

$$D = \{(t, s) \mid 0 \leq s \leq t \leq T\}, h_1 \in C[D \times X, X], \ \ h_2 \in C[J \times J \times X, X].$$

In this paper, we utilize the Mönch fixed point theorem, the non-compact measure, and the method of stepwise estimation to eliminate the Lipschitz condition for the nonlinear function f. Additionally, we do not require the compressibility coefficient to be less than 1. The key findings in this study enhance and extend some previous results.

2. Preliminaries

We suppose that $(X, \| \cdot \|)$ is a Banach space, $J = [0, T]$, $J_0 = [0, t_1]$, $J_i = (t_i, t_{i+1}], i = 1, 2, ...n - 1$, $J_n = (t_n, T]$, and

$$PC[J, X] = \{u : J \to X \mid u \in C[(t_i, t_{i+1}], X],$$

and there exist

$$u(t_i^-), u(t_i^+) \text{ with } u(t_i^+) = u(t_i^-), i = 1, 2, .., n\}.$$

Obviously, $PC[J, X]$ is a Banach space endowed with PC-norm $\|u\|_{pc} = \sup_{t \in J} u(t)$.

Definition 2.1. [2] The Caputo fractional derivative of order $\beta \in (0, 1]$ for a function $u(t)$ is given by

$$^{C}_{t_0}D^{\beta}_{t} = \frac{1}{\Gamma(p - \beta)} \int_{t_0}^{t} \frac{u^{(p)}(s)}{(t - s)^{1+\beta-p}} ds,$$

where $p = [\beta] + 1$.

Definition 2.2. [24] We assume that $A(t) : D(A) \subset E \to E$ is a closed linear operator. We define the resolvent set of $A(t)$ by $\rho[A(t)]$. If there exist a nonnegative constant $\omega \geq 0$ and a strongly continuous function $U_\beta : R_+^2 \to B(E)$ satisfying

$$(\lambda^{\beta}I - A(s))^{-1}x = \int_0^{\infty} e^{-\lambda(t-s)}U_\beta(t, s)x dt, Re\lambda > \omega, x \in E,$$

then we call $A(t)$ the generator of a β-resolvent family. $U_\beta(t,s)$ is called the β-resolvent family generated by $A(t)$.

Property 2.1. [24] $U_\beta(x,y)$ has the following properties:

(1) $U_\beta(x,x) = I, U_\beta(x,y) = U_\beta(x,r)U_\beta(r,y)$ for $0 \le x \le r \le y \le T$;

(2) For $0 \le x \le y \le T$, $(x,y) \to U_\beta(x,y)$ is strongly continuous.

Definition 2.3. [25] We suppose that D is a space and mapping $T : D \to D$. If there exists $x_0 \in D$ such that $Tx_0 = x_0$, then $x_0 \in D$ is called the fixed point of operator T.

Lemma 2.1. [14] Suppose $B \subset PC[J,X]$ is equi-continuous and bounded, then for $t \in J_i$, $\alpha(\{u(t)|u \in B\})$ is continuous and

$$\alpha\left(\left\{\int_J u(t)dt : u \in B\right\}\right) \le \int_J \alpha\left(B(t)\right) dt.$$

Lemma 2.2. [26] Suppose $B \subset PC[J,X]$ is equi-continuous and bounded, then $\overline{CoB} \subset PC[J,X]$ is equi-continuous and bounded.

Lemma 2.3. [27] For any $R > 0$, the function f is uniformly continuous and bounded on $J \times B_R^4$ where $B_R = \{u \in X : \|u\| \le R\}$, if $B \subset PC[J,X]$ is equi-continuous and bounded, then $\{f(t,u(t),u(\tau(t)),(H_1u)(t),(H_2u)(t) : u \in B\} \subset PC[J,X]$ is also equi-continuous and bounded.

Lemma 2.4. [27] Suppose $B \subset PC[J,X]$ is bounded and equi-continuous, then

$$\alpha_{PC}(B) = \sup_{t \in J} \alpha(B(t)).$$

Lemma 2.5. [6] Suppose $B \subset PC[J,X]$ is a bounded set, then

$$\alpha(B(J) \le \alpha_{PC}(B)).$$

Lemma 2.6. [27] Suppose that $r \in C[J,R^+]$ meets

$$r(t) \le N_1 \int_0^t r(s)ds + N_2 t \int_0^t r(s)ds + N_3 t \int_0^T r(s)ds, t \in J,$$

where $N_1 > 0, N_2 \ge 0, N_3 \ge 0$ are constants. If one of the following conditions holds

(i) $TN_3(e^{T(N_1+TN_2)} - 1) < N_1 + TN_2$,

(ii) $T(2N_1 + TN_2 + TN_3) < 2$,

then $r(t) \equiv 0, t \in J$.

Lemma 2.7. (Mönch fixed point theorem) [7] We assume that X is a Banach space, $\Omega \subset X$ is open and bounded, $\theta \in \Omega$, and $Q : \overline{\Omega} \to \overline{\Omega}$ is a continuous operator and meets the following condition:

(i) $u \ne \lambda Qu, 0 \le \lambda \le 1, u \in \partial\Omega$;

(ii) If $W \subset \overline{\Omega}$ is countable and $W \subset \overline{co}(\{\theta\} \cup Q(W))$, then W is relatively compact.

Thus, Q exists at least a fixed point on Ω.

Definition 2.4. $u \in PC[J, X]$ is called a mild solution of (1.1) if $u(t)$ satisfies equations:

$$u(t) = \begin{cases} U_\beta(t, 0)(u_0 - g_1(u)) + \displaystyle\int_0^t U_\beta(t, s) f(s, u(s), (H_1 u)(s), (H_2 u)(s)) ds, & t \in [0, t_1], \\[3mm] l_i(t, u(t)), & t \in (t_i, s_i], i = 1, 2, ..., n, \\[3mm] U_\beta(t, s_i) l_i(s_i, u(s_i)) + \displaystyle\int_{s_i}^t U_\beta(t, s) f(s, u(s), (H_1 u)(s), (H_2 u)(s)) ds, & t \in (s_i, t_{i+1}], i = 1, 2, ..., n. \end{cases}$$

3. Main results

We first give the following hypotheses.

(H$_1$) $f : J \times T_R^3 \to X$ is uniformly continuous and bounded, where $T_R = \{u \in PC[J, X] : \|u\|_{PC} \leq R\}$.

(H$_2$) There exist $p(t, \cdot), q(t, \cdot) \in L[J, R^+]$, such that

$$\| h_1(t, s, u(s)) \| \leq p(t, s) \| u(s) \| + e_h, (t, s) \in D, u \in X,$$

$$\| h_2(t, s, u(s)) \| \leq q(t, s) \| u(s) \| + e_g, (t, s) \in J \times J, u \in X,$$

$$h_0 = \max_{0 \leq s \leq t \leq T} \int_0^t p(t, s) ds < +\infty, g_0 = \max_{0 \leq s, t \leq T} \int_0^T q(t, s) ds < +\infty.$$

(H$_3$) There exist $b_1 \in L[J, R^+], b_2, b_3 \in L^2[J, R^+]$ satisfying

$$\| f(s, u(s), u_1, u_2) \| \leq b_1(s) + b_2(s) \| u_1 \| + b_3(s) \| u_2 \|.$$

(H$_4$) Suppose $l_i \in C[J \times X, X], g_1 \in C[PC[J, X], X]$. For $u \in PC[J, X]$, there are nonnegative constants $d \geq 0, e \geq 0$ such that

$$\| g_1(u) \| \leq d \| u \|_{PC} + e.$$

(H$_5$) There exist nonnegative constants $L_h, L_g, L_{l_i}(i = 1, 2, ..., n)$ such that for any bounded subset $B \subset X$, we have

$$\alpha(h_1(t, s, B)) \leq L_h \alpha(B), \alpha(h_2(t, s, B)) \leq L_g \alpha(B), \alpha(l_i(t, B)) \leq L_{l_i} \int_0^t \alpha(B) ds.$$

(H$_6$) There exist constants $L_k \geq 0(k = 1, 2, 3)$ satisfying one of the following conditions:

(i) $TM^*(2(L_1 + L_{l_i}) + TL_h L_2 + TL_g L_3) < 2,$

(ii) $TL_g L_3(e^{TM^*(L_1 + L_{l_i} + TL_2 L_h)} - 1) < (L_1 + L_{l_i}) + TL_2 L_h,$

and for any bounded subset $S_k \subset X(k = 1, 2, 3)$, we have

$$\alpha(f(t, S_1, S_2, S_3) \leq \sum_{k=1}^3 L_i \alpha(S_k).$$

(H_7) There exists a constant $L_{g_1} > 0$ such that for any bounded subset $B \subset PC[J, X]$, we obtain

$$\alpha(g_1(B)) \le L_{g_1} \int_0^t \alpha(B) ds.$$

(H_8) There exist constants $Y_k \ge 0 (k = 1, 2, 3)$ meeting conditions (i) or (ii):

(i) $TL_g Y_3 (e^{TM^*(Y_1 + L_{l_i} + L_{g_1} + TY_2 L_h)} - 1) < (Y_1 + L_{l_i} + L_{g_1}) + TY_2 L_h$,

(ii) $TM^*(2(Y_1 + L_{l_i} + L_{g_1}) + TL_h Y_2 + TL_g Y_3) < 2$,

and for any bounded subset $S_k \subset X (k = 1, 2, 3)$,

$$\alpha(f(t, S_1, S_2, S_3)) \le \sum_{i=1}^3 Y_i \alpha(S_k).$$

Theorem 3.1. We assume that conditions $(H_1) - (H_6)$ are fulfilled, and g_1 is compact, then the problem (1.1) exists at least one mild solution.

Proof. We initially define the operator $Q : PC[J, X] \to PC[J, X]$ by

$$Qu(t) = \begin{cases} U_\beta(t, 0)(u_0 - g_1(u)) + \displaystyle\int_0^t U_\beta(t, s) f(s, u(s), (H_1 u)(s), (H_2 u)(s)) ds, & t \in [0, t_1], \\[2ex] l_i(t, u(t)), & t \in (t_i, s_i], i = 1, 2, ..., n, \\[2ex] U_\beta(t, s_i) l_i(s_i, u(s_i)) + \displaystyle\int_{s_i}^t U_\beta(t, s) f(s, u(s), (H_1 u)(s), (H_2 u)(s)) ds, & t \in (s_i, t_{i+1}], i = 1, 2, ..., n. \end{cases}$$

(3.1)

The fixed point of operator Q in $PC[J, X]$ is a solution of problem (1.1). Next, we will prove that operator Q has a fixed point in $PC[J, X]$.

As f is uniformly continuous, g_1, l_i are continuous, and $U_\beta(t, s)$ is strongly continuous, we can get that Q is continuous.

Next, we prove that $\Omega_0 = \{u \in PC[J, X] | u = \lambda Qu, \lambda \in [0, 1]\}$ is bounded. In fact, if $y_0 \in \Omega_0$, then there exists $0 \le \lambda_0 \le 1$ satisfying $y_0(t) = \lambda_0 (Qy_0)(t), t \in J$.

When $t \in [0, t_1]$, we have

$$\begin{aligned} y_0(t) =& \lambda_0 (Qy_0)(t) = \lambda_0 U_\beta(t, 0)[u_0 - g_1(y_0)] \\ &+ \lambda_0 \int_0^t U_\beta(t, s) f(s, y_0(s), (H_1 y_0)(s), (H_2 y_0)(s)) ds. \end{aligned}$$

(3.2)

Let $y(t) = \| y_0(t) \|$. By $(H_2) - (H_4)$ and (3.2), we get

$$\begin{aligned} y(t) \le& M^*(\| u_0 \| + \| g_1(y_0) \|) \\ &+ M^* \int_0^t [b_1(s) + b_2(s) \| H_1 y_0(s) \| + b_3(s) \| H_2 y_0(s) \|] ds, \end{aligned}$$

where $M^* = \max_{0 \le s \le t \le a} \| U_\beta(t, s) \|$.

Since

$$\| H_1 y_0(t) \| \leq \int_0^t \| h_1(t, s, y_0(s)) \| \, ds$$

$$\leq (p(t, s) \| y_0(s) \| + e_h) t_1$$

$$\leq t_1 h_0 \| y_0(s) \| + t_1 e_h,$$

$$= t_1 h_0 y(s) + t_1 e_h,$$

similarly, we can get $\| H_2 y_0(t) \| \leq t_1 g_0 y(s) + t_1 e_g$, then

$$y(t) \leq M^*(\| u_0 \| + \| g_1(y_0) \|) + M^* \int_0^t [b_3(s) a e_h + b_4(s) a e_g] ds$$

$$+ M^* \int_0^t [b_1(s) + t_1 b_2(s) y(s) h_0 + t_1 b_3(s) y(s) g_0] ds$$

$$\leq M_1 + M_2 + M^* F y(t),$$

where

$$M_1 = M^*(\| u_0 \| + \| g_1(y_0) \|) + M^* \int_0^t [b_2(s) t_1 e_h + b_3(s) t_1 e_g] ds,$$

$$M_2 = M^* \int_0^{t_1} b_1(s) ds, \quad F y(t) = \int_0^t [t_1 b_2(s) h_0 + t_1 b_3(s) g_0] y(s) ds.$$

Therefore $F : C[J_0, R^+] \to C[J_0, R^+]$ is completely continuous.

As $b_2, b_3, b_4 \in L^2[J, R^+]$, then

$$M = (\int_0^T [(t_1 b_2(s) h_0 + t_1 b_3(s) g_0)]^2 ds)^{\frac{1}{2}} < +\infty.$$

Next, we will demonstrate $r_\sigma(F) = 0$. Because

$$\| F y(t) \| \leq \int_0^t \| [t_1 b_2(s) h_0 + t_1 b_3(s) g_0] y(s) \| \, ds$$

$$\leq (\int_0^t [t_1 b_2(s) h_0 + t_1 b_3(s) g_0]^2 ds)^{\frac{1}{2}} (\int_0^t \| y \|^2 \, ds)^{\frac{1}{2}}$$

$$\leq M \| y \| t^{\frac{1}{2}},$$

$$\| F^2 y(t) \| \leq \int_0^t \| (t_1 b_2(s) h_0 + t_1 b_3(s) g_0) B y(s) \| \, ds$$

$$\leq (\int_0^t [(t_1 b_2(s) h_0 + t_1 b_3(s) g_0)]^2 ds)^{\frac{1}{2}} (\int_0^t (M \| y \| t^{\frac{1}{2}})^2 ds)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{2}} M^2 \| y \| t,$$

$$\| F^3 y(t) \| \leq \int_0^t \| (t_1 b_2(s) h_0 + t_1 b_3(s) g_0) F^2 y(s) \| \, ds$$

$$\leq (\int_0^t [(t_1 b_2(s) h_0 + t_1 b_3(s) g_0)]^2 ds)^{\frac{1}{2}} (\int_0^t (\frac{1}{\sqrt{2}} M^2 \| y \| t)^2 ds)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{3}\sqrt{2}} M^3 \| y \| t^{\frac{3}{2}},$$

for any $n \in N$, by mathematical induction, we can obtain

$$\| F^n y(t) \| \leq \frac{1}{\sqrt{n!}} M^n \| y \| t^{\frac{n}{2}},$$

then

$$\| F^n \| \leq \frac{1}{\sqrt{n!}} M^n a^{\frac{n}{2}},$$

$$r_\sigma(F) = \lim_{n \to +\infty} \| F^n \|^{\frac{1}{n}} = 0.$$

There is a nonnegative constant $C_1 \geq 0$ independent of function u satisfying

$$\|y_0(t)\| \leq C_1, t \in J_0.$$

Since l_1 is continuous, there is a nonnegative constant $\beta_1 \geq 0$ satisfying

$$\| l_1(t, u(t)) \| \leq \beta_1, \forall t \in J, \|u\| \leq C_1.$$

Let

$$z_0(t) = \begin{cases} l_1(t, y_0(t)), & t \in (t_1, s_1], \\ U_\beta(t, s_1)l_1(s_1, y_0(s_1)) + \displaystyle\int_{s_1}^t U_\beta(t, s)f(s, y_0(s), (H_1 y_0)(s), (H_2 y_0)(s))ds, & t \in (s_1, t_2], \\ l_1(t, y_0(t_1^+)), & t = t_1, \end{cases}$$

then $z_0 \in C[[t_1, t_2], X]$ and

$$
\begin{aligned}
\| z_0(t) \| \leq{} & \| l_1(t, y_0(t)) \| + \| U_\beta(t, s_1)l_1(s_1, y_0(s_1)) \\
& + \int_{s_1}^t U_\beta(t, s)f(s, y_0(s), (H_1 y_0)(s), (H_2 y_0)(s))ds \| \\
\leq{} & \| \beta_1 \| + M^* \| \beta_1 \| + \int_{s_1}^t \| U_\beta(t, s)f(s, y_0(s), (H_1 y_0)(s), (H_2 y_0)(s)) \| ds \\
\leq{} & (1 + M^*) \| \beta_1 \| + M^* \int_{s_1}^t b_1(s)ds + M^* \int_0^t [b_2(s)Te_h + b_3(s)Te_g]ds \\
& + M^* \int_0^t [t_1 b_2(s)y(s)h_0 + Tb_3(s)y(s)g_0]ds.
\end{aligned}
$$

Similar to the method from the previous step, we get that there exists $C_2 \geq 0$ satisfying

$$\|z_0(t)\| \leq C_2, t \in [t_1, t_2],$$

therefore,

$$\|y_0(t)\| \leq C_2, t \in J_1.$$

Likewise, we obtain that there exist $C_{n+1} \geq 0$ satisfying

$$\|y_0(t)\| \leq C_{n+1}, t \in J_n.$$

Let

$$C = \max\{C_j | 1 \le j \le n+1\},$$

then

$$\|y_0(t)\| \le C, t \in J.$$

Hence

$$\|y_0\|_{PC} \le C, y_0 \in \Omega_0,$$

so Ω_0 is bounded.

We take a constant R satisfying $R > C$ and $\Omega = \{u \in PC[J, X] | \|u\|_{PC} \le R\}$, then Ω is bounded and open. When $u \in \partial\Omega, 0 \le \lambda \le 1$, we obtain

$$u(t) \ne \lambda(Qu(t)).$$

We suppose that $W \subset \overline{\Omega}$ is a countable set and $W \subset \overline{co}(\{\theta\} \cup Q(W))$, $\theta \in \Omega$, then W is relatively compact.

In fact, as $U_\beta(t, s)$ is strong continuity, by the assumption (H_1), we obtain that $Q(W)$ is equi-continuous on $J_i(i = 0, 1, ...n)$, then W is equi-continuous on J_i. $\forall u \in T_R, u_1, u_2 \in [0, t_1]$.

Case 1: $\forall t \in J_0 = [0, t_1]$. Since g_1 is compact, by the assumption (H_6) and Lemma 2.1, we obtain

$$
\begin{aligned}
\alpha(W(t)) &\le \alpha(QW(t)) \\
&= \alpha(\int_0^t U_\beta(t, s) f(s, W(s), (H_1 W)(s), (H_2 W)(s)) ds) \\
&\le M^* \int_0^t (L_1 \alpha(W(s)) + L_2 \alpha((H_1 W)(s)) + L_3 \alpha((H_2 W)(s))) ds,
\end{aligned}
$$

because

$$
\begin{aligned}
\alpha(H_1 W(t)) &= \alpha(\int_0^t h_1(t, \tau, W(\tau)) d\tau) \le \int_0^t \alpha(h_1(t, \tau, W(\tau)) d\tau \le \int_0^t L_h \alpha(W(\tau)) d\tau \\
&= L_h \int_0^t \alpha(W(\tau)) d\tau,
\end{aligned}
$$

then

$$
\begin{aligned}
\int_0^t \alpha(H_1 W(s)) ds &\le L_h \int_0^t (\int_0^s \alpha(W(\tau)) d\tau) ds \\
&= L_h \int_0^t d\tau \int_\tau^t \alpha(W(\tau)) ds \\
&= L_h \int_0^t \alpha(W(\tau))(t - \tau) d\tau \\
&\le L_h t \int_0^t \alpha(W(\tau)) d\tau.
\end{aligned}
$$

Similarly, we can obtain

$$\int_0^t \alpha(H_2 W(s)) ds \le L_g \int_0^t \alpha(\int_0^T W(\tau) d\tau) ds$$

$$= L_g \int_0^t ds \int_0^T \alpha(W(\tau)) d\tau$$

$$= L_g t \int_0^T \alpha(W(\tau)) d\tau,$$

hence

$$\alpha(W(t)) \le M^* L_1 \int_0^t \alpha(W(s)) ds$$

$$+ M^* L_h L_2 t \int_0^t \alpha(W(s)) ds \qquad (6)$$

$$+ M^* L_g L_3 t \int_0^T \alpha((W(s))) ds.$$

By assumption (H_6) and Lemma 2.6, we get $\alpha(W(t)) = 0, t \in J_0 = [0, t_1]$, and especially $\alpha(W(t_1)) = 0$, then W is relatively compact on $C[J_0, X]$.

Case 2: $\forall t \in (t_1, s_1]$. We obtain

$$\alpha(W(t)) \le \alpha(QW(t)) = \alpha(l_i(t, u(t))) \le L_{l_i} \int_0^t \alpha(W(s)) ds.$$

By the assumption (H_6) and Lemma 2.6, we have $\alpha(W(t)) = 0, t \in (t_1, s_1]$.

Case 3: $\forall t \in (s_1, t_2]$. We obtain

$$\alpha(W(t)) \le M^*[L_{l_i} \int_0^t \alpha(W(s)) ds + \alpha(\int_{s_i}^t f(s, W(s), (H_1 W)(s), (H_2 W)(s)) ds)]$$

$$\le M^*[L_{l_i} \int_0^t \alpha(W(s)) ds + \int_{s_i}^t \alpha(f(s, W(s), (H_1 W)(s), (H_2 W)(s))) ds]$$

$$\le M^*[L_{l_i} \int_0^t \alpha(W(s)) ds + \int_{s_i}^t L_1 \alpha(W(s) + L_2 \alpha(H_1 W(s)) + L_3 \alpha(H_2 W(s))) ds]$$

$$\le M^*[L_{l_i} \int_0^t \alpha(W(s)) ds + L_1 \int_{s_i}^t \alpha(W(s)) ds$$

$$+ L_2 L_h \int_{s_i}^t \alpha(W(s)) ds + L_3 L_g \int_{s_i}^t \alpha(W(s)) ds],$$

since

$$\int_{s_i}^t \alpha(H_1 W(s)) ds \le \int_0^t \alpha(H_2 W(s)) ds$$

$$\le L_h t \int_0^t \alpha(W(\tau)) d\tau,$$

$$\int_{s_i}^t \alpha(H_2 W(s)) ds \le \int_0^t \alpha(H_2 W(s)) ds$$

$$\le L_g t \int_0^T \alpha(W(\tau)) d\tau.$$

Thus

$$\alpha(W(t)) \le M^* [L_{l_i} \int_0^t \alpha(W(s)) ds + L_1 \int_0^t \alpha(W(s)) ds$$

$$+ L_2 L_h t \int_0^t \alpha(W(s)) ds + L_3 L_g t \int_0^T \alpha(W(s)) ds]$$

$$= M^* (L_{l_k} + L_1) \int_0^t \alpha(W(s)) ds$$

$$+ M^* L_2 L_h t \int_0^t \alpha(W(s)) ds + M^* L_3 L_g t \int_0^T \alpha(W(s)) ds.$$

By (H_6) and Lemma 2.6, we get $\alpha(W(t)) = 0$, $t \in (s_1, t_2]$, then $\alpha(W(t)) = 0$, $t \in J_1 = (t_1, t_2]$, especially $\alpha(W(t_2)) = 0$; then W is relatively compact on $C[J_1, X]$.

Similarly, we can prove that W is relatively compact on $C[J_i, X](i = 2, 3, ..., n)$, so W is relatively compact on $C[J, X]$.

By Lemma 2.7, Q exists a fixed point, thus problem (1.1) exists a solution $u^* \in PC[J, X]$.

Theorem 3.2. We suppose that $(H_1 - H_5), (H_7), (H_8)$ are fulfilled, then problem (1.1) exists at least a mild solution.

Proof. By Theorem 3.1, we obtain that $\Omega_0 = \{u \in PC[J, X] | u = \lambda Q u, \lambda \in [0, 1]\}$ is bounded. $\forall t \in J_0$, by (H_8). We obtain

$$\alpha(W(t)) \le \alpha(QW(t))$$

$$\le M^* \alpha(g_1 W(t)) + \alpha(\int_0^t U_\beta(t, s) f(s, W(s), (H_1 W)(s), (H_2 W)(s)) ds)$$

$$\le M^* [L_{g_1} \int_0^t \alpha(W(s)) ds + \int_0^t (Y_1 \alpha(W(s)) + Y_2 \alpha((H_1 W)(s)) + Y_3 \alpha((H_2 W)(s))) ds],$$

since

$$\int_0^t \alpha(H_1 W(s)) ds \le L_h t \int_0^t \alpha(W(\tau)) d\tau,$$

$$\int_0^T \alpha(H_2 W(s)) ds \le L_g t \int_0^T \alpha(W(\tau)) d\tau,$$

then

$$\alpha(W(t)) \le M^* (L_{g_1} + Y_1) \int_0^t \alpha(W(s)) ds$$

$$+ M^* L_h Y_2 t \int_0^t \alpha(W(s)) ds$$

$$+ M^* L_g Y_3 t \int_0^T \alpha(W(s)) ds.$$

By assumption (H_8) and Lemma 2.6, we get $\alpha(W(t)) = 0, t \in J_0 = [0, t_1]$, especially $\alpha(W(t_1)) = 0$, then W is relatively compact on $C[J_0, X]$. $\forall t \in J_1 = (t_1, s_1]$,

$$\alpha(W(t)) \le \alpha(QW(t)) = \alpha(l_i(t, u(t))) \le L_{l_i} \int_0^t \alpha(W(s))ds,$$

by assumption (H_8) and Lemma 2.6, we get $\alpha(W(t)) = 0, t \in (t_1, s_1]$.

$\forall t \in (s_1, t_2]$, resembling Theorem 3.1. We obtain

$$\alpha(W(t)) \le M^*(L_{l_i} + Y_1) \int_0^t \alpha(W(s))ds$$
$$+ M^* Y_2 L_h t \int_0^t \alpha(W(s))ds + M^* Y_3 L_g t \int_0^T \alpha(W(s))ds.$$

By (H_8) and Lemma 2.6, we get $\alpha(W(t)) = 0, t \in (s_1, t_2]$; hence, $\alpha(W(t)) = 0, t \in J_1 = (t_1, t_2]$, especially $\alpha(W(t_2)) = 0$, then W is relatively compact on $C[J_1, X]$.

Similarly, we obtain that W is relatively compact on $C[J_i, X], (i = 2, 3, ..., n)$, then W is relatively compact on $C[J, X]$.

By Lemma 2.7, Q exists a fixed point; therefore, problem (1.1) exists a solution.

Remark 3.1. When $L_4 = 0$, the condition (ii) in assumption (H_6) is clearly valid. In this situation, the assumption (H_6) can be removed.

4. Example

We study the following problem:

$$\begin{cases} {}^C_0 D_t^{\frac{1}{2}} u(t) = \dfrac{u(t)}{1 + \sqrt{t}} + \dfrac{e^{-t}}{1 + e^t} \sin(u(t - \tau)) + \dfrac{1}{2(1 + t)} \displaystyle\int_0^t tu(s) \cos s \, ds \\ \qquad + \dfrac{1}{2(1 + e^t)} \displaystyle\int_0^1 \dfrac{s \sin t}{2} u(s) ds, \quad t \in [0, 1/4] \cup (1/2, 3/4], \\ u(t) = \dfrac{u(t)}{5 + 4t}, \quad t \in (1/4, 1/2], \\ u(0) + \dfrac{1}{4} \sin u = u_0 \in X. \end{cases} \quad (4.1)$$

Define

$$g_1(u) = \frac{1}{4} \sin u, \tau(t) = t - \tau \le t, (H_1 u)(t) = \int_0^t t \cos s \, u(s) ds, l_i(t, u(t)) = \frac{u(t)}{5 + 4t},$$

$$(H_2 u)(t) = \int_0^1 \frac{s \sin t}{2} u(s) ds, h_1(t, s, u) = tu \cos s, h_2(t, s, u) = \frac{s \sin t}{2} u,$$

$$f(t, u_1, u_2, u_3, u_4) = \frac{u_1}{1 + \sqrt{t}} + \frac{e^{-t}}{1 + e^t} \sin u_2 + \frac{1}{2(1 + t)} u_3 + \frac{1}{2(1 + e^t)} u_4,$$

then Eq (4.1) can be transformed into the following form:

$$\begin{cases} {}_0^C D_t^{\frac{1}{2}} u(t) = A(t)u(t) + f(t, u(t), u(\tau(t)), (H_1 u)(t), (H_2 u)(t)), & t \in [0, 1/4] \cup (1/2, 3/4], \\ u(t) = l_i(t, u(t)), & t \in (1/4, 1/2], \\ u(0) + g_1(u) = u_0 \in X. \end{cases} \quad (4.2)$$

For any $x_k, y_k \in X(k = 1, 2, 3, 4), t \in [0, 1]$, we obtain

$$\|f(t, y_1, y_2, y_3, y_4) - f(t, x_1, x_2, x_3, x_4)\| \leq \frac{1}{(1 + \sqrt{t})} \| y_1 - x_1 \| + \frac{e^{-t}}{1 + e^t} \| y_2 - x_2 \|$$
$$+ \frac{1}{2(1 + t)} \| y_3 - x_3 \| + \frac{1}{2(1 + e^t)} \| y_4 - x_4 \|,$$

$$\|h_1(t, s, y_1) - h_1(t, s, x_1)\| \leq \|y_1 - x_1\|, \|h_2(t, s, y_1) - h_2(t, s, x_1)\| \leq \frac{1}{2}\|y_1 - x_1\|,$$

$$\|l_i(t, y_1) - l_k(t, x_1)\| \leq \frac{1}{9}\|y_1 - x_1\|, \|g_1(y_1) - g_1(x_1)\| \leq \frac{1}{4}\|y_1 - x_1\|.$$

If other assumptions of Theorem 3.1 are fulfilled, then system (4.1) exists a mild solution.

5. Conclusions

The literatures [15,17] studied the problem of non-instantaneous impulses of integer order. In process of proving the theorem, the author used the Lipschitz condition and required the compression coefficient to be less than 1. By employing the non-compact measure, the method of stepwise estimation, and the Mönch fixed point theorem, we eliminated the need for the Lipschitz condition and the requirement of a compressibility coefficient less than 1. The literature [18] studied the problem of non-instantaneous impulses of fractional order. They gave the mild solutions via the Banach contraction mapping principle without the condition of compressibility coefficient less than 1, but did not consider the impulses and nonlocal conditions. Through the use of the β-order solution operator, the Mönch fixed point theorem, and the method of stepwise estimation, we eliminated the Lipschitz condition and the requirement of the compressibility coefficient less than 1. This work discusses the existence of mild solutions, with our future focus directed toward investigating the stability outcomes of fractional non-instantaneous impulse integro-differential equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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