Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(5): 11941–11957.
DOI: 10.3934/math.2024583
Received: 29 January 2024
Revised: 17 March 2024
Accepted: 21 March 2024
Published: 27 March 2024

Research article

A faster iterative scheme for common fixed points of G-nonexpansive mappings via directed graphs: application in split feasibility problems

Maryam Iqbal[1], Afshan Batool[1], Aftab Hussain[2,*] and Hamed Al-Sulami[2]

[1] Department of Mathematical Sciences, Fatima Jinnah Women University, Rawalpindi, Pakistan
[2] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: aniassuirathka@kau.edu.sa; Tel: +966531937156.

Abstract: We have suggested a new modified iterative scheme for approximating a common fixed point of two G-nonexpansive mappings. Our approach was based on an iterative scheme in the context of Banach spaces via directed graphs. First, we proved a weak convergence theorem using the Opial's property of the underlying space. A weak convergence result without the Opial's property was also given. After this, we established several strong convergence theorems using various mild conditions. We also carried out some numerical simulations to examine the main techniques. Eventually, we obtained an application of our result to solve split feasibility problems (SFP) in the context of G-nonexpansive mappings.

Keywords: K^*-iteration; common fixed points; G-nonexpansive mappings; convergence; condition (B); directed graph
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

Let D be a Banach space. A mapping Υ defined on $U \subseteq D$ is called nonexpansive (sometimes, it is called a mapping that does not increase distances) whenever for any $u, v \in U$,

$$\|\Upsilon u - \Upsilon v\| \leq \|u - v\|.$$

A fixed point of Υ is an element $q_0 \in U$ that satisfies the equation $q_0 = \Upsilon q_0$. Often, we specify the fixed point set of Υ as F_Υ. In 1965, Browder [1] was the first mathematician who obtained an elementary existence theorem of fixed points for nonexpansive mappings on the convex closed bounded sets in Hilbert spaces. Very soon, this result was extended to other studies done by Kirk [2], Gohde [3], and Browder [4] to uniformly convex Banach spaces (UCBSs).

The simplest and most basic iterative scheme in the theory of fixed points is attributed to Picard [5]. The Picard iterative scheme can used for finding fixed points of contraction-type mappings, but it is not applicable under nonexpansive mappings. Hence, when a mapping is nonexpansive, we try to use the Mann iterative algorithm [6], which is more general than the mentioned Picard scheme. The convergence rate of both the Picard and Mann iterative schemes is slow and often cannot be applied to obtain a common fixed point. In [7], Ishikawa introduced a new iteration for finding fixed points of a certain category of nonlinear mappings for which the Mann [6], iteration fails to converge. After this, Agarwal et al. [8] provided a new iteration, called the S-iteration and proved that it was faster than all of the above iterative schemes. Ullah et al. [9] gave a faster iterative scheme, called the K^* iteration and proved that it had a fast convergence in comparison to many other leading iterative schemes, including the above iterative schemes. Fixed point theory of nonexpansive and contraction mappings find many useful applications in various fields of applied sciences (see [10–19]). Recently, Debnath [20, 21] worked on a Górnicki-type pair of mappings and F-contractive mappings. He established a criterion for existence and uniqueness of common fixed points for such a pair without assuming continuity of the underlying mappings. Thus, it is very natural to investigate some extensions of the class of these mappings in order to expand its area of application. To achieve this aim, in [22], Jachymski first combined the graph theory with the theory of fixed points, and obtained the BCP in the context of a complete metric space furnished with a directed graph. After that, Aleomraninejad et al. [23] suggested iterative schemes to obtain fixed points of G-contractions and G-nonexpansive mappings in the framework of Banach spaces equipped with graphs. Later, Tiammee et al. [24] obtained Browder-type convergence result for G-nonexpansive mappings in the context of Hilbert spaces with directed graphs.

In 2016, Tripak [25] introduced a modified Ishikawa-type iterative algorithm to obtain common fixed points of G-nonexpansive maps as follows:

$$\begin{cases} u_0 \in U, \\ v_n = (1-\beta_n)u_n + \beta_n \Upsilon_1 u_n, \\ u_{n+1} = (1-\alpha_n)u_n + \alpha_n \Upsilon_2 v_n, \end{cases} \tag{1.1}$$

where Υ_1 and Υ_2 are two G-nonexpansive mappings and $\alpha_n, \beta_n \in (0,1)$. Suparatulatorn et al. [26], in 2018, constructed a modified S-type iteration to obtain common fixed points of G-nonexpansive maps as follows:

$$\begin{cases} u_0 \in U, \\ v_n = (1-\beta_n)u_n + \beta_n \Upsilon_1 u_n, \\ u_{n+1} = (1-\alpha_n)\Upsilon_2 u_n + \alpha_n \Upsilon_2 v_n. \end{cases} \tag{1.2}$$

With the above algorithm, the authors proved numerically that (1.2) converges better than (1.1) under G-nonexpansive maps.

Inspired by the above works, recently, Thianwan and Yambangwai [27] proposed a new iteration method for finding common fixed points of G-nonexpansive maps and analyzed its convergence in the context of a uniformly convex Banach space furnished with a graph:

$$\begin{cases} u_0 \in U, \\ v_n = (1-\beta_n)u_n + \beta_n \Upsilon_1 u_n, \\ u_{n+1} = (1-\alpha_n)\Upsilon_1 v_n + \alpha_n \Upsilon_2 v_n. \end{cases} \tag{1.3}$$

They proved numerically that the iterative algorithm (1.3) converges better than both iterative schemes (1.1) and (1.2) under G-nonexpansive mappings.

Now, it is very natural to ask the following question:

Is there an iteration technique that can be used to find common fixed points of a G-nonexpansive mapping and converges faster than all of the above iterative schemes?

To answer the above question, we have proposed the following new iterative scheme based on the K^*- iterative scheme of Ullah and Arshad [9] as

$$\begin{cases} u_0 \in U, \\ w_n = (1-\beta_n)u_n + \beta_n \Upsilon_2 u_n, \\ v_n = \Upsilon_2\left((1-\alpha_n)w_n + \alpha_n \Upsilon_2 w_n\right), \\ w_{n+1} = \Upsilon_1 v_n, \quad n \in \mathbb{N}. \end{cases} \tag{1.4}$$

The purpose of this research was to establish weak and strong convergence of our suggested iterative algorithm (1.4) toward a common fixed point of two G-nonexpansive mappings. In order to support the main goal, we have offered an example and have shown that our new iteration suggests highly accurate numerical results in comparison to the above iterative schemes. Therefore, we have extended several famous results of the current literatures. Eventually, we have given an application of one result for solving split feasibility problems (SFPs).

2. Preliminaries

In order to establish our main theorems, we have collected the elementary concepts, lemmas, and notions.

Regard a nonempty set U in a Banach space and define the set

$$\Delta = \{(u,u) : u \in U\}.$$

Also, we shall use the notation V_G to represents the set of all vertices that coincide with set U in a directed graph G. Moreover, the set E_G stands for edges that have essentially all loops, that is, $\Delta \subseteq E_G$. Now we suppose that G contains no parallel edge for identifying the graph G having the pair (V_G, E_G). Assume that G^{-1} is the conversion of G. In this case, set

$$E_G^{-1} = \{(v,u) \in U^2 : (u,v) \in E_G\}. \tag{2.1}$$

Now, we suggest the concept of a dominated set and dominated elements as follows. Notice that a given set U is said to be dominated by the element u_0 if for any choice of $u \in u$, the pair (u_0, u) is in the set E_G. On the other hand, the element u_0 is said to be dominated by the set U if for any choice of $u \in U$, the pair (u, u_0) is in the set E_G.

In the sequel, we consider a selfmap Υ of U. Assume that Υ is an edge preserving map, that is, Υ satisfies the condition $(u,v) \in E_G \Rightarrow (\Upsilon u, \Upsilon v) \in E_G$. Then Υ is called G-nonexpansive if the following estimate holds:

$$\|\Upsilon u - \Upsilon v\| \leq \|u - v\|, \quad \text{for every choice of } (u,v) \in E_G. \tag{2.2}$$

The concept of G-demiclosedness is given in the following.

Definition 2.1. *A selfmap Υ on a set U is called G-demiclosed at the point* 0 *if and only if for every weakly convergent sequence, namely,* $\{u_n\}$ *in U whose weak limit is* $q_0 \in U$, *one has*

$$(u_n, u_{n+1}) \in E_G, \ \ \Upsilon u_n \to 0 \ \Rightarrow \Upsilon q_0 = 0.$$

The following definition provides a property for certain Banach spaces. The mentioned property is termed as the Opial's property, which was first introduced by Opial in [28].

Definition 2.2. *A Banach space D is said to be equipped with the Opial's property if for arbitrary weakly convergent sequence* $\{u_n\} \subseteq D$ *with limit* q_0, *the following estimate is fulfilled*

$$\limsup_{n\to\infty} \|u_n - q_0\| < \limsup_{n\to\infty} \|u_n - p_0\|,$$

for all $p_0 \in D - \{q_0\}$.

The following definition is about the semi-compactness of the given mapping.

Definition 2.3. *[29] A selfmap Υ on a subset U of a Banach space is known as semi-compact on U if for any choice of a convergent sequence* $\{u_n\}$ *in the set U that satisifies the condition* $\lim_{n\to\infty} \|\Upsilon u_n - u_n\| = 0$, *one can extract a convergent subsequence, namely,* $\{u_{n_k}\}$ *of* $\{u_n\}$.

Definition 2.4. *Let U be a subset of a Banach space and* $G = (V_G, E_G)$ *denotes the directed graph in such a way that* $V_G = U$. *In this case, the set U is said to be equipped with the property WG (resp. equipped with the property SG) if for any choice of sequence* $\{u_n\}$ *in the set U that is weakly convergent (resp. strongly convergent) to a point, namely,* $q_0 \in U$ *and* $(u_n, u_{n+1}) \in E_G$, *one can find a subsequence, namely,* $\{u_{n_k}\}$ *of* $\{u_n\}$ *with the property* $(u_{n_k}, q_0) \in E_G$.

Lemma 2.5. *[26] Assume that D denotes a Banach space that is enriched with the Opial's property. In this case, if* $U \subseteq D$ *is enriched with the property WG and Υ is a self G-nonexpansive map on U, then* $I - \Upsilon$ *is G–demiclosed at the point* 0*; that is, for any sequence,* $\{u_n\} \subseteq U$ *such that* $u_n \to q_0$ *and* $\|u_n - \Upsilon u_n\| \to 0$, *it follows that* $q_0 \in F_\Upsilon$.

Every UCBS possesses the following useful characterization. This characterization was suggested for the first time by Schu in [30].

Lemma 2.6. *[30] Suppose that a UCBS D is given. If the sequence* $\{\alpha_n\}$ *is such that* $0 < c \leq \alpha_n \leq r < 1$ *and the two sequences, namely,* $\{u_n\}$ *and* $\{v_n\}$ *in D, satisfy the conditions* $\limsup_{n\to\infty} \|u_n\| \leq z$, $\limsup_{n\to\infty} \|v_n\| \leq z$, *and* $\lim_{n\to\infty} \|\alpha_n u_n + (1 - \alpha_n)v_n\| = z$, *where z is any real number in the interval* $[0, \infty)$, *then the estimate* $\lim_{n\to\infty} \|u_n - v_n\| = 0$ *always holds.*

The following lemma is also necessary for establishing our weak convergence result.

Lemma 2.7. *[31] D denotes a Banach space that is enriched with the Opial's property and* $\{u_n\} \subseteq D$. *Assume that some pair of two points* $u, v \in D$ *for which* $\limsup_{n\to\infty} \|u_n - u\|$ *and* $\limsup_{n\to\infty} \|u_n - v\|$ *exists. If* $\{u_{n_j}\}$ *and* $\{u_{n_k}\}$ *denote any arbitrary weakly convergent subsequences of* $\{u_n\}$ *with weak limits u and v, then the equation* $u = v$ *is to be held.*

Since a UCBS is reflexive, we will also need the following lemma.

Lemma 2.8. *[32] Assume that D denotes a reflexive Banach space and $\{u_n\} \subseteq D$. If this sequence is bounded in D and there are some weakly convergent subsequences, namely, $\{u_{n_j}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$, and both admit the same weak limit, namely, $q_0 \in D$, then $\{u_n\}$ itself is weakly convergent to q_0.*

Lemma 2.9. *[27] Suppose a selfmap Υ on a subset U of a UCBS is G-nonexpansive. If U is enriched with the property WG, then the operator $I - \Upsilon$ is eventually G-demiclosed at the point* 0.

3. Main results

Now, we are able to start our main results. Before going to convergence results, we need a key proposition as follows. Note that, throughout the section, we may use the notation F for the set $F_{\Upsilon_1} \cap F_{\Upsilon_2}$.

Proposition 3.1. *Suppose that D is a UCBS enriched with a directed graph and $\emptyset \neq U \subseteq D$ is convex and closed. Assume that Υ_1 and Υ_1 are G-nonexpansive selfmaps on U with $F \neq \emptyset$. In addition, we assume $V_G = U$ and the set E_G is convex, G is transitive, and we set the sequence of iterates $\{u_n\}$ by using (1.4) for any starting guess $u_0 \in U$. If q_0 in the set F is such that $(u_0, q_0), (q_0, u_0) \in E_G$, then all the pairs $(u_n, q_0), (v_n, q_0), (w_n, q_0), (q_0, u_n), (q_0, v_n), (q_0, w_n)$, $(u_n, v_n), (u_n, w_n)$, and (u_n, u_{n+1}) are also in the set E_G.*

Proof. The proof will be completed using induction. To do this, since $(u_0, q_0) \in E_G$, it follows from the edge preserving property of the mapping Υ_2 that $(\Upsilon_2 u_0, q_0) \in E_G$. Due to the convexity of the set E_G, we get $(w_0, q_0) \in E_G$. Now since $(w_0, q_0) \in E_G$, it follows from the edge preserving property of the mapping Υ_2 that $(\Upsilon_2 w_0, q_0) \in E_G$. Now E_G is convex,Υ_2 is edge preserving, and

$$(\Upsilon_1 u_0, q_0), (\Upsilon_2 w_0, q_0) \in E_G.$$

It follows that $(v_0, q_0) \in E_G$. Again, since $(v_0, q_0) \in E_G$ and the fact that mapping Υ is edge preserving, we obtain $(\Upsilon_1 v_0, q_0) \in E_G$. Similarly, Υ_2 is edge preserving, $(\Upsilon_2 u_1, q_0) \in E_G$, and we obtain $(w_1, q_0) \in E_G$, because the set E_G is convex. Hence due to the edge preserving property of the mapping Υ_2, so one has $(\Upsilon_2 w_1, q_0) \in E_G$. Again due to the convexity of E_G, $(\Upsilon_1 w_1, q_0), (\Upsilon_2 w_1, q_0) \in E_G$ and the mapping Υ_2 is edge preserving, one has $(y_1, q_0) \in E_G$. Also by the edge preserving property of the mapping Υ_1, one has $(\Upsilon_1 v_1, q_0) \in E_G$, and hence we obtain $(w_2, q_0) \in E_G$. Next, we suppose that $(u_k, q_0) \in E_G$. Now, the set E_G is convex and Υ_2 is edge preserving, one has $(\Upsilon_2 u_k, q_0) \in E_G$ and $(w_k, q_0) \in E_G$. On the other hand, one can apply the edge preserving property of the mapping Υ_2 on $(w_k, q_0) \in E_G$, and thus we get $(\Upsilon_2 w_k, q_0) \in E_G$. Since the set E_G is convex and $(\Upsilon_1 u_k, q_0), (T_2 w_k, q_0) \in E_G$ and the mapping Υ_2 is edge preserving, we get $(v_k, q_0) \in E_G$. Also the mapping Υ_1 is edge preserving; it follows that $(u_{k+1}, q_0) \in E_G$. Due to the edge preserving property of the mapping Υ_2, one has $(\Upsilon_2 u_{k+1}, q_0) \in E_G$ and thus $(w_{k+1}, q_0) \in E_G$, since the set E_G is convex. Now, E_G is convex and $(\Upsilon_1 u_{k+1}, q_0), (\Upsilon_2 w_{k+1}, q_0) \in E_G$ and the mapping Υ_2 is edge preserving. We can write $(v_{k+1}, q_0) \in E_G$. Hence, we conclude that $(u_n, q_0), (v_n, q_0), (w_n, q_0) \in E_G$ for any choice of $n \geq 0$. In a way similar to the above, one can prove that $(q_0, w_n), (q_0, v_n), (q_0, w_n) \in E_G$. But the set G is transitive. So, one can write

$$(u_n, v_n), (u_n, w_n), (v_n, w_n), (u_n, v_{n+1}) \in E_G.$$

Thus, the proof is finished. □

Lemma 3.2. *Suppose that $D, U, \Upsilon_1, \Upsilon_2, F$, and the sequence of iterates $\{u_n\}$ be the same as what is given in Proposition 3.1. If the pairs $(u_0, q_0), (q_0, u_0)$ are in the set E_G for every choice of $u_0 \in U$ and $q_0 \in F$, then we have:*

(i) $\lim_{n\to\infty} \|u_n - q_0\|$ *exists;*
(ii) $\lim_{n\to\infty} \|\Upsilon_1 u_n - u_n\| = 0 = \lim_{n\to\infty} \|\Upsilon_2 u_n - u_n\|$.

Proof. (i) Since both the mappings Υ_1 and Υ_2 are G-nonexpansive, from Proposition 3.1, one can write $(u_n, q_0), (v_n, q_0), (w_n, q_0) \in E_G$. Accordingly, we have

$$
\begin{aligned}
\|u_{n+1} - q_0\| &= \|\Upsilon_1 v_n - q_0\| \\
&\leq \|v_n - q_0\| \\
&= \|\Upsilon_2((1-\alpha_n)w_n + \alpha_n \Upsilon_2 w_n) - q_0\| \\
&\leq \|(1-\alpha_n)w_n + \alpha_n \Upsilon_2 w_n - q_0\| \\
&\leq (1-\alpha_n)\|w_n - q_0\| + \alpha_n \|\Upsilon_2 w_n - q_0\| \\
&\leq (1-\alpha_n)\|w_n - q_0\| + \alpha_n \|w_n - q_0\| \\
&= \|w_n - q_0\| \\
&= \|(1-\beta_n)u_n + \beta_n \Upsilon_2 u_n - q_0\| \\
&\leq (1-\beta_n)\|u_n - q_0\| + \beta_n \|\Upsilon_2 u_n - q_0\| \\
&\leq (1-\beta_n)\|u_n - q_0\| + \beta_n \|u_n - q_0\| \\
&\leq \|u_n - q_0\|.
\end{aligned}
$$

Eventually, we observe that $\|u_{n+1} - q_0\| \leq \|u_n - q_0\|$ for any choice of $n \geq 0$. It yields that the real sequence $\{\|u_n - q_0\}$ is non-increasing and, accordingly, is bounded. Therefore, we conclude that $\lim_{n\to\infty} \|u_n - q_0\|$ exists.

Now we prove (ii). We first take $F \neq \emptyset$ and fix any $q_0 \in F$. By (i), $\lim_{n\to\infty} \|u_n - q_0\|$ exists. Put

$$\lim_{n\to\infty} \|u_n - q_0\| = z. \tag{3.1}$$

As we proved above, we write

$$\limsup_{n\to\infty} \|w_n - q_0\| \leq \limsup_{n\to\infty} \|u_n - q_0\| = z. \tag{3.2}$$

Υ_2 is G-nonexpansive. So

$$\limsup_{n\to\infty} \|\Upsilon_2 u_n - q_0\| \leq \limsup_{n\to\infty} \|u_n - q_0\| = z. \tag{3.3}$$

Again, as proved above, we estimate

$$z = \liminf_{n\to\infty} \|u_{n+1} - q_0\| \leq \liminf_{n\to\infty} \|w_n - q_0\|. \tag{3.4}$$

From (3.2) and (3.4), we have

$$z = \lim_{n\to\infty} \|w_n - q_0\|. \tag{3.5}$$

Using (3.5), one has

$$z = \lim_{n\to\infty} \|w_n - p\| = \lim_{n\to\infty} \|(1-\beta_n)(u_n - q_0) + \beta_n(\Upsilon_2 u_n - q_0)\|.$$

Hence,

$$z = \lim_{n\to\infty} \|(1-\beta_n)(u_n - q_0) + \beta_n(\Upsilon_2 u_n - q_0)\|. \tag{3.6}$$

By Lemma 2.6, we have

$$\lim_{n\to\infty} \|\Upsilon_2 u_n - u_n\| = 0.$$

In a similar way,

$$\lim_{n\to\infty} \|\Upsilon_1 u_n - u_n\| = 0.$$

Subsequently, we obtain

$$\lim_{n\to\infty} \|\Upsilon_1 u_n - u_n\| = 0 = \lim_{n\to\infty} \|\Upsilon_2 u_n - u_n\|.$$

Both proofs are complete. □

We will now consider the assumption that the space is enriched with the Opial's property, and provide a weak convergence result for G-nonexpansive maps by applying the iterative scheme (1.4).

Theorem 3.3. *Suppose that $D, U, \Upsilon_1, \Upsilon_2, F$, and the sequence of iterates $\{u_n\}$ be the same as what is given in Proposition 3.1. Assume that U has the property WG and the pairs $(u_0, q_0), (q_0, u_0)$ are in the set E_G for each choice of $u_0 \in U$ and $q_0 \in F$. Then $\{u_n\}$ weakly converges to a point of F if D is enriched with the Opial's property.*

Proof. According to Lemma 3.2, the sequence of iterates $\{u_n\}$ is bounded in U and $\lim_{n\to\infty} \|u_n - p_0\|$ exists. Since D is UCBS, it follows that D is reflexive. Suppose $\{u_n\}$ is weakly convergent to a point $u \in U$. But in Lemma 3.2(ii),

$$\lim_{n\to\infty} \||\Upsilon_1 u_n - u_n\| = 0 = \lim_{n\to\infty} \|\Upsilon_2 u_n - u_n\|.$$

Hence, from Lemma 2.5, we have $u \in F$. Now, we take two subsequences, namely, $\{u_{n_k}\}$ and $\{u_{n_j}\}$ of the sequence of iterates $\{u_n\}$ such that both are convergent to u and v, respectively. Applying Lemma 2.5, one gets $u, v \in F$. Hence using Lemma 3.2(i), $\lim_{n\to\infty} \|u_n - u\|$ and $\lim_{n\to\infty} \|u_n - v\|$ exist. By Lemma 2.7, we obtain $u = v$. Subsequently, the sequence of iterates $\{u_n\}$ is weakly convergent to a point of F. □

In the following result, we do not need the Opial's property of the space D.

Theorem 3.4. *Let $D, U, \Upsilon_1, \Upsilon_2, F$, and the sequence of iterates $\{u_n\}$ be the same as what is given in Proposition 3.1. Assume that U has the property WG and the pairs $(u_0, q_0), (q_0, u_0)$ are in the set E_G for every choice of $u_0 \in U$ and $u_0 \in F$. Then $\{u_n\}$ converges weakly to a point of F if F is dominated by u_0 and F dominates u_0.*

Proof. The proof is clear. □

The next theorem is based on the following condition (B).

Definition 3.5. *[29] Suppose that Υ_1 and Υ_2 are two selfmaps of the subset U in a UCBS D. In this case, Υ_1 and Υ_2 are said to be with the condition (B) if one has a nondecreasing map g such that $g(0) = 0$ and $g(\alpha) > 0$ for any $\alpha \in (0, \infty)$ and $\max\{\|u - \Upsilon_1 u\|, \|u - \Upsilon_2 u\|\} \geq \psi(d_s(u, F))$ for every choice of $u \in U$, where $d_s(u, F)$ stands for the norm distance between u and the set F.*

Theorem 3.6. *Suppose that $D, U, \Upsilon_1, \Upsilon_2, F$, and the sequence of iterates $\{u_n\}$ are the same as what is given in Proposition 3.1. Assume that the pairs $(u_0, q_0), (q_0, u_0)$ are in the set E_G for every choice of $u_0 \in U$ and $q_0 \in F$. If F is dominated by u_0, F dominates u_0, then $\{u_n\}$ converges strongly to a point of F provided that the mappings Υ_1 and Υ_2 are equipped with condition (B).*

Proof. In view of Lemma 3.2(i), $\lim_{n\to\infty} \|u_n - q_0\|$ exists. It follows that $\lim_{n\to\infty} d_s(u_n, F)$ exists for any choice of $q_0 \in F$.Thus, using Lemma 3.2(ii),

$$\lim_{n\to\infty} \|\Upsilon_1 u_n - u_n\| = 0 = \lim_{n\to\infty} \|\Upsilon_2 u_n - u_n\|.$$

Now condition (B) gives

$$g(d_s(u_n, F)) \leq \max\{\|u_n - \Upsilon_1 u_n\|, \|u_n - \Upsilon_2 u_n\|\}.$$

In any case, we get $\lim_{n\to\infty} g(d_s(u_n, F)) = 0$. It follows that $\lim_{n\to\infty} d_s(u_n, F)) = 0$. Therefore, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ and $\{q_s\}$ in F such that $\|u_{n_s} - q_s\| \leq \frac{1}{2^s}$ for all choices of $s \geq 0$. Since the proof of Lemma 3.2(i) provides that $\{u_n\}$ is non-increasing, then

$$\|u_{n_{s+1}} - q_j\| \leq \|u_{n_s} - q_s\| \leq \frac{1}{2^s}.$$

Therefore,

$$\begin{aligned} \|q_{s+1} - q_s\| &\leq \|q_{s+1} - u_{n_{s+1}}\| + \|u_{n_{s+1}} - q_s\| \\ &\leq \frac{1}{2^{s+1}} + \frac{1}{2^s} \\ &\leq \frac{1}{2^{s-1}} \to 0, \text{ when } s \to \infty. \end{aligned}$$

Hence, we proved that $\{q_s\}$ form a Cauchy sequence in F and thus, it has a limit, namely, p_0. Since F is a closed subset of U, we must have $p_0 \in F$. Now, applying Lemma 3.2(i) on p_0, we get that $\lim_{n\to\infty} \|u_n - q_0\|$ exists. This proves that the point $p_0 \in F$ is the strong limit of $\{u_n\}$. Hence, the proof is finished. □

We close this section by giving a strong convergence theorem using the semi-compactness assumption.

Theorem 3.7. *Suppose that $D, U, \Upsilon_1, \Upsilon_2, F$, and the sequence of iterates $\{u_n\}$ are the same as what is given in Proposition 3.1. Assume that U has the property SG and the pairs $(u_0, q_0), (q_0, u_0)$ are in the set E_G for every choice of $u_0 \in U$ and $q_0 \in F$. If F is dominated by u_0, F dominates u_0, then $\{u_n\}$ converges strongly to a point of F provided that the mappings Υ_1 and Υ_2 are semi-compact.*

Proof. From Lemma 3.2(i), the iterative sequence $\{u_n\}$ is essentially bounded and

$$\lim_{n\to\infty} \|\Upsilon_1 u_n - u_n\| = 0 = \lim_{n\to\infty} \|\Upsilon_2 u_n - u_n\|.$$

By semi-compactness of Υ_1 or Υ_2, one has a subsequence $\{u_{n_k}\}$ such that

$$\lim_{k\to\infty} \|u_{n_k} - q_0\| = 0, \tag{3.7}$$

for some element $q_0 \in U$. By property SG of U, and keeping in mind the transitivity of G, one has $(u_{n_k}, q_0) \in E_G$. Hence

$$\lim_{k\to\infty} \|u_{n_k} - \Upsilon_i u_{n_k}\| = 0, \quad (i = 1, 2). \tag{3.8}$$

Using (3.7) and (3.8) and being G-nonexpansive of Υ_i, one has

$$\begin{aligned} \|q_0 - \Upsilon_i q_0\| &\le \|q_0 - u_{n_k}\| + \|u_{n_k} - \Upsilon_i u_{n_k}\| + \|\Upsilon_i u_{n_k} - \Upsilon_i q_0\| \\ &\le \|q_0 - u_{n_k}\| + \|u_{n_k} - \Upsilon_i u_{n_k}\| + \|u_{n_k} - q_0\| \\ &= 2\|q_0 - u_{n_k}\| + \|u_{n_k} - \Upsilon_i u_{n_k}\| \to 0. \end{aligned}$$

Subsequently, we obtained $q_0 = \Upsilon_i q_0$. This shows that q_0 is a point of F. Accordingly, $\lim_{n\to\infty} d_s(u_{n_k}, F)$ exists by Theorem 3.6. But $d_s(u_{n_k}, F) \le d_s(u_{n_k}, q_0) \to 0$, that is,

$$\lim_{n\to\infty} d_s(u_{n_k}, F) = 0.$$

In view of the proof of Theorem 3.6, we conclude that $\{u_n\}$ converges strongly to a common fixed point of Υ_1 and Υ_2. □

4. Numerical example

Next, we will discuss an example of G-nonexpansive mappings that are not nonexpansive. We will connect our new modified iterative scheme and other iterative schemes from the literature to show the effectiveness of our results. We will use some numerical tables for this purpose.

Example 4.1. *Define two mappings Υ_1 and Υ_2 as follows:*

$$\Upsilon_1 u = u + \frac{12 \arcsin^{-1}(1-u)}{18},$$

and

$$\Upsilon_2 u = u^{\frac{1}{4}}.$$

In this case, both Υ_1 and Υ_2 are G-nonexpansive and admit a common fixed point 1.

We now take $\alpha_n = \beta_n = 0.5$ and obtain Tables 1–3 for various values of u_0. The graphical comparison is given in these cases in Figures 1–3.

Table 1. Comparison of various iterations for $u_0 = 1.2$.

n	**New** (1.4)	**Thianwan** (1.3)	**S** (1.2)	**Ishikawa** (1.1)
0	1.2	1.2	1.2	1.2
1	1.0123859386	1.0756643868	1.0868146410	1.1196953341
2	1.0008037129	1.0287446908	1.0377867331	1.0718861233
3	1.0000523138	1.0109537143	1.0164894180	1.0432711048
4	1.0000034058	1.0041799009	1.0072056196	1.0260838084
5	1.0000002217	1.0015959266	1.0031507977	1.0157372183
6	1.0000000144	1.0006094727	1.0013781533	1.0094999288
7	1.0000000009	1.0002327726	1.0006028804	1.0057366219
8	1.0000000001	1.0000889044	1.0002637484	1.0034648088
9	**1**	1.0000339563	1.0001153877	1.0020929321
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
21	1	1.0000000003	1.0000000057	1.0000049489
22	1	1.0000000001	1.0000000025	1.0000029900
23	1	**1**	1.0000000011	1.0000018065
24	1	1	1.0000000005	1.0000010914
25	1	1	1.0000000002	1.0000006594
26	1	1	1.0000000001	1.0000003984
27	1	1	**1**	1.0000002407
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
44	1	1	1	**1**

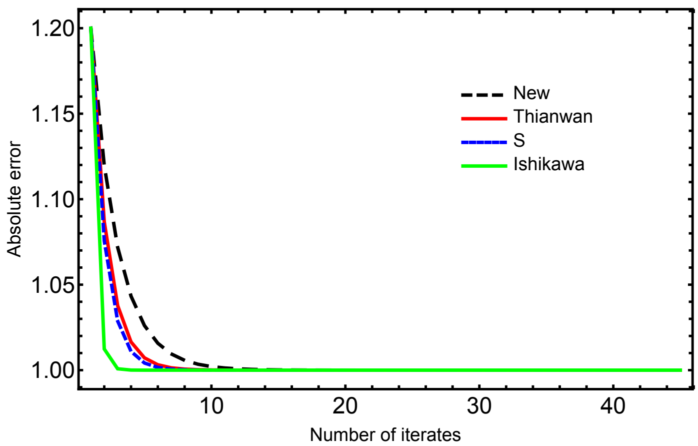

Figure 1. Graphical comparison of different iterative schemes for $u_0 = 1.2$.

Table 2. Comparison of various iterations for $u_0 = 1.5$.

n	**New** (1.4)	**Thianwan** (1.3)	**S** (1.2)	**Ishikawa** (1.1)
0	1.5	1.5	1.5	1.5
1	1.0290417199	1.1923944719	1.2207787459	1.2962458207
2	1.0018762181	1.0727931850	1.0958340352	1.1766367539
3	1.0000523138	1.0276584880	1.0416965272	1.1058166689
4	1.0000079482	1.0105406503	1.0181912777	1.0635918574
5	1.0000005175	1.004022412	1.0079483819	1.0382940901
6	1.0000000337	1.001535816	1.0034753998	1.0230895279
7	1.0000000022	1.0005865199	1.0015200975	1.0139328453
8	1.0000000001	1.0002240068	1.0006649677	1.0084115026
9	**1**	1.0000855565	1.000290907	1.0050796598
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
21	1	1.0000000008	1.0000000143	1.0000120065
22	1	1.0000000003	1.0000000063	1.0000072539
23	1	1.0000000001	1.0000000027	1.0000043826
24	1	**1**	1.0000000012	1.0000026478
25	1	1	1.0000000005	1.0000015997
26	1	1	1.0000000002	1.0000009665
27	1	1	1.0000000001	1.0000005839
28	1	1	**1**	1.0000003528
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
46	1	1	1	**1**

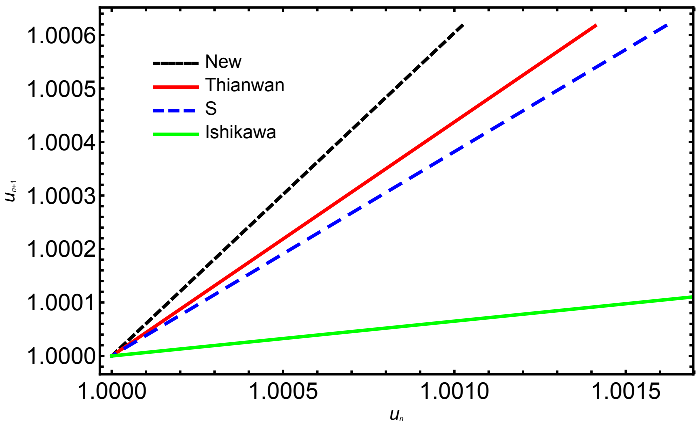

Figure 2. Graphical comparison of different iterative schemes for $u_0 = 1.5$.

Table 3. Comparison of various iterations for $u_0 = 1.9$.

n	**New** (1.4)	**Thianwan** (1.3)	**S** (1.2)	**Ishikawa** (1.1)
0	1.9	1.9	1.9	1.9
1	1.0486666838	1.4033824472	1.4542642842	1.5310077792
2	1.0031280319	1.1536373468	1.1995308532	1.3143734715
3	1.0002034772	1.058174495	1.0866111569	1.1873240273
4	1.0000132465	1.0221235889	1.0376985031	1.112168284
5	1.0000008624	1.0084348414	1.0164510048	1.0673889846
6	1.0000000561	1.0032193688	1.0071888526	1.0405730575
7	1.0000000037	1.0012292851	1.0031434698	1.0244607882
8	1.0000000002	1.0004694696	1.0013749488	1.0147592445
9	**1**	1.0001793041	1.0006014788	1.0089100251
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
21	1	1.0000000017	1.0000000296	1.0000210493
22	1	1.00000000007	1.0000000129	1.0000127172
23	1	1.00000000003	1.0000000057	1.0000076833
24	1	1.00000000001	1.0000000025	1.000004642
25	1	**1**	1.0000000011	1.0000028045
26	1	1	1.0000000005	1.0000016944
27	1	1	1.0000000002	1.0000010237
28	1	1	1.0000000001	1.0000006185
29	1	1	**1**	1.0000003737
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
47	.	.	.	**1**

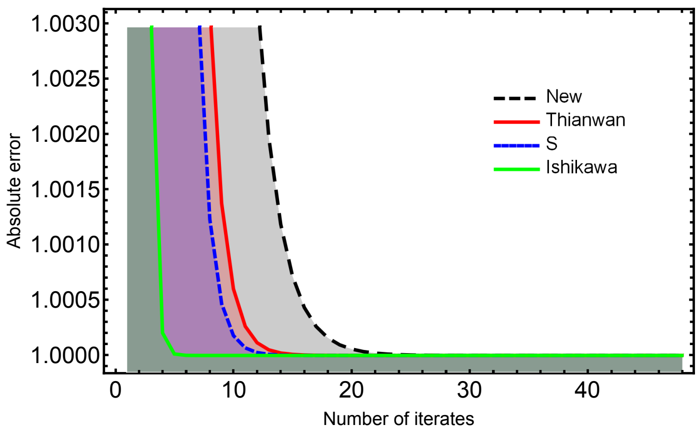

Figure 3. Graphical comparison of different iterative schemes for $u_0 = 1.9$.

Now we suggest different values for the initial points and the parameters α_n, β_n, and also, we set stopping criterion as $\|u_n - q^*\| < 10^{-10}$, and note to keep in mind that $q^* = 1$ is a common fixed point of the selfmaps Υ_1 and Υ_2. The numerical results are then given in Tables 4–6.

Table 4. When $\alpha_n = \frac{n}{n+8}$ and $\beta_n = \frac{1}{n+6}$.

The required number of iterations for obtaining the fixed point				
u_0	**Ishikawa** (1.1)	**S** (1.2)	**Thianwan** (1.3)	**New** (1.4)
1.2	32	24	23	**10**
1.3	32	24	24	**10**
1.4	32	24	24	**10**
1.5	33	24	24	**10**
1.6	33	25	24	**10**
1.7	33	25	24	**10**
1.8	33	25	25	**10**

Table 5. When $\alpha_n = \frac{n+1}{5n+4}$ and $\beta_n = \frac{n+3}{9n+6}$.

The required number of iterations for obtaining the fixed point				
u_0	**Ishikawa** (1.1)	**S** (1.2)	**Thianwan** (1.3)	**New** (1.4)
1.2	118	36	33	**9**
1.3	121	36	34	**9**
1.4	122	37	34	**9**
1.5	124	37	34	**9**
1.6	125	38	35	**9**
1.7	125	38	35	**9**
1.8	126	39	35	**9**

Table 6. When $\alpha_n = 1 - (\frac{1}{\sqrt{5n+3}})$ and $\beta_n = \frac{1}{(n+1)^3}$.

The required number of iterations for obtaining the fixed point				
u_0	**Ishikawa** (1.1)	**S** (1.2)	**Thianwan** (1.3)	**New** (1.4)
1.2	20	17	17	**7**
1.3	20	17	17	**7**
1.4	20	18	17	**7**
1.5	21	18	18	**7**
1.6	21	18	18	**7**
1.7	21	18	18	**7**
1.8	21	18	18	**7**

5. Application to split feasibility problems (SFPs)

Suppose that D_1 and D_2 are Hilbert spaces with directed graphs. Assume that $C \subseteq D_1$ and $Q \subseteq D_2$ are any two nonempty convex and closed sets such that the mapping $\mathcal{A} : \mathcal{D}_1 \to \mathcal{D}_2$ is any given linear and bounded function. First, we give the basic concept about a SFP [18] as follows. Mathematically, a SFP is defined as:

$$\text{Compute } q^* \in C : \mathcal{A}q^* \in Q. \tag{5.1}$$

As we know from [19], the concept of SFP is applicable to many type of problems in applied sciences. Thus, the concept of SFP is more important than the many other type of concepts in nonlinear analysis.

In the present study, first we considered this assumption that our SFP (5.1) possessed one solution and we wrote $\mathcal{S}$ to denote its solution set. By [19], we know that any point $q^* \in C$ eventually solves the Problem (5.1) if and only if q^* is the solution for the equation

$$u = P_C(I - \xi\mathcal{A}^*(I - P_Q)\mathcal{A})u,$$

in which P_C and P_Q stand for the nearest point projections onto C and Q, respectively. Also, $\xi > 0$ and the mapping $\mathcal{A}^*$ specifies the adjoint operator of $\mathcal{A}$. In [10], Byrne used the concept of nonexpansive mappings. That is, he first proved that for any ξ and $0 < \xi < \frac{2}{\eta}$,

$$\Upsilon = P_C(I - \xi\mathcal{A}^*(I - P_Q)\mathcal{G}),$$

is essentially nonexpansive and its CQ iterative scheme given as

$$u_{n+1} = P_C(I - \xi\mathcal{A}^*(I_{id} - P_Q)\mathcal{A})u_n, n \geq 0$$

is weakly convergent to a point of $\mathcal{S}$.

When a weak convergence is established, naturally we would like to investigate a result for the case of strong convergence. To do this, we require some more conditions (see [19]) to conduct an analysis on the recent research about the Halpern-type algorithms.

We have adopted a new method to solve SFPs by applying the concept of G-nonexpansive operators whose nature is more general than the concept of nonexpansive mappings (we saw this in the example provided in this paper). We shall examine and confirm that the proposed scheme is convergent to the solution of the SFP (5.1).

Theorem 5.1. *Consider the SFP (5.1) with $\mathcal{S} \neq \emptyset$, $0 < \xi < \frac{2}{\eta}$, $P_C(I - \xi\mathcal{A}^*(I - P_Q)\mathcal{A})$ is a G-nonexpansive operator and satisfies the condition (B). In this case, the sequence of iterates $\{u_n\}$ given by (1.4) is strongly convergent to some solution, namely, q^* of the SFP (5.1).*

Proof. We can set

$$\Upsilon = \Upsilon_1 = \Upsilon_2 = P_C(I - \xi\mathcal{A}^*(I - P_Q)\mathcal{A}).$$

Clearly, Υ will be a G-nonexpansive operator. The conclusion of Theorem 3.6 gives the fact that $\{u_n\}$ is strongly convergent in F. As $F = \mathcal{S}$, we deduce that $\{u_n\}$ is strongly convergent to some solution, namely, q^* of the SFP introduced in (5.1). □

6. Conclusions

(i) We introduced a new modified iterative scheme based on the K^*-iterative scheme for finding common fixed points of G-nonexpansive mappings.

(ii) We obtained several weak and strong convergence results the for new iterative scheme under possible mild conditions.
(iii) A comparative numerical experiment was performed which proves the high accuracy of our new scheme in comparison with the already existing iterative schemes.
(iv) Eventually, we applied our main results to solve SFPs.
(v) Our findings extended and improved the corresponding results of Tripak [25], Suparatulatorn et al. [26], and Thianwan and Yambangwai [27] with a faster iterative scheme. Moreover, our theorems unified the main result of the paper written by Ullah et al. [9] to the case of G-nonexpansive mappings.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We are thankful to the employer of King Abdulaziz University for their financial support and encouragement.

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. F. E. Browder, Fixed-point theorems for noncompact mappings in Hilbert space, *Proc. Natl. Acad. Sci. USA*, **53** (1965), 1272–1276. https://doi.org/10.1073/pnas.53.6.1272
2. W. A. Kirk, A fixed point theorem for mappings which do not increase distance, *Amer. Math. Mon.*, **72** (1965), 1004–1006. https://doi.org/10.2307/2313345
3. D. Göhde, Zum prinzip der kontraktiven abbildung, *Math. Nachr.*, **30** (1965), 251–258.
4. F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Natl. Acad. Sci. USA*, **54** (1965), 1041–1044. https://doi.org/10.1073/pnas.54.4.1041
5. E. Picard, Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives, *J. Math. Pures Appl.*, **6** (1890), 145–210.
6. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510.
7. S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1974), 147–150. https://doi.org/10.1090/S0002-9939-1974-0336469-5
8. R. P. Agarwal, D. O'Regon, D. R. Sahu, Iterative construction of fixed points of nearly asymtotically nonexpansive mappings, *J. Nonlinear Convex Anal.*, **8** (2007), 61–79.
9. K. Ullah, M. Arshad, New three-step iteration process and fixed point approximation in Banach spaces, *J. Linear Topol. Algebra*, **7** (2018), 87–100.

10. C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Probl.*, **20** (2004), 103–120. https://doi.org/10.1088/0266-5611/20/1/006

11. A. Moussaoui, N. Hussain, S. Melliani, N. Hayel, M. Imdad, Fixed point results via extended $\mathcal{FZ}$-simulation functions in fuzzy metric spaces, *J. Inequal. Appl.*, **2022** (2022), 69. https://doi.org/10.1186/s13660-022-02806-z

12. A. Moussaoui, S. Radenović, S. Melliani, Fixed point theorems involving $\mathcal{FZ}$-ϑ_f-contractions in GV-fuzzy metrics, *Filomat*, **6** (2024), 1973–1985.

13. A. Moussaoui, S. Melliani, S. Radenović, A nonlinear fuzzy contraction principle via control functions, *Filomat*, **6** (2024), 1963–1972.

14. M. Iqbal, A. Ali, H. A. Sulami, A. Hussain, Iterative stability analysis for generalized α-nonexpensive mappings with fixed points, *Axioms*, **13** (2024), 156. https://doi.org/10.3390/axioms13030156

15. A. Hussain, N. Hussain, D. Ali, Estimation of newly established iterative scheme for generalized nonexpansive mappings, *J. Funct. Spaces*, **2021** (2021), 6675979. https://doi.org/10.1155/2021/6675979

16. D. Ali, A. Hussain, E. Karapinar, P. Cholamjiak, Efficient fixed-point iteration for generalized nonexpansive mappings and its stability in Banach spaces, *Open Math.*, **20** (2023), 1753–1769. https://doi.org/10.1515/math-2022-0461

17. A. Hussain, D. Ali, E. Karapinar, Stability data dependency and errors estimation for a general iteration method, *Alex. Eng. J.*, **60** (2021), 703–710. https://doi.org/10.1016/j.aej.2020.10.002

18. Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numer. Algor.*, **8** (1994), 221–239. https://doi.org/10.1007/BF02142692

19. G. Lopez, V. Martin-Marquez, H. K. Xu, Halpern's iteration for nonexpansive mappings, *Contemp. Math.*, **513** (2010), 211–231.

20. P. Debnath, Results on discontinuity at fixed point for a new class of F-contractive mappings, *Sahand Commun. Math. Anal.*, **20** (2023), 21–32. https://doi.org/10.22130/scma.2023.560141.1161

21. P. Debnath, New common fixed point theorems for Górnicki-type mappings and enriched contractions, *São Paulo J. Math. Sci.*, **16** (2022), 1401–1408. https://doi.org/10.1007/s40863-022-00283-2

22. J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Amer. Math. Soc.*, **136** (2008), 1359–1373. https://doi.org/10.1090/S0002-9939-07-09110-1

23. J. Tiammee, A. Kaewkhao, S. Suantai, On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs, *Fixed Point Theory Appl.*, **2015** (2015), 187. https://doi.org/10.1186/s13663-015-0436-9

24. S. M. A. Aleomraninejad, S. Rezapour, N. Shahzad, Some fixed point results on a metric space with a graph, *Topol. Appl.*, **159** (2012), 659–663. https://doi.org/10.1016/j.topol.2011.10.013

25. O. Tripak, Common fixed points of G-nonexpansive mappings on Banach spaces with a graph, *Fixed Point Theory Appl.*, **2016** (2016), 83. https://doi.org/10.1186/s13663-016-0578-4

26. R. Suparatulatorn, W. Cholamjiak, S. Suantai, A modified S-iteration process for G-nonexpansive mappings in Banach spaces with graphs, *Numer. Algor.*, **77** (2018), 479–490. https://doi.org/10.1007/s11075-017-0324-y
27. T. Thianwan, D. Yambangwai, Convergence analysis for a new two-step iteration process for G-nonexpansive mappings with directed graphs, *J. Fixed Point Theory Appl.*, **21** (2019), 44. https://doi.org/10.1007/s11784-019-0681-3
28. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591–597. https://doi.org/10.1090/S0002-9904-1967-11761-0
29. S. Shahzad, R. Al-Dubiban, Approximating common fixed points of nonexpansive mappings in Banach spaces, *Georgian Math. J.*, **13** (2006), 529–537. https://doi.org/10.1515/GMJ.2006.529
30. J. Schu, Weak and strong convergence to fixed points of asymtotically nonexpansive mappings, *Bull. Aust. Math. Soc.*, **43** (1991), 153–159. https://doi.org/10.1017/S0004972700028884
31. S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, **311** (2015), 506–517. https://doi.org/10.1016/j.jmaa.2005.03.002
32. M. G. Sangago, Convergence of iterative schemes for nonexpansive mappings, *Asian-Eur. J. Math.*, **4** (2011) 671–682. https://doi.org/10.1142/S1793557111000551

AIMS Press