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Research article

Existence, and Ulam's types stability of higher-order fractional Langevin equations on a star graph

Gang Chen, Jinbo Ni* **and Xinyu Fu**

School of mathematics and big data, Anhui University of Science and Technology, Huainan 232001, China

* **Correspondence:** Email: nijinbo2005@126.com.

Abstract: A study was conducted on the existence of solutions for a class of nonlinear Caputo type higher-order fractional Langevin equations with mixed boundary conditions on a star graph with $k + 1$ nodes and k edges. By applying a variable transformation, a system of fractional differential equations with mixed boundary conditions and different domains was converted into an equivalent system with identical boundary conditions and domains. Subsequently, the existence and uniqueness of solutions were verified using Krasnoselskii's fixed point theorem and Banach's contraction principle. In addition, the stability results of different types of solutions for the system were further discussed. Finally, two examples are illustrated to reinforce the main study outcomes.

Keywords: fractional Langevin equation; mixed boundary condition; existence and uniqueness; Ulam stability
Mathematics Subject Classification: 34A08, 34A34, 34B15

1. Introduction

The field of fractional calculus, which focuses on the application of fractional derivatives, has been rapidly developing in recent years. Compared to traditional integer calculus, fractional calculus has a broader range of applicability in both temporal and spatial scales, enabling more accurate descriptions of real-world problems. As an important branch of fractional calculus, fractional differential equations (FDEs) contain fractional derivatives and have been widely applied in various fields, including chemistry, physics, electrical engineering, economics, and biology [1–3]. These equations hold significant practical value and have had a profound impact on the theoretical development of calculus, serving as one of the key foundations in its research. Currently, boundary value problems (BVPs) of FDEs are a hot topic of study. Scholars have mainly used fixed point theory [4–8], such as the Banach fixed point theorem, Schaefer fixed point theorem, and Krasnoselskii fixed point theorem, to establish sufficient conditions for the existence and uniqueness (EU) of FDEs solutions. These achievements have been highly important for deepening our understanding of FDEs and promoting their practical applications.

Fractional differential equations are of great significance in describing actual network structures, such as pipelines, gas pipelines, molecular structures, and computer network extensions. In 1980, Lumer [9] proposed the theory of differential equations on graphs based on the branching space framework, and this field has since been widely developed for application in multiple disciplines such as chemical engineering, biology, and physics. Therefore, researchers have a strong interest in the existence and stability of solutions to BVPs of differential equations and their fractional mathematical models on graphs.

Graet et al. [10] published their first work in the field in 2014, where they utilized known fixed point theorems to prove the EU of solutions to fractional BVPs on star graphs. That is, the authors studied a star graph $\tilde{\mathcal{G}} = \tilde{\mathcal{V}}(\tilde{\mathcal{G}}) \cup \tilde{\mathcal{E}}(\tilde{\mathcal{G}})$, where $\tilde{\mathcal{V}} = \{\epsilon_0, \epsilon_1, \epsilon_2\}$ and $\tilde{\mathcal{E}} = \{\overrightarrow{\epsilon_1\epsilon_0}, \ \overrightarrow{\epsilon_2\epsilon_0}\}$ are sets of three vertices and of two edges, respectively; ϵ_0 is the junction node and $\overrightarrow{\epsilon_i\epsilon_0}$ are the edges connecting nodes ϵ_i to ϵ_0 with length $\rho_i = \left|\overrightarrow{\epsilon_i\epsilon_0}\right|$ for $i = 1, 2$. The nonlinear fractional BVPs system on each edge $\overrightarrow{\epsilon_i\epsilon_0}$ in their work [10] is defined as follows:

$$\begin{cases} -\mathfrak{D}_0^\alpha \varphi_i = \mathfrak{m}_i(t)\mathcal{F}_i(t, \varphi_i), \ t \in (0, \rho_i), \ i = 1, 2, \\ \varphi_1(0) = \varphi_2(0) = 0, \ \varphi_1(\rho_1) = \varphi_2(\rho_2), \ \mathfrak{D}_0^\beta \varphi_1(\rho_1) + \mathfrak{D}_0^\beta \varphi_2(\rho_2) = 0, \end{cases} \tag{1.1}$$

where $\mathfrak{D}_0^\alpha$ and $\mathfrak{D}_0^\beta$ represent the Riemann-Liouville fractional derivative of orders $\alpha \in (1, 2]$ and $\beta \in (0, \alpha)$, respectively; $\mathfrak{m}_i : [0, \rho_i] \to \mathbb{R}$ are continuous functions with $\mathfrak{m}_i(t) \neq 0$ on $[0, \rho_i]$, and $\mathcal{F}_i : [0, \rho_i] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. The EU of solutions for the BVPs (1.1) are derived by applying Scheafer's fixed point theorem and the Banach contraction principle. In 2019, Mehandirata et al. [11] expanded upon Graef's earlier work by generalizing it to apply to star graphs comprising $k + 1$ nodes and k edges as follows:

$$\begin{cases} {}^C\mathfrak{D}_0^\alpha \varphi_i(t) = \mathcal{F}_i(t, \varphi_i(t), {}^C\mathfrak{D}_0^\beta \mathfrak{u}_i(t)), \ t \in (0, \rho_i), \ i = 1, 2, \cdots, k, \\ \varphi_i(0) = 0, \ \sum_{i=1}^k \varphi_i'(\rho_i) = 0, \ i = 1, 2, \cdots, k, \\ \varphi_i(\rho_i) = \varphi_j(\rho_j), \ i, j = 1, 2, \cdots, k, \ i \neq j, \end{cases} \tag{1.2}$$

where ${}^C\mathfrak{D}_0^\alpha$ and ${}^C\mathfrak{D}_0^\beta$ represent the Caputo fractional derivative of orders $\alpha \in (1, 2]$ and $\beta \in (0, \alpha - 1)$, $\mathcal{F}_i : [0, \rho_i] \times \mathbb{R} \to \mathbb{R}$ are continuous functions; $\tilde{\mathcal{G}} = \tilde{\mathcal{V}}(\tilde{\mathcal{G}}) \cup \tilde{\mathcal{E}}(\tilde{\mathcal{G}})$ with $\tilde{\mathcal{V}}(\tilde{\mathcal{G}}) = \{\epsilon_0, \epsilon_1, \cdots, \epsilon_k\}$ and $\tilde{\mathcal{E}}(\tilde{\mathcal{G}}) = \{e_i = \overrightarrow{\epsilon_i\epsilon_0}, \ i = 1, 2, \cdots, k\}$, $\rho_i = \left|\overrightarrow{\epsilon_i\epsilon_0}\right|, i = 1, 2, \cdots, k$ (a collection of k edges incident to a single node point (see Figure 1)). The EU of the solution to the BVPs (1.2) was proven using the same fixed-point theorem as in prior work [10].

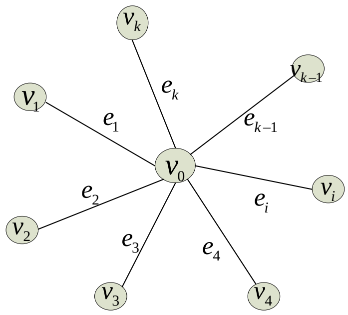

Figure 1. A sketch of the star graph with k edges.

The BVPs of FDEs on graphs have attracted widespread attention from scholars, and some interesting research results have been achieved [12–19]. For instance, in 2020, Etemad [12] proved the existence of solutions to fractional BVPs on ethane graphs using Schaefer's fixed point theorem and Krasnoselskii's fixed point theorem. In the same year, Mophou and Leugering [13] proved the EU of the solution through their study of the optimal control of fractional Sturm-Liouville BVPs on star graphs. In 2021, Turab [14] verified the existence of solutions to fractional BVPs on hexagonal graphs using Krasnoselskii's fixed point theorem and Schaefer's fixed point theorem. In Han's study [15], the Banach contraction mapping principle and Schaefer's fixed point theorem were applied to the EU of solutions for BVPs of nonlinear fractional differential equations on star graphs. In 2021, Ali [16] considered the existence of solutions to fractional BVPs on cyclohexane graphs. Additionally, Zhang et al. [17] studied the BVPs of fractional Langevin equations on star graphs as follows:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}_{0,t}(\mathcal{D}+\lambda_i)\varphi_i(t) = \mathcal{F}_i(t,\varphi_i(t),\ {}^{C}\mathfrak{D}^{\beta}_{0,t}\varphi_i(t)),\ t\in(0,\rho_i),\ i=1,2,\cdots,k, \\ \varphi_i(0)=0,\ \sum_{i=1}^{k}\varphi_i'(\rho_i)=0,\ i=1,2,\cdots,k, \\ \varphi_i(\rho_i)=\varphi_j(\rho_j),\ i,j=1,2,\cdots,k,\ i\neq j, \end{cases} \tag{1.3}$$

where $\alpha\in(0,1)$ and $\beta\in(0,\alpha)$, $\lambda_i\in\mathbb{R}^+$, $\mathcal{D}$ is the ordinary derivative, $\mathcal{F}_i:[0,\rho_i]\times\mathbb{R}\to\mathbb{R}$ are continuous functions, and the two fractional operators ${}^{C}\mathfrak{D}^{\alpha}_{0}$ and ${}^{C}\mathfrak{D}^{\beta}_{0}$ denote the Caputo fractional derivatives. The [17] studied a star graph $\tilde{\mathcal{G}}=\tilde{\mathcal{V}}(\tilde{\mathcal{G}})\cup\tilde{\mathcal{E}}(\tilde{\mathcal{G}})$, where $\tilde{\mathcal{V}}=\{\epsilon_0,\epsilon_1,\cdots,\epsilon_k\}$ and $\tilde{\mathcal{E}}=\{e_i=\overrightarrow{\epsilon_i\epsilon_0},\ i=1,2,\cdots,k\}$ are sets of $k+1$ nodes and set of k edges, with $\rho_i=\left|\overrightarrow{\epsilon_i\epsilon_0}\right|$. The author proved the EU of the BVP (1.3) solution by utilizing Schaefer's fixed point theorem and Banach contraction principle.

On the other hand, Ulam stability analysis was proposed by Ulam in the 1940s and further developed by Hyers. This analysis primarily studies whether the behavior of a system remains stable when there are slight changes in its parameters. Regarding Banach spaces, S. Banach introduced the famous fixed-point theorem in 1932. Later, Gleason, Ricciardi, and Ulam provided the theoretical foundation for Ulam stability by extending this theorem to additive mappings in metric spaces. In the field of FDEs, scholars began to study the stability of Ulam in the late 20th century and have achieved some notable results [20–26]. In 2017, Khan et al. [20] studied the Ulam stability of solutions to fractional differential equation systems using topological degree methods. In 2019, the same author [21] utilized the Guo-Krasnoselskii theorem to study the uniqueness of solutions and Ulam stability for differential equations containing Atangana-Baleanu-Caputo fractional derivatives. The same year, Devi et al. [22] proved the Ulam stability of specific FDEs and provided examples illustrating their application. More recently, Zhang et al. [23] studied the existence and Ulam-type stability of solutions to BVPs containing Caputo fractional derivatives on star graphs. These research studies show that the analysis of stability for solutions of FDEs is an active and important field of study with significant implications for understanding the long-term behavior of complex dynamic systems.

Inspired by the above-mentioned research [15, 17, 23], it can be seen that how to solve the mixed boundary conditions for a class of nonlinear higher-order fractional Langevin equations on a star graph consisting of $k+1$ nodes and k edges has not yet been examined. Specifically, this work examines the following problems:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}_{0,t}(\mathcal{D}^2+\lambda_i)\varphi_i(t) = \mathcal{F}_i(t,\varphi_i(t),{}^{C}\mathfrak{D}^{\beta}_{0,t}\varphi_i(t)),\ t\in(0,\rho_i),\ i=1,2,\cdots,k, \\ \varphi_i'(0)=\varphi_i(\rho_i)=0,\ \sum_{i=1}^{k}\varphi_i''(\rho_i)=0,\ i=1,2,\cdots k, \\ \varphi_i''(\rho_i)=\varphi_j''(\rho_j),\ i,j=1,2,\cdots k,\ i\neq j, \end{cases} \tag{1.4}$$

where $\alpha \in (0,1)$ and $\beta \in (0,\alpha)$, $\lambda_i \in \mathbb{R}^+$, $\mathcal{D}^2$ is the ordinary second-order derivative, $\mathcal{F}_i : [0,\rho_i]\times\mathbb{R} \to \mathbb{R}$ are continuous functions, and the two fractional operators ${}^{C}\mathfrak{D}_0^{\alpha}$ and ${}^{C}\mathfrak{D}_0^{\beta}$ denote the Caputo fractional derivatives. The star graph is $\tilde{\mathcal{G}} = \tilde{\mathcal{V}}(\tilde{\mathcal{G}}) \cup \tilde{\mathcal{E}}(\tilde{\mathcal{G}})$, where $\tilde{\mathcal{V}} = \{\epsilon_0, \epsilon_1, \cdots, \epsilon_k\}$ and $\tilde{\mathcal{E}} = \{e_i = \overrightarrow{\epsilon_i\epsilon_0}, i = 1, 2, \cdots, k\}$ are sets of $k+1$ nodes and set of k edges, and $\rho_i = \left|\overrightarrow{\epsilon_i\epsilon_0}\right|$. We consider a local coordinate system with ϵ_i as the origin and coordinates t in the interval $(0, \rho_i)$.

This study investigates the existence, uniqueness, and Ulam stability of solutions to nonlinear fractional BVP (1.4). The EU of the solution to problem (1.4) can be demonstrated by utilizing Krasnoselskii's fixed point theorem and Banach's contraction mapping principle. Meanwhile, the Ulam stability of this system was verified using the matrix eigenvalue method. Compared with existing research results, the innovative results presented in this article can be summarized as follows: First, we extend the fractional Langevin equation on a star graph to higher-order fractional cases. Second, compared with other research [10, 11, 15, 23], the Langevin equation in this paper introduces nonlocal terms, which increases the difficulty of prior estimation. Third, compared with another study [17], we not only investigated the existence and uniqueness of solutions to higher-order fractional Langevin equations, but also extended the relevant results on their Ulam stability. Fourth, compared with prior research [10, 11, 15, 17, 23], the fractional BVP (1.4) considered in this paper are more generalized and complex, as its nonlinear terms and boundary conditions depend not only on unknown functions, but also on fractional derivatives. Finally, Theorem 3.1 (see Section 3) proves that the problem (1.4) has at least one solution. This theorem proposes a linear growth condition for nonlinear terms, reduces the existence conditions, and thus makes the required existence condition more relaxed than condition (H_3) used in previous literature [11, 15].

The remaining parts of this manuscript are structured as follows. Section 2 proposes the auxiliary Lemma 2.5, which transforms the BVP (1.4) into an equivalent system (2.1), while reviewing the main relationships in fractional calculus. Section 3 proves the uniqueness and existence of solutions to the fractional differential BVP (1.4). Section 4 establishes sufficient conditions for the Ulam stability of the solution to system (2.1). Section 5 illustrates the main results of this paper regarding the existence, uniqueness, and Ulam stability of solutions through two examples. Finally, some conclusions are given in the last section.

2. Preliminaries

In this section, we will revisit the concept of fractional calculus and outline some basic results that will provide a basis for the subsequent discussions in this paper.

Definition 2.1. *[1]. Let $f \in C([a,b],\mathbb{R})$. Then the Riemann-Liouville fractional integral is given by*

$$I_a^{\alpha} f(\tau) = \frac{1}{\Gamma(\alpha)} \int_a^{\tau} (\tau - s)^{\alpha-1} f(s)ds, \ \alpha > 0, \ \tau > a,$$

where $\Gamma(\cdot)$ is the classical Euler gamma function.

Definition 2.2. *[1]. Let $f \in C^n([a,b],\mathbb{R})$. Then the Caputo fractional derivative operator of order $\alpha > 0$ is defined by*

$${}^{C}\mathfrak{D}_{a,\tau}^{\alpha} f(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_a^{\tau} (\tau - s)^{n-\alpha-1} f^{(n)}(s)ds, \ \tau > a,$$

where $n-1<\alpha\le n$ and $n\in N$.

Lemma 2.1. *[1]. Let $\alpha>0$. Suppose that $u\in AC^n[0,1]$. Then*

$$I_0^{\alpha C}\mathfrak{D}_{0,\tau}^{\alpha}u(\tau)=u(\tau)+c_1+c_2\tau+c_3\tau^2+\cdots+c_n\tau^{n-1},$$

where $c_i\in\mathbb{R}$, $i=1,2,\cdots n$, $n=[\alpha]+1$.

Lemma 2.2. *[27]. Suppose that $(\mathfrak{D}^n t)(\tau)$ and $({}^C\mathfrak{D}_{a,\tau}^{\alpha+n}t)(\tau)$ exist. Then*

$$({}^C\mathfrak{D}_{a,\tau}^{\alpha}\mathfrak{D}^n t)(\tau)=({}^C\mathfrak{D}_{a,\tau}^{\alpha+n}t)(\tau),\ \alpha>0,$$

where $n\in\mathbb{N}$ and $\mathfrak{D}=d/d\tau$.

Lemma 2.3. *[1]. If $\alpha>0$, $\beta>\alpha-1$, $\tau>0$, then*

$${}^C\mathfrak{D}_{0,\tau}^{\alpha}\tau^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\tau^{\beta-\alpha}.$$

Theorem 2.1 (Urs)**.** *[28] Suppose $\mathcal{A}$ is a square matrix of order n with positive real entries, i.e., $\mathcal{A}\in M_{nn}(\mathbb{R}^+)$. Then, the following statements are equivalent:*

(i) *The eigenvalues of matrix $\mathcal{A}$ (in the open unit disc), denoted by λ, $\forall\lambda\in\mathbb{C}$ with det($\lambda I-\mathcal{A}$)=0, i.e., $|\lambda|<1$;*
(ii) *The matrix $(I-\mathcal{A})$ is nonsingular;*
(iii) *The matrix $(I-\mathcal{A})^{-1}$ has nonnegative elements and $(I-\mathcal{A})^{-1}=I+\mathcal{A}+\cdots+\mathcal{A}^n+\cdots$.*

Theorem 2.2 (Krasnoselskii's fixed point theorem)**.** *[29]. Let P be a closed, bounded, convex, and nonempty subset of a Banach space Z. Let A and B be two operators such that*

(a) *$Az+B\bar{z}\in P$ for all $z,\bar{z}\in P$,*
(b) *A is compact and continuous on P,*
(c) *B is a contraction mapping on P.*

Then there exists $\hbar\in P$ such that $\hbar=A\hbar+B\hbar$.

Lemma 2.4. *[11]. Let $\alpha>0$, φ be a function defined on $[0,\rho]$, and $z(\tau)=\varphi(\rho\tau)$. Suppose that ${}^C\mathfrak{D}_{0,t}^{\alpha}\varphi$ exists on $[0,\rho]$, $t\in[0,\rho]$, then*

$${}^C\mathfrak{D}_{0,t}^{\alpha}\varphi(t)=\rho^{-\alpha}({}^C\mathfrak{D}_{0,t}^{\alpha}z(\tau)),\ \tau=t/\rho\in[0,1].$$

Lemma 2.5. *Let $\alpha\in(n-1,n)$, φ be a function defined on $[0,\rho]$, and $z(\tau)=\varphi(\rho\tau)$. Suppose that ${}^C\mathfrak{D}_{0,t}^{\alpha}\varphi$ exists on $[0,\rho]$, $t\in[0,\rho]$, then*

$${}^C\mathfrak{D}_{0,t}^{\alpha}(\mathcal{D}^2+\lambda)\varphi(t)=\rho^{-\alpha-2C}\mathfrak{D}_{0,\tau}^{\alpha}(\mathcal{D}^2+\lambda\rho^2)z(\tau),\ \tau=t/\rho\in[0,1].$$

Proof. By means of Definition 2.2 and Lemma 2.2, we obtain

$$\begin{aligned}&{}^C\mathfrak{D}_{0,t}^{\alpha}(\mathcal{D}^2+\lambda)\varphi(t)={}^C\mathfrak{D}_{0,t}^{\alpha+2}\varphi(t)+\lambda^C\mathfrak{D}_{0,t}^{\alpha}\varphi(t)\\&=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}\varphi^{(n+2)}(s)ds+\frac{\lambda}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}\varphi^{(n)}(s)ds\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{\Gamma(n-\alpha)}\int_0^{\rho\tau}(\rho\tau-s)^{n-\alpha+1}\varphi^{(n+2)}(s)ds+\frac{\lambda}{\Gamma(n-\alpha)}\int_0^{\rho\tau}(\rho\tau-s)^{n-\alpha-1}\varphi^{(n)}(s)ds \quad (t=\rho\tau)\\
&=\frac{\rho^{n-\alpha}}{\Gamma(n+2-\alpha)}\int_0^{\tau}(\tau-\hat{s})^{n-\alpha+1}\varphi^{(n+2)}(\rho\hat{s})d\hat{s}+\frac{\lambda\rho^{n-\alpha}}{\Gamma(n-\alpha)}\int_0^{\tau}(\tau-\hat{s})^{n-\alpha-1}\varphi^{(n)}(\rho\hat{s})d\hat{s} \quad (\hat{s}=s/\rho)\\
&=\frac{\rho^{-\alpha-2}}{\Gamma(n+2-\alpha)}\int_0^{\tau}(\tau-\hat{s})^{n-\alpha+1}z^{(n+2)}(\hat{s})d\hat{s}\\
&\quad+\frac{\lambda\rho^{-\alpha}}{\Gamma(n-\alpha)}\int_0^{\tau}(\tau-\hat{s})^{n-\alpha-1}z^{(n)}(\hat{s})d\hat{s} \quad (z^{(n)}(\tau)=\rho^n\varphi^{(n)}(\rho\tau))\\
&=\rho^{-\alpha-2}\,{}^C\mathfrak{D}_{0,\tau}^{\alpha+2}z(\tau)+\lambda\rho^{-\alpha}\,{}^C\mathfrak{D}_{0,\tau}^{\alpha}z(\tau)\\
&=\rho^{-\alpha-2}\,{}^C\mathfrak{D}_{0,\tau}^{\alpha}(\mathcal{D}^2+\lambda\rho^2)z(\tau).
\end{aligned}
$$

This concludes the proof of Lemma 2.5. □

Under the direct application of Lemmas 2.4 and 2.5, BVP (1.4) is equivalently transformed into system (2.1) defined on [0,1], as given by

$$
\begin{cases}
{}^C\mathfrak{D}_{0,\tau}^{\alpha}(\mathcal{D}^2+\lambda_i\rho_i^2)z_i(\tau)=\rho_i^{\alpha+2}f_i(\tau,z_i(\tau),\rho_i^{-\beta}\,{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i(\tau)),\ \tau\in(0,1),\\
z_i'(0)=z_i(1)=0,\ \sum_{i=1}^k\rho_i^{-2}z_i''(1)=0,\ i=1,2,\cdots k,\\
z_i''(1)=z_j''(1),\ i,j=1,2,\cdots k,\ i\neq j,
\end{cases}
\tag{2.1}
$$

where $z_i(\tau)=\varphi_i(\rho_i\tau)$, $f_i(\tau,u,v)=\mathcal{F}_i(\rho_i\tau,u,v)$.

Lemma 2.6. *Let $h_i(\tau)\in C[0,1]$, $\alpha\in(0,1)$. Then $z_i(\tau)$ is a solution of the BVP*

$$
\begin{cases}
{}^C\mathfrak{D}_{0,\tau}^{\alpha}(\mathcal{D}^2+\lambda_i\rho_i^2)z_i(\tau)=h_i(\tau),\ \tau\in(0,1),\ i=1,2,\cdots k,\\
z_i'(0)=z_i(1)=0,\ \sum_{i=1}^k\rho_i^{-2}z_i''(1)=0,\ i=1,2,\cdots k,\\
z_i''(1)=z_j''(1),\ i,j=1,2,\cdots k,\ i\neq j,
\end{cases}
\tag{2.2}
$$

which is given by

$$
\begin{aligned}
z_i(\tau)=&\frac{1}{\Gamma(\alpha+2)}\int_0^{\tau}(\tau-s)^{\alpha+1}h_i(s)ds-\frac{1}{\Gamma(\alpha+2)}\int_0^{1}(1-s)^{\alpha+1}h_i(s)ds\\
&+\lambda_i\rho_i^2\int_0^1(1-s)z_i(s)ds-\lambda_i\rho_i^2\int_0^{\tau}(\tau-s)z_i(s)ds\\
&+(1-\tau^2)\sum_{j=1}^k\ell_j\Big(\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}h_j(s)ds-\lambda_j\rho_j^2z_j(1)\Big)\\
&+(1-\tau^2)\sum_{j=1,j\neq i}^k\ell_j(\lambda_j\rho_j^2z_j(1)-\lambda_i\rho_i^2z_i(1))\\
&-(1-\tau^2)\sum_{j=1,j\neq i}^k\ell_j\Big(\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}(h_j(s)-h_i(s))ds\Big),
\end{aligned}
\tag{2.3}
$$

where $\ell_j=\frac{\rho_j^{-2}}{2\sum_{j=1}^k\rho_j^{-2}}$, $j=1,2,\cdots k$.

Proof. Applying the integral operator $I_0^{\alpha+2}$ to (2.2) and using Lemma 2.1, we have

$$z_i(\tau) = -\lambda_i\rho_i^2 \int_0^\tau (\tau - s)z_i(s)ds + I_0^{\alpha+2}h_i(\tau) - a_i^{(1)} - a_i^{(2)}\tau - a_i^{(3)}\tau^2,\ i = 1, 2, \cdots, k, \tag{2.4}$$

where a_i^j $(i = 1, 2, \cdots, k,\ j = 1, 2, 3)$ are some constants. Deriving both sides of Eq (2.4) from 0 to τ gives

$$z_i'(\tau) = -\lambda_i\rho_i^2 \int_0^\tau z_i(s)ds + I_0^{\alpha+1}h_i(\tau) - a_i^{(2)} - 2\tau a_i^{(3)}$$

and

$$z_i''(\tau) = -\lambda_i\rho_i^2 z_i(\tau) + I_0^{\alpha}h_i(\tau) - 2a_i^{(3)}.$$

$z_i'(0) = 0$ implies that $a_i^{(2)} = 0$, which leads to

$$z_i'(\tau) = -\lambda_i\rho_i^2 \int_0^\tau z_i(s)ds + I_0^{\alpha+1}h_i(\tau) - 2\tau a_i^{(3)}.$$

Since $z_i''(1) = z_j''(1)$ and $\sum_{i=1}^k \rho_i^{-2} z_i''(1) = 0$, we obtain

$$-\lambda_i\rho_i^2 z_i(1) + I_0^{\alpha}h_i(\tau)|_{\tau=1} - 2a_i^{(3)} = -\lambda_j\rho_j^2 z_j(1) + I_0^{\alpha}h_j(\tau)|_{\tau=1} - 2a_j^{(3)},\ i \neq j \tag{2.5}$$

and

$$\sum\nolimits_{i=1}^k \rho_i^{-2}(-\lambda_i\rho_i^2 z_i(1) + I_0^{\alpha}h_i(\tau)|_{\tau=1} - 2a_i^{(3)}) = 0. \tag{2.6}$$

According to (2.5) and (2.6), there is

$$\begin{aligned} 2\sum_{j=1}^k \rho_j^{-2} a_i^{(3)} = &\sum_{j=1}^k \rho_j^{-2}\left(-\lambda_j\rho_j^2 z_j(1) + I_0^{\alpha}h_j(\tau)|_{\tau=1}\right) \\ &- \sum_{j=1, j\neq i}^k \rho_j^{-2}\left(-\lambda_j\rho_j^2 z_j(1) + \lambda_i\rho_i^2 z_i(1) + I_0^{\alpha}h_j(\tau)|_{\tau=1} - I_0^{\alpha}h_i(\tau)|_{\tau=1}\right), \end{aligned}$$

which implies

$$\begin{aligned} a_i^{(3)} = &- \sum_{j=1, j\neq i}^k \ell_j\left(-\lambda_j\rho_j^2 z_j(1) + \lambda_i\rho_i^2 z_i(1) + I_0^{\alpha}h_j(\tau)|_{\tau=1} - I_0^{\alpha}h_i(\tau)|_{\tau=1}\right) \\ &+ \sum_{j=1}^k \ell_j\left(-\lambda_j\rho_j^2 z_j(1) + I_0^{\alpha}h_j(\tau)|_{\tau=1}\right). \end{aligned} \tag{2.7}$$

By substituting $z_i(1) = 0$ and $a_i^{(3)}$ into in (2.4), we get

$$\begin{aligned} a_i^{(1)} = &I_0^{\alpha+2}h_i(\tau)|_{\tau=1} - \lambda_i\rho_i^2 \int_0^1 (1 - s)z_i(s)ds - \sum_{j=1}^k \ell_j\left(-\lambda_j\rho_j^2 z_j(1) + I_0^{\alpha}h_j(\tau)|_{\tau=1}\right) \\ &+ \sum_{j=1, j\neq i}^k \ell_j\left(-\lambda_j\rho_j^2 z_j(1) + \lambda_i\rho_i^2 z_i(1) + I_0^{\alpha}h_j(\tau)\Big|_{\tau=1} - I_0^{\alpha}h_i(\tau)\Big|_{\tau=1}\right). \end{aligned} \tag{2.8}$$

Inserting the values from (2.7) and (2.8) into (2.4), we get the solution (2.3). This completes the proof. □

3. Existence and uniqueness results

In this section, we prove the existence and uniqueness of the solution to system (2.1). We first define a Banach space $X = \{z : z, {}^{C}\mathfrak{D}_{0,\tau}^{\beta} z \in C[0,1]\}$, with the supremum norm

$$\|z\|_X = \|z\| + \|{}^{C}\mathfrak{D}_{0,\tau}^{\beta} z\|,$$

where $\|z\| = \max\limits_{\tau\in[0,1]} \left|z(\tau)\right|$, $\|{}^{C}\mathfrak{D}_{0,\tau}^{\beta} z\| = \max\limits_{\tau\in[0,1]} |{}^{C}\mathfrak{D}_{0,\tau}^{\beta} z(\tau)|$. It is obvious that the product space $(X^k = X_1 \times X_2 \times \cdots \times X_k, \|\cdot\|_{X^k})$ is a Banach space, where the norm is defined by

$$\|(z_1, z_2, \cdots, z_k)\|_{X^k} = \sum\nolimits_{i=1}^{k} \|z_i\|_X,\ (z_1, z_2, \cdots, z_k) \in X^k.$$

By considering Lemma 2.6, we introduce the operator $T : X^k \to X^k$, related to system (2.1) by

$$T(z_1, z_2, \cdots, z_k)(\tau) = (T_1(z_1, z_2, \cdots, z_k)(\tau), T_2(z_1, z_2, \cdots, z_k)(\tau) \cdots, T_k(z_1, z_2, \cdots, z_k)(\tau)),$$

for $\tau \in [0,1]$ and $z_i \in X,\ i = 1, 2, \cdots k$, where

$$\begin{aligned}
&T_i(z_1, z_2, \cdots, z_k)(\tau)\\
=&\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+2)} \int_0^\tau (\tau - s)^{\alpha+1} f_i(s, z_i(s), \rho_i^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_i(s))ds\\
&- \frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+2)} \int_0^1 (1 - s)^{\alpha+1} f_i(s, z_i(s), \rho_i^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_i(s))ds\\
&+ \lambda_i\rho_i^2 \int_0^1 (1 - s) z_i(s)ds - \lambda_i\rho_i^2 \int_0^\tau (\tau - s) z_i(s)ds\\
&+ (1 - \tau^2) \sum_{j=1, j\neq i}^{k} \ell_j(\lambda_j\rho_j^2 z_j(1) - \lambda_i\rho_i^2 z_i(1))\\
&+ (1 - \tau^2) \sum_{j=1}^{k} \ell_j\Big(\frac{\rho_j^{\alpha+2}}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f_j(s, z_j(s), \rho_j^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_j(s))ds - \lambda_j\rho_j^2 z_j(1)\Big)\\
&+ (1 - \tau^2) \sum_{j=1, j\neq i}^{k} \frac{\ell_j\rho_i^{\alpha+2}}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f_i(s, z_i(s), \rho_i^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_i(s))ds\\
&- (1 - \tau^2) \sum_{j=1, j\neq i}^{k} \frac{\ell_j\rho_j^{\alpha+2}}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f_j(s, z_j(s), \rho_j^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_j(s))ds.
\end{aligned} \tag{3.1}$$

Assume that the following conditions hold:

(H_1) The functions $f_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, $i = 1, 2, \cdots, k$.
(H_2) There exist functions $\xi_i(\tau), \eta_i(\tau), \psi_i(\tau) \in C([0,1], [0,+\infty))$, such that

$$|f_i(\tau, u, v)| \leq \xi_i(\tau) + \eta_i(\tau)\,|u(\tau)| + \psi_i(\tau)\,|v(\tau)|\,,\ i = 1, 2, \cdots, k,$$

for all $\tau \in [0,1]$ and $u, v \in \mathbb{R}^2$.

(H_3) There exist functions $w_i(\tau) \in C([0,1],[0,+\infty))$, such that

$$|f_i(\tau,u,v) - f_i(\tau,u_1,v_1)| \leq w_i(\tau)(|u-u_1| + |v-v_1|),\ i = 1,2,\cdots,k,$$

for all $\tau \in [0,1]$ and $(u,v),(u_1,v_1) \in \mathbb{R}^2$.

For the convenience of calculation, the following symbols are given:

$$Q_1 = \frac{2}{\Gamma(\alpha+3)} + \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+3)} + \frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)},$$
$$Q_2 = \frac{2}{\Gamma(\alpha+1)} + \frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}.$$

Theorem 3.1. *Assume that (H_1) and (H_2) hold, then system (2.1) allows at least one solution on [0,1], provided that*

$$\sum_{i=1}^{k} \theta_i < 1,$$

where

$$\theta_i = \left[\Delta_i(\eta_i^* + \rho_i^{-\beta}\psi_i^*) + \Omega_i + \sum_{j=1,j\neq i}^{k} (\tilde{\Delta}_j(\eta_j^* + \rho_j^{-\beta}\psi_j^*) + \tilde{\Omega}_j)\right]$$

and

$$\Delta_i = Q_1\rho_i^{\alpha+2},\ \Omega_i = 3\lambda_i\rho_i^2 + \frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)},\ \tilde{\Omega}_j = 2\lambda_j\rho_j^2 + \frac{4\lambda_j\rho_j^2}{\Gamma(3-\beta)},$$
$$\tilde{\Delta}_j = Q_2\rho_j^{\alpha+2},\ \xi_i^* = \max_{\tau\in[0,1]} |\xi_i(\tau)|,\ \eta_i^* = \max_{\tau\in[0,1]} |\eta_i(\tau)|,\ \psi_i^* = \max_{\tau\in[0,1]} |\psi_i(t)|.$$

Proof. Let $\Theta = \left\{z = (z_1, z_2, \cdots z_k) \in X^k : \|z_i\|_X \leq r\right\}$, where r is chosen such that

$$r \geq \frac{\sum_{i=1}^{k} (\Delta_i\xi_i^* + \sum_{j=1,j\neq i}^{k} \tilde{\Delta}_j\xi_j^*)}{(1 - \sum_{i=1}^{k} \theta_i)}, \tag{3.2}$$

then Θ is a bounded and closed convex subset of the Banach space X^k. We define A_i and B_i on Θ as

$$A_i(z_1,z_2,\cdots,z_k)(\tau) = (A_1(z_1,z_2,\cdots,z_k)(\tau), A_2(z_1,z_2,\cdots,z_k)(\tau)\cdots, A_k(z_1,z_2,\cdots,z_k)(\tau)),$$
$$B_i(z_1,z_2,\cdots,z_k)(\tau) = (B_1(z_1,z_2,\cdots,z_k)(\tau), B_2(z_1,z_2,\cdots,z_k)(\tau)\cdots, B_k(z_1,z_2,\cdots,z_k)(\tau)),$$

where

$$\begin{aligned}
&A_i(z_1,z_2,\cdots,z_k)(\tau)\\
=&\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+2)}\int_0^{\tau} (\tau-s)^{\alpha+1} f_i(s, z_i(s), \rho_i^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_i(s))ds\\
&-\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+2)}\int_0^{1} (1-s)^{\alpha+1} f_i(s, z_i(s), \rho_i^{-\beta C}D_{0,s}^{\beta} z_i(s))ds
\end{aligned}$$

$$+ (1-\tau^2)\sum_{j=1}^{k}\frac{\ell_j\rho_j^{\alpha+2}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f_j(s, z_j(s), \rho_j^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_j(s))ds$$
$$+ (1-\tau^2)\sum_{j=1, j\neq i}^{k}\frac{\ell_j\rho_i^{\alpha+2}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f_i(s, z_i(s), \rho_i^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_i(s))ds$$
$$- (1-\tau^2)\sum_{j=1, j\neq i}^{k}\frac{\ell_j\rho_j^{\alpha+2}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f_j(s, z_j(s), \rho_j^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_j(s))ds$$

and

$$\begin{aligned} B_i(z_1, z_2, \cdots, z_k)(\tau) =& \lambda_i\rho_i^2\int_0^1 (1-s)z_i(s)ds - \lambda_i\rho_i^2\int_0^\tau (\tau-s)z_i(s)ds \\ &+ (1-\tau^2)\sum_{j=1, j\neq i}^{k}\ell_j(\lambda_j\rho_j^2 z_j(1) \\ &- \lambda_i\rho_i^2 z_i(1)) - (1-\tau^2)\sum_{j=1}^{k}\ell_j\lambda_j\rho_j^2 z_j(1), \end{aligned}$$

for all $\tau \in [0, 1]$ and $z = (z_1, z_2, \cdots, z_k) \in \Theta$.

Now, for every $z = (z_1, z_2, \cdots, z_k)$, $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_k) \in X^k$, we have

$$\begin{aligned} &|(A_i z + B_i\bar{z})(\tau)| \\ \leq& \frac{2\rho_i^{\alpha+2}}{\Gamma(\alpha+2)}\int_0^1 (1-s)^{\alpha+1}\left|f_i(s, z_i(s), \rho_i^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_i(s))\right|ds \\ &+ \sum_{j=1}^{k}\ell_j\Big(\frac{\rho_j^{\alpha+2}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f_j(s, z_j(s), \rho_j^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_j(s))\right|ds + \lambda_j\rho_j^2\left|\bar{z}_j(1)\right|\Big) \\ &+ 2\lambda_i\rho_i^2\int_0^1 (\tau-s)\left|\bar{z}_i(s)\right|ds + \sum_{j=1, j\neq i}^{k}\ell_j\left(\lambda_j\rho_j^2\left|\bar{z}_j(1)\right| + \lambda_i\rho_i^2\left|\bar{z}_j(s)\right|\right) \\ &+ \sum_{j=1, j\neq i}^{k}\frac{\ell_j\rho_j^{\alpha+2}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f_j(s, z_j(s), \rho_j^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_j(s))\right|ds \\ &+ \sum_{j=1, j\neq i}^{k}\frac{\ell_j\rho_i^{\alpha+2}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f_i(s, z_i(s), \rho_i^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_i(s))\right|ds \\ \leq& \frac{2\rho_i^{\alpha+2}}{\Gamma(\alpha+3)}(\xi_i^* + \eta_i^*\|z_i\| + \rho_i^{-\beta}\psi_i^*\left\|{}^C\mathfrak{D}_{0,\tau}^{\beta} z_i\right\|) + \lambda_i\rho_i^2\|\bar{z}_i\| + \sum_{j=1}^{k}\lambda_j\rho_j^2\left\|\bar{z}_j\right\| \\ &+ \sum_{j=1}^{k}\frac{\rho_j^{\alpha+2}}{\Gamma(\alpha+1)}(\xi_j^* + \eta_j^*\left\|z_j\right\| + \rho_j^{-\beta}\psi_j^*\left\|{}^C\mathfrak{D}_{0,\tau}^{\beta} z_j\right\|) + \lambda_i\rho_i^2\|\bar{z}_i\| \\ &+ \frac{1}{\Gamma(\alpha+1)}\sum_{j=1, j\neq i}^{k}\rho_j^{\alpha+2}(\xi_j^* + \eta_j^*\left\|z_j\right\| + \rho_j^{-\beta}\psi_j^*\left\|{}^C\mathfrak{D}_{0,\tau}^{\beta} z_j\right\|) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\left\|\bar{z}_j\right\|+\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+1)}(\xi_i^*+\eta_i^*\left\|z_i\right\|+\rho_i^{-\beta}\psi_i^*\left\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i\right\|)\\
\leq&\frac{2\rho_i^{\alpha+2}}{\Gamma(\alpha+3)}(\xi_i^*+(\eta_i^*+\rho_i^{-\beta}\psi_i^*)\|z_i\|_X)+\lambda_i\rho_i^2\|\bar{z}_i\|_X+\sum_{j=1}^{k}\lambda_j\rho_j^2\left\|\bar{z}_j\right\|_X\\
&+\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\left\|\bar{z}_j\right\|_X+\lambda_i\rho_i^2\|\bar{z}_i\|_X+\sum_{j=1}^{k}\frac{\rho_j^{\alpha+2}}{\Gamma(\alpha+1)}(\xi_j^*+(\eta_j^*+\rho_j^{-\beta}\psi_j^*)\left\|z_j\right\|_X)\\
&+\sum_{j=1,j\neq i}^{k}\frac{\rho_j^{\alpha+2}}{\Gamma(\alpha+1)}(\xi_j^*+(\eta_j^*+\rho_j^{-\beta}\psi_j^*)\left\|z_j\right\|_X)\\
&+\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+1)}(\xi_i^*+(\eta_i^*+\rho_i^{-\beta}\psi_i^*)\|z_i\|_X).
\end{aligned}
$$

From this we can deduce the following:

$$
\begin{aligned}
&\|(A_iz+B_i\bar{z})(\tau)\|\\
\leq&\Big(\frac{2}{\Gamma(\alpha+1)}+\frac{2}{\Gamma(\alpha+3)}\Big)\rho_i^{\alpha+2}(\xi_i^*+(\eta_i^*+\rho_i^{-\beta}\psi_i^*)\|z_i\|_X)+3\lambda_i\rho_i^2\|\bar{z}_i\|_X\\
&+\sum_{j=1,j\neq i}^{k}\frac{2\rho_j^{\alpha+2}}{\Gamma(\alpha+1)}(\xi_j^*+(\eta_j^*+\rho_j^{-\beta}\psi_j^*)\left\|z_j\right\|_X)+2\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\left\|\bar{z}_j\right\|_X\\
\leq&\Big[\Big(\frac{2}{\Gamma(\alpha+1)}+\frac{2}{\Gamma(\alpha+3)}\Big)\rho_i^{\alpha+2}(\eta_i^*+\rho_i^{-\beta}\psi_i^*)+3\lambda_i\rho_i^2\\
&+\sum_{j=1,j\neq i}^{k}\Big(\frac{2\rho_j^{\alpha+2}}{\Gamma(\alpha+1)}(\eta_j^*+\rho_j^{-\beta}\psi_j^*)+2\lambda_j\rho_j^2\Big)\Big]r\\
&+\Big(\frac{2}{\Gamma(\alpha+1)}+\frac{2}{\Gamma(\alpha+3)}\Big)\rho_i^{\alpha+2}\xi_i^*+\sum_{j=1,j\neq i}^{k}\frac{2\rho_j^{\alpha+2}\xi_j^*}{\Gamma(\alpha+1)}.
\end{aligned}\tag{3.3}
$$

By Lemma 2.3 and (H_2), we also can get

$$
\begin{aligned}
&\left|{}^C\mathfrak{D}_{0,\tau}^{\beta}A_iz(\tau)+{}^C\mathfrak{D}_{0,\tau}^{\beta}B_i\bar{z}(\tau)\right|\\
\leq&\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha-\beta+2)}\int_0^{\tau}(\tau-s)^{\alpha-\beta+1}\left|f_i(s,z_i(s),\rho_i^{-\beta C}\mathfrak{D}_{0,s}^{\beta}z_i(s))\right|ds\\
&+\frac{\lambda_i\rho_i^2}{\Gamma(2-\beta)}\int_0^{\tau}(\tau-s)^{1-\beta}\left|\bar{z}_i(s)\right|ds+\frac{2\tau^{2-\beta}}{\Gamma(3-\beta)}\sum_{j=1}^{k}\ell_j\lambda_j\rho_j^2\left|\bar{z}_j(1)\right|\\
&+\frac{2\tau^{2-\beta}}{\Gamma(\alpha)\Gamma(3-\beta)}\sum_{j=1}^{k}\ell_j\rho_j^{\alpha+2}\int_0^1(1-s)^{\alpha-1}\left|f_j(s,z_j(s),\rho_j^{-\beta C}\mathfrak{D}_{0,s}^{\beta}z_j(s))\right|ds\\
&+\frac{2t^{2-\beta}}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\ell_j\lambda_j\rho_j^2\left|\bar{z}_j\right|+\frac{2\tau^{2-\beta}}{\Gamma(3-\beta)}\lambda_i\rho_i^2\left|\bar{z}_i\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\tau^{2-\beta}}{\Gamma(\alpha)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\ell_j\rho_j^{\alpha+2}\int_0^1(1-s)^{\alpha-1}\left|f_j(s,z_j(s),\rho_j^{-\beta C}\mathfrak{D}_{0,s}^{\beta}z_j(s))\right|ds\\
&+\frac{2\tau^{2-\beta}}{\Gamma(\alpha)\Gamma(3-\beta)}\ell_i\rho_i^{\alpha+2}\int_0^1(1-s)^{\alpha-1}\left|f_i(s,z_i(s),\rho_i^{-\beta C}\mathfrak{D}_{0,s}^{\beta}z_i(s))\right|ds\\
\leq&\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha-\beta+3)}(\xi_i^*+\eta_i^*\|z_i\|+\rho_i^{-\beta}\psi_i^*\left\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i\right\|)+\frac{\lambda_i\rho_i^2}{\Gamma(3-\beta)}\|\bar{z}_i\|\\
&+\frac{2}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1}^{k}\rho_j^{\alpha+2}(\xi_j^*+\eta_j^*\left\|z_j\right\|+\rho_j^{-\beta}\psi_j^*\left\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_j\right\|)\\
&+\frac{2}{\Gamma(3-\beta)}\sum_{j=1}^{k}\lambda_j\rho_j^2\left\|\bar{z}_j\right\|+\frac{2}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\left\|\bar{z}_j\right\|+\frac{2\lambda_i\rho_i^2\|\bar{z}_i\|}{\Gamma(3-\beta)}\\
&+\frac{2}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha+2}(\xi_j^*+\eta_j^*\left\|z_j\right\|+\rho_j^{-\beta}\psi_j^*\left\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_j\right\|)\\
&+\frac{2\rho_i^{\alpha+2}}{\Gamma(\alpha+1)\Gamma(3-\beta)}(\xi_i^*+\eta_i^*\|z_i\|+\rho_i^{-\beta}\psi_i^*\left\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i\right\|).
\end{aligned}
$$

By using similar computations, we obtain

$$
\begin{aligned}
&\left\|{}^C\mathfrak{D}_{0,\tau}^{\beta}(A_iz+B_i\bar{z})(\tau)\right\|\\
\leq&\Big[\Big(\frac{1}{\Gamma(\alpha-\beta+3)}+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)\rho_i^{\alpha+2}(\eta_i^*+\rho_i^{-\beta}\psi_i^*)+\frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\Big]\|z_i\|_X\\
&+\sum_{j=1,j\neq i}^{k}\Big[\Big(\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)\rho_j^{\alpha+2}(\eta_j^*+\rho_j^{-\beta}\psi_j^*)+\frac{4\lambda_j\rho_j^2}{\Gamma(3-\beta)}\Big]\left\|z_j\right\|_X\\
&+\Big(\frac{1}{\Gamma(\alpha-\beta+3)}+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)\rho_i^{\alpha+2}\xi_i^*+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha+2}\xi_j^*\\
\leq&\Big[\Big(\frac{1}{\Gamma(\alpha-\beta+3)}+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)\rho_i^{\alpha+2}(\eta_i^*+\rho_i^{-\beta}\psi_i^*)+\frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\\
&+\sum_{j=1,j\neq i}^{k}\Big(\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\rho_j^{\alpha+2}(\eta_j^*+\rho_j^{-\beta}\psi_j^*)+\frac{4\lambda_j\rho_j^2}{\Gamma(3-\beta)}\Big)\Big]r\\
&+\Big(\frac{1}{\Gamma(\alpha-\beta+3)}+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)\rho_i^{\alpha+2}\xi_i^*+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha+2}\xi_j^*.
\end{aligned}\tag{3.4}
$$

From (3.3) and (3.4), we get

$$
\begin{aligned}
&\|Tz\|_X=\left\|(A_iz+B_i\bar{z})(\tau)\right\|+\left\|{}^C\mathfrak{D}_{0,\tau}^{\beta}(A_iz+B_i\bar{z})(\tau)\right\|\\
\leq&\Big[Q_1\rho_i^{\alpha+2}(\eta_i^*+\rho_i^{-\beta}\psi_i^*)+\Omega_i+\sum_{j=1,j\neq i}^{k}(Q_2\rho_j^{\alpha+2}(\eta_j^*+\rho_j^{-\beta}\psi_j^*)+\tilde{\Omega}_j))\Big]r+Q_1\rho_i^{\alpha+2}\xi_i^*+\sum_{j=1,j\neq i}^{k}Q_2\rho_j^{\alpha+2}\xi_j^*
\end{aligned}
$$

$$
\begin{aligned}
&\leq\Big[\Delta_i(\eta_i^*+\rho_i^{-\beta}\psi_i^*)+\Omega_i+\sum_{j=1,j\neq i}^{k}(\tilde{\Delta}_j(\eta_j^*+\rho_j^{-\beta}\psi_j^*)+\tilde{\Omega}_j)\Big]r+\Delta_i\xi_i^*+\sum_{j=1,j\neq i}^{k}\tilde{\Delta}_j\xi_j^*\\
&\leq\theta_i r+\mathrm{N}_i,
\end{aligned}
$$

where

$$
\mathrm{N}_i=\Delta_i\xi_i^*+\sum_{j=1,j\neq i}^{k}\tilde{\Delta}_j\xi_j^*,\ i=1,2\cdots k.
$$

Hence,

$$
\|Tz\|_{X^k}=\sum_{i=1}^{k}\|T_i z\|_X\leq\sum_{i=1}^{k}\theta_i r+\sum_{i=1}^{k}N_i\leq r,
$$

and so $A_i(z)+B_i(\bar{z})\in\Theta$.

On the other hand, the continuity of A_i follows from the continuity of functions f_i $(i=1,2,\cdots,k)$. Now, we show that the operator A_i is uniformly bounded. For this, note that

$$
\begin{aligned}
&\Big|(A_i z(\tau)+{}^C\mathfrak{D}_{0,\tau}^{\beta}A_i z(\tau)\Big|\\
&\leq\Big[Q_1\rho_i^{\alpha+2}(\eta_i^*+\rho_i^{-\beta}\psi_i^*)+\sum_{j=1,j\neq i}^{k}(Q_2\rho_j^{\alpha+2}(\eta_j^*+\rho_j^{-\beta}\psi_j^*))\Big]r+Q_1\rho_i^{\alpha+2}\xi_i^*+\sum_{j=1,j\neq i}^{k}Q_2\rho_j^{\alpha+2}\xi_j^*\\
&\leq\Big[\Delta_i(\eta_i^*+\rho_i^{-\beta}\psi_i^*)+\sum_{j=1,j\neq i}^{k}\tilde{\Delta}_j(\eta_j^*+\rho_j^{-\beta}\psi_j^*)\Big]r+\Delta_i\xi_i^*+\sum_{j=1,j\neq i}^{k}\tilde{\Delta}_j\xi_j^*.
\end{aligned}
$$

This shows that the operator A_i is uniformly bounded on Θ.

Now, we show that the operator A_i is compact on Θ. Let $\tau_1,\tau_2\in(0,1)$, $\tau_1<\tau_2$, then we have

$$
\begin{aligned}
&\Big|A_i z(\tau_2)-A_i z(\tau_1)\Big|\\
=&\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+2)}\int_0^{\tau_1}((\tau_2-s)^{\alpha+1}-(\tau_1-s)^{\alpha+1})ds(\xi_i^*+\eta_i^*\|z_i\|+\rho_i^{-\beta}\psi_i^*\Big\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i\Big\|)\\
&+\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+2)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha+1}ds(\xi_i^*+\eta_i^*\|z_i\|+\rho_i^{-\beta}\psi_i^*\Big\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i\Big\|)\\
&+(\tau_2^2-\tau_1^2)\sum_{j=1}^{k}\Big(\frac{\rho_j^{\alpha+2}}{\Gamma(\alpha+1)}(\xi_j^*+\eta_j^*\Big\|z_j\Big\|+\rho_j^{-\beta}\psi_j^*\Big\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_j\Big\|)\Big)\\
&+(\tau_2^2-\tau_1^2)\sum_{j=1,j\neq i}^{k}\frac{\rho_j^{\alpha+2}}{\Gamma(\alpha+1)}(\xi_j^*+\eta_j^*\Big\|z_j\Big\|+\rho_j^{-\beta}\psi_j^*\Big\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_j\Big\|)\\
&+(\tau_2^2-\tau_1^2)\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+1)}(\xi_i^*+\eta_i^*\|z_i\|+\rho_i^{-\beta}\psi_i^*\Big\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i\Big\|)\\
\leq&\frac{\rho_i^{\alpha+2}(\xi_i^*+(\eta_i^*+\rho_i^{-\beta}\psi_i^*)\|z_i\|_X)}{\Gamma(\alpha+3)}(\tau_2^{\alpha+2}-\tau_1^{\alpha+2})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha+1)}\sum_{j=1}^{k}\rho_j^{\alpha+2}(\xi_j^*+(\eta_j^*+\rho_j^{-\beta}\psi_j^*)\|z_j\|_X)(\tau_2^2-\tau_1^2)\\
&+\frac{1}{\Gamma(\alpha+1)}\rho_j^{\alpha+2}(\xi_j^*+(\eta_j^*+\rho_j^{-\beta}\psi_j^*)\|z_j\|_X)(\tau_2^2-\tau_1^2)\\
&+\frac{\rho_i^{\alpha+2}(\xi_i^*+(\eta_i^*+\rho_i^{-\beta}\psi_i^*)\|z_i\|_X)}{\Gamma(\alpha+1)}(\tau_2^2-\tau_1^2)
\end{aligned}\tag{3.5}
$$

and

$$
\begin{aligned}
&\left|{}^C\mathfrak{D}_{0,\tau}^{\beta}A_iz(\tau_2)-{}^C\mathfrak{D}_{0,\tau}^{\beta}A_iz(\tau_1)\right|\\
\le&\frac{\rho_i^{\alpha+2}(\xi_i^*+\eta_i^*\|z_i\|+\rho_i^{-\beta}\psi_i^*\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i\|)}{\Gamma(\alpha-\beta+2)}\int_0^{\tau_1}((\tau_2-s)^{\alpha-\beta+1}-(\tau_1-s)^{\alpha-\beta+1})ds\\
&-\frac{\rho_i^{\alpha+2}(\xi_i^*+\eta_i^*\|z_i\|+\rho_i^{-\beta}\psi_i^*\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i\|)}{\Gamma(\alpha-\beta+2)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-\beta+1}ds\\
&+\frac{2(\tau_2^{2-\beta}-\tau_1^{2-\beta})}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1}^{k}\rho_j^{\alpha+2}(\xi_j^*+\eta_j^*\|z_j\|+\rho_j^{-\beta}\psi_j^*\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_j\|)\\
&+\frac{2(\tau_2^{2-\beta}-\tau_1^{2-\beta})}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha+2}(\xi_j^*+\eta_j^*\|z_j\|+\rho_j^{-\beta}\psi_j^*\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_j\|)\\
&+\frac{2(\tau_2^{2-\beta}-\tau_1^{2-\beta})}{\Gamma(\alpha+1)\Gamma(3-\beta)}\rho_i^{\alpha+2}(\xi_i^*+\eta_i^*\|z_i\|+\rho_i^{-\beta}\psi_i^*\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i\|)\\
\le&\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha-\beta+3)}(\xi_i^*+(\eta_i^*+\rho_i^{-\beta}\psi_i^*)\|z_i\|_X)(\tau_2^{\alpha-\beta+2}-\tau_1^{\alpha-\beta+2})\\
&+\frac{2}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1}^{k}\rho_j^{\alpha+2}(\xi_j^*+(\eta_j^*+\rho_j^{-\beta}\psi_j^*)\|z_j\|_X)(\tau_2^{2-\beta}-\tau_1^{2-\beta})\\
&+\frac{2\rho_i^{\alpha+2}}{\Gamma(\alpha+1)\Gamma(3-\beta)}(\xi_i^*+(\eta_i^*+\rho_i^{-\beta}\psi_i^*)\|z_i\|_X)(\tau_2^{2-\beta}-\tau_1^{2-\beta})\\
&+\frac{2}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha+2}(\xi_j^*+(\eta_j^*+\rho_j^{-\beta}\psi_j^*)\|z_j\|_X)(\tau_2^{2-\beta}-\tau_1^{2-\beta}).
\end{aligned}\tag{3.6}
$$

From (3.5) and (3.6), we get

$$
\begin{aligned}
&\|A_iz(\tau_2)-A_iz(\tau_1)\|_X\\
\le&\Big(\frac{2\rho_i^{\alpha+2}(\xi_i^*+(\eta_i^*+\rho_i^{-\beta}\psi_i^*)\|z_i\|_X)}{\Gamma(\alpha+1)}\Big)(\tau_2^2-\tau_1^2)\\
&+\frac{\rho_i^{\alpha+2}(\xi_i^*+(\eta_i^*+\rho_i^{-\beta}\psi_i^*)\|z_i\|_X)}{\Gamma(\alpha+3)}(\tau_2^{\alpha+2}-\tau_1^{\alpha+2})\\
&+\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha-\beta+3)}(\xi_i^*+(\eta_i^*+\rho_i^{-\beta}\psi_i^*)\|z_i\|_X)(\tau_2^{\alpha-\beta+2}-\tau_1^{\alpha-\beta+2})
\end{aligned}
$$

$$+\frac{4\rho_i^{\alpha+2}}{\Gamma(\alpha+1)\Gamma(3-\beta)}(\xi_i^*+(\eta_i^*+\rho_i^{-\beta}\psi_i^*)\|z_i\|_X)(\tau_2^{2-\beta}-\tau_1^{2-\beta})$$
$$+\frac{2}{\Gamma(\alpha+1)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha+2}(\xi_j^*+(\eta_j^*+\rho_j^{-\beta}\psi_j^*)\left\|z_j\right\|_X)(\tau_2^2-\tau_1^2)$$
$$+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\rho_j^{\alpha+2}(\xi_j^*+(\eta_j^*+\rho_j^{-\beta}\psi_j^*)\left\|z_j\right\|_X)(\tau_2^{2-\beta}-\tau_1^{2-\beta}),$$

which implies $\|A_iz(\tau_2)-A_iz(\tau_1)\|_X\to 0$ as $\tau_2\to\tau_1$, and so $\|A_iz(\tau_2)-A_iz(\tau_1)\|_{X^k}\to 0$ as $\tau_2\to\tau_1$. Thus, A_i is equicontinuous and, by the Arzelá-Ascoli theorem, we conclude that the operator A_i is a completely continuous operator.

Next, we prove that the operator B_i is a contraction. Letting $z,\bar{z}\in\Theta$, we have

$$\begin{aligned}
&|B_iz(\tau)-B_i\bar{z}(\tau)|\\
\leq&\lambda_i\rho_i^2\int_0^\tau(\tau-s)\,|\bar{z}_i(s)-\bar{z}_i(s)|\,ds+\lambda_i\rho_i^2\int_0^1(1-s)\,|\bar{z}_i(s)-\bar{z}_i(s)|\,ds\\
&+(1-\tau^2)\sum_{j=1}^{k}\ell_j\lambda_j\rho_j^2\left|z_j(1)-\bar{z}_j(1)\right|+(1-\tau^2)\sum_{j=1,j\neq i}^{k}\ell_j\lambda_j\rho_j^2\left|z_j(1)-\bar{z}_j(1)\right|\\
&+(1-\tau^2)\sum_{j=1,j\neq i}^{k}\ell_j\lambda_i\rho_i^2\,|z_i(1)-\bar{z}_i(1)|\\
\leq&3\lambda_i\rho_i^2\,\|z_i-\bar{z}_i\|+2\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\left\|z_j-\bar{z}_j\right\|
\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}
&\left|{}^C\mathfrak{D}_{0,\tau}^{\beta}B_iz(\tau)-{}^C\mathfrak{D}_{0,\tau}^{\beta}B_i\bar{z}(\tau)\right|\\
\leq&\frac{\lambda_i\rho_i^2}{\Gamma(2-\beta)}\int_0^\tau(\tau-s)^{1-\beta}\,|z_i(s)-\bar{z}_i(s)|\,ds+\frac{2\tau^{2-\beta}}{\Gamma(3-\beta)}\sum_{j=1}^{k}\ell_j\lambda_j\rho_j^2\left|z_j-\bar{z}_j\right|\\
&+\frac{2\tau^{2-\beta}}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\ell_j\lambda_j\rho_j^2\left|z_j-\bar{z}_j\right|+\frac{2\tau^{2-\beta}}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\ell_j\lambda_i\rho_i^2\,|z_i-\bar{z}_i|\\
\leq&\frac{\lambda_i\rho_i^2}{\Gamma(3-\beta)}\,\|z_i-\bar{z}_i\|+\frac{2}{\Gamma(3-\beta)}\sum_{j=1}^{k}\lambda_j\rho_j^2\left\|z_j-\bar{z}_j\right\|\\
&+\frac{2}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\left\|z_j-\bar{z}_j\right\|+\frac{2\lambda_i\rho_i^2}{\Gamma(3-\beta)}\,\|z_i-\bar{z}_i\|.
\end{aligned}\tag{3.8}$$

Thus, from (3.7) and (3.8), we get

$$\|B_iz-B_i\bar{z}\|_X=\|B_iz-B_i\bar{z}\|+\left\|{}^C\mathfrak{D}_{0,\tau}^{\beta}B_iz-{}^C\mathfrak{D}_{0,\tau}^{\beta}B_i\bar{z}\right\|$$

$$\leq \Big(3\lambda_i\rho_i^2 + \frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\Big)\|z_i - \bar{z}_i\|_X + \sum_{j=1, j\neq i}^{k} \Big(2\lambda_j\rho_j^2 + \frac{4\lambda_j\rho_j^2}{\Gamma(3-\beta)}\Big)\big\|z_j - \bar{z}_j\big\|_X$$
$$\leq L_i \sum_{j=1}^{k} \big\|z_j - \bar{z}_j\big\|_X.$$

Furthermore, we obtain

$$\|B_i z - B_i \bar{z}\|_{X^k} = \sum_{i=1}^{k} \|B_i z - B_i \bar{z}\|_X \leq \sum_{i=1}^{k} L_i \big\|z_j - \bar{z}_j\big\|_{X^k}.$$

$\sum_{i=1}^{k} L_i < 1$, which means that Θ is bounded. We use Theorem 2.2 to show that the operator T at least has one fixed point, and then system (2.1) has at least one solution. □

Theorem 3.2. *Assume that* (H_1) *and* (H_3) *hold. Then system (2.1) has a unique solution on [0,1], that is,*

$$\Big(\sum_{i=1}^{k} K_i\Big)\Big(\sum_{i=1}^{k} W_i\Big) + \sum_{i=1}^{k} L_i < 1, \tag{3.9}$$

where

$$K_i = Q_1(\rho_i^{\alpha+2} + \rho_i^{\alpha-\beta+2}) + Q_2 \sum_{j=1, j\neq i}^{k} (\rho_j^{\alpha+2} + \rho_j^{\alpha-\beta+2}),$$
$$\mathrm{W}_i = \max_{\tau\in[0,1]} |w_i(\tau)|, \; L_i = 3\lambda_i\rho_i^2 + \frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)} + \sum_{j=1, j\neq i}^{k} \Big(2\lambda_j\rho_j^2 + \frac{4\lambda_j\rho_j^2}{\Gamma(3-\beta)}\Big).$$

Proof. We will prove that T is a contraction mapping. For any $z = (z_1, z_2, \cdots, z_k)$, $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_k) \in X^k$, $\tau \in [0, 1]$. By Eq (3.1), we get

$$|T_i z(\tau) - T_i \bar{z}(\tau)|$$
$$\leq \frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+2)} \int_0^\tau (\tau - s)^{\alpha+1} \big|\tilde{f}_i(s)\big| ds + \frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+2)} \int_0^1 (1-s)^{\alpha+1} \big|\tilde{f}_i(s)\big| ds$$
$$+ \lambda_i\rho_i^2 \int_0^\tau (\tau - s)\, |z_i(s) - \bar{z}_i(s)|\, ds + \lambda_i\rho_i^2 \int_0^1 (1-s)\, |z_i(s) - \bar{z}_i(s)|\, ds$$
$$+ (1-\tau^2) \sum_{j=1}^{k} \frac{\ell_j\rho_j^{\alpha+2}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \big|\tilde{f}_j(s)\big| ds + (1-\tau^2) \sum_{j=1}^{k} \ell_j\lambda_j\rho_j^2 \big|z_j(1) - \bar{z}_j(1)\big|$$
$$+ (1-\tau^2) \sum_{j=1, j\neq i}^{k} \ell_j\lambda_j\rho_j^2 \big|z_j(1) - \bar{z}_j(1)\big| + (1-\tau^2) \sum_{j=1, j\neq i}^{k} \ell_j\lambda_i\rho_i^2\, |z_i(1) - \bar{z}_i(1)|$$
$$+ (1-\tau^2) \sum_{j=1, j\neq i}^{k} \frac{\ell_j\rho_j^{\alpha+2}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \big|\tilde{f}_j(s)\big| ds$$

$$+ (1-\tau^2) \sum_{j=1, j\neq i}^{k} \frac{\ell_j \rho_i^{\alpha+2}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left|\tilde{f}_j(s)\right| ds,$$

where

$$\tilde{f}_i = f_i(s, z_i(s), \rho_i^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_i(s)) - f_i(s, \bar{z}_i(s), \rho_i^{-\beta C}\mathfrak{D}_{0,s}^{\beta} \bar{z}_i(s)),$$
$$\tilde{f}_j = f_j(s, z_j(s), \rho_j^{-\beta C}\mathfrak{D}_{0,s}^{\beta} z_j(s)) - f_j(s, \bar{z}_j(s), \rho_j^{-\beta C}\mathfrak{D}_{0,s}^{\beta} \bar{z}_j(s)),\ i = 1, 2, \cdots, k.$$

Now using assumption (H_3) and $1-\tau^2 < 1$ $(0 < \tau < 1)$, $\ell_j \in (0,1)$, $j = 1, 2, \cdots, k$, we have

$$\begin{aligned}
&|T_i z(\tau) - T_i \bar{z}(\tau)| \\
\leq & \frac{2\rho_i^{\alpha+2}}{\Gamma(\alpha+3)} W_i \|z_i - \bar{z}_i\| + \frac{2\rho_i^{\alpha-\beta+2}}{\Gamma(\alpha+3)} W_i \left\|{}^C\mathfrak{D}_{0,s}^{\beta} z_i - {}^C\mathfrak{D}_{0,s}^{\beta} \bar{z}_i\right\| \\
& + \lambda_i \rho_i^2 \|z_i - \bar{z}_i\| + \sum_{j=1}^{k} \lambda_j \rho_j^2 \left\|z_j - \bar{z}_j\right\| + \sum_{j=1}^{k} \frac{\rho_j^{\alpha+2} W_j}{\Gamma(\alpha+1)} \left\|z_j - \bar{z}_j\right\| \\
& + \sum_{j=1}^{k} \frac{\rho_j^{\alpha-\beta+2} W_j}{\Gamma(\alpha+1)} \left\|{}^C\mathfrak{D}_{0,s}^{\beta} z_j - {}^C\mathfrak{D}_{0,s}^{\beta} \bar{z}_j\right\| + \sum_{j=1, j\neq i}^{k} \lambda_j \rho_j^2 \left\|z_j - \bar{z}_j\right\| \\
& + \sum_{j=1, j\neq i}^{k} \lambda_i \rho_i^2 \|z_i - \bar{z}_i\| + \sum_{j=1, j\neq i}^{k} \frac{\rho_j^{\alpha+2} W_j}{\Gamma(\alpha+1)} \left\|z_j - \bar{z}_j\right\| + \sum_{j=1, j\neq i}^{k} \frac{\rho_i^{\alpha+2} W_i}{\Gamma(\alpha+1)} \|z_i - \bar{z}_i\| \\
& + \sum_{j=1, j\neq i}^{k} \frac{\rho_j^{\alpha-\beta+2} W_j}{\Gamma(\alpha+1)} \left\|{}^C\mathfrak{D}_{0,s}^{\beta} z_j - {}^C\mathfrak{D}_{0,s}^{\beta} \bar{z}_j\right\| + \sum_{j=1, j\neq i}^{k} \frac{\rho_i^{\alpha-\beta+2} W_i}{\Gamma(\alpha+1)} \left\|{}^C\mathfrak{D}_{0,s}^{\beta} z_i - {}^C\mathfrak{D}_{0,s}^{\beta} \bar{z}_i\right\| \\
\leq & \frac{2W_i}{\Gamma(\alpha+3)} (\rho_i^{\alpha+2} + \rho_i^{\alpha-\beta+2}) (\|z_i - \bar{z}_i\| + \left\|{}^C\mathfrak{D}_{0,s}^{\beta} z_i - {}^C\mathfrak{D}_{0,s}^{\beta} \bar{z}_i\right\|) \\
& + 3\lambda_i \rho_i^2 \|z_i - \bar{z}_i\| + 2 \sum_{j=1, j\neq i}^{k} \lambda_j \rho_j^2 \left\|z_j - \bar{z}_j\right\| \\
& + \frac{W_i}{\Gamma(\alpha+1)} (\rho_i^{\alpha+2} + \rho_i^{\alpha-\beta+2}) (\|z_i - \bar{z}_i\| + \left\|{}^C\mathfrak{D}_{0,s}^{\beta} z_i - {}^C\mathfrak{D}_{0,s}^{\beta} \bar{z}_i\right\|) \\
& + \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^{k} (\rho_j^{\alpha+2} + \rho_j^{\alpha-\beta+2}) W_j (\left\|z_j - \bar{z}_j\right\| + \left\|{}^C\mathfrak{D}_{0,s}^{\beta} z_j - {}^C\mathfrak{D}_{0,s}^{\beta} \bar{z}_j\right\|) \\
& + \frac{1}{\Gamma(\alpha+1)} \sum_{j=1, j\neq i}^{k} (\rho_j^{\alpha+2} + \rho_j^{\alpha-\beta+2}) W_j (\left\|z_j - \bar{z}_j\right\| + \left\|{}^C\mathfrak{D}_{0,s}^{\beta} z_j - {}^C\mathfrak{D}_{0,s}^{\beta} \bar{z}_j\right\|).
\end{aligned}$$

Hence, for any $z, \bar{z} \in X^k$, we obtain

$$\begin{aligned}
&\|T_i z - T_i \bar{z}\| \\
\leq & \Big(\frac{2}{\Gamma(\alpha+3)} + \frac{2}{\Gamma(\alpha+1)}\Big) (\rho_i^{\alpha+2} + \rho_i^{\alpha-\beta+2}) W_i \|z_i - \bar{z}_i\|_X + 3\lambda_i \rho_i^2 \|z_i - \bar{z}_i\|_X \\
& + 2 \sum_{j=1, j\neq i}^{k} \lambda_j \rho_j^2 \left\|z_j - \bar{z}_j\right\|_X + \frac{2}{\Gamma(\alpha+1)} \sum_{j=1, j\neq i}^{k} (\rho_j^{\alpha+2} + \rho_j^{\alpha-\beta+2}) W_j \left\|z_j - \bar{z}_j\right\|_X. \qquad (3.10)
\end{aligned}$$

On the other hand, using Lemma 2.3,

$$\begin{aligned}
&\left|{}^{C}\mathfrak{D}^{\beta}_{0,\tau}T_i z(\tau)-{}^{C}\mathfrak{D}^{\beta}_{0,\tau}T_i\bar{z}(\tau)\right|\\
\leq&\frac{\rho_i^{\alpha+2}}{\Gamma(\alpha-\beta+2)}\int_0^{\tau}(\tau-s)^{\alpha-\beta+1}\left|\tilde{f}_i(s)\right|ds+\frac{\lambda_i\rho_i^2}{\Gamma(2-\beta)}\int_0^{\tau}(\tau-s)^{1-\beta}\left|z_i(s)-\bar{z}_i(s)\right|ds\\
&+\frac{2\tau^{2-\beta}}{\Gamma(3-\beta)}\sum_{j=1}^{k}\ell_j\lambda_j\rho_j^2\left|z_j-\bar{z}_j\right|+\frac{2\tau^{2-\beta}}{\Gamma(\alpha)\Gamma(3-\beta)}\sum_{j=1}^{k}\ell_j\rho_j^{\alpha+2}\int_0^1(1-s)^{\alpha-1}\left|\tilde{f}_j(s)\right|ds\\
&+\frac{2\tau^{2-\beta}}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\ell_j\lambda_j\rho_j^2\left|z_j-\bar{z}_j\right|+\frac{2\tau^{2-\beta}}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\ell_j\lambda_i\rho_i^2\left|z_i-\bar{z}_i\right|\\
&+\frac{2\tau^{2-\beta}}{\Gamma(\alpha)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\ell_j\rho_j^{\alpha+2}\int_0^1(1-s)^{\alpha-1}\left|\tilde{f}_j(s)\right|ds\\
&+\frac{2\tau^{2-\beta}\ell_j\rho_i^{\alpha+2}}{\Gamma(\alpha)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\int_0^1(1-s)^{\alpha-1}\left|\tilde{f}_i(s)\right|ds,
\end{aligned}$$

by using similar computations, we get

$$\begin{aligned}
&\left|{}^{C}\mathfrak{D}^{\beta}_{0,\tau}T_i z(\tau)-{}^{C}\mathfrak{D}^{\beta}_{0,\tau}T_i\bar{z}(\tau)\right|\\
\leq&\frac{(\rho_i^{\alpha+2}+\rho_i^{\alpha-\beta+2})}{\Gamma(\alpha-\beta+3)}W_i(\|z_i-\bar{z}_i\|+\left\|{}^{C}\mathfrak{D}^{\beta}_{0,s}z_i-{}^{C}\mathfrak{D}^{\beta}_{0,s}\bar{z}_i\right\|)+\frac{\lambda_i\rho_i^2}{\Gamma(3-\beta)}\|z_i-\bar{z}_i\|\\
&+\frac{2}{\Gamma(3-\beta)}\sum_{j=1}^{k}\lambda_j\rho_j^2\left\|z_j-\bar{z}_j\right\|+\frac{2}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\left\|z_j-\bar{z}_j\right\|+\frac{2\lambda_i\rho_i^2}{\Gamma(3-\beta)}\|z_i-\bar{z}_i\|\\
&+\frac{2}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1}^{k}(\rho_j^{\alpha+2}+\rho_j^{\alpha-\beta+2})W_j(\left\|z_j-\bar{z}_j\right\|+\left\|{}^{C}\mathfrak{D}^{\beta}_{0,s}z_j-{}^{C}\mathfrak{D}^{\beta}_{0,s}\bar{z}_j\right\|)\\
&+\frac{2}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2}+\rho_j^{\alpha-\beta+2})W_j(\left\|z_j-\bar{z}_j\right\|+\left\|{}^{C}\mathfrak{D}^{\beta}_{0,s}z_j-{}^{C}\mathfrak{D}^{\beta}_{0,s}\bar{z}_j\right\|)\\
&+\frac{2}{\Gamma(\alpha+1)\Gamma(3-\beta)}(\rho_i^{\alpha+2}+\rho_i^{\alpha-\beta+2})W_i(\|z_i-\bar{z}_i\|+\left\|{}^{C}\mathfrak{D}^{\beta}_{0,s}z_i-{}^{C}\mathfrak{D}^{\beta}_{0,s}\bar{z}_i\right\|),
\end{aligned}$$

this implies that, for any $z,\bar{z}\in X^k$,

$$\begin{aligned}
&\left\|{}^{C}\mathfrak{D}^{\beta}_{0,\tau}T_i z-{}^{C}\mathfrak{D}^{\beta}_{0,\tau}T_i\bar{z}\right\|\\
\leq&\Big(\frac{1}{\Gamma(\alpha-\beta+3)}+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)(\rho_i^{\alpha+2}+\rho_i^{\alpha-\beta+2})W_i\|z_i-\bar{z}_i\|_X\\
&+\frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\|z_i-\bar{z}_i\|_X+\frac{4}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\left\|z_j-\bar{z}_j\right\|_X\\
&+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2}+\rho_j^{\alpha-\beta+2})W_j\left\|z_j-\bar{z}_j\right\|_X. \qquad (3.11)
\end{aligned}$$

It follows from (3.10) and (3.11) that

$$
\begin{aligned}
&\|T_iz - T_i\bar{z}\|_X = \|T_iz - T_i\bar{z}\| + \left\|{}^C\mathfrak{D}^{\beta}_{0,\tau}T_iz - {}^C\mathfrak{D}^{\beta}_{0,\tau}T_i\bar{z}\right\| \\
\le&\Big(\frac{2}{\Gamma(\alpha+3)} + \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+3)} + \frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)(\rho_i^{\alpha+2} + \rho_i^{\alpha-\beta+2})W_i\|z_i - \bar{z}_i\|_X \\
&+ \Big(3\lambda_i\rho_i^2 + \frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\Big)\|z_i - \bar{z}_i\|_X + \sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j^2 + \frac{4\lambda_j\rho_j^2}{\Gamma(3-\beta)}\Big)\left\|z_j - \bar{z}_j\right\|_X \\
&+ \Big(\frac{2}{\Gamma(\alpha+1)} + \frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2} + \rho_j^{\alpha-\beta+2})W_j\left\|z_j - \bar{z}_j\right\|_X \\
\le&Q_1(\rho_i^{\alpha+2} + \rho_i^{\alpha-\beta+2})W_i\|z_i - \bar{z}_i\|_X + \Big(3\lambda_i\rho_i^2 + \frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\Big)\|z_i - \bar{z}_i\|_X \\
&+ \sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j^2 + \frac{4\lambda_j\rho_j^2}{\Gamma(3-\beta)}\Big)\left\|z_j - \bar{z}_j\right\|_X + Q_2\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2} + \rho_j^{\alpha-\beta+2})W_j\left\|z_j - \bar{z}_j\right\|_X.
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
&\|T_iz - T_i\bar{z}\|_X \\
\le&\Big(Q_1(\rho_i^{\alpha+2} + \rho_i^{\alpha-\beta+2}) + Q_2\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2} + \rho_j^{\alpha-\beta+2})\Big)\Big(\sum_{i=1}^{k}W_i\Big)\sum_{j=1}^{k}\left\|z_j - \bar{z}_j\right\|_X \\
&+ \Big(3\lambda_i\rho_i^2 + \frac{3\lambda_i\rho_i^2}{\Gamma(3-\beta)} + \sum_{j=1,j\neq i}^{k}(2\lambda_j\rho_j^2 + \frac{4\lambda_j\rho_j^2}{\Gamma(3-\beta)})\Big)\sum_{j=1}^{k}\left\|z_j - \bar{z}_j\right\|_X \\
=&\Big(K_i\sum_{i=1}^{k}W_i + L_i\Big)\sum_{j=1}^{k}\left\|z_j - \bar{z}_j\right\|_X.
\end{aligned}
$$

As a consequence, we obtain

$$
\|T_iz - T_i\bar{z}\|_{X^k} = \sum_{i=1}^{k}\|T_iz - T_i\bar{z}\|_X \le \Big(\sum_{i=1}^{k}K_i\Big(\sum_{i=1}^{k}W_i\Big) + \sum_{i=1}^{k}L_i\Big)\left\|z_j - \bar{z}_j\right\|_{X^k},
$$

which, given condition (3.9), proves that operator T is a contraction. This implies that T has a unique fixed point on X^k, that is, system (2.1) has a unique solution on [0,1]. □

4. Ulam type stability analysis

In this section, we introduce Ulam type stability concepts for system (2.1). Let $\varepsilon_i > 0$, $f_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, and $\Phi_i(\tau) : [0,1] \to \mathbb{R}^+$ ($\tau \in [0,1]$) be nondecreasing. Consider the following inequalities:

$$
\left|{}^C\mathfrak{D}^{\alpha}_{0,\tau}(\mathcal{D}^2 + \lambda_i\rho_i^2)z_i(\tau) - \rho_i^{\alpha+2}f_i(\tau, z_i(\tau), \rho_i^{-\beta C}\mathfrak{D}^{\beta}_{0,\tau}z_i(\tau))\right| \le \varepsilon_i,\ i = 1, 2, \cdots, k, \tag{4.1}
$$

$$
\left|{}^C\mathfrak{D}^{\alpha}_{0,\tau}(\mathcal{D}^2 + \lambda_i\rho_i^2)z_i(\tau) - \rho_i^{\alpha+2}f_i(\tau, z_i(\tau), \rho_i^{-\beta C}\mathfrak{D}^{\beta}_{0,\tau}z_i(\tau))\right| \le \Phi_i(\tau)\varepsilon_i,\ i = 1, 2, \cdots, k, \tag{4.2}
$$

$$
\left|{}^C\mathfrak{D}^{\alpha}_{0,\tau}(\mathcal{D}^2 + \lambda_i\rho_i^2)z_i(\tau) - \rho_i^{\alpha+2}f_i(\tau, z_i(\tau), \rho_i^{-\beta C}\mathfrak{D}^{\beta}_{0,\tau}z_i(\tau))\right| \le \Phi_i(\tau),\ i = 1, 2, \cdots, k. \tag{4.3}
$$

Definition 4.1. *[23] System (2.1) is Ulam-Hyers stable if there exists a real number $c_{f_1,f_2,\cdots,f_k} > 0$ such that, for each $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k) > 0$ and for each solution $z = (z_1, z_2, \cdots, z_k) \in X$ of inequalities (4.1), there exists a solution $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_k) \in X$ of system (2.1) with*

$$\|z - \bar{z}\|_X \leq c_{f_1,f_2,\cdots,f_k}\varepsilon, \ \tau \in [0, 1].$$

Definition 4.2. *[23] System (2.1) is generalized Ulam-Hyers stable if there exists a function $\Upsilon_{f_1,f_2,\cdots,f_k} \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\Upsilon_{f_1,f_2,\cdots,f_k}(0) = 0$ such that, for each $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k) > 0$ and for each solution $v = (z_1, z_2, \cdots, z_k) \in X$ of inequalities (4.2), there exists a solution $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_k) \in X$ of system (2.1) with*

$$\|z - \bar{z}\|_X \leq \Upsilon_{f_1,f_2,\cdots,f_k}(\varepsilon), \ \tau \in [0, 1].$$

Definition 4.3. *[23] System (2.1) is Ulam-Hyers-Rassias stable with respect to $\Phi = \Phi(\Phi_1, \Phi_2, \cdots, \Phi_k) \in C([0, 1], \mathbb{R}^+)$ if there exists a real number $c_{f_1,f_2,\cdots,f_k}\Phi > 0$ such that, for every $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k) > 0$ and for each solution $z = (z_1, z_2, \cdots, z_k) \in X$ of inequalities (4.2), there exists a solution $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_k) \in X$ of system (2.1) with*

$$\|z - \bar{z}\|_X \leq c_{f_1,f_2,\cdots,f_{k,\varphi}}\varepsilon\Phi(\tau), \ \tau \in [0, 1].$$

Definition 4.4. *[23] System (2.1) is generalized Ulam-Hyers-Rassias stable with respect to $\Phi = \Phi(\Phi_1, \Phi_2, \cdots, \Phi_k) \in C([0, 1], \mathbb{R}^+)$ if there exists a real number $c_{f_1,f_2,\cdots,f_k}\Phi > 0$ such that, for each solution $z = (z_1, z_2, \cdots, z_k) \in X$ of inequalities (4.2), there exists a solution $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_k) \in X$ of system (2.1) with*

$$\|z - \bar{z}\|_X \leq c_{f_1,f_2,\cdots,f_{k,\Phi}}\Phi(\tau), \ \tau \in [0, 1].$$

Remark 4.1. *A function $z = (z_1, z_2, \cdots, z_k) \in X$ is a solution of (4.1), if there exist functions $\phi_i \in C([0, 1], \mathbb{R})$ which depend on z_i such that*

(*i*) $|\phi_i(\tau)| \leq \varepsilon_i, \ \tau \in [0, 1], \ i = 1, 2, \cdots, k,$
(*ii*) ${}^C\mathfrak{D}^\alpha_{0,\tau}(\mathcal{D}^2 + \lambda_i\rho_i^2)z_i(\tau) = \rho_i^{\alpha+2}f_i(\tau, z_i(\tau), \rho_i^{-\beta C}\mathfrak{D}^\beta_{0,\tau}z_i(\tau)) + \phi_i(\tau), \ \tau \in [0, 1], \ i = 1, 2, \cdots, k.$

One can make similar Remarks for inequalities (4.2) and (4.3).

Lemma 4.1. *If $z = (z_1, z_2, \cdots, z_k) \in X$ is a solution of inequality (4.1), then the following inequalities hold:*

$$|z_i(\tau) - \theta_i(\tau)| \leq \Big(\frac{2}{\Gamma(\alpha + 3)} + \frac{2}{\Gamma(\alpha + 1)}\Big)\varepsilon_i + \frac{2}{\Gamma(\alpha + 1)} \sum_{j=1, j\neq i}^{k} \varepsilon_j, \ i = 1, 2, \cdots, k$$

and

$$\begin{aligned}\left|{}^C\mathfrak{D}^\beta_{0,\tau}z_i(\tau) - {}^C\mathfrak{D}^\beta_{0,\tau}\theta_i(\tau)\right| \leq &\Big(\frac{1}{\Gamma(\alpha - \beta + 3)} + \frac{4}{\Gamma(\alpha + 1)\Gamma(3 - \beta)}\Big)\varepsilon_i \\ &+ \frac{4}{\Gamma(\alpha + 1)\Gamma(3 - \beta)} \sum_{j=1, j\neq i}^{k} \varepsilon_j, \ i = 1, 2, \cdots, k,\end{aligned}$$

where

$$\theta_i(\tau) = \frac{1}{\Gamma(\alpha + 2)} \int_0^\tau (\tau - s)^{\alpha+1}h_i(s)ds + \frac{1}{\Gamma(\alpha + 2)} \int_0^1 (1 - s)^{\alpha+1}h_i(s)ds$$

$$+ \lambda_i\rho_i^2\int_0^1 (1-s)\theta_i(s)ds - \lambda_i\rho_i^2\int_0^\tau (\tau-s)\theta_i(s)ds$$
$$+ (1-\tau^2)\sum_{j=1}^k \ell_j\Big(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h_j(s)ds - \lambda_j\rho_j^2\theta_j(1)\Big)$$
$$+ (1-\tau^2)\sum_{j=1,j\neq i}^k \ell_j(\lambda_j\rho_j^2\theta_j(1) - \lambda_i\rho_i^2\theta_i(1))$$
$$- (1-\tau^2)\sum_{j=1,j\neq i}^k \ell_j\Big(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(h_j(s)-h_i(s))ds\Big),\ i=1,2,\cdots,k$$

and

$$h_i(s) = \rho_i^{\alpha+2}f_i(s, z_i(s), \rho_i^{-\beta C}\mathfrak{D}_{0,\tau}^{\beta}z_i(s)),\ i=1,2,\cdots,k,$$
$$h_j(s) = \rho_j^{\alpha+2}f_j(s, z_j(s), \rho_j^{-\beta C}\mathfrak{D}_{0,\tau}^{\beta}z_j(s)),\ i=1,2,\cdots,k.$$

Proof. Given z is the solution of (4.1), then according to Remark 4.1 we have

$$\begin{cases} {}^C\mathfrak{D}_{0,\tau}^{\alpha}(\mathcal{D}^2+\lambda_i\rho_i^2)z_i(\tau) = \rho_i^{\alpha+2}f_i(\tau, z_i(\tau), \rho_i^{-\beta C}\mathfrak{D}_{0,\tau}^{\beta}z_i(\tau)) + \phi_i(\tau),\ i=1,2,\cdots k, \\ z_i'(0)=z_i(1)=0,\ \sum_{i=1}^k \rho_i^{-2}z_i''(1)=0,\ i=1,2,\cdots k, \\ \bar{z}_i''(1)=z_j''(1),\ i,j=1,2,\cdots k,\ i\neq j. \end{cases} \tag{4.4}$$

By Lemma 2.6, the solution of (4.4) is given by

$$z_i(\tau) = \frac{1}{\Gamma(\alpha+2)}\int_0^\tau (\tau-s)^{\alpha+1}(h_i(s)+\phi_i(s))ds - \frac{1}{\Gamma(\alpha+2)}\int_0^1 (1-s)^{\alpha+1}(h_i(s)+\phi_i(s))ds$$
$$+ \lambda_i\rho_i^2\int_0^1 (1-s)z_i(s)ds - \lambda_i\rho_i^2\int_0^\tau (\tau-s)z_i(s)ds$$
$$+ (1-\tau^2)\sum_{j=1}^k \ell_j\Big(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(h_j(s)+\phi_j(s))ds - \lambda_j\rho_j^2 z_j(1)\Big)$$
$$+ (1-\tau^2)\sum_{j=1,j\neq i}^k \ell_j\Big(\lambda_j\rho_j^2 z_j(1) - \lambda_i\rho_i^2 z_i(1)\Big)$$
$$+ (1-\tau^2)\sum_{j=1,j\neq i}^k \ell_j\Big(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(h_i(s)+\phi_i(s))ds\Big)$$
$$- (1-\tau^2)\sum_{j=1,j\neq i}^k \ell_j\Big(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(h_j(s)+\phi_j(s))ds\Big). \tag{4.5}$$

From (4.5), we deduce that

$$|z_i(\tau)-\theta_i(\tau)| \le \frac{2}{\Gamma(\alpha+3)}\varepsilon_i + \frac{1}{\Gamma(\alpha+1)}\sum_{j=1}^k \ell_j\varepsilon_j + \frac{1}{\Gamma(\alpha+1)}\sum_{j=1,j\neq i}^k \ell_j\varepsilon_i + \frac{1}{\Gamma(\alpha+1)}\sum_{j=1,j\neq i}^k \ell_j\varepsilon_j$$

$$\leq \Big(\frac{2}{\Gamma(\alpha+3)} + \frac{2}{\Gamma(\alpha+1)}\Big)\varepsilon_i + \frac{2}{\Gamma(\alpha+1)} \sum_{j=1, j\neq i}^{k} \varepsilon_j,\ i = 1, 2, \cdots, k.$$

Similarly, applying the operator ${}^{C}\mathfrak{D}^{\beta}_{0,\tau}$ on (4.5), we have

$$\begin{aligned}
{}^{C}\mathfrak{D}^{\beta}_{0,\tau} z_i(\tau) \leq & \frac{1}{\Gamma(\alpha-\beta+2)} \int_0^\tau (\tau-s)^{\alpha-\beta+1}(h_i(s)+\phi_i(s))ds + \frac{\lambda_i \rho_i^2}{\Gamma(2-\beta)} \int_0^\tau (\tau-s)^{1-\beta} z_i(s)ds \\
& + \frac{2\tau^{2-\beta}}{\Gamma(3-\beta)} \sum_{j=1}^{k} \ell_j \lambda_j \rho_j^2 z_j(1) + \frac{2\tau^{2-\beta}}{\Gamma(\alpha)\Gamma(3-\beta)} \sum_{j=1}^{k} \ell_j \int_0^1 (1-s)^{\alpha-1}(h_j(s)+\phi_j(s))ds \\
& + \frac{2\tau^{2-\beta}}{\Gamma(3-\beta)} \lambda_i \rho_i^2 z_i(1) + \frac{2\tau^{2-\beta}}{\Gamma(\alpha)\Gamma(3-\beta)} \ell_i \int_0^1 (1-s)^{\alpha-1}(h_i(s)+\phi_i(s))ds \\
& + \frac{2\tau^{2-\beta}}{\Gamma(\alpha)\Gamma(3-\beta)} \sum_{j=1, j\neq i}^{k} \ell_j \int_0^1 (1-s)^{\alpha-1}(h_j(s)+\phi_j(s))ds \\
& + \frac{2\tau^{2-\beta}}{\Gamma(3-\beta)} \sum_{j=1, j\neq i}^{k} \ell_j \lambda_j \rho_j^2 z_j(1),\ i = 1, 2, \cdots, k.
\end{aligned}$$

Then we have

$$\begin{aligned}
& \left|{}^{C}\mathfrak{D}^{\beta}_{0,\tau} z_i(\tau) - {}^{C}\mathfrak{D}^{\beta}_{0,\tau} \theta_i(\tau)\right| \\
\leq & \frac{1}{\Gamma(\alpha-\beta+3)}\varepsilon_i + \frac{2}{\Gamma(\alpha+1)\Gamma(3-\beta)} \sum_{j=1}^{k} \ell_j \varepsilon_j + \frac{2}{\Gamma(\alpha+1)\Gamma(3-\beta)} \sum_{j=1, j\neq i}^{k} \ell_j \varepsilon_j + \frac{2}{\Gamma(\alpha+1)\Gamma(3-\beta)} \ell_i \varepsilon_i \\
\leq & \Big(\frac{1}{\Gamma(\alpha-\beta+3)} + \frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)\varepsilon_i + \frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)} \sum_{j=1, j\neq i}^{k} \varepsilon_j,\ i = 1, 2, \cdots, k.
\end{aligned}$$

The proof is completed. □

Lemma 4.2. *If $z = (z_1, z_2, \cdots, z_k) \in X$ is the solution of inequality (4.2). then, the following inequalities hold:*

$$|z_i(\tau) - \theta_i(\tau)| \leq \Big(\frac{2\Phi_i(\tau)}{\Gamma(\alpha+3)} + \frac{2\Phi_j(1)}{\Gamma(\alpha+1)}\Big)\varepsilon_i + \frac{2}{\Gamma(\alpha+1)} \sum_{j=1, j\neq i}^{k} \varepsilon_j \Phi_j(1),\ i = 1, 2, \cdots, k$$

and

$$\begin{aligned}
\left|{}^{C}\mathfrak{D}^{\beta}_{0,\tau} z_i(\tau) - {}^{C}\mathfrak{D}^{\beta}_{0,\tau} \theta_i(\tau)\right| \leq & \Big(\frac{\Phi_i(\tau)}{\Gamma(\alpha-\beta+3)} + \frac{4\Phi_j(1)}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)\varepsilon_i \\
& + \frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)} \sum_{j=1, j\neq i}^{k} \Phi_j \varphi_j(1),\ i = 1, 2, \cdots, k.
\end{aligned}$$

Proof. The proof can be obtained using a similar analysis as in Lemma 4.1 and the fact that $\Phi_i(\tau)$ $i = 1, 2, \cdots, k$ are nondecreasing functions. Therefore, the proof is omitted. □

Theorem 4.1. *Suppose that* (H_1), (H_3), *and (3.9) hold. If* $|\lambda| < 1$, *for every* $\lambda \in \mathbb{C}$ *with* $\det(\lambda I - \mathcal{A}) = 0$, *then system (2.1) is Ulam-Hyers stable, where*

$$\mathcal{A} = \begin{pmatrix} Q_1\sigma_1 + \pi_1 & Q_2\sigma_2 + \pi_2 & \cdots & Q_2\sigma_k + \pi_k \\ Q_2\sigma_1 + \pi_1 & Q_1\sigma_2 + \pi_2 & \cdots & Q_2\sigma_k + \pi_k \\ \vdots & \vdots & \ddots & \vdots \\ Q_2\sigma_1 + \pi_1 & Q_2\sigma_2 + \pi_2 & \cdots & Q_1\sigma_k + \pi_k \end{pmatrix}$$

and

$$\sigma_i = (\rho_i^{\alpha+2} + \rho_i^{\alpha-\beta+2})W_i,\ \pi_i = \Big(3\lambda_i\rho_i^2 + \frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\Big).$$

Proof. Let $z = (z_1, z_2, \cdots, z_k) \in X$ be the solution of inequality (4.1) and $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_k) \in X$ be the solution of the following BVP:

$$\begin{cases} {}^C\mathfrak{D}^{\alpha}_{0,\tau}(\mathcal{D}^2 + \lambda_i\rho_i^2)\bar{z}_i(\tau) = \rho_i^{\alpha+2} f_i(\tau, \bar{z}_i(\tau), \rho_i^{-\beta C}\mathfrak{D}^{\beta}_{0,\tau}\bar{z}_i(\tau)),\ i = 1, 2, \cdots k, \\ \bar{z}_i'(0) = \bar{z}_i(1) = 0,\ \sum_{i=1}^k \rho_i^{-2} z_i''(1) = 0,\ i = 1, 2, \cdots k, \\ \bar{z}_i''(1) = \bar{z}_j''(1),\ i, j = 1, 2, \cdots k,\ i \neq j. \end{cases} \tag{4.6}$$

According to Lemma 2.6 and Theorem 3.1, the solution to Eq (4.6) can be expressed as

$$\begin{aligned} \bar{z}_i(\tau) =& \frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+2)} \int_0^\tau (\tau - s)^{\alpha+1} f_i(s, \bar{z}_i(s), \rho_i^{-\beta C}\mathfrak{D}^{\beta}_{0,s}\bar{z}_i(s))ds \\ &- \frac{\rho_i^{\alpha+2}}{\Gamma(\alpha+2)} \int_0^1 (1 - s)^{\alpha+1} f_i(s, \bar{z}_i(s), \rho_i^{-\beta C}\mathfrak{D}^{\beta}_{0,s}\bar{z}_i(s))ds \\ &+ \lambda_i\rho_i^2 \int_0^1 (1 - s)\bar{z}_i(s)ds - \lambda_i\rho_i^2 \int_0^t (\tau - s)\bar{z}_i(s)ds \\ &+ (1 - \tau^2) \sum_{j=1}^k \ell_j\Big(\frac{\rho_j^{\alpha+2}}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f_j(s, \bar{z}_j(s), \rho_j^{-\beta C}\mathfrak{D}^{\beta}_{0,s}\bar{z}_j(s))ds - \lambda_j\rho_j^2\bar{z}_j(1)\Big) \\ &+ (1 - \tau^2) \sum_{j=1, j\neq i}^k \ell_j(\lambda_j\rho_j^2\bar{z}_j(1) - \lambda_i\rho_i^2\bar{z}_i(1)) \\ &+ (1 - \tau^2) \sum_{j=1, j\neq i}^k \frac{\ell_j\rho_i^{\alpha+2}}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f_i(s, \bar{z}_i(s), \rho_i^{-\beta C} D^{\beta}_{0,s}\bar{z}_i(s))ds \\ &- (1 - \tau^2) \sum_{j=1, j\neq i}^k \frac{\ell_j\rho_j^{\alpha+2}}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f_j(s, \bar{z}_j(s), \rho_j^{-\beta C} D^{\beta}_{0,s}\bar{z}_j(s))ds,\ i = 1, 2, \cdots k. \end{aligned}$$

Now, by Lemma 4.1, for $\tau \in [0, 1]$, we have

$$\begin{aligned} |z_i(\tau) - \bar{z}_i(\tau)| \leq& |z_i(\tau) - \theta_i(\tau)| + |\theta_i(\tau) - \bar{z}_i(\tau)| \\ \leq& \Big(\frac{2}{\Gamma(\alpha+3)} + \frac{2}{\Gamma(\alpha+1)}\Big)\varepsilon_i + \frac{2}{\Gamma(\alpha+1)} \sum_{j=1, j\neq i}^k \varepsilon_j \end{aligned}$$

$$+\Big(\frac{2}{\Gamma(\alpha+3)}+\frac{2}{\Gamma(\alpha+1)}\Big)(\rho_i^{\alpha+2}+\rho_i^{\alpha-\beta+2})W_i\|z_i-\bar{z}_i\|_X$$
$$+3\lambda_i\rho_i^2\|z_i-\bar{z}_i\|_X+2\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\big\|z_j-\bar{z}_j\big\|_X$$
$$+\frac{2}{\Gamma(\alpha+1)}\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2}+\rho_j^{\alpha-\beta+2})W_j\big\|z_i-\bar{z}_j\big\|_X \tag{4.7}$$

and

$$\Big|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i(\tau)-{}^C\mathfrak{D}_{0,\tau}^{\beta}\bar{z}_i(\tau)\Big|\le\Big|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i(\tau)-{}^C\mathfrak{D}_{0,\tau}^{\beta}\theta_i(\tau)\Big|+\Big|{}^C\mathfrak{D}_{0,\tau}^{\beta}\theta_i(\tau)-{}^C\mathfrak{D}_{0,t\tau}^{\beta}\bar{z}_i(\tau)\Big|$$
$$\le\Big(\frac{1}{\Gamma(\alpha-\beta+3)}+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)\varepsilon_i+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\varepsilon_j$$
$$+\Big(\frac{1}{\Gamma(\alpha-\beta+3)}+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)(\rho_i^{\alpha+2}+\rho_i^{\alpha-\beta+2})W_i\|z_i-\bar{z}_i\|_X$$
$$+\frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\|z_i-\bar{z}_i\|_X+\frac{4}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\big\|z_j-\bar{z}_j\big\|_X$$
$$+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2}+\rho_j^{\alpha-\beta+2})W_j\big\|z_j-\bar{z}_j\big\|_X. \tag{4.8}$$

Therefore, from (4.7) and (4.8), it follows that

$$\|z_i(\tau)-\bar{z}_i(\tau)\|_X=\big\|z_i(\tau)-\bar{z}_i(\tau)\big\|+\big\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i(\tau)-{}^C\mathfrak{D}_{0,\tau}^{\beta}\bar{z}_i(\tau)\big\|$$
$$\le Q_1\varepsilon_i+Q_2\sum_{j=1,j\neq i}^{k}\varepsilon_j+Q_1(\rho_i^{\alpha+2}+\rho_i^{\alpha-\beta+2})W_i\|z_i-\bar{z}_i\|_X+\Big(3\lambda_i\rho_i^2+\frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\Big)\|z_i-\bar{z}_i\|_X$$
$$+\sum_{j=1,j\neq i}^{k}\Big(2\lambda_j\rho_j^2+\frac{4\lambda_j\rho_j^2}{\Gamma(3-\beta)}\Big)\big\|z_j-\bar{z}_j\big\|_X+Q_2\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2}+\rho_j^{\alpha-\beta+2})W_j\big\|z_j-\bar{z}_j\big\|_X$$
$$\le Q_1\varepsilon_i+Q_2\sum_{j=1,j\neq i}^{k}\varepsilon_j+Q_1(\rho_i^{\alpha+2}+\rho_i^{\alpha-\beta+2})W_i\|z_i-\bar{z}_i\|_X+\Big(3\lambda_i\rho_i^2+\frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\Big)\|z_i-\bar{z}_i\|_X$$
$$+\sum_{j=1,j\neq i}^{k}\Big(3\lambda_j\rho_j^2+\frac{5\lambda_j\rho_j^2}{\Gamma(3-\beta)}\Big)\big\|z_j-\bar{z}_j\big\|_X+Q_2\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2}+\rho_j^{\alpha-\beta+2})W_j\big\|z_j-\bar{z}_j\big\|_X$$
$$\le Q_1\varepsilon_i+Q_2\sum_{j=1,j\neq i}^{k}\varepsilon_j+Q_1\sigma_i\|z_i-\bar{z}_i\|_X+\pi_i\|z_i-\bar{z}_i\|_X+\sum_{j=1,j\neq i}^{k}\pi_j\big\|z_j-\bar{z}_j\big\|_X$$
$$+Q_2\sum_{j=1,j\neq i}^{k}\sigma_j\big\|z_j-\bar{z}_j\big\|_X. \tag{4.9}$$

Meanwhile, inequality (4.9) can also have the form

$$(\|z_1-\bar{z}_1\|_X,\|z_2-\bar{z}_2\|_X,\cdots,\|z_k-\bar{z}_k\|_X)^T$$

$$\leq \mathcal{B}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k)^T + \mathcal{A}(\|z_1 - \bar{z}_1\|_X, \cdots, \|z_k - \bar{z}_k\|_X)^T,$$

where

$$\mathcal{B} = (b_{ij})_{k\times k},\ b_{ij} = \begin{cases} Q_1,\ i = j, \\ Q_2,\ i \neq j. \end{cases}$$

Using matrix $\mathcal{A}$ and Theorem 2.1, we get

$$\begin{aligned} &(\|z_1 - \bar{z}_1\|_X, \|z_2 - \bar{z}_2\|_X, \cdots, \|z_k - \bar{z}_k\|_X)^T \\ &\leq (I - \mathcal{A})^{-1}\mathcal{B}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k)^T, \end{aligned} \tag{4.10}$$

set

$$C = (I - \mathcal{A})^{-1}\mathcal{B} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{21} & \cdots & c_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kk} \end{pmatrix}.$$

Obviously, $c_{ij} \geq 0$. Choose $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k\}$. Then it follows from (4.10) that

$$\|z - \bar{z}\|_X \leq \Big(\sum_{i=1}^{k}\sum_{i=1}^{k} c_{ij}\Big)\varepsilon = \Xi_\varepsilon. \tag{4.11}$$

Thus, system (2.1) is Ulam-Hyers stable. □

Remark 4.2. *Take $\Upsilon_{f_1,f_2,\cdots,f_k}(\varepsilon) = \Xi_\varepsilon$ in (4.11). Obviously, we have $\Upsilon_{f_1,f_2,\cdots,f_k}(0) = 0$. Then, using Definition 4.2, we conclude that system (2.1) is generalized Ulam-Hyers stable.*

Theorem 4.2. *Suppose that (H_1), (H_3), and (3.9) hold. Let $\Phi_i(\tau) \in C([0,1], \mathbb{R}^+)$ $(i = 1, 2, \cdots, k)$ be nondecreasing. If $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(\lambda I - \mathcal{A}) = 0$, where $\mathcal{A}$ is defined as before and the function Φ is defined by*

$$\Phi = \Phi(\Phi_1, \Phi_2, \cdots, \Phi_k) \in C([0,1], \mathbb{R}^+), \Phi(\tau) = \max\{g_i(\tau), i = 1, 2, \cdots, k\}$$

and

$$g_i(\tau) = \Big(\frac{\Phi_i(\tau)}{\Gamma(\alpha - \beta + 3)} + \frac{4\Phi_j(1)}{\Gamma(\alpha + 1)\Gamma(3 - \beta)} + \frac{2\Phi_i(\tau)}{\Gamma(\alpha + 3)} + \frac{2\Phi_j(\tau)}{\Gamma(\alpha + 1)}\Big),$$

then system (2.1) is Ulam-Hyers-Rassias stable with respect to Φ.

Proof. Assume that $z = (z_1, z_2, \cdots, z_k) \in X$ is a solution of inequality (4.2). Also, let $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_k) \in X$ be the unique solution of system (2.1). Then, combining Lemma 4.2 with an analysis similar to that used to prove Theorem 4.1, we obtain

$$\begin{aligned} &|z_i(\tau) - \bar{z}_i(\tau)| \leq |z_i(\tau) - \theta_i(\tau)| + |\theta_i(\tau) - \bar{z}_i(\tau)| \\ \leq&\Big(\frac{2\Phi_i(\tau)}{\Gamma(\alpha + 3)} + \frac{2\Phi_j(1)}{\Gamma(\alpha + 1)}\Big)\varepsilon_i + \frac{2}{\Gamma(\alpha + 1)}\sum_{j=1, j\neq i}^{k} \varepsilon_j\Phi_j(1) \end{aligned}$$

$$+\Big(\frac{2}{\Gamma(\alpha+3)}+\frac{2}{\Gamma(\alpha+1)}\Big)(\rho_i^{\alpha+2}+\rho_i^{\alpha-\beta+2})W_i\|z_i-\bar{z}_i\|_X$$
$$+3\lambda_i\rho_i^2\|z_i-\bar{z}_i\|_X+2\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\big\|z_j-\bar{z}_j\big\|_X$$
$$+\frac{2}{\Gamma(\alpha+1)}\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2}+\rho_j^{\alpha-\beta+2})W_j\big\|z_i-\bar{z}_j\big\|_X,\ i=1,2,\cdots,k \tag{4.12}$$

and

$$|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i(\tau)-{}^C\mathfrak{D}_{0,\tau}^{\beta}\bar{z}_i(\tau)|\leq|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i(\tau)-{}^C\mathfrak{D}_{0,\tau}^{\beta}\theta_i(\tau)|+|{}^C\mathfrak{D}_{0,\tau}^{\beta}\theta_i(\tau)-{}^C\mathfrak{D}_{0,\tau}^{\beta}\bar{z}_i(\tau)|$$
$$\leq\Big(\frac{\Phi_i(\tau)}{\Gamma(\alpha-\beta+3)}+\frac{4\Phi_j(1)}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)\varepsilon_i+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\varepsilon_j\Phi_j(1)$$
$$+\Big(\frac{1}{\Gamma(\alpha-\beta+3)}+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)(\rho_i^{\alpha+2}+\rho_i^{\alpha-\beta+2})W_i\|z_i-\bar{z}_i\|_X$$
$$+\frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\|z_i-\bar{z}_i\|_X+\frac{4}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\big\|z_j-\bar{z}_j\big\|_X$$
$$+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2}+\rho_j^{\alpha-\beta+2})W_j\big\|z_j-\bar{z}_j\big\|_X,\ i=1,2,\cdots,k. \tag{4.13}$$

Hence, from (4.12) and (4.13), we get

$$\|z_i(\tau)-\bar{z}_i(\tau)\|$$
$$\leq\Big(\frac{2\Phi_i(\tau)}{\Gamma(\alpha+3)}+\frac{2\Phi_j(1)}{\Gamma(\alpha+1)}\Big)\varepsilon_i+\frac{2}{\Gamma(\alpha+1)}\sum_{j=1,j\neq i}^{k}\varepsilon_j\Phi_j(1)$$
$$+\Big(\frac{2}{\Gamma(\alpha+3)}+\frac{2}{\Gamma(\alpha+1)}\Big)(\rho_i^{\alpha+2}+\rho_i^{\alpha-\beta+2})W_i\|z_i-\bar{z}_i\|_X$$
$$+3\lambda_i\rho_i^2\|z_i-\bar{z}_i\|_X+2\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\big\|z_j-\bar{z}_j\big\|_X$$
$$+\frac{2}{\Gamma(\alpha+1)}\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2}+\rho_j^{\alpha-\beta+2})W_j\big\|z_i-\bar{z}_j\big\|_X,\ i=1,2,\cdots,k. \tag{4.14}$$

Similarly, one can obtain

$$\big\|{}^C\mathfrak{D}_{0,\tau}^{\beta}z_i(\tau)-{}^C\mathfrak{D}_{0,\tau}^{\beta}\bar{z}_i(\tau)\big\|$$
$$\leq\Big(\frac{\Phi_i(\tau)}{\Gamma(\alpha-\beta+3)}+\frac{4\Phi_j(1)}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)\varepsilon_i+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\varepsilon_j\Phi_j(1)$$
$$+\Big(\frac{1}{\Gamma(\alpha-\beta+3)}+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\Big)(\rho_i^{\alpha+2}+\rho_i^{\alpha-\beta+2})W_i\|z_i-\bar{z}_i\|_X$$

$$+\frac{5\lambda_i\rho_i^2}{\Gamma(3-\beta)}\|z_i-\bar{z}_i\|_X+\frac{4}{\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}\lambda_j\rho_j^2\left\|z_j-\bar{z}_j\right\|_X$$
$$+\frac{4}{\Gamma(\alpha+1)\Gamma(3-\beta)}\sum_{j=1,j\neq i}^{k}(\rho_j^{\alpha+2}+\rho_j^{\alpha-\beta+2})W_j\left\|z_j-\bar{z}_j\right\|_X,\ i=1,2,\cdots,k. \tag{4.15}$$

From (4.14) and (4.15), we find that

$$\|z_i(\tau)-\bar{z}_i(\tau)\|_X=\|z_i(\tau)-\bar{z}_i(\tau)\|+\left\|{}^CD_{0,\tau}^{\beta}z_i(\tau)-{}^CD_{0,\tau}^{\beta}\bar{z}_i(\tau)\right\|$$
$$\leq g_i(\tau)\varepsilon_i+Q_2\sum_{j=1,j\neq i}^{k}\varphi_j(1)\varepsilon_j+Q_1\sigma_i\|z_i-\bar{z}_i\|_X+\pi_i\|z_i-\bar{z}_i\|_X$$
$$+\sum_{j=1,j\neq i}^{k}\pi_j\left\|z_j-\bar{z}_j\right\|_X+Q_2\sum_{j=1,j\neq i}^{k}\sigma_j\left\|z_j-\bar{z}_j\right\|_X. \tag{4.16}$$

We rewrite (4.16) as:

$$(\|z_1-\bar{z}_1\|_X,\|z_2-\bar{z}_2\|_X,\cdots,\|z_k-\bar{z}_k\|_X)^T$$
$$\leq\mathcal{B}(\tau)(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k)^T+\mathcal{A}(\|z_1-\bar{z}_1\|_X,\|z_2-\bar{z}_2\|_X,\cdots,\|z_k-\bar{z}_k\|_X)^T,$$

where

$$\mathcal{B}=(b_{ij})_{k\times k},\ b_{ij}=\begin{cases}g_i(\tau),\ i=j,\\ Q_2,\ i\neq j.\end{cases}$$

Using matrix $\mathcal{A}$ and Theorem 2.1, we get

$$(\|z_1-\bar{z}_1\|_X,\|z_2-\bar{z}_2\|_X,\cdots,\|z_k-\bar{z}_k\|_X)^T\leq(I-\mathcal{A})^{-1}\mathcal{B}(\tau)(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k)^T, \tag{4.17}$$

and further, we define

$$(I-\mathcal{A})^{-1}=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1k}\\a_{21}&a_{22}&\cdots&a_{2k}\\\vdots&\vdots&\ddots&\vdots\\a_{k1}&a_{k2}&\cdots&a_{kk},\end{pmatrix},C(\tau)=\begin{pmatrix}c_{11}(\tau)&c_{12}(\tau)&\cdots&c_{1k}(\tau)\\c_{21}(\tau)&c_{22}(\tau)&\cdots&c_{2k}(\tau)\\\vdots&\vdots&\ddots&\vdots\\c_{k1}(\tau)&c_{k2}(\tau)&\cdots&c_{kk}(\tau)\end{pmatrix}.$$

It is easy to verify that

$$a_{ij}\geq 0,\ c_{ij}(\tau)\geq 0,\ i,j=1,2,\cdots,k$$

and

$$c_{ij}(\tau)=a_{ij}g_j(\tau)+Q_2\Phi_j(1)\sum_{r=1,r\neq j}^{k}a_{ir}\leq\Big(a_{ij}+Q_2\frac{\Phi_j(1)}{g_j(0)}\sum_{r=1,r\neq j}^{k}a_{ir}\Big)g_j(\tau),\ i,j=1,2,\cdots,k.$$

Setting $\varepsilon=\max\{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k\}$, we have

$$\|z-\bar{z}\|_X\leq\sum_{j=1}^{k}\sum_{i=1}^{k}\Big(a_{ij}+Q_2\frac{\Phi_j(1)}{g_j(0)}\sum_{r=1,r\neq j}^{k}a_{ir}\Big)\Phi(\tau)\varepsilon. \tag{4.18}$$

Let

$$c_{f_1,f_2,\cdots,f_k,\Phi} = \sum_{j=1}^{k}\sum_{i=1}^{k}\Big(a_{ij} + Q_2\Phi_j(1)g^{-1}{}_j(0)\sum_{r=1,r\neq j}^{k} a_{ir}\Big).$$

By Definition 4.3, system (2.1) is Ulam-Hyers-Rassias stable with respect to Φ. □

Remark 4.3. *Taking $\varepsilon = 1$ in (4.18), we conclude using Definition 4.4 that system (2.1) is generalized Ulam-Hyers-Rassias stable with respect to Φ.*

5. Example

Example 5.1. Consider the BVP (1.4) with $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $k = 3$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{4}$, $\lambda_3 = \frac{1}{8}$, $\rho_1 = \rho_2 = \frac{1}{10}$, $\rho_3 = \frac{1}{5}$, and

$$\begin{cases} f_1(t,u,v) = \frac{t}{5} + \frac{u}{4(t+3)^2} + \frac{5}{3\sqrt{2}(t+2)}v, \ (t,u,v) \in [0,\rho_1]\times\mathbb{R}\times\mathbb{R}, \\ f_2(t,u,v) = \frac{\cos x}{3} + \frac{u}{2(t+4)^3} + \frac{5t}{6\sqrt{2}}v, \ (t,u,v) \in [0,\rho_2]\times\mathbb{R}\times\mathbb{R}, \\ f_3(t,u,v) = 2t^2 + \frac{u}{2(t+5)^2} + \frac{v}{6(t+2)}, \ (t,u,v) \in [0,\rho_3]\times\mathbb{R}\times\mathbb{R}. \end{cases}$$

Using Lemma 2.5, we obtain the equivalent system

$$\begin{cases} {}^C\mathfrak{D}_{0,\tau}^{1/2}(\mathcal{D}^2 + \frac{1}{200})z_1(\tau) = (\frac{1}{2})^{5/2}\Big(\frac{\tau}{5} + \frac{z_1(\tau)}{4(\tau+3)^2} + (\frac{1}{2})^{-1/3}\frac{5}{3\sqrt{2}(\tau+2)}{}^C\mathfrak{D}_{0,\tau}^{1/3}z_1(\tau)\Big), \\ {}^C\mathfrak{D}_{0,\tau}^{1/2}(\mathcal{D}^2 + \frac{1}{400})z_2(\tau) = (\frac{1}{4})^{5/2}\Big(\frac{\cos\tau}{3} + \frac{z_2(\tau)}{2(\tau+4)^3} + (\frac{1}{4})^{-1/3}\frac{5}{6\sqrt{2}}{}^C\mathfrak{D}_{0,\tau}^{1/3}z_2(\tau)\Big), \\ {}^C\mathfrak{D}_{0,\tau}^{1/2}(\mathcal{D}^2 + \frac{1}{200})z_3(\tau) = (\frac{1}{8})^{5/2}\Big(2\tau^2 + \frac{z_3(\tau)}{2(\tau+5)^2} + (\frac{1}{8})^{-1/3}\frac{1}{6(\tau+2)}{}^C D_{0,\tau}^{1/3}z_3(\tau)\Big), \\ z_1'(0) = z_2'(0) = z_3'(0) = z_1(0) = z_2(0) = z_3(0) = 0, \\ z_1''(1) = z_2''(1) = z_3''(1), \ 9z_1''(1) + 16z_2''(1) + 25z_3''(1) = 0, \end{cases} \tag{5.1}$$

then

$$\xi_1(\tau) = \frac{\tau}{5}, \ \xi_2(\tau) = \frac{\cos\tau}{3}, \ \xi_3(\tau) = 2\tau^2,$$
$$\eta_1(\tau) = \frac{1}{4(\tau+3)^2}, \ \eta_2(\tau) = \frac{1}{2(\tau+4)^3}, \ \eta_3(\tau) = \frac{1}{2(\tau+5)^2},$$
$$\psi_1(\tau) = (\frac{1}{2})^{-1/3}\frac{5}{3\sqrt{2}(\tau+2)}, \ \psi_2(\tau) = (\frac{1}{4})^{-1/3}\frac{5}{6\sqrt{2}}, \ \ \psi_3(\tau) = (\frac{1}{8})^{-1/3}\frac{1}{6(\tau+2)}.$$

For $\tau \in [0,1]$, we have $\xi_1^* = \frac{1}{5}$, $\xi_2^* = \frac{1}{3}$, $\xi_3^* = 2$, $\eta_1^* = \frac{1}{36}$, $\eta_2^* = \frac{1}{128}$, $\eta_3^* = \frac{1}{50}$, $\psi_1^* = \psi_2^* = \frac{5}{3}$, $\psi_3^* = \frac{1}{4}$. By calculation, we get

$$L_1 = 0.06674, \ \Delta_1 = 0.022, \ \tilde{\Delta}_1 = 0.01664, \ \Omega_1 = 0.0317, \ \tilde{\Omega}_1 = 0.01168, \ N_1 = 0.1982,$$
$$L_2 = 0.06257, \ \Delta_2 = 0.022, \ \tilde{\Delta}_2 = 0.01664, \ \Omega_2 = 0.01586, \ \tilde{\Omega}_2 = 0.01168, \ N_2 = 0.1988,$$
$$L_3 = 0.07842, \ \Delta_3 = 0.1242, \ \tilde{\Delta}_3 = 0.09410, \ \Omega_3 = 0.0317, \ \tilde{\Omega}_3 = 0.0234, \ N_3 = 0.2573,$$

so, we have

$$\theta_1 = \Delta_1(\eta_1^* + \rho_1^{-\beta}\psi_1^*) + \Omega_1 + \tilde{\Delta}_2(\eta_2^* + \rho_2^{-\beta}\psi_2^*) + \tilde{\Omega}_2 + \tilde{\Delta}_3(\eta_3^* + \rho_3^{-\beta}\psi_3^*) + \tilde{\Omega}_3 = 0.2487,$$

$$\theta_2 = \Delta_2(\eta_2^* + \rho_2^{-\beta}\psi_2^*) + \Omega_2 + \tilde{\Delta}_1(\eta_1^* + \rho_1^{-\beta}\psi_1^*) + \tilde{\Omega}_1 + \tilde{\Delta}_3(\eta_3^* + \rho_3^{-\beta}\psi_3^*) + \tilde{\Omega}_3 = 0.2445,$$
$$\theta_3 = \Delta_3(\eta_3^* + \rho_3^{-\beta}\psi_3^*) + \Omega_3 + \tilde{\Delta}_1(\eta_1^* + \rho_1^{-\beta}\psi_1^*) + \tilde{\Omega}_1 + \tilde{\Delta}_2(\eta_2^* + \rho_2^{-\beta}\psi_2^*) + \tilde{\Omega}_2 = 0.2448,$$

and

$$\sum_{i=1}^{3}\theta_i = 0.736 < 1, \ \sum_{i=1}^{3} N_i = 0.6543.$$

Thus,

$$r > \frac{\sum_{i=1}^{3} N_i}{1 - \sum_{i=1}^{3}\theta_i} = \frac{0.6543}{1 - 0.736} \doteq 2.479.$$

According to Theorem 3.1, BVP (5.1) has at least one solution on [0,1].

Example 5.2. Consider the BVP (1.4) with $\alpha = \frac{1}{2},\ \beta = \frac{1}{3},\ k = 3,\ \lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{10},\ \rho_1 = \rho_2 = \rho_3 = \frac{1}{5}$, and

$$\begin{cases} f_1(t,u,v) = \cos t + \frac{1}{3(\sqrt{t}+3)^2}(\sin u + v),\ (t,u,v) \in [0,\rho_1] \times \mathbb{R} \times \mathbb{R}, \\ f_2(t,u,v) = \frac{1}{t} + \frac{\sqrt{\pi}}{(t+27)}(|u| + |v|),\ (t,u,v) \in [0,\rho_2] \times \mathbb{R} \times \mathbb{R}, \\ f_3(t,u,v) = 1 + 2t^2 + \frac{2\sqrt{2}}{9(t+3)}(\frac{u}{1+u} + v),\ (t,u,v) \in [0,\rho_3] \times \mathbb{R} \times \mathbb{R}. \end{cases}$$

Using Lemma 2.5, we obtain the equivalent system

$$\begin{cases} {}^{C}\mathfrak{D}_{0,\tau}^{1/2}(\mathcal{D}^2 + \frac{1}{250})z_1(\tau) = (\frac{1}{5})^{5/2}\Big[\cos\tau + \frac{1}{3(\sqrt{\tau}+3)^2}(\sin z_1(\tau) + (\frac{1}{5})^{-1/3}{}^{C}\mathfrak{D}_{0,\tau}^{1/3}z_1(\tau))\Big], \\ {}^{C}\mathfrak{D}_{0,\tau}^{1/2}(\mathcal{D}^2 + \frac{1}{250})z_2(\tau) = (\frac{1}{5})^{5/2}\Big[\frac{1}{\tau} + \frac{\sqrt{\pi}}{(\tau+27)}(|z_2(\tau)| + (\frac{1}{5})^{-1/3}\big|{}^{C}\mathfrak{D}_{0,\tau}^{1/3}z_2(\tau)\big|)\Big], \\ {}^{C}\mathfrak{D}_{0,\tau}^{1/2}(\mathcal{D}^2 + \frac{1}{250})z_3(\tau) = (\frac{1}{5})^{5/2}\Big[1 + 2\tau^2 + \frac{2\sqrt{2}}{9(\tau+3)}(\frac{\tau}{1+\tau} + (\frac{1}{5})^{-1/3}{}^{C}\mathfrak{D}_{0,\tau}^{1/3}z_3(\tau))\Big], \\ z_1'(0) = z_2'(0) = z_3'(0) = z_1(0) = z_2(0) = z_3(0) = 0, \\ z_1''(1) = z_2''(1) = z_3''(1),\ 9z_1''(1) + 16z_2''(1) + 25z_3''(1) = 0, \end{cases} \tag{5.2}$$

and for $\tau \in [0,1]$, $u, v, u_1, v_1 \in \mathbb{R}$, we can conclude that

$$\big|f_1(\tau,u,v) - f_1(\tau,u_1,v_1)\big| \le \frac{1}{3(\sqrt{\tau}+3)^2}(\big|u - v\big| + \big|u_1 - v_1\big|),$$
$$\big|f_2(\tau,u,v) - f_2(\tau,u_1,v_1)\big| \le \frac{\sqrt{\pi}}{(\tau+27)}(\big|u - v\big| + \big|u_1 - v_1\big|),$$
$$\big|f_3(\tau,u,v) - f_3(\tau,u_1,v_1)\big| \le \frac{2\sqrt{2}}{9(\tau+3)}(\big|u - v\big| + \big|u_1 - v_1\big|),$$

and so, we get

$$w_1(\tau) = \frac{1}{3(\sqrt{\tau}+3)^2},\ w_2(\tau) = \frac{\sqrt{\pi}}{(\tau+27)},\ w_3(\tau) = \frac{2\sqrt{2}}{9(\tau+3)}.$$

By simple calculation, we obtain

$$a = Q_1\sigma_i + \pi_i = 0.1652,\ b = Q_2\sigma_i + \pi_i = 0.1316,\ i = 1,2,3,$$

$$W_1 = W_2 = W_3 = \max_{\tau\in[0,1]} |w_1(\tau)| = \frac{1}{27},\ \sigma_1 = \sigma_2 = \sigma_3 = 0.0202,$$
$$K_1 = K_2 = K_3 \doteq 0.8467,\ L_1 = L_2 = L_3 \doteq 0.04396,\ \pi_1 = \pi_2 = \pi_3 = 0.02531.$$

Then

$$\Big(\sum_{i=1}^{k} K_i\Big)\Big(\sum_{i=1}^{k} W_i\Big) + \sum_{i=1}^{k} L_i \doteq 0.4450 < 1.$$

Since all conditions of Theorem 3.2 have been satisfied, therefore, the system (5.2) has a unique solution in [0,1]. Using the given value, we also have

$$\mathcal{A} = \begin{pmatrix} 0.1652 & 0.1316 & 0.1316 \\ 0.1316 & 0.1652 & 0.1316 \\ 0.1316 & 0.1316 & 0.1652 \end{pmatrix}.$$

Let

$$0 = \det(\lambda E - \mathcal{A}) = (\lambda - 0.4288)(\lambda - 0.034)^2. \tag{5.3}$$

Solving Eq (5.3) gives

$$\lambda_1 = 0.4288 < 1,\ \lambda_2 = \lambda_3 = 0.034 < 1.$$

From Theorem 4.1 and Remark 4.2, it can be seen that the system (5.2) is Ulam-Hyers stable and generalized Ulam-Hyers stable. Similarly, we obtain that system (5.2) is Ulam-Hyers-Rassia stable and generalized Ulam-Hyers-Rassia stable.

6. Conclusions

This article discussed a class of nonlinear Caputo type higher-order fractional Langevin equations on a star graph. By utilizing Lemmas 2.4 and 2.5, BVP (1.4) was transformed into system (2.1) defined on the interval [0,1]. The existence and uniqueness of solutions are proven using fixed point theorems, specifically the Krasnoselskii fixed point theorem and the Banach contraction mapping principle. Furthermore, the Ulam-Hyers stability, Ulam-Hyers-Rassias stability, and their generalized forms are explored based on Definitions 4.1–4.4, which may provide researchers with a new approach to analyzing the Ulam stability of higher-order fractional differential equations. The results presented in this article are new and extend some existing literature on this topic (see prior references [10, 11, 15, 17, 23]). Finally, two examples demonstrated the application of the main results. One promising avenue for future research is to explore fractional differential equations on star graphs, including the fractional Sturm-Liouville equation, the fractional Langevin equation with the p-Laplacian operator, and fractional integral-differential equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier Science Ltd., 2006.
2. S. Alizadeh, D. Baleanu, S. Rezapour, Analyzing transient response of the parallel RCL circuit by using the Caputo-Fabrizio fractional derivative, *Adv. Differ. Equ.*, **2020** (2020), 55. https://doi.org/10.1186/s13662-020-2527-0
3. S. Rezapour, S. Etemad, H. Mohammadi, A mathematical analysis of a system of Caputo-Fabrizio fractional differential equations for the anthrax disease model in animals, *Adv. Differ. Equ.*, **2020** (2020), 481. https://doi.org/10.1186/s13662-020-02937-x
4. C. Li, R. Wu, R. Ma, Existence of solutions for caputo fractional iterative equations under several boundary value conditions, *AIMS Mathematics*, **8** (2023), 317–339. https://doi.org/10.3934/math.2023015
5. B. Ahmad, M. Alghanmi, A. Alsaedi, J. Nieto, Existence and uniqueness results for a nonlinear coupled system involving caputo fractional derivatives with a new kind of coupled boundary conditions, *Appl. Math. Lett.*, **116** (2021), 107018. https://doi.org/10.1016/j.aml.2021.107018
6. J. Ni, J. Zhang, W. Zhang, Existence of solutions for a coupled system of p-Laplacian Caputo-Hadamard fractional Sturm-Liouville-Langevin equations with antiperiodic boundary conditions, *J. Math.*, **2022** (2022), 3346115. https://doi.org/10.1155/2022/3346115
7. A. Salem, B. Alghamdi, Multi-strip and multi-point boundary conditions for fractional Langevin equation, *Fractal Fract.*, **4** (2020), 18. https://doi.org/10.3390/fractalfract4020018
8. Y. Adjabi, M. Samei, M. Matar, J. Alzabut, Langevin differential equation in frame of ordinary and Hadamard fractional derivatives under three point boundary conditions, *AIMS Mathematics*, **6** (2021), 2796–2843. https://doi.org/10.3934/math.2021171
9. G. Lumer, Connecting of local operators and evolution equations on networks, In: *Potential theory Copenhagen 1979*, Berlin: Springer, 2006, 219–234. https://doi.org/10.1007/BFb0086338
10. J. Graef, L. Kong, M. Wang, Existence and uniqueness of solutions for a fractional boundary value problem on a graph, *FCAA*, **17** (2014), 499–510. https://doi.org/10.2478/s13540-014-0182-4
11. V. Mehandiratta, M. Mehra, G. Leugering, Existence and uniqueness results for a nonlinear Caputo fractional boundary value problem on a star graph, *J. Math. Anal. Appl.*, **477** (2019), 1243–1264. https://doi.org/10.1016/j.jmaa.2019.05.011

12. S. Etemad, S. Rezapour, On the existence of solutions for fractional boundary value problems on the ethane graph, *Adv. Differ. Equ.*, **2020** (2020), 276. https://doi.org/10.1186/s13662-020-02736-4
13. G. Mophou, G. Leugering, P. Fotsing, Optimal control of a fractional Sturm-Liouville problem on a star graph, *Optimization*, **70** (2021), 659–687. https://doi.org/10.1080/02331934.2020.1730371
14. A. Turab, Z. Mitrović, A. Savić, Existence of solutions for a class of nonlinear boundary value problems on the hexasilinane graph, *Adv. Differ. Equ.*, **2021** (2021), 494. https://doi.org/10.1186/s13662-021-03653-w
15. X. Han, H. Cai, H. Yang, Existence and uniqueness of solutions for the boundary value problems of nonlinear fractional differential equations on star graph (Chinese), *Acta Math. Sci.*, **42** (2022), 139–156.
16. W. Ali, A. Turab, J. Nieto, On the novel existence results of solutions for a class of fractional boundary value problems on the cyclohexane graph, *J. Inequal. Appl.*, **2022** (2022), 5. https://doi.org/10.1186/s13660-021-02742-4
17. W. Zhang, J. Zhang, J. Ni, Existence and uniqueness results for fractional Langevin equations on a star graph, *Math. Biosci. Eng.*, **19** (2022), 9636–9657. https://doi.org/10.3934/mbe.2022448
18. D. Baleanu, S. Etemad, H. Mohammadi, S. Rezapour, A novel modeling of boundary value problems on the glucose graph, *Commun. Nonlinear Sci.*, **100** (2021), 105844. https://doi.org/10.1016/j.cnsns.2021.105844
19. V. Mehandiratta, M. Mehra, G. Leugering, Existence results and stability analysis for a nonlinear fractional boundary value problem on a circular ring with an attached edge: A study of fractional calculus on metric graph, *Netw. Heterog. Media.*, **16** (2021), 155–185. https://doi.org/10.3934/nhm.2021003
20. H. Khan, Y. Li, W. Chen, D. Baleanu, A. Khan, Existence theorems and Hyers-Ulam stability for a coupled system of fractional differential equations with p-Laplacian operator, *Bound. Value. Probl.*, **2017** (2017), 157. https://doi.org/10.1186/s13661-017-0878-6
21. H. Khan, F. Jarad, T. Abdeljawad, A. Khan, A singular ABC-fractional differential equation with p-Laplacian operator, *Chaos Soliton. Fract.*, **129** (2019), 56–61. https://doi.org/10.1016/j.chaos.2019.08.017
22. A. Devi, A. Kumar, T. Abdeljawad, A. Khan, Stability analysis of solutions and existence theory of fractional Lagevin equation, *Alex. Eng. J.*, **60** (2021), 3641–3647. https://doi.org/10.1016/j.aej.2021.02.011
23. W. Zhang, W. Liu, Existence and Ulam's type stability results for a class of fractional boundary value problems on a star graph, *Math. Method. Appl. Sci.*, **43** (2020), 8568–8594. https://doi.org/10.1002/mma.6516
24. A. Devi, A. Kumar, Hyers-Ulam stability and existence of solution for hybrid fractional differential equation with p-Laplacian operator, *Chaos Soliton. Fract.*, **156** (2022), 111859. https://doi.org/10.1016/j.chaos.2022.111859
25. M. Abbas, Ulam stability and existence results for fractional differential equations with hybrid proportional-Caputo derivatives, *J. Interdiscip. Math.*, **25** (2022), 213–231. https://doi.org/10.1080/09720502.2021.1889156

26. I. Ahmad, K. Shah, G. Ur Rahman, D. Baleanu, Stability analysis for a nonlinear coupled system of fractional hybrid delay differential equations, *Math. Method. Appl. Sci.*, **43** (2020), 8669–8682. https://doi.org/10.1002/mma.6526
27. I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, San Diego: Academic Press, Inc., 1999.
28. C. Urs, Coupled fixed point theorems and applications to periodic boundary value problems, *Miskolc Math. Notes*, **14** (2013), 323–333. https://doi.org/10.18514/MMN.2013.598
29. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer, 2003. https://doi.org/10.1007/978-0-387-21593-8