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Research article

A generalization of convexity via an implicit inequality

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Abstract: We unified several kinds of convexity by introducing the class $\mathcal{A}_{\zeta,w}([0,1]\times I^2)$ of (ζ,w)-admissible functions $F:[0,1]\times I\times I\to\mathbb{R}$. Namely, we proved that most types of convexity from the literature generate functions $F\in\mathcal{A}_{\zeta,w}([0,1]\times I^2)$ for some $\zeta\in C([0,1])$ and $w\in C^1(I)$ with $w(I)\subset I$ and $w'>0$. We also studied some properties of (ζ,w)-admissible functions and established some integral inequalities that unify various Hermite-Hadamard-type inequalities from the literature.

Keywords: convexity; implicit inequality; (ζ,w)-admissible functions; integral inequalities; Hermite-Hadamard-type inequalities
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1. Introduction

Let I be an interval of $\mathbb{R}$ and $f:I\to\mathbb{R}$. The function f is said to be convex on I, if $f(tx+(1-t)y)\le tf(x)+(1-t)f(y)$ for all $t\in[0,1]$ and $x,y\in I$. The class of convex functions is widely used in pure and applied mathematics. Several works related to the study of convex functions can be found in the literature, (see e.g., [2, 15, 24, 25, 28, 29]). Convex functions satisfy nice properties that are very useful for the study of various mathematical problems. A natural question is to ask whether such properties can be extended to other classes of functions. This question motivated the generalization of convexity in various directions. We recall below some interesting generalizations from the literature. Throughout this paper, by I we mean an interval of $\mathbb{R}$. The class of convex functions on I is denoted by $CV(I)$.

Breckner [3] introduced the class of s-convex functions in the second sense. Namely, a function $f : I \to \mathbb{R}$ is called s-convex in the second sense on I, where $0 < s \leq 1$, if

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y)$$

for all $t \in [0, 1]$ and $x, y \in I$. The class of s-convex functions in the second sense on I is denoted by $K_s^2(I)$. Observe that, if $f \in K_s^2(I)$, then $f \geq 0$ (it can be easily seen by taking $x = y$ and $t = \frac{1}{2}$). Clearly, we have $K_1^2(I) = CV(I)$. Remark also that, if $f \in CV(I)$ and $f \geq 0$, then $f \in K_s^2(I)$ for all $0 < s \leq 1$. Some related works to s-convexity in the second sense can be found in [7, 8, 16].

Dragomir et al. [9] introduced the class of P-functions. Recall that $f : I \to \mathbb{R}$ is called a P-function on I, if

$$f(tx + (1 - t)y) \leq f(x) + f(y)$$

for all $t \in [0, 1]$ and $x, y \in I$. The class of P-functions on I is denoted by $P(I)$. Observe that, if $f \in K_s^2(I)$ for some $s \in (0, 1]$, then $f \in P(I)$. For some studies related to P-functions, see e.g., [17, 19, 23].

In [27], Toader introduced the class of m-convex functions. Namely, for $0 < m \leq 1$, a function $f : [0, \infty) \to \mathbb{R}$ is called m-convex on $[0, \infty)$, if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)mf\left(\frac{y}{m}\right)$$

for all $t \in [0, 1]$ and $x, y \geq 0$. We denote by $K_m([0, \infty))$ the class of m-convex on $[0, \infty)$. Remark that $K_1([0, \infty)) = CV([0, \infty))$. Some contributions related to m-convex functions can be found in [5, 10, 22].

By combining the concepts of s-convexity and m-convexity, Park [21] introduced the class of (s, m)-convex functions. Namely, a function $f : [0, \infty) \to \mathbb{R}$ is called (s, m)-convex on $[0, \infty)$ for some $s, m \in (0, 1]$, if

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s mf\left(\frac{y}{m}\right)$$

for all $t \in [0, 1]$ and $x, y \geq 0$. We denote by $K_{s,m}^2([0, \infty))$ the class of (s, m)-convex functions on $[0, \infty)$. Observe that $K_{s,1}^2([0, \infty)) = K_s^2([0, \infty))$ and $K_{1,m}^2([0, \infty)) = K_m([0, \infty))$. Remark also that, if $f \in K_m([0, \infty))$ and $f \geq 0$, then $f \in K_{s,m}^2([0, \infty))$ for all $0 < s \leq 1$. We refer to [1, 4, 12, 30, 31], for some works related to (s, m)-convex functions.

Our aim in this paper is to provide a unification of all the above types of convexity via an implicit inequality involving three functions $F : [0, 1] \times I \times I \to \mathbb{R}$, $\zeta \in C(I)$ ($C(I)$ is the class of continuous functions on I) and $w \in C^1(I)$ ($C^1(I)$ is the class of differentiable functions whose derivatives are continuous on I). Our main idea is motivated by the following observation. Assume that $f \in CV(I)$, that is,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y), \quad 0 \leq t \leq 1,\ x, y \in I. \tag{1.1}$$

If we consider the function $F : [0, 1] \times I \times I \to \mathbb{R}$ defined by

$$F(t, x, y) = f(tx + (1 - t)y), \quad 0 \leq t \leq 1,\ x, y \in I,$$

then (1.1) reduces to the inequality

$$F(t, x, y) \leq \zeta(t)F(1, x, y) + \zeta(1 - t)\frac{F(0, x, w(y))}{w'(y)}, \quad 0 \leq t \leq 1,\ x, y \in I,$$

where $\zeta(t) = t$, $0 \leq t \leq 1$, and $w(x) = x$, $x \in I$.

The rest of this paper is arranged as follows. In Section 2, we introduce the class of (ζ, w)-admissible functions $\mathcal{A}_{\zeta,w}\left([0,1] \times I^2\right)$, where $\zeta \in C([0,1])$ and $w \in C^1(I)$ with $w(I) \subset I$ and $w' > 0$. We show that the definitions of several kinds of convexity from the literature can be reduced to an implicit inequality involving a mapping $F \in \mathcal{A}_{\zeta,w}\left([0,1] \times I^2\right)$ for some $\zeta \in C([0,1])$ and $w \in C^1(I)$ with $w(I) \subset I$ and the derivative $w' > 0$. We also establish some properties of the class of functions $\mathcal{A}_{\zeta,w}\left([0,1] \times I^2\right)$. In Section 3, we establish new integral inequalities involving (ζ, w)-admissible functions. We show that several Hermite-Hadamard-type inequalities from the literature can be deduced from our obtained inequalities.

2. The class of (ζ, w)-admissible functions

2.1. Definition and examples

Definition 2.1. *Let $\zeta \in C([0,1])$ and $w \in C^1(I)$ with $w(I) \subset I$ and $w' > 0$. A function*

$$F : [0,1] \times I \times I \to \mathbb{R}$$

is said to be (ζ, w)-admissible on $[0,1] \times I \times I$, if

$$F(t,x,y) \leq \zeta(t)F(1,x,y) + \zeta(1-t)\frac{F(0,x,w(y))}{w'(y)}$$

for all $t \in [0,1]$ and $x, y \in I$. The class of (ζ, w)-admissible functions on $[0,1] \times I \times I$ is denoted by $\mathcal{A}_{\zeta,w}\left([0,1] \times I^2\right)$.

We show below that the class of (ζ, w)-admissible functions generalizes various kinds of convexity.

Proposition 2.1. *For all $f \in CV(I)$, there exists $F = F_f \in \mathcal{A}_{\zeta,w}\left([0,1] \times I^2\right)$ for some $\zeta \in C([0,1])$ and $w \in C^1(I)$ with $w(I) \subset I$ and $w' > 0$.*

Proof. Let $f \in CV(I)$. We introduce the functions ζ and w defined by

$$\zeta(t) = t, \quad 0 \leq t \leq 1,$$
$$w(x) = x, \quad x \in I.$$

We also introduce the function F defined by

$$F(t,x,y) = f(tx + (1-t)y), \quad t \in [0,1],\ x, y \in I.$$

Observe that for all $x, y \in I$, we have

$$F(1,x,y) = f(x),$$
$$F(0,x,y) = f(y).$$

Then, by convexity of f, for all $t \in [0,1]$ and $x, y \in I$, we obtain

$$\begin{aligned} F(t,x,y) &= f(tx + (1-t)y) \\ &\leq tf(x) + (1-t)f(y) \\ &= \zeta(t)F(1,x,y) + \zeta(1-t)F(0,x,y) \\ &= \zeta(t)F(1,x,y) + \zeta(1-t)\frac{F(0,x,w(y))}{w'(y)}, \end{aligned}$$

which shows that F is (ζ, w)-admissible. □

Proposition 2.2. *Let* $0 < s \leq 1$. *For all* $f \in K_s^2(I)$, *there exists* $F = F_f \in \mathcal{A}_{\zeta,w}\left([0,1] \times I^2\right)$ *for some* $\zeta \in C([0,1])$ *and* $w \in C^1(I)$ *with* $w(I) \subset I$ *and* $w' > 0$.

Proof. Let $f \in K_s^2(I)$. We introduce the functions ζ and w defined by

$$\begin{aligned} \zeta(t) &= t^s, \quad 0 \leq t \leq 1, \\ w(x) &= x, \quad x \in I. \end{aligned}$$

We also introduce the function F defined by

$$F(t,x,y) = f(tx + (1-t)y), \quad t \in [0,1],\ x,y \in I.$$

Then, by s-convexity of f, for all $t \in [0,1]$ and $x, y \in I$, we obtain

$$\begin{aligned} F(t,x,y) &= f(tx + (1-t)y) \\ &\leq t^s f(x) + (1-t)^s f(y) \\ &= \zeta(t)F(1,x,y) + \zeta(1-t)F(0,x,y) \\ &= \zeta(t)F(1,x,y) + \zeta(1-t)\frac{F(0,x,w(y))}{w'(y)}, \end{aligned}$$

which shows that F is (ζ, w)-admissible. □

Proposition 2.3. *For all* $f \in P(I)$, *there exists* $F = F_f \in \mathcal{A}_{\zeta,w}\left([0,1] \times I^2\right)$ *for some* $\zeta \in C([0,1])$ *and* $w \in C^1(I)$ *with* $w(I) \subset I$ *and* $w' > 0$.

Proof. Let $f \in P(I)$. We introduce the functions ζ and w defined by

$$\begin{aligned} \zeta(t) &= 1, \quad 0 \leq t \leq 1, \\ w(x) &= x, \quad x \in I. \end{aligned}$$

We also introduce the function F defined by

$$F(t,x,y) = f(tx + (1-t)y), \quad t \in [0,1],\ x,y \in I.$$

Then, since $f \in P(I)$, for all $t \in [0,1]$ and $x, y \in I$, we obtain

$$\begin{aligned} F(t,x,y) &= f(tx + (1-t)y) \\ &\leq f(x) + f(y) \\ &= \zeta(t)F(1,x,y) + \zeta(1-t)F(0,x,y) \\ &= \zeta(t)F(1,x,y) + \zeta(1-t)\frac{F(0,x,w(y))}{w'(y)}, \end{aligned}$$

which shows that F is (ζ, w)-admissible. □

Proposition 2.4. *Let* $0 < m \leq 1$. *For all* $f \in K_m([0,\infty))$, *there exists* $F = F_f \in \mathcal{A}_{\zeta,w}\left([0,1] \times [0,\infty)^2\right)$ *for some* $\zeta \in C([0,1])$ *and* $w \in C^1([0,\infty))$ *with* $w \geq 0$ *and* $w' > 0$.

Proof. Let $f \in K_m([0,\infty))$. We introduce the functions ζ and w defined by

$$\zeta(t) = t, \quad 0 \le t \le 1,$$
$$w(x) = \frac{x}{m}, \quad x \ge 0.$$

We also introduce the function F defined by

$$F(t,x,y) = f(tx + (1-t)y), \quad t \in [0,1],\ x,y \ge 0.$$

Then, since $f \in K_m([0,\infty))$, for all $t \in [0,1]$ and $x,y \ge 0$, we obtain

$$\begin{aligned} F(t,x,y) &= f(tx+(1-t)y) \\ &\le tf(x) + (1-t)mf\left(\frac{y}{m}\right) \\ &= \zeta(t)F(1,x,y) + \zeta(1-t)\frac{F(0,x,w(y))}{w'(y)}, \end{aligned}$$

which shows that F is (ζ,w)-admissible. □

Proposition 2.5. *Let $0 < s,m \le 1$. For all $f \in K^2_{s,m}([0,\infty))$, there exists $F = F_f \in \mathcal{A}_{\zeta,w}\left([0,1]\times[0,\infty)^2\right)$ for some $\zeta \in C([0,1])$ and $w \in C^1([0,\infty))$ with $w \ge 0$ and $w' > 0$.*

Proof. Let $f \in K^2_{s,m}([0,\infty))$. We introduce the functions ζ and w defined by

$$\zeta(t) = t^s, \quad 0 \le t \le 1,$$
$$w(x) = \frac{x}{m}, \quad x \ge 0.$$

We also introduce the function F defined by

$$F(t,x,y) = f(tx + (1-t)y), \quad t \in [0,1],\ x,y \ge 0.$$

Then, since $f \in K^2_{s,m}([0,\infty))$, for all $t \in [0,1]$ and $x,y \ge 0$, we obtain

$$\begin{aligned} F(t,x,y) &= f(tx+(1-t)y) \\ &\le t^s f(x) + (1-t)^s mf\left(\frac{y}{m}\right) \\ &= \zeta(t)F(1,x,y) + \zeta(1-t)\frac{F(0,x,w(y))}{w'(y)}, \end{aligned}$$

which shows that F is (ζ,w)-admissible. □

2.2. Basic properties

Proposition 2.6. *Let $F,G \in \mathcal{A}_{\zeta,w}\left([0,1]\times I^2\right)$ for some $\zeta \in C([0,1])$ and $w \in C^1(I)$ with $w(I) \subset I$ and $w' > 0$. Then, for all $\alpha \ge 0$, $\alpha F \in \mathcal{A}_{\zeta,w}\left([0,1]\times I^2\right)$ and $F+G \in \mathcal{A}_{\zeta,w}\left([0,1]\times I^2\right)$.*

Proof. Let $t \in [0,1]$ and $x, y \in I$. Since F is (ζ, w)-admissible, we have

$$F(t,x,y) \le \zeta(t)F(1,x,y) + \zeta(1-t)\frac{F(0,x,w(y))}{w'(y)}. \tag{2.1}$$

For all $\alpha \ge 0$, multiplying the above inequality by α, we get

$$\alpha F(t,x,y) \le \zeta(t)\left[\alpha F(1,x,y)\right] + \zeta(1-t)\frac{\alpha F(0,x,w(y))}{w'(y)},$$

which shows that αF is (ζ, w)-admissible. Furthermore, since G is (ζ, w)-admissible, we have

$$G(t,x,y) \le \zeta(t)G(1,x,y) + \zeta(1-t)\frac{G(0,x,w(y))}{w'(y)}. \tag{2.2}$$

Summing (2.1) and (2.2), we obtain

$$(F+G)(t,x,y) \le \zeta(t)(F+G)(1,x,y) + \zeta(1-t)\frac{(F+G)(0,x,w(y))}{w'(y)},$$

which shows that $F+G$ is (ζ, w)-admissible. □

Proposition 2.7. *Let $\zeta_1, \zeta_2 \in C([0,1])$ be such that*

$$\zeta_1(t) \le \zeta_2(t), \quad 0 \le t \le 1.$$

Let $w \in C^1(I)$ with $w(I) \subset I$ and $w' > 0$. If $F \in \mathcal{A}_{\zeta_1,w}\left([0,1]\times I^2\right)$ and $F \ge 0$, then $F \in \mathcal{A}_{\zeta_2,w}\left([0,1]\times I^2\right)$.

Proof. Let $F \in \mathcal{A}_{\zeta_1,w}\left([0,1]\times I^2\right)$. Let $t \in [0,1]$ and $x, y \in I$. Then

$$F(t,x,y) \le \zeta_1(t)F(1,x,y) + \zeta_1(1-t)\frac{F(0,x,w(y))}{w'(y)}. \tag{2.3}$$

Since $\zeta_1 \le \zeta_2$, $w' > 0$ and $F \ge 0$, we have

$$\zeta_1(t)F(1,x,y) + \zeta_1(1-t)\frac{F(0,x,w(y))}{w'(y)} \le \zeta_2(t)F(1,x,y) + \zeta_2(1-t)\frac{F(0,x,w(y))}{w'(y)}. \tag{2.4}$$

Hence, from (2.3) and (2.4), we deduce that F is (ζ_2, w)-admissible. □

Proposition 2.8. *Let $\zeta \in C([0,1])$, $\zeta \ge 0$ and $w_1, w_2 \in C^1(I)$ with $w_i(I) \subset I$ and $w_i' > 0$, $i = 1,2$. Let $F \in \mathcal{A}_{\zeta,w_2}\left([0,1]\times I^2\right)$ with*

$$\frac{F(0,x,w_2(y))}{w_2'(y)} \le \frac{F(0,x,w_1(y))}{w_1'(y)} \tag{2.5}$$

for all $x, y \in I$. Then $F \in \mathcal{A}_{\zeta,w_1}\left([0,1]\times I^2\right)$.

Proof. Let $t \in [0,1]$ and $x, y \in I$. Since F is (ζ, w_2)-admissible, we have

$$F(t,x,y) \le \zeta(t)F(1,x,y) + \zeta(1-t)\frac{F(0,x,w_2(y))}{w_2'(y)}. \tag{2.6}$$

Furthermore, since $\zeta \ge 0$ and using (2.5), we obtain

$$\zeta(t)F(1,x,y) + \zeta(1-t)\frac{F(0,x,w_2(y))}{w_2'(y)} \le \zeta(t)F(1,x,y) + \zeta(1-t)\frac{F(0,x,w_1(y))}{w_1'(y)}. \tag{2.7}$$

Hence, from (2.6) and (2.7), we deduce that F is (ζ, w_1)-admissible. □

We provide below an application of Proposition 2.8.

Example 2.1. *Let $0 < m_1 \le m_2 < 1$ and $f \in K_{m_2}([0,\infty))$, that is,*

$$f(tx + (1-t)y) \le tf(x) + (1-t)m_2 f\left(\frac{y}{m_2}\right) \tag{2.8}$$

for all $t \in [0,1]$ and $x, y \ge 0$. We shall use Proposition 2.8 to show that $f \in K_{m_1}([0,\infty))$.

Remark that for $t = y = 0$, the above inequality reduces to

$$(1-m_2)f(0) \le 0,$$

which implies (since $m_2 < 1$) that

$$f(0) \le 0. \tag{2.9}$$

Now, taking $y = 0$ in (2.8), we get that for all $t \in [0,1]$ and $x \ge 0$,

$$f(tx) \le tf(x) + (1-t)m_2 f(0),$$

which implies by (2.9) that

$$f(tx) \le tf(x), \quad t \in [0,1],\ x \ge 0.$$

In particular, for $t = \frac{m_1}{m_2}$ and $x = \frac{y}{m_1}$, $y \ge 0$, we have

$$m_2 f\left(\frac{y}{m_2}\right) \le m_1 f\left(\frac{y}{m_1}\right). \tag{2.10}$$

On the other hand, from Proposition 2.4 (see also its proof), we know that the function

$$F(t,x,y) = f(tx + (1-t)y), \quad t \in [0,1],\ x \ge 0$$

is (ζ, w_2)-admissible on $[0,1] \times [0,\infty) \times [0,\infty)$, where $\zeta(t) = t$ and

$$w_2(x) = \frac{x}{m_2}, \quad x \ge 0.$$

Let us introduce the function

$$w_1(x) = \frac{x}{m_1}, \quad x \ge 0.$$

By (2.10)*, for all* $x, y \geq 0$*, we have*

$$\begin{aligned}\frac{F(0,x,w_2(y))}{w_2'(y)} &= m_2 f\left(\frac{y}{m_2}\right)\\ &\leq m_1 f\left(\frac{y}{m_1}\right)\\ &= \frac{F(0,x,w_1(y))}{w_1'(y)}.\end{aligned}$$

Hence, by Proposition 2.8, F *is also* (ζ, w_1)*-admissible on* $[0,1]\times[0,\infty)\times[0,\infty)$*, that is,*

$$F(t,x,y) \leq \zeta(t)F(1,x,y) + \zeta(1-t)\frac{F(0,x,w_1(y))}{w_1'(y)}, \quad t\in[0,1],\ x,y\geq 0,$$

which is equivalent to

$$f(tx+(1-t)y) \leq tf(x) + (1-t)m_1 f\left(\frac{y}{m_1}\right), \quad t\in[0,1],\ x,y\geq 0.$$

Consequently, f *is also* m_1*-convex.*

Proposition 2.9. *Let* $\zeta_1, \zeta_2 \in C([0,1])$ *be two nonnegative functions and* $w(x) = x$ *for all* $x \in I$*. Let*

$$\zeta^* = \max_{0\leq t\leq 1} \zeta(t),$$

where $\zeta(t) = \max\{\zeta_1(t), \zeta_2(t)\}$*,* $t \in [0,1]$*. Let* $F, G : [0,1]\times I\times I \to [0,\infty)$ *be two functions satisfying the following properties:*

(i) $F \in \mathcal{A}_{\zeta_1,w}\left([0,1]\times I^2\right)$*;*

(ii) $G \in \mathcal{A}_{\zeta_2,w}\left([0,1]\times I^2\right)$*;*

(iii) *For all* $x, y \in I$*,*
$$(F(1,x,y) - F(0,x,y))(G(1,x,y) - G(0,x,y)) \geq 0.$$

Then $FG \in \mathcal{A}_{2\zeta^*\zeta,w}\left([0,1]\times I^2\right)$*.*

Proof. Let $t \in [0,1]$ and $x, y \in I$. By (i) and (ii), we have

$$\begin{aligned}(FG)(t,x,y) &= F(t,x,y)G(t,x,y)\\ &\leq (\zeta_1(t)F(1,x,y) + \zeta_1(1-t)F(0,x,y))\,(\zeta_2(t)G(1,x,y) + \zeta_2(1-t)G(0,x,y))\\ &= \zeta_1(t)\zeta_2(t)(FG)(1,x,y) + \zeta_1(t)\zeta_2(1-t)F(1,x,y)G(0,x,y)\\ &\quad + \zeta_1(1-t)\zeta_2(t)F(0,x,y)G(1,x,y) + \zeta_1(1-t)\zeta_2(1-t)(FG)(0,x,y)\\ &\leq \zeta^2(t)(FG)(1,x,y) + \zeta(t)\zeta(1-t)F(1,x,y)G(0,x,y)\\ &\quad + \zeta(1-t)h(t)F(0,x,y)G(1,x,y) + \zeta^2(1-t)(FG)(0,x,y),\end{aligned}$$

that is,

$$\begin{aligned}(FG)(t,x,y) \leq\ & \zeta^2(t)(FG)(1,x,y) + \zeta(t)\zeta(1-t)F(1,x,y)G(0,x,y)\\ & + \zeta(1-t)\zeta(t)F(0,x,y)G(1,x,y) + \zeta^2(1-t)(FG)(0,x,y).\end{aligned} \tag{2.11}$$

On the other hand, by (iii), we have

$$F(1,x,y)G(0,x,y)+F(0,x,y)G(1,x,y)\leq (FG)(0,x,y)+(FG)(1,x,y),$$

which implies by (2.11) that

$$\begin{aligned}(FG)(t,x,y) &\leq \zeta^2(t)(FG)(1,x,y)+\zeta(t)\zeta(1-t)(FG)(0,x,y)+\zeta(t)\zeta(1-t)(FG)(1,x,y)\\ &\quad +\zeta^2(1-t)(FG)(0,x,y)\\ &= \zeta(t)\left(\zeta(t)(FG)(1,x,y)+\zeta(1-t)(FG)(0,x,y)\right)\\ &\quad +\zeta(1-t)\left(\zeta(t)(FG)(1,x,y)+\zeta(1-t)(FG)(0,x,y)\right)\\ &= \left(\zeta(t)+\zeta(1-t)\right)\left(\zeta(t)(FG)(1,x,y)+\zeta(1-t)(FG)(0,x,y)\right)\\ &\leq 2\zeta^*\zeta(t))(FG)(1,x,y)+2\zeta^*\zeta(1-t)(FG)(0,x,y),\end{aligned}$$

which proves that FG is $(2\zeta^*\zeta, w)$-admissible. □

3. Hermite-Hadamard-type inequalities

One of the most famous inequalities involving convex functions is the double Hermite-Hadamard inequality [13, 14]:

$$f\left(\frac{a+b}{2}\right)\leq \frac{1}{b-a}\int_a^b f(x)\,dx\leq \frac{f(a)+f(b)}{2},$$

which holds for all $f\in CV(I)$ and $a,b\in I$ with $a<b$. The above double-inequality has been generalized and extended in various directions, see e.g., [4, 5, 7, 9, 11, 18, 23, 26, 30] and the references therein.

In this section, using the class of (ζ, w)-admissible functions, we provide generalizations of several Hermite-Hadamard-type inequalities from the literature.

Theorem 3.1. *Let $F\in \mathcal{A}_{\zeta,w}\left([0,1]\times I^2\right)$ for some $\zeta\in C([0,1])$ and $w\in C^1(I)$ with $w(I)\subset I$ and $w'>0$. Then, for all $x,y\in I$, we have*

$$\int_0^1 F(t,x,y)\,dt\leq I_\zeta\left(F(1,x,y)+\frac{F(0,x,w(y))}{w'(y)}\right), \tag{3.1}$$

where $I_\zeta=\int_0^1\zeta(t)\,dt$.

Proof. Let $x,y\in I$. Since F is (ζ,w)-admissible, then

$$F(t,x,y)\leq \zeta(t)F(1,x,y)+\zeta(1-t)\frac{F(0,a,w(y))}{w'(y)},\quad t\in(0,1).$$

Integrating the above inequality over $(0,1)$, we obtain

$$\int_0^1 F(t,x,y)\,dt\leq\left(\int_0^1\zeta(t)\,dt\right)F(1,x,y)+\left(\int_0^1\zeta(1-t)\,dt\right)\frac{F(0,x,w(y))}{w'(y)}.$$

Remarking that

$$\int_0^1\zeta(1-t)\,dt=\int_0^1\zeta(t)\,dt,$$

we get

$$\int_0^1 F(t,x,y)\,dt \leq \left(\int_0^1 \zeta(t)\,dt\right)\left(F(1,x,y) + \frac{F(0,x,w(y))}{w'(y)}\right),$$

which proves (3.1). □

Theorem 3.2. *Let $F \in \mathcal{A}_{\zeta,w}\left([0,1]\times I^2\right)$ for some $\zeta \in C([0,1])$ and $w \in C^1(I)$ with $w(I) \subset I$ and $w' > 0$. Assume also that F is continuous on $[0,1]\times I\times I$. Then, for all $x,y \in I$, we have*

$$\begin{aligned}
&\int_0^1 F\left(\frac{1}{2}, tx+(1-t)y, ty+(1-t)x\right) dt \\
&\leq \zeta\left(\frac{1}{2}\right)\int_0^1 F(1, tx+(1-t)y, ty+(1-t)x)\,dt \\
&\quad + \zeta\left(\frac{1}{2}\right)\int_0^1 \frac{F(0, tx+(1-t)y, w(ty+(1-t)x))}{w'(ty+(1-t)x)}\,dt.
\end{aligned} \tag{3.2}$$

Proof. Let $x,y \in I$. Since F is (ζ,w)-admissible, then

$$F(t,u,v) \leq \zeta(t)F(1,u,v) + \zeta(1-t)\frac{F(0,u,w(v))}{w'(v)}, \quad t \in (0,1), u,v \in I.$$

In particular, for $t = \frac{1}{2}$, we have

$$F\left(\frac{1}{2},u,v\right) \leq \zeta\left(\frac{1}{2}\right)F(1,u,v) + \zeta\left(\frac{1}{2}\right)\frac{F(0,u,w(v))}{w'(v)}, \quad u,v \in I. \tag{3.3}$$

Taking $u = tx+(1-t)y$ and $v = ty+(1-t)x$, where $t \in (0,1)$, (3.3) reduces to

$$\begin{aligned}
&F\left(\frac{1}{2}, tx+(1-t)y, ty+(1-t)x\right) \\
&\leq \zeta\left(\frac{1}{2}\right)F(1, tx+(1-t)y, ty+(1-t)x) + \zeta\left(\frac{1}{2}\right)\frac{F(0, tx+(1-t)y, w(ty+(1-t)x))}{w'(ty+(1-t)x)}.
\end{aligned}$$

Integrating the above inequality over $t \in (0,1)$, we get

$$\begin{aligned}
&\int_0^1 F\left(\frac{1}{2}, tx+(1-t)y, ty+(1-t)x\right) dt \\
&\leq \zeta\left(\frac{1}{2}\right)\int_0^1 F(1, tx+(1-t)y, ty+(1-t)x)\,dt \\
&\quad + \zeta\left(\frac{1}{2}\right)\int_0^1 \frac{F(0, tx+(1-t)y, w(ty+(1-t)x))}{w'(ty+(1-t)x)}\,dt,
\end{aligned}$$

which proves (3.2). □

Theorem 3.3. *Let $\zeta \in C([0,1])$, $\zeta \geq 0$ and $w(x) = x$, $x \in I$. Let $F : [0,1]\times I\times I \to \mathbb{R}$ be such that for all $x,y \in I$, the function $F(\cdot,x,y) : [0,1] \ni t \mapsto F(t,x,y)$ is differentiable on $[0,1]$. Assume* *that* $|\partial_1 F| \in \mathcal{A}_{\zeta,w}\left([0,1]\times I^2\right)$*, where* $\partial_1 F$ *is the partial derivative of* F *with respect to its first variable. Then, for all* $x, y \in I$*, we have*

$$\begin{aligned}&\left|\frac{F(1,x,y)+F(0,x,y)}{2}-\int_0^1 F(t,x,y)\,dt\right|\\&\leq\left(\int_0^1|2t-1|\zeta(t)\,dt\right)\frac{|\partial_1F(1,x,y)|+|\partial_1F(0,x,y)|}{2}.\end{aligned}\tag{3.4}$$

Proof. Let $x, y \in I$. For all $c \in \mathbb{R}$, integrating by parts, we obtain

$$\begin{aligned}\int_0^1 F(t,x,y)\,dt&=[(t+c)F(t,x,y)]_0^1-\int_0^1(t+c)\partial_1F(t,x,y)\,dt\\&=(c+1)F(1,x,y)-cF(0,x,y)-\int_0^1(t+c)\partial_1F(t,x,y)\,dt,\end{aligned}$$

that is,

$$(c+1)F(1,x,y)-cF(0,x,y)-\int_0^1 F(t,x,y)\,dt=\int_0^1(t+c)\partial_1F(t,x,y)\,dt.$$

In particular, for $c=-\frac{1}{2}$, we get

$$\frac{F(1,x,y)+F(0,x,y)}{2}-\int_0^1 F(t,x,y)\,dt=\frac{1}{2}\int_0^1(2t-1)\partial_1F(t,x,y)\,dt,$$

which implies that

$$\left|\frac{F(1,x,y)+F(0,x,y)}{2}-\int_0^1 F(t,x,y)\,dt\right|\leq\frac{1}{2}\int_0^1|2t-1||\partial_1F(t,x,y)|\,dt.\tag{3.5}$$

On the other hand, since $|\partial_1 F| \in \mathcal{A}_{\zeta,w}\left([0,1]\times I^2\right)$, then for all $t\in(0,1)$, we have

$$|2t-1||\partial_1F(t,x,y)|\leq|2t-1|\zeta(t)|\partial_1F(1,x,y)|+|2t-1|\zeta(1-t)|\partial_1F(0,x,y)|,$$

which implies after integration over $t\in(0,1)$ that

$$\begin{aligned}\int_0^1|2t-1||\partial_1F(t,x,y)|\,dt\leq&\left(\int_0^1|2t-1|\zeta(t)\,dt\right)|\partial_1F(1,x,y)|\\&+\left(\int_0^1|2t-1|\zeta(1-t)\,dt\right)|\partial_1F(0,x,y)|.\end{aligned}$$

Remarking that

$$\int_0^1|2t-1|\zeta(t)\,dt=\int_0^1|2t-1|\zeta(1-t)\,dt,$$

we obtain

$$\int_0^1|2t-1||\partial_1F(t,x,y)|\,dt\leq\left(\int_0^1|2t-1|\zeta(t)\,dt\right)\left(|\partial_1F(1,x,y)|+|\partial_1F(0,x,y)|\right).\tag{3.6}$$

Finally, (3.4) follows from (3.5) and (3.6). □

We now study some special cases of the above results.

3.1. The case of convex functions

From Theorems 3.1 and 3.2, we deduce the double Hermite-Hadamard inequality for convex functions.

Corollary 3.1. *Let $f \in CV(I)$ and $a, b \in \overset{\circ}{I}$ (interior of I) with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{3.7}$$

Proof. From Proposition 2.1 (see also its proof), we know that $F = F_f \in \mathcal{A}_{\zeta,w}\left([0,1]\times I^2\right)$, where

$$\begin{aligned} F(t,x,y) &= f(tx+(1-t)y), \quad t\in[0,1],\ x,y\in I,\\ \zeta(t) &= t, \quad 0\le t\le 1,\\ w(x) &= x, \quad x\in I. \end{aligned}$$

In this case, we have

$$\int_0^1 F(t,a,b)\,dt = \frac{1}{b-a}\int_a^b f(x)\,dx,\ \ F(1,a,b)+\frac{F(0,a,w(b))}{w'(b)} = f(a)+f(b),\ \ I_\zeta = \frac{1}{2}.$$

Hence, (3.1) with $x = a$ and $y = b$ reduces to

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{3.8}$$

On the other hand, we have

$$\begin{aligned} &\int_0^1 F\left(\frac{1}{2}, ta+(1-t)b, tb+(1-t)a\right) dt\\ &= \int_0^1 f\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right) dt\\ &= f\left(\frac{a+b}{2}\right) \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 F(1, ta+(1-t)b, tb+(1-t)a)\,dt\\ &= \int_0^1 \frac{F(0, ta+(1-t)b, w(tb+(1-t)a))}{w'(tb+(1-t)a)}\,dt\\ &= \frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned}$$

Then (3.2) with $x = a$ and $y = b$ reduces to

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{3.9}$$

Hence, from (3.8) and (3.9), we obtain (3.7). □

From Theorem 3.3, we deduce the following result, which was previously established in [6].

Corollary 3.2. *Let $f : I \to \mathbb{R}$ be a differentiable function such that $|f'| \in CV(I)$. Let $a, b \in I$ with $a < b$. Then*

$$\left|\frac{a+b}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \tag{3.10}$$

Proof. We consider the same functions F, ζ and w introduced in the proof of Corollary 3.1. For all $t \in [0,1]$ and $x, y \in I$, we have

$$\partial_1 F(t,x,y) = (x-y)f'(tx + (1-t)y).$$

Since $|f'| \in CV(I)$, then for all $t \in [0,1]$ and $x, y \in I$,

$$\begin{aligned}|\partial_1 F(t,x,y)| &\leq |x-y| t |f'(x)| + |x-y|(1-t)|f'(y)| \\ &= \zeta(t)|\partial_1 F(1,x,y)| + \zeta(1-t)|\partial_1 F(0,x,y)|,\end{aligned}$$

which shows that $|\partial_1 F| \in \mathcal{A}_{\zeta,w}([0,1] \times I^2)$. On the other hand, we have

$$\left|\frac{F(1,a,b) + F(0,a,b)}{2} - \int_0^1 F(t,a,b)\,dt\right| = \left|\frac{a+b}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{3.11}$$

and

$$\left(\int_0^1 |2t-1|\zeta(t)\,dt\right)\frac{|\partial_1 F(1,a,b)| + |\partial_1 F(0,a,b)|}{2} = \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}. \tag{3.12}$$

Finally, from (3.4) with $x = a$ and $y = b$, (3.11) and (3.12), we obtain (3.10). □

3.2. The case of s-convex functions in the second sense

From Theorems 3.1 and 3.2, we deduce the double Hermite-Hadamard inequality for s-convex functions, which was previously obtained in [7].

Corollary 3.3. *Let $0 < s \leq 1$, $f \in K_s^2(I)$ be a continuous function and $a, b \in I$ with $a < b$. Then*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a) + f(b)}{s+1}. \tag{3.13}$$

Proof. From Proposition 2.2 (see also its proof), we know that $F = F_f \in \mathcal{A}_{\zeta,w}\left([0,1] \times I^2\right)$, where

$$\begin{aligned}F(t,x,y) &= f(tx + (1-t)y), \quad t \in [0,1],\ x, y \in I, \\ \zeta(t) &= t^s, \quad 0 \leq t \leq 1, \\ w(x) &= x, \quad x \in I.\end{aligned}$$

In this case, we have

$$\int_0^1 F(t,a,b)\,dt = \frac{1}{b-a}\int_a^b f(x)\,dx,\ \ F(1,a,b) + \frac{F(0,a,w(b))}{w'(b)} = f(a) + f(b),\ \ I_\zeta = \frac{1}{s+1}.$$

Hence, (3.1) with $x = a$ and $y = b$ reduces to

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{s+1}. \tag{3.14}$$

On the other hand, we have

$$\int_0^1 F\left(\frac{1}{2}, ta+(1-t)b, tb+(1-t)a\right) dt = f\left(\frac{a+b}{2}\right)$$

and

$$\begin{aligned}&\int_0^1 F(1, ta+(1-t)b, tb+(1-t)a)\,dt\\ &= \int_0^1 \frac{F(0, ta+(1-t)b, w(tb+(1-t)a))}{w'(tb+(1-t)a)}\,dt\\ &= \frac{1}{b-a}\int_a^b f(x)\,dx.\end{aligned}$$

Then (3.2) with $x = a$ and $y = b$ reduces to

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{3.15}$$

Finally, from (3.14) and (3.15), we get (3.13). □

From Theorem 3.3, we deduce the following result, which was previously established in [20] (Corollary 3.8 with $r_1 = 1$).

Corollary 3.4. *Let $f : I \to \mathbb{R}$ be a differentiable function such that $|f'| \in K_s^2(I)$ for some $0 < s \le 1$. Let $a, b \in I$ with $a < b$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(s+2^{-s})(b-a)(|f'(a)|+|f'(b)|)}{2(s+1)(s+2)}. \tag{3.16}$$

Proof. Let $\zeta(t) = t^s$, $t \in [0,1]$. We consider the same functions F and w introduced in the proof of Corollary 3.1. For all $t \in [0,1]$ and $x, y \in I$, we have

$$\partial_1 F(t,x,y) = (x-y) f'(tx+(1-t)y).$$

Since $|f'| \in K_s^2(I)$, then for all $t \in [0,1]$ and $x, y \in I$,

$$\begin{aligned}|\partial_1 F(t,x,y)| &\le |x-y| t^s |f'(x)| + |x-y|(1-t)^s |f'(y)|\\ &= \zeta(t)|\partial_1 F(1,x,y)| + \zeta(1-t)|\partial_1 F(0,x,y)|,\end{aligned}$$

which shows that $|\partial_1 F| \in \mathcal{A}_{\zeta,w}([0,1]\times I^2)$. On the other hand, we have

$$\left|\frac{F(1,a,b)+F(0,a,b)}{2} - \int_0^1 F(t,a,b)\,dt\right| = \left|\frac{a+b}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|, \tag{3.17}$$

$$\frac{|\partial_1 F(1,a,b)| + |\partial_1 F(0,a,b)|}{2} = \frac{(b-a)(|f'(a)|+|f'(b)|)}{2},$$

$$\int_0^1 |2t-1|\zeta(t)\,dt = \int_0^1 |2t-1| t^s\,dt = \frac{s+2^{-s}}{(s+1)(s+2)}.$$

Then, from (3.4) with $x = a$ and $y = b$, we obtain (3.16). □

3.3. The case of m-convex functions

From Theorem 3.1, we deduce the following Hermite-Hadamard-type inequality for m-convex functions, which was previously obtained in [5].

Corollary 3.5. *Let* $0 < m \leq 1$, $f \in K_m([0,\infty))$ *and* $0 \leq a < b$. *We have*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}, \frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{3.18}$$

Proof. From Proposition 2.4 (see also its proof), we know that $F = F_f \in \mathcal{A}_{\zeta,w}\left([0,1]\times[0,\infty)^2\right)$, where

$$\begin{aligned} F(t,x,y) &= f(tx+(1-t)y), \quad t\in[0,1],\ x,y\geq 0,\\ \zeta(t) &= t, \quad 0\leq t\leq 1,\\ w(x) &= \frac{x}{m}, \quad x\geq 0. \end{aligned}$$

In this case, for all $x, y \geq 0$ with $x \neq y$, we have

$$\begin{aligned} \int_0^1 F(t,x,y)\,dt &= \frac{1}{y-x}\int_a^b f(z)\,dz,\\ F(1,x,y)+\frac{F(0,x,w(y))}{w'(y)} &= f(x)+mf\left(\frac{y}{m}\right). \end{aligned}$$

Hence, (3.1) with $x = a$ and $y = b$ reduces to

$$\frac{1}{b-a}\int_a^b f(z)\,dz \leq \frac{f(a)+mf\left(\frac{b}{m}\right)}{2}. \tag{3.19}$$

We now apply (3.1) with $x = b$ and $y = a$ to obtain

$$\frac{1}{b-a}\int_a^b f(z)\,dz \leq \frac{f(b)+mf\left(\frac{a}{m}\right)}{2}. \tag{3.20}$$

Hence, from (3.19) and (3.20), we obtain (3.18). □

From Theorem 3.2, we deduce the following Hermite-Hadamard-type inequality for m-convex functions, which was also previously obtained in [5].

Corollary 3.6. *Let* $0 < m \leq 1$ *and* $f \in K_m([0,\infty))$ *be a continuous function. Let* $0 \leq a < b$. *We have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}\,dx. \tag{3.21}$$

Proof. We consider the same functions F, ζ and w introduced in the proof of Corollary 3.5. We have

$$\int_0^1 F\left(\frac{1}{2}, ta+(1-t)b, tb+(1-t)a\right)dt = f\left(\frac{a+b}{2}\right),$$

$$\int_0^1 F(1, ta + (1-t)b, tb + (1-t)a)\, dt = \frac{1}{b-a} \int_a^b f(x)\, dx$$

and

$$\begin{aligned}\int_0^1 \frac{F(0, ta + (1-t)b, w(tb + (1-t)a))}{w'(tb + (1-t)a)}\, dt &= m \int_0^1 f\left(t\frac{b}{m} + (1-t)\frac{a}{m}\right) dt \\ &= \frac{m}{b-a} \int_a^b f\left(\frac{x}{m}\right) dx.\end{aligned}$$

Then (3.2) with $x = a$ and $y = b$ reduces to (3.21). □

3.4. The case of (s, m)-convex functions

From Theorem 3.1, we deduce the following Hermite-Hadamard-type inequality for (s, m)-convex functions, which was previously obtained in [4].

Corollary 3.7. *Let* $0 < s, m \leq 1$, $f \in K^2_{s,m}([0, \infty))$ *and* $0 \leq a < b$. *Then*

$$\frac{1}{b-a} \int_a^b f(x)\, dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{s+1}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{s+1} \right\}. \tag{3.22}$$

Proof. From Proposition 2.5, the function $F = F_f \in \mathcal{A}_{\zeta,w}([0, 1] \times [0, \infty)^2)$, where

$$\begin{aligned} F(t, x, y) &= f(tx + (1-t)y), \quad t \in [0, 1],\ x, y \geq 0, \\ \zeta(t) &= t^s, \quad 0 \leq t \leq 1, \\ w(x) &= \frac{x}{m}, \quad x \geq 0. \end{aligned}$$

In this case, for all $x, y \geq 0$ with $x \neq 0$, we have

$$\begin{aligned} &\int_0^1 F(t, x, y)\, dt = \frac{1}{y-x} \int_a^b f(z)\, dz, \\ &F(1, x, y) + \frac{F(0, x, w(y))}{w'(y)} = f(x) + mf\left(\frac{y}{m}\right), \\ &I_\zeta = \frac{1}{s+1}. \end{aligned}$$

Hence, (3.1) with $x = a$ and $y = b$ reduces to

$$\frac{1}{b-a} \int_a^b f(z)\, dz \leq \frac{f(a) + mf\left(\frac{b}{m}\right)}{s+1}. \tag{3.23}$$

Similarly, (3.1) with $x = b$ and $y = a$ reduces to

$$\frac{1}{b-a} \int_a^b f(z)\, dz \leq \frac{f(b) + mf\left(\frac{a}{m}\right)}{s+1}. \tag{3.24}$$

Hence, (3.22) follows from (3.23) and (3.24). □

Similarly, from Theorem 3.2, we obtain [4, Theorem 8] (with $g \equiv 1$).

Corollary 3.8. *Let* $0 < s, m \leq 1$ *and* $f \in K^2_{s,m}([0, \infty))$ *be a continuous function. Let* $0 \leq a < b$. *Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2^s} \, dx. \tag{3.25}$$

Proof. We consider the same functions F, ζ and w introduced in the proof of Corollary 3.7. In this case, we have

$$\int_0^1 F\left(\frac{1}{2}, ta + (1-t)b, tb + (1-t)a\right) dt = f\left(\frac{a+b}{2}\right),$$

$$\int_0^1 F(1, ta + (1-t)b, tb + (1-t)a) \, dt = \frac{1}{b-a} \int_a^b f(x) \, dx$$

and

$$\int_0^1 \frac{F(0, ta + (1-t)b, w(tb + (1-t)a))}{w'(tb + (1-t)a)} \, dt = m \int_0^1 f\left(t\frac{b}{m} + (1-t)\frac{a}{m}\right) dt$$
$$= \frac{m}{b-a} \int_a^b f\left(\frac{x}{m}\right) dx.$$

Hence, (3.2) with $x = a$ and $y = b$ reduces to (3.25). □

4. Conclusions

We generalized several kinds of convexity from the literature using an implicit inequality involving three functions $F : [0, 1] \times I \times I \to \mathbb{R}$, $\zeta \in C(I)$ and $w \in C^1(I)$ with $w(I) \subset I$ and $w' > 0$, where I is an interval of $\mathbb{R}$. After studying some properties of this class of functions, we established new integral inequalities unifying several Hermite-Hadamard-type inequalities from the literature.

It would be interesting to continue the study of this new class of functions. For instance, several known inequalities from the literature (such as Jensen-type inequalities, Ostrowski-type inequalities, Simpson-type inequalities) can be studied using the class of functions $\mathcal{A}_{\zeta,w}([0, 1] \times I^2)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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