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Research article

Analytic approximations for European-style Asian spread options

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Abstract: Spread option is a exotic option, which allows investors to simultaneously take positions in two correlated underlying assets and profit from their price difference over some spread. This option provides stable investment opportunities for practitioners in unpredictable and complex financial markets. However, investors of the spread option may face problems caused by manipulating the two underlying assets' prices near the expiry, compared to plain vanilla options. To overcome such disadvantages, we propose Asian-spread options, which are linked to the price difference between two average prices of two underlying assets over the life of the option, and exhibit the original properties of standard spread options. In this paper, using distribution-approximating and moment-matching approaches, lower bounds of prices for the European spread option on the geometric average Asian option and arithmetic average Asian option are obtained in the classical Black-Scholes model. We verified the pricing accuracy of the proposed Asian-spread options by comparing our solutions with those obtained by Monte Carlo simulations. Finally, we analyzed the influence of stock price, maturity date, and some model parameters on option price and delta value through numerical examples. Numerical results showed that the lower bounds had a very high precision.

Keywords: spread options; Asian option; closed-from approximation; distribution-approximating method; moment-matching approach
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1. Introduction

In this paper, we extend the horizon of exotic options by incorporating an activating average condition into the payoff of spread options. As one of the most popular exotic options, spread options are a type of so-called rainbow option whose payoff relies on the price difference (or so-called the spread) between two underlying assets. Spread options are widely traded nowsday both on organized exchanges and over the counter in equity, interest rate, currency, foreign exchange, commodity markets, and energy markets nowadays. For instance, in the energy markets, crack spread options, which either exchange crude oil and unleaded gasoline or exchange crude oil and heating oil, are traded on the New York Mercantile Exchange (NYMEX). They are extensively used to speculate, hedge correlation risks, and even evaluate real assets (see Dempster et al. [23] and Luciano [47]). For a detailed review of different spread option types and their applications, we refer to Carmona and Durrleman [12] and Caldana and Fusai [11].

Despite their popularity, valuation on spread options written on two underlying assets is an especially challenging problem in quantitative finance. One of the difficulties in pricing spread options is that the exercise boundary is non-linear when the spread is not zero. Obviously, when the spread is zero, this spread option reduces to an exchange option, which allows the holders to exchange one asset for another. Margrabe [48] first deduced the European exchange option pricing formula under the bivariate geometric Brownian motion (GBM) paradigm. Bjerskund and Stensland [6] considered the pricing of American exchange options in the GBM setting. Further analysis and extension to power exchange options is given by Blenman and Clark [5]. The underlying asset pricing model mentioned above assumes that the return of the asset is normally distributed and that its variance is a constant. Recently, there have been a lot of researche on pricing the exchange options under the modified Black-Scholes model (see Black and Scholes [3]) by incorporating with various other factors such as a stochastic interest rate (see Liu and Wang [45]), stochastic volatility (e.g., Antonelli et al. [2], Alos and Rheinlander [1], Kim and Park [35], and Pasricha and Goel [52]), credit risk (e.g., Kim and Koo [33], Wang et al. [61], Pasricha and Goel [51], Xu et al. [66], and Wang et al. [60]), skew-Brownian motion (see Pasricha and He [54]), fractional Brownian motion (e.g., Kim et al. [34] and Kim et al. [36]), jump-diffusion and/or stochastic volatility (e.g., Chen and Wan [17], Cheang and Chaiarella [15], Caldana et al. [10], Wang [58], Li et al. [41], Cufaro-Petroni and Sabino [19], Cheang and Garces [16], Pasricha and Goel [53], and Lian et al. [44]), and so on.

However, when the spread is not zero, the exercise boundary is non-linear, and it is difficult to obtain an closed-form solutions for these spread options. Instead, we have to resort to analytical approximations or numerical methods. However, practitioners often prefer to use analytical approximations rather than numerical methods because of their computational ease. The pricing issues of the spread options have been investigated in the literature. For instance, Kirk [37] presented an analytical approximation by approximating the sum of the second asset with the fixed strike by a log normal random variable. His method can be thought of as a linear approximation of the exercise boundary. Later, Lo [46] improved Kirk's approximation with an operator splitting method. Pearson [55], Poitras [57], and later Carmona and Durrleman [12], Deng et al. [24], and Bjerksund and Stensland [7] provided lower and upper bounds for the spread option price using suitable approximations of the corresponding discounted expected payoff in log-normal asset models. Caldana and Fusai [11] obtained new lower and upper bounds for the spread option price by using characteristic function and univariate Fourier inversion based on the work of Bjerksund and Stensland [7]. Venkatrmana and Alexander [64] expressed the closed-form price of the spread option as the sum of the prices of two compound exchange options. Kao and Xie [32] proposed a bivariate generalized Edgeworth expansion for pricing spread options. Amongst numerical methods, approaches based on the discrete fast Fourier transform (FFT, see Carr and Madan [13]) have met with large success. For instance, Dempster and Hong [22] introduced a numerical integration method for spread options based on Fourier transforms when the two assets follow Heston's (see Heston [30]) stochastic volatility model. Based on the work of Dempster and Hong [22], there have been many extended results on spread options with models, such as the exponential Levy model (see Hurd and Zhou [31]), GARCH model (see Wang [59]), and Heston stochastic volatility model with jumps (e.g., Olivares and Cane [50] and Hainaut [28]). Furthermore, some researchers have studied other spread option types such as the basket spread option (e.g., Deelstra et al. [20], Pellegrino and Sabono [56], and Lau and Lo [38]) and spread options with credit risk (see Li and Wang [42]).

As noted above, the payoff at maturity date for the spread option depends on the price at maturity of the two underlying assets alone, which exposes the holder to the risk that the writer may manipulate the two underlying assets' prices such that the payoff of the spread option being benefits according to the writer favorable way near the expiry (see, Deelstra et al. [20]). On the other hand, contracts such as spread options are very common in energy, power, and commodity markets. Especially in energy markets, various forms of average underlying prices are traded, often on the temporal average or multiple underlying assets. Additionally, the average feature can smooth the randomness, or the "noise", inherent in the stock price so that the risk-managers can be evaluated more fundamentally. Therefore, this paper extends the spread options to Asian-spread options using average-rate as the underlying. The payoff of the Asian-spread option depends on the difference between two averages of the underlying asset prices over some predetermined time interval, and has generally the effect of decreasing the variance and offering simpler hedging strategies than regular spread options. Therefore, the price of the spread option combined with the average-rate will be cheaper than that of the regular spread option. That is, this combination would open a wider spectrum of spread payoffs, while making them more accessible to investors or traders at a lower cost. Actually, Asian spread options have recently gained more popularity in the energy market (see, e.g., Carmona and Durrleman [12], Caldana and Fusai [11], Benth and Kruner [4], Deelstra et al. [21], Wang and Zhang [62], and Li et al. [43], and their references therein). The spread part may, for example, be the cost of converting fuel into energy. While the Asian part (the temporal average) avoids the problem common to the European options, namely that speculators can increase gain from the option by manipulating the price of the asset near maturity. The most prominent examples of such contracts is basket spread options, Asian basket spread options, and calendar spread options. These options have been investigated in the options pricing literature. For instance, a multi-asset spread option, such as the basket spread option, resembles the Asian spread option with discrete sampling arithmetic average (see Deelstra et al. [20], Li et al. [40], Pellegrino and Sabino [56], and Lau and Lo [38]) and the moving average exchange options (see, e.g., Han et al. [29]). Choi [18] proposed an efficient and unified method for pricing options such as the basket, spread, and Asian options under multivariate GBM models. A simple, accurate, and efficient method to price and hedge Asian spread options is therefore inevitable.

The average-rate options, or Asian options, are popular and commonly employed in fields like currency, interest rate, energy, and insurance markets, among others. In general, the average considered in option contracts can be a geometric or arithmetic one, and it can be observed continuously or discretely. Practically, most Asian derivatives on the markets are settled on the arithmetic average price. Usually, the geometric average option can be priced by exact analytical formulas, whereas the arithmetic one does not have closed-form solutions. This is because the probability distribution of the average prices of the underlying asset generally does not have a simple analytical expression. Extensive literature investigates the pricing of the Asian options (see, for example, Levy [39], Castellacci and Siclari [14], Fusai and Kyriakou [26], Willems [63], and Malhotra et al. [49], and their references therein). We refer to Boyle and Boyle [9] for a brief introduction to the development of Asian options.

In this paper, we propose a theoretical framework for pricing Asian spread options, and derive analytical approximations, which are often preferred to use by spread option traders for their computational ease and the availability of closed-form formulae for hedge ratios. We extend approximation methods of Levy [39], Bjerksund and Stensland [7], and Lau and Lo [38] to the Asian spread case. The main contribution of the present work is twofold. First, we extend spread option to Asian spread options that help those investors who like to mitigate the adverse movements of two underlying assets and hedge both the risk of financial assets over the periods of time. Second, we obtain the derivation of a lower bound, as in Bjerksund and Stensland [7] and Lau and Lo [38], but for spread options with Asian features. Indeed, the only quantity we need to know explicitly is the joint probability density function of the log-returns of the two averages.

The remainder of this paper is organized as follows: Section 2 describes the market model and the Asian spread options. In Section 3, the closed-form approximate formulas of the spread options with geometric averaging and arithmetic averaging are derived respectively. Numerical examples are presented in Section 4 to show the accuracy and efficiency of the proposed method, while the effects of some parameters on options and their deltas are analyzed. Finally, Section 5 concludes this paper.

2. Model setup

Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, Q)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Moreover, denote by $\mathrm{E}(\cdot)$ the expectation operator with respect to a risk neutral equivalent martingale measure Q. A continuous-time financial market is considered with a finite time horizon $[0, T]$, where $T < \infty$ under the complete probability space $(\mathcal{F}_t)_{t\in[0,T]}$. Assume that there exists two risky assets and the risk-free asset traded continuously in this financial market over a finite time interval $[0, T]$. Let the process of the risk-free asset is governed by

$$\mathrm{d}B_t = rB_t\mathrm{d}t, \quad B_0 = 1,$$

where r is the risk-free interest rate. Under the risk neutral probability Q, the two risky assets whose prices are denoted by S_{1t} and S_{2t} are governed by the following stochastic differential equations

$$\frac{\mathrm{d}S_{1t}}{S_{1t}} = r\mathrm{d}t + \sigma_1\mathrm{d}W_{1t}, \tag{2.1}$$

$$\frac{\mathrm{d}S_{2t}}{S_{2t}} = r\mathrm{d}t + \sigma_2\mathrm{d}W_{2t}, \tag{2.2}$$

where $\sigma_i's \ \ (i = 1, 2)$ are the volatilities of both assets. In addition, W_{1t} and W_{2t} are two standard Brownian motions defined on this filtered probability space. We assume that the correlation coefficient between W_{1t} and W_{2t} is given by ρ. Assume that r, σ_1, σ_2 and ρ are constants, and $(\mathcal{F}_t)_{t\geq 0}$ is produced by the σ-algebra of the price pair $(S_{1t}, S_{2t})_{t\geq 0}$.

Now, we present the payoff of Asian spread options (ASO). The payoff of these options is based on the difference between two average asset prices of two underlying assets for the period up to T. In the following, we introduce the payoff function of the Asian spread options as follows:

$$h(x_1, x_2, K) = (x_1 - x_2 - K)^+, \tag{2.3}$$

where x_i can be either G_{iT} or A_{iT} depending on the geometric average price or arithmetic average price, respectively. The notation $x^+ = \max\{x, 0\}$ and $K \geq 0$ is the strike price of this option. Additionally, G_{it} $(i = 1, 2)$ is the continuously monitored geometric average of S_{iu} $(i = 1, 2)$ over time $[0, t]$, that is,

$$G_{it} = \exp(\frac{1}{t}\int_0^t \ln S_{iu}\mathrm{d}u),$$

and A_{it} $(i = 1, 2)$ is the continuously monitored arithmetic average of S_{iu} over time $[0, t]$, i.e.,

$$A_{it} = \frac{1}{t}\int_0^t S_{iu}\mathrm{d}u.$$

There is no known closed form for this case defined in (2.3) despite the use of the more tractable geometric average.

From (2.1) and (2.2), we have for any $t \in [0, T]$,

$$\ln S_{it} = \ln S_{i0} + (r - \frac{1}{2}\sigma_i^2)t + \sigma_i W_{it}, \quad i = 1, 2,$$

and

$$\begin{aligned}\ln G_{iT} &= \frac{1}{T}\int_0^T \ln S_{it}\mathrm{d}t \\ &= \ln S_{i0} + (r - \frac{1}{2}\sigma_i^2)\frac{T}{2} + \frac{\sigma_i}{T}\int_0^T (T - t)\mathrm{d}W_{it}. \end{aligned} \tag{2.4}$$

Accordingly, for any constant α, one gets that

$$\mathrm{E}(G_{iT}^{\alpha}) = S_{i0}^{\alpha} \exp\{\alpha(r - \frac{1}{2}\sigma_i^2)\frac{T}{2} + \frac{\alpha^2}{6}\sigma_i^2 T\}, \tag{2.5}$$

and from the result of Geman and Yor [27],

$$\mathrm{E}(A_{iT}) = \frac{S_{i0}(e^{rT} - 1)}{rT}, \tag{2.6}$$

$$\mathrm{E}(A_{iT}^2) = \frac{2S_{i0}^2[r\mathrm{e}^{(2r+\sigma_i^2)T} - (2r + \sigma_i^2)\mathrm{e}^{rT} + (r + \sigma_i^2)]}{rT^2(r + \sigma_i^2)(2r + \sigma_i^2)}, \quad i = 1, 2. \tag{2.7}$$

In the following, we investigate the pricing problem for the Asian spread options under the payoff functions in the geometric average and arithmetic average cases described previously.

3. Closed-form approximations for Asian spread options

Without loss of generality, we focus on the option price at the inception $(t = 0)$. Let $GASO(S_1, S_2, K)$ be the price at time $t = 0$ for the ASO whose payoff function is $h(G_{1T}, G_{2T}, K)$, and $AASO(S_1, S_2, K)$ be the price at time $t = 0$ for the ASO whose payoff function is $h(A_{1T}, A_{2T}, K)$. According to the risk-neutral pricing theory, the option prices are thus given by

$$GASO(S_1, S_2, K) = \mathrm{E}\{e^{-rT}(G_{1T} - G_{2T} - K)^+\}, \tag{3.1}$$

$$AASO(S_1, S_2, K) = \mathrm{E}\{e^{-rT}(A_{1T} - A_{2T} - K)^+\}, \tag{3.2}$$

respectively. Obviously, a more difficult problem for pricing the Asian spread options defined in the above work is the unknown joint distribution of two arithmetic average prices (A_{1T}, A_{2T}) and the non-linear exercise boundary with the spread K being not zero. In this paper, we will use approximation approaches to solve this problem. The next result is well-known.

Proposition 1. Assume that random variables X_1 and X_2 satisfy $X_1 \sim N(\mu_1, \delta_1^2)$, $X_2 \sim N(\mu_2, \delta_2^2)$, and $\varrho = Corr(X_1, X_2)$. In addition, $a, b, c, d, e, \mu_1, \mu_2, \delta_1, \delta_2$ and ϱ are assumed to be constants, where at least one of c and d is non-zero. Then,

$$\begin{aligned}\mathrm{E}\Big[\mathrm{e}^{aX_1+bX_2}\mathbf{1}_{(cX_1+dX_2\geq e)}\Big] &= \exp[a\mu_1 + b\mu_2 + \frac{1}{2}(a^2\delta_1^2 + 2\varrho ab\delta_1\delta_2 + b^2\delta_2^2)] \\ &\cdot \mathrm{N}\Big(\frac{c\mu_1 + d\mu_2 - e + ac\delta_1^2 + \varrho(ad+cb)\delta_1\delta_2 + bd\delta_2^2}{\sqrt{c^2\delta_1^2 + 2\varrho cd\delta_1\delta_2 + d^2\delta_2^2}}\Big),\end{aligned} \tag{3.3}$$

where $\mathrm{N}(\cdot)$ denotes the standard normal cumulative distribution function and $\mathbf{1}_A$ is an indicator function for any event A.

Proof: It is obvious that $\frac{(aX_1+bX_2)-(a\mu_1+b\mu_2)}{\sqrt{a^2\delta_1^2+2\varrho ab\delta_1\delta_2+b^2\delta_2^2}} \sim N(0,1)$ and $\frac{(cX_1+dX_2)-(c\mu_1+d\mu_2)}{\sqrt{c^2\delta_1^2+2\varrho cd\delta_1\delta_2+d^2\delta_2^2}} \sim N(0,1)$. Then, it follows from the lemma in Dravid et al. [25] that the proof is completed.

In order to value the Asian spread options defined in (2.3), we first requires to determine the linear exercise boundary in logarithmic variables from the exercise region $\{G_{1T} - G_{2T} \geq K\}$ (or $\{A_{1T} - A_{2T} \geq K\}$). In this paper, along with the method used in Bjerksund and Stensland [7] for options written on the spread between two assets, and Lau and Lo [38] for multi-asset basket spread options, we derive an approximated closed-form formula by modifying the origin exercise region slightly. Second, we need to know the joint probability distribution of two arithmetic average prices. Here, we will apply distribution-approximating and moment-matching (see, e.g., Brignone et al. [8]) methods to derive the joint probability distribution and extend the work in Levy [39] from one-dimensional unknown arithmetic average distribution by the corresponding log-normal distribution to two-dimensional cases. Actually, our approximated formula is always a little less than the fair value of spread options so that it can be seen as a lower bound. Now, we are in the position to state the main theoretical results of this paper.

Proposition 2. Based on the proposed model specification (2.1) and (2.2), the price at time $t = 0$ for the ASO on geometric average is given by

$$\begin{aligned}GASO(S_1, S_2, K) &= S_{10}\mathrm{e}^{-(r+\frac{1}{6}\sigma_1^2)\frac{T}{2}}\mathrm{N}(d_1) \\ &-S_{20}\mathrm{e}^{-(r+\frac{1}{6}\sigma_2^2)\frac{T}{2}}\mathrm{N}(d_2) - K\mathrm{e}^{-rT}\mathrm{N}(d_3),\end{aligned} \tag{3.4}$$

where $d_i, i = 1, 2, 3$ are given by

$$d_1 = \frac{\ln\frac{S_{10}}{k} + [(r + \frac{1}{6}\sigma_1^2)\frac{1}{2} + \frac{1}{6}\alpha^2\sigma_2^2 - \frac{1}{3}\rho\alpha\sigma_1\sigma_2]T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\ln\frac{S_{10}}{k} + [\frac{1}{6}\alpha^2\sigma_2^2 - \frac{1}{3}\alpha\sigma_2^2 + (r - \frac{1}{2}\sigma_1^2)\frac{1}{2} + \frac{1}{3}\rho\sigma_1\sigma_2]T}{\sigma\sqrt{T}},$$

$$d_3 = \frac{\ln\frac{S_{10}}{k} + [(r - \frac{1}{2}\sigma_1^2)\frac{1}{2} + \frac{1}{6}\alpha^2\sigma_2^2]T}{\sigma\sqrt{T}},$$

in which the parameters σ, α and k are

$$\sigma = \sqrt{\frac{1}{3}(\sigma_1^2 - 2\rho\alpha\sigma_1\sigma_2 + \alpha^2\sigma_2^2)},\ \ \alpha = \frac{\mathrm{E}(G_{2T})}{\mathrm{E}(G_{2T}) + K},\ \ k = \mathrm{E}(G_{2T}) + K.$$

Proof: See the Appendix.

Proposition 3. Based on the proposed model specification (2.1) and (2.2), the price at time $t = 0$ for the ASO on arithmetic average is given by

$$AASO(S_1, S_2, K) = \mathrm{e}^{-rT}\Big[e^{\mu_1+\frac{1}{2}\delta_1^2}\mathrm{N}(\hat{d}_1) - e^{\mu_2+\frac{1}{2}\delta_2^2}\mathrm{N}(\hat{d}_2) - K\mathrm{N}(\hat{d}_3)\Big], \tag{3.5}$$

where $\hat{d}_i, i = 1, 2, 3$ are given by

$$\hat{d}_1 = \frac{\mu_1 - \hat{k} + (\delta_1^2 - \hat{\rho}\beta\delta_1\delta_2 + \frac{1}{2}\beta^2\delta_2^2)}{\sqrt{\delta_1^2 - 2\hat{\rho}\beta\delta_1\delta_2 + \beta^2\delta_2^2}},$$
$$\hat{d}_2 = \frac{\mu_1 - \hat{k} + (\hat{\rho}\delta_1\delta_2 - \beta\delta_2^2 + \frac{1}{2}\beta^2\delta_2^2)}{\sqrt{\delta_1^2 - 2\hat{\rho}\beta\delta_1\delta_2 + \beta^2\delta_2^2}},$$
$$\hat{d}_3 = \frac{\mu_1 - \hat{k} + \frac{1}{2}\beta^2\delta_2^2}{\sqrt{\delta_1^2 - 2\hat{\rho}\beta\delta_1\delta_2 + \beta^2\delta_2^2}},$$

in which the parameters above are

$$\beta = \frac{\mathrm{E}(A_{2T})}{\mathrm{E}(A_{2T}) + K},\ \ \hat{k} = \ln[\mathrm{E}(A_{2T}) + K],$$
$$\mu_i = 2\ln\mathrm{E}(A_{iT}) - \frac{1}{2}\ln\mathrm{E}(A_{iT}^2),$$
$$\delta_i^2 = \ln\mathrm{E}(A_{iT}^2) - 2\ln\mathrm{E}(A_{iT}),\ \ i = 1, 2,$$
$$\hat{\rho} = \frac{2[\ln\mathrm{E}(A_{1T}A_{2T}) - (\mu_1 + \mu_2)] - (\delta_1^2 + \delta_2^2)}{2\delta_1\delta_2},$$
$$\mathrm{E}(A_{1T}A_{2T}) = \frac{2S_{10}S_{20}}{rT^2}\Big[\frac{\mathrm{e}^{(2r+\rho\sigma_1\sigma_2)T} - \mathrm{e}^{rT}}{(r + \rho\sigma_1\sigma_2)} - \frac{\mathrm{e}^{(2r+\rho\sigma_1\sigma_2)T} - 1}{(2r + \rho\sigma_1\sigma_2)}\Big].$$

Proof: See the Appendix in detail.

Remark 1. First, the formulas (3.4) and (3.5) are the lower bounds of the exact Asian spread option prices, and in practice they are so tight that they could be seen as an accurate approximation of the true value. These will be shown through numerical experiments in Section 4. Second, the values of parameter α (or β) and k (or $\hat{k}$) given in the above expressions are proposed by Bjerksund and Stensland [7] in the log-normal setting within the underlying assets. They are also effective when we take average-rate prices into consideration (see Lau and Lo [38] for multi-asset basket spread options whose payoff is the linear combination of several underlying assets). The above formula is very easy to implement by the software such as Matlab. In the numerical section, we shall show the accuracy and efficiency of the approximation pricing formulas.

Remark 2. When $K < 0$ by using the fact that $\mathrm{E}[X^+] = \mathrm{E}[(-X)^+] + \mathrm{E}[X]$, we can obtain that

$$\begin{aligned}
&\mathrm{E}\{e^{-rT}(G_{1T} - G_{2T} - K)^+\} = \mathrm{E}\{e^{-rT}[-(G_{2T} - G_{1T} - \tilde{K})^+]\}\\
&= \mathrm{E}\{e^{-rT}(G_{2T} - G_{1T} - \tilde{K})^+\} - \mathrm{E}\{e^{-rT}(G_{2T} - G_{1T} - \tilde{K})\}\\
&= GASO(S_2, S_1, -K) - S_{20}\exp\{-\frac{1}{2}(r + \frac{1}{6}\sigma_2^2)T\}\\
&\quad + S_{10}\exp\{-\frac{1}{2}(r + \frac{1}{6}\sigma_1^2)T\} - K\mathrm{e}^{-rT},
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{E}\{e^{-rT}(A_{1T} - A_{2T} - K)^+\} = AASO(S_2, S_1, -K) - \frac{S_{20}(1 - \mathrm{e}^{-rT})}{rT}\\
&\quad + \frac{S_{10}(1 - \mathrm{e}^{-rT})}{rT} - K\mathrm{e}^{-rT},
\end{aligned}$$

where $\tilde{K} = -K > 0$.

A special case of the above formula has been extensively studied. For example, if we set $\sigma_2 = 0$ and $\rho = 0$, we have an Asian option. In addition, the formulas (3.4) and (3.5) have two simple forms when $K = 0$. In fact, when $K = 0$, the formulas (11) and (12) reduce to the pricing formulas for the Asian exchange options which are similar to the works by Margrabe [48] and Han et al. [29].

Corollary 1. If we let $K = 0$ in (11) and (12), we obtain the following formulas for the Asian exchange options:

$$\begin{aligned}
GAEO(S_1, S_2) = {}& S_{10}\mathrm{e}^{-(r+\frac{1}{6}\sigma_1^2)\frac{T}{2}}\mathrm{N}\Big(\frac{\ln\frac{S_{10}}{S_{20}} + (r + \frac{1}{12}\sigma_1^2 - \frac{1}{4}\sigma_2^2)T}{\sqrt{\frac{T}{3}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)}}\Big)\\
&- S_{20}\mathrm{e}^{-(r+\frac{1}{6}\sigma_2^2)\frac{T}{2}}\mathrm{N}\Big(\frac{\ln\frac{S_{10}}{S_{20}} - (\frac{1}{4}\sigma_1^2 + \frac{1}{12}\sigma_2^2 - \frac{1}{3}\rho\sigma_1\sigma_2)T}{\sqrt{\frac{T}{3}(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)}}\Big), \qquad (3.6)
\end{aligned}$$

and

$$\begin{aligned}
AAEO(S_1, S_2) = {}& \mathrm{e}^{-rT}\Big[e^{\mu_1+\frac{1}{2}\delta_1^2}\mathrm{N}\Big(\frac{\mu_1 - \mu_2 + \frac{1}{2}(\delta_1^2 - \hat{\rho}\delta_1\delta_2)}{\sqrt{\delta_1^2 - 2\hat{\rho}\delta_1\delta_2 + \delta_2^2}}\Big)\\
&- e^{\mu_2+\frac{1}{2}\delta_2^2}\mathrm{N}\Big(\frac{\mu_1 - \mu_2 + \frac{1}{2}(\hat{\rho}\delta_1\delta_2 - \delta_2^2)}{\sqrt{\delta_1^2 - 2\hat{\rho}\delta_1\delta_2 + \delta_2^2}}\Big), \qquad (3.7)
\end{aligned}$$

respectively.

Proof. When $K = 0$, then $\alpha = 1$, $\beta = 1$, $k = \mathrm{E}(G_{2T})$, and $\hat{k} = \ln \mathrm{E}(A_{2T})$ defined in formulas (3.4) and (3.5). In this case, $k = S_{20}\exp\{(r - \frac{1}{6}\sigma_2^2)\frac{T}{2}\}$ and $\hat{k} = \mu_2 + \frac{1}{2}\delta_2^2$. Hence, it follows that (3.6) and (3.7) hold.

The approximation can be applied to the Greeks computation, as well. In particular, we can derive the explicit expressions for the delta, which is defined as the rate of change of the option value with respect to the underlying asset price, of the ASO priced by our two approximations, GASO and AASO. Similar formulas can be computed for the other Greeks, such as Gamma, Vega, Theta, and Rho. The following corollary summarizes our findings.

Corollary 2. The hedging parameter Δ of the ASO, based on the GASO and AASO pricing formulas, are respectively

$$\Delta_1^{(g)} = \frac{\partial GASO}{\partial S_1} = e^{-(r+\frac{1}{6}\sigma_1^2)\frac{T}{2}}N(d_1) + \frac{e^{-(r+\frac{1}{6}\sigma_1^2)\frac{T}{2}-\frac{d_1^2}{2}}}{\sigma\sqrt{2\pi T}}$$
$$-\frac{S_2 e^{-(r+\frac{1}{6}\sigma_2^2)\frac{T}{2}-\frac{d_2^2}{2}}}{S_1\sigma\sqrt{2\pi T}} - \frac{Ke^{-rT-\frac{d_3^2}{2}}}{S_1\sigma\sqrt{2\pi T}}, \tag{3.8}$$

$$\Delta_2^{(g)} = \frac{\partial GASO}{\partial S_2} = -\frac{\alpha S_1 e^{-(r+\frac{1}{6}\sigma_1^2)\frac{T}{2}-\frac{d_1^2}{2}}}{S_2\sigma\sqrt{2\pi T}}\Big\{-1 + \Big[1 + \frac{d_1}{\sigma\sqrt{T}}\Big]$$
$$\frac{(1-\alpha)(\alpha\sigma_2^2 - \rho\sigma_1\sigma_2)T}{3}\Big\} - e^{-(r+\frac{1}{6}\sigma_2^2)\frac{T}{2}}N(d_2)$$

$$+\frac{\alpha e^{-(r+\frac{1}{6}\sigma_2^2)\frac{T}{2}-\frac{d_2^2}{2}}}{\sigma\sqrt{2\pi T}}\Big[1 + \frac{(1-\alpha)^2}{3}\sigma_2^2 T$$
$$+\frac{(1-\alpha)(\alpha\sigma_2^2 - \rho\sigma_1\sigma_2)d_2\sqrt{T}}{3\sigma}\Big] + \frac{\alpha K e^{-(rT+\frac{d_3^2}{2})}}{S_2\sigma\sqrt{2\pi T}}\Big[1 - \frac{\alpha(1-\alpha)}{3}\sigma_2^2 T$$
$$+\frac{(1-\alpha)(\alpha\sigma_2^2 - \rho\sigma_1\sigma_2)d_3\sqrt{T}}{3\sigma}\Big], \tag{3.9}$$

$$\Delta_1^{(a)} = \frac{\partial AASO}{\partial S_1} = e^{-rT}\Big\{\frac{1}{S_1}e^{\mu_1+\frac{1}{2}\delta_1^2}N(\hat{d}_1) + \frac{S_1}{\sqrt{2\pi}\delta}\Big[e^{\mu_1+\frac{1}{2}(\delta_1^2-\hat{d}_1^2)}$$
$$\cdot\Big(1 - \frac{\hat{d}_1[1-\frac{1}{S_1^2}]}{\delta}\Big) + e^{\mu_2+\frac{1}{2}(\delta_2^2-\hat{d}_2^2)}\Big(1 - \frac{2}{S_1^2} + \frac{\hat{d}_2[1-\frac{1}{S_1^2}]}{\delta}\Big)$$
$$+Ke^{-\frac{1}{2}\hat{d}_3^2}\Big(1 - \frac{2}{S_1^2} + \frac{\hat{d}_3[1-\frac{1}{S_1^2}]}{\delta}\Big)\Big]\Big\}, \tag{3.10}$$

$$\Delta_2^{(a)} = \frac{\partial AASO}{\partial S_2} = \frac{e^{-rT}}{S_2}\Big\{-e^{\mu_2+\frac{1}{2}\delta_2^2}N(\hat{d}_2) + \frac{\beta}{\sqrt{2\pi}\delta}\Big[-e^{\mu_1+\frac{1}{2}(\delta_1^2-\hat{d}_1^2)}$$
$$\cdot\Big(1 - [\beta(S_2^2-1) - \hat{\rho}(1-\beta)\delta_1\delta_2 + \beta(1-\beta)\delta_2^2][1 - \frac{\hat{d}_1}{\delta}]\Big)$$
$$+e^{\mu_2+\frac{1}{2}(\delta_2^2-\hat{d}_2^2)}\Big(1 + [1-\beta][(S_2^2-1) + (1-\beta)\delta_2^2] + \frac{\hat{d}_2}{\delta}[\beta(S_2^2-1)$$
$$+\beta(1-\beta)\delta_2^2 - \hat{\rho}(1-\beta)\delta_1\delta_2]\Big) + Ke^{-\frac{1}{2}\hat{d}_3^2}\Big(1 - \beta[(S_2^2-1) + (1-\beta)\delta_2^2]$$

$$+\frac{\hat{d}_3}{\delta}[\beta(S_2^2-1)+\beta(1-\beta)\delta_2^2-\hat{\rho}(1-\beta)\delta_1\delta_2]\Big)\Big]\Big\}, \tag{3.11}$$

where $\delta = \sqrt{\delta_1^2 - 2\hat{\rho}\beta\delta_1\delta_2 + \beta^2\delta_2^2}$.
Proof: Using the results of Propositions 2 and 3, straightforward calculations lead to the formulas (3.8)–(3.11).

4. Numerical analysis

In this section, we present a few numerical examples of our main results in Propositions 2 and 3 above. We first compare the performance of Propositions 2 and 3 with the Monte Carlo (MC) simulation in Table 1 below. The MC simulation values were used for the benchmark, and accuracy was measured by the root of mean-squared errors (RMSE) and maximum absolute error (MAE). In our comparison, we continued to use the same model parameters as the numerical examples in [7]. More specifically, we picked the following model parameters in our numerical tests: $r = 0.05, S_{10} = 100, S_{20} = 96, \sigma_1 = 0.2$, and $\sigma_2 = 0.1$. The MC simulation was implemented by means of the Euler-Maruyama discretization method. In our numerical experiments, we generated $N = 1,000,000$ sample paths with running on a daily basis. Second, we compared the option prices for the ASOs with those of the commom spread options. Finally, in order to see the impact of underlying parameters on approximated prices and delta values, sensitivity analysis was conducted. All the algorithms were implemented in MatLab (R2018b). The codes for the examples were run in MATLAB R2017a on a PC with the configuration: Intel(R)Core(TM) i7-8550UCPU@1.80GHz 1.99GHz and 8.0GB RAM.

Table 1 provides the computational results for 33 ASOs with different strike prices. In Table 1, GASO and AASO are the values obtained by our pricing formulas (3.4) and (3.5), and 'G^{MC}' and 'A^{MC}' are the values by MC simulations for the ASO on geometric average and arithmetic average, respectively. The columns labeled "95% C.I." is the 95% confidence interval for the MC simulation method. From Table 1, we can see that the approximated formulas in Propositions 2 and 3 gave highly accurate Asian spread option prices as they were very close to the prices obtained by the MC simulation. Concretely, the RMSE was less than 3% and the MAE was less than 7% for all cases. In addition, we observed that the speed of this approximated method was faster than the MC simulation method. Concretely, the average of the computational time of the MC methods was over 8 seconds, while the approximated approach took less than 0.008 second. As a result, the approximated formulas in Propositions 2 and 3 were pretty tight so that they could be used as an excellent approximation.

Table 2 reports comparisons between the ASOs prices and the plain-vanilla spread option prices derived by Bjerksund and Stensland [7]. As expected, it can be seen from Table 2 that the ASOs prices were all less than those of the plain-vanilla spread option, and the prices of the ASO with arithmetic averaging were higher than those of the ASO with geometric averaging under the same parameter values. In addition, higher strike prices K and higher correlation coefficients ρ led to lower Asian spread call option prices. This finding was similar to that for the plain-vanilla spread option in Hurd and Zhou [31], Bjerksund and Stensland [7], and Lo [46]. On the other hand, we also observed that the prices of the spread options including the ASOs and plain-vanilla spread option increased as the maturity time T increased, meaning that the maturity had a significant effect on the values of Asian spread options. Intuitively, the values of Asian spread options depended on the relative performances between two underlying assets. This may be explained by the fact that the volatilities of the two underlying assets in a long time period were larger, so the option price rose.

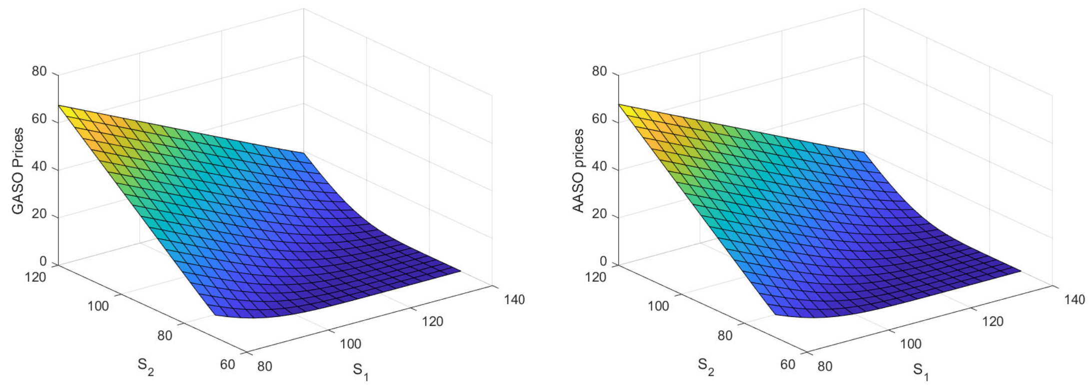
Table 1. Accuracy comparison of pricing formulas of ASO for the case $T = 1$ (*year*).

ρ	K	GASO	G^{MC}	95% C.I.	AASO	A^{MC}	95% C.I.
-0.5	0.0	7.8154	7.7813	(7.7550, 7.8176)	8.0166	7.9646	(7.9377, 8.0916)
	0.4	7.5943	7.5850	(7.5589, 7.6111)	7.7933	7.7667	(7.7400, 7.7934)
	0.8	7.3771	7.3602	(7.3346, 7.3859)	7.5740	7.5417	(7.5155, 7.5780)
	1.2	7.1638	7.1871	(7.1617, 7.2125)	7.3585	7.3671	(7.3411, 7.3932)
	1.6	6.9545	6.9524	(6.9272, 6.9775)	7.1470	7.1317	(7.1059, 7.1575)
	2.0	6.7492	6.7358	(6.7111, 6.7605)	6.9393	6.9129	(6.8876, 6.9403)
	2.4	6.5478	6.5173	(6.4930, 6.5486)	6.7356	6.6926	(6.6677, 6.7376)
	2.8	6.3504	6.3386	[6.3145, 6.3627)	6.5358	6.5136	(6.4888, 6.5383)
	3.2	6.1569	6.1572	(6.1335, 6.1809)	6.3400	6.3299	(6.3056, 6.3543)
	3.6	5.9674	5.9611	(5.9377, 5.9846)	6.1480	6.1328	(6.1088, 6.1569)
	4.0	5.7819	5.7969	(5.7737, 5.8200)	5.9600	5.9673	(5.9435, 5.9911)
0.0	0.0	6.9506	6.9027	(6.8798, 6.9555)	7.1500	7.0812	(7.0578, 7.1546)
	0.4	6.7239	6.7127	(6.6900, 6.7353)	6.9209	6.8900	(6.8668, 6.9232)
	0.8	6.5019	6.4958	(6.4735, 6.5181)	6.6963	6.6719	(6.6490, 6.6988)
	1.2	6.2845	6.2959	(6.2739, 6.3178)	6.4763	6.4711	(6.4485, 6.4936)
	1.6	6.0717	6.0639	(6.0423, 6.0855)	6.2609	6.2372	(6.2149, 6.2694)
	2.0	5.8637	5.8213	(5.8000, 5.8726)	6.0502	5.9939	(5.9720, 6.0558)
	2.4	5.6603	5.6620	(5.6409, 5.6831)	5.8441	5.8320	(5.8103, 5.8537)
	2.8	5.4616	5.5168	(5.4960, 5.5376)	5.6426	5.6879	(5.6665, 5.7093)
	3.2	5.2676	5.3129	(5.2924, 5.3333)	5.4457	5.4820	(5.4609, 5.5031)
	3.6	5.0783	5.0836	(5.0635, 5.1037)	5.2535	5.2491	(5.2284, 5.2699)
	4.0	4.8936	4.9114	(4.8916, 4.9311)	5.0659	5.0757	(5.0553, 5.0961)
0.5	0.0	5.9065	5.9083	(5.8896, 5.9270)	6.1050	6.0865	(6.0673, 6.1058)
	0.4	5.6695	5.6527	(5.6343, 5.6710)	5.8649	5.8273	(5.8084, 5.8663)
	0.8	5.4384	5.4560	(5.4377, 5.4742)	5.6307	5.6294	(5.6106, 5.6482)
	1.2	5.2133	5.2160	(5.1981, 5.2339)	5.4023	5.3861	(5.3676, 5.4045)
	1.6	4.9942	4.9812	(4.9637, 4.9987)	5.1798	5.1509	(5.1328, 5.1890)
	2.0	4.7811	4.7857	(4.7684, 4.8029)	4.9633	4.9523	(4.9345, 4.9702)
	2.4	4.5740	4.5887	(4.5718, 4.6056)	4.7527	4.7525	(4.7350, 4.7700)
	2.8	4.3729	4.3362	(4.3197, 4.3827)	4.5480	4.4977	(4.4806, 4.5547)
	3.2	4.1779	4.1472	(4.1311, 4.1833)	4.3493	4.3057	(4.2890, 4.3524)
	3.6	3.9888	3.9790	(3.9632, 3.9949)	4.1566	4.1354	(4.1189, 4.1578)
	4.0	3.8057	3.8129	(3.7973, 3.8285)	3.9697	3.9691	(3.9529, 3.9853)
RMSE		0.0222			0.0298		
MAE		0.0552			0.0688		
Time (se)		0.0192	246.46		0.0273	258.75	

Table 2. Comparison of ASOs and spread options.

		T=1			T=3		
ρ	K	GASO	AASO	Spread option	GASO	AASO	Spread option
-0.5	0.0	7.8154	8.0166	12.4356	11.0883	11.7499	19.8298
	0.4	7.5943	7.7933	12.2317	10.9077	11.5660	19.6595
	0.8	7.3771	7.5740	12.0301	10.7290	11.3841	19.4903
	1.2	7.1638	7.3585	11.8307	10.5524	11.2041	19.3221
	1.6	6.9545	7.1470	11.6336	10.3778	11.0261	19.1551
	2.0	6.7492	6.9393	11.4387	10.2052	10.8500	18.9891
	2.4	6.5478	6.7356	11.2461	10.0345	10.6759	18.8242
	2.8	6.3504	6.5358	11.0557	9.8659	10.5037	18.6604
	3.2	6.1569	6.3400	10.8676	9.6993	10.3334	18.4977
	3.6	5.9674	6.1480	10.6817	9.5346	10.1651	18.3361
	4.0	5.7819	5.9600	10.4981	9.3720	9.9987	18.1756
0.0	0.0	6.9506	7.1500	10.8684	9.6365	10.2755	17.1303
	0.4	6.7239	6.9209	10.6620	9.4544	10.0895	16.9603
	0.8	6.5019	6.6963	10.4583	9.2745	9.9058	16.7916
	1.2	6.2845	6.4763	10.2572	9.0971	9.7244	16.6242
	1.6	6.0717	6.2609	10.0589	8.9221	9.5453	16.4581
	2.0	5.8637	6.0502	9.8632	8.7494	9.3686	16.2933
	2.4	5.6603	5.8441	9.6702	8.5791	9.1941	16.1298
	2.8	5.4616	5.6426	9.4798	8.4111	9.0220	15.9675
	3.2	5.2676	5.4457	9.2921	8.2455	8.8521	15.8066
	3.6	5.0783	5.2535	9.1071	8.0822	8.6845	15.6469
	4.0	4.8936	5.0659	8.9247	7.9213	8.5192	15.4885
0.5	0.0	5.9065	6.1050	8.9497	7.8549	8.4637	13.7923
	0.4	5.6695	5.8649	8.7386	7.6698	8.2737	13.6229
	0.8	5.4384	5.6307	8.5311	7.4877	8.0867	13.4552
	1.2	5.2133	5.4023	8.3269	7.3088	7.9027	13.2891
	1.6	4.9942	5.1798	8.1263	7.1329	7.7217	13.1248
	2.0	4.7811	4.9633	7.9290	6.9601	7.5436	12.9621
	2.4	4.5740	4.7527	7.7352	6.7903	7.3685	12.8011
	2.8	4.3729	4.5480	7.5449	6.6235	7.1964	12.6417
	3.2	4.1779	4.3493	7.3579	6.4598	7.0272	12.4839
	3.6	3.9888	4.1566	7.1743	6.2991	6.8609	12.3278
	4.0	3.8057	3.9697	6.9942	6.1413	6.6975	12.1733

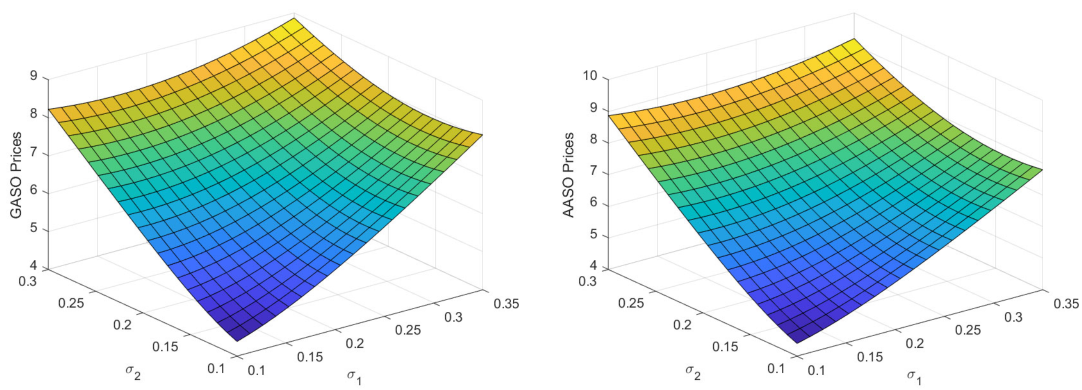
The impacts of basic parameters on the ASOs prices or their deltas are shown in Figures 1–3, including the initial values and volatilities of the underlying assets. Figure 1 shows that a larger S_2 results in a larger option price with other parameters being fixed, but the impact of the parameter S_1 on the ASOs price is opposite with a larger S_1 corresponding to a lower option price, where other parameters are fixed. In other words, a larger difference $(S_1 - S_2)$ based on the two underlying asset prices results in a lower price for the ASOs.



(a) GASO (b) AASO

Figure 1. Prices of ASOs against different initial values S_1 and S_2.

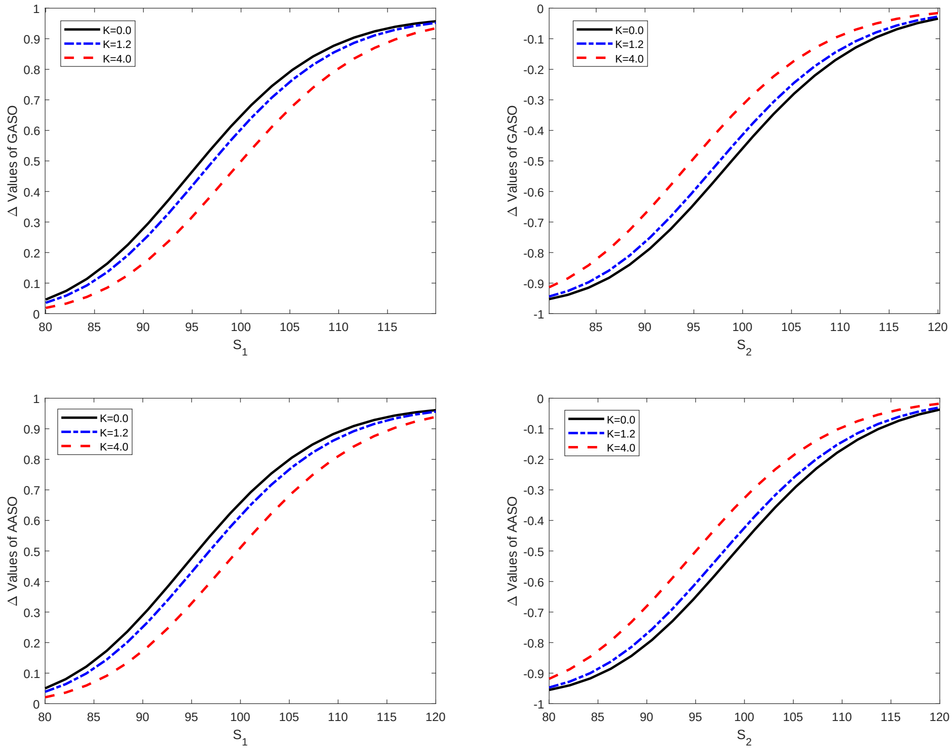
Figure 2 depicts the influence of the volatilities σ_1 and σ_2 of the two underlying assets. The prices of the ASO options increase with the volatilities. This increases the volatility of the spread and hence the spread option value.



(a) GASO (b) AASO

Figure 2. Impact of option price for different volatilities σ_1 and σ_2.

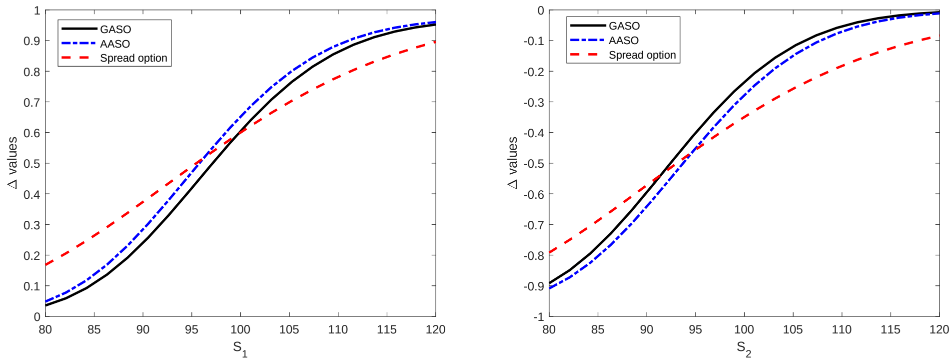
Figure 3 displays the Δ values of the GASO and AASO against the underlying asset prices S_1 or S_2. From Figure 3(a), one can observe that the Δ values of GASO increase as S_1 increases, and become larger as the strike price K decreases. Contrary to the case of the underlying asset price S_2, the Δ values of GASO decrease as S_2 increases and become larger as the strike price K increases. Also, there is a similar pattern for the Δ values of AASO with respect to S_2 and K from Figure 3(b).



(a) GASO (b) AASO

Figure 3. Δ values against different initial values S_1 or S_2.

Figure 4 illustrates the changes in the Δ value of GASO, AASO, and the spread option with respect to S_1 or S_2, respectively. We observed that an increase in the initial underlying asset value led to a rise in the option's delta value. However, the delta value increased at a slower pace for the spread option compared to the Asian spread options. This is a result of the average value of the underlying asset over the entire period, which affected a portion of the option's delta value and made it less sensitive than in the case where it was based on the underlying asset's value at the expiration date.



(a) S_1 (b) S_2

Figure 4. Comparison of Δ values against different initial values S_1 or S_2.

5. Conclusions

This paper presents a new variety of financial instruments-spread option on two average asset prices. In virtue of the moment-matching and distribution-approximating techniques, we generalized the Bjerksund and Stensland [7] approximate of two asset spread option formulas to the case of the Asian spread options, and obtained the analytical valuation formulas for the Asian spread options on geometric averaging and arithmetic averaging. Finally, we presented some numerical results. The main contribution of this paper is to provide practitioners with a pricing formula, which can be used for real-time pricing of Asian spread options. For example, practitioners might apply the above option pricing formulas (3.4) and (3.5) to calibrate the model and estimate the model parameters on a set of market data of European-style Asian spread options by minimizing the difference between market prices and model prices in a least-squared error fit.

At present, although trading of the options proposed in this paper is very limited in the real market, financial innovations have become increasingly important to risk management. Creating an option is unusual and is more likely to occur when academics are involve. Therefore, the Asian spread options that non-trivially combine the spread option with Asian options obviously provide great value-added potentials in financial markets. In particular, Asian spread options not only have wide applications in risk management but also appropriately enter the payoff function of incentive contracts for management compensation.

There are a few directions that one can take to extend and improve the results in this paper. First, in the geometric Brownian motions case, our results can be easily extended to incorporate jumps or stochastic volatility (for example, the Merton jump-diffusion model or the Hull-White stochastic volatility model) in the price processes of the assets. Second, the approximation method might be improved for Asian spread options such as using the Edgeworth expansion approach by Kao and Xie [32], or the method considered in Castellaccia and Siclari [14], who approximate an arithmetic average with a geometric one by adjusting the strike for the discrepancy. In addition, the formulas of continuously monitored average Asian spread options can be a good proxy for the corresponding discrete-type option in cases that are a kind of basket option. We leave these interesting topics for future research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Appendix

Proof of Proposition 2: We define the following event,

$$A = \Big\{\omega : \frac{G_{1T}}{G_{2T}^{\alpha}} > \frac{\mathrm{e}^{k}}{\mathrm{E}(G_{2T}^{\alpha})}\Big\}, \tag{A.1}$$

and then from (2.4) and (2.5)

$$\begin{aligned} A &= \Big\{\omega : \frac{S_{10}\mathrm{e}^{(r-\frac{1}{2}\sigma_1^2)\frac{T}{2}+\frac{\sigma_1}{T}\int_0^T (T-t)\mathrm{d}W_{1t}}}{S_{20}^{\alpha}\mathrm{e}^{(r-\frac{1}{2}\sigma_2^2)\frac{\alpha T}{2}+\frac{\alpha\sigma_2}{T}\int_0^T (T-t)\mathrm{d}W_{2t}}} > \frac{\mathrm{e}^{k}}{\mathrm{E}(G_{2T}^{\alpha})}\Big\} \\ &= \Big\{\omega : \frac{\sigma_1}{T}X_1 - \frac{\alpha\sigma_2}{T}X_2 > \ln\frac{k}{S_{10}} - \frac{1}{6}\alpha^2\sigma_2^2 T - (r - \frac{1}{2}\sigma_1^2)\frac{T}{2}\Big\}, \end{aligned} \tag{A.2}$$

where $X_1 = \int_0^T (T-t)\mathrm{d}W_{1t} \sim \mathrm{N}\Big(0, \frac{T^3}{3}\Big)$ and $X_2 = \int_0^T (T-t)\mathrm{d}W_{2t} \sim \mathrm{N}\Big(0, \frac{T^3}{3}\Big)$. Therefore, following the idea of Bjerksund and Stensland [7], and [38], we have $(G_{1T} - G_{2T} - K)^+ \geq (G_{1T} - G_{2T} - K)\mathbf{1}_A$, and get a lower bound to the Asian spread options

$$\begin{aligned} GASO(S_1, S_2, K) &= \mathrm{E}\{\mathrm{e}^{-rT}(G_{1T} - G_{2T} - K)\mathbf{1}_A\} \\ &= \mathrm{e}^{-rT}(I_1 - I_2 - I_3), \end{aligned}$$

where $I_1 = \mathrm{E}\{G_{1T}\mathbf{1}_A\}$, $I_2 = \mathrm{E}\{G_{21T}\mathbf{1}_A\}$ and $I_3 = K\mathrm{E}\{\mathbf{1}_A\}$. In the following, we adopt Proposition 1 to derive the expressions of I_1–I_3 in turn. First, we derive I_1 as follows:

$$\begin{aligned} I_1 &= \mathrm{E}\{G_{1T}\mathbf{1}_A\} = \mathrm{E}\Big[S_{10}\mathrm{e}^{(r-\frac{1}{2}\sigma_1^2)\frac{T}{2}+\frac{\sigma_1}{T}X_1}\mathbf{1}_A\Big] \\ &= \mathrm{E}\Big[S_{10}\mathrm{e}^{(r-\frac{1}{2}\sigma_1^2)\frac{T}{2}+\frac{\sigma_1}{T}X_1}\mathbf{1}_{\left(\frac{\sigma_1}{T}X_1-\frac{\alpha\sigma_2}{T}X_2>\ln\frac{k}{S_{10}}-\frac{1}{6}\alpha^2\sigma_2^2T-(r-\frac{1}{2}\sigma_1^2)\frac{T}{2}\right)}\Big] \\ &= S_{10}\mathrm{e}^{(r-\frac{1}{6}\sigma_1^2)\frac{T}{2}}\mathrm{N}(d_1). \end{aligned} \tag{A.3}$$

Similarly, it holds that

$$\begin{aligned} I_2 &= \mathrm{E}\{G_{2T}\mathbf{1}_A\} = \mathrm{E}\Big[S_{20}\mathrm{e}^{(r-\frac{1}{2}\sigma_2^2)\frac{T}{2}+\frac{\sigma_2}{T}X_2}\mathbf{1}_A\Big] \\ &= \mathrm{E}\Big[S_{20}\mathrm{e}^{(r-\frac{1}{2}\sigma_2^2)\frac{T}{2}+\frac{\sigma_2}{T}X_2}\mathbf{1}_{\left(\frac{\sigma_1}{T}X_1-\frac{\alpha\sigma_2}{T}X_2>\ln\frac{k}{S_{10}}-\frac{1}{6}\alpha^2\sigma_2^2T-(r-\frac{1}{2}\sigma_1^2)\frac{T}{2}\right)}\Big] \\ &= S_{20}\mathrm{e}^{(r-\frac{1}{6}\sigma_2^2)\frac{T}{2}}\mathrm{N}(d_2), \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} I_3 &= K\mathrm{E}\{\mathbf{1}_A\} \\ &= KQ\Big(\frac{\sigma_1}{T}X_1 - \frac{\alpha\sigma_2}{T}X_2 > \ln\frac{k}{S_{10}} - \frac{1}{6}\alpha^2\sigma_2^2T - (r - \frac{1}{2}\sigma_1^2)\frac{T}{2}\Big) \\ &= K\mathrm{N}(d_3). \end{aligned} \tag{A.5}$$

Now, we have completed the proof.

Proof of Proposition 3: Due to the fact that the integral of lognormal distribution no longer obeys lognormal distribution, its probability distribution is difficult to determine. In the following, we adopt the wilkinson approximation of Levy [39] for one-dimensional $\ln A_T$ with a two-parameter lognormal distribution to extend the two-dimensional case $(\ln A_{1T}, \ln A_{2T})$ with five-parameter adjoint lognormal distribution approximation. In this case, we assume that

$$(\ln A_{1T}, \ln A_{2T}) \sim N_2(\mu_1, \mu_2, \delta_1^2, \delta_2^2, \hat{\rho}), \tag{A.6}$$

where $N_2(\cdot)$ denotes the bivariate normal cumulative distribution function. It is obvious that the moment generating function for the two-dimensional random vector $(\ln A_{1T}, \ln A_{2T})$ is

$$\mathrm{E}(e^{\lambda_1 \ln A_{1T} + \lambda_2 \ln A_{2T}}) = \exp\{\lambda_1\mu_1 + \lambda_2\mu_2 + \frac{1}{2}(\lambda_1^2\delta_1^2 + 2\hat{\rho}\lambda_1\lambda_2\delta_1\delta_2 + \lambda_2^2\delta_2^2)\},$$

and it holds then that

$$\begin{aligned}
\mathrm{E}(e^{\ln A_{1T}}) &= \mathrm{E}(A_{1T}) = \exp\{\mu_1 + \frac{1}{2}\delta_1^2\},\\
\mathrm{E}(e^{2\ln A_{1T}}) &= \mathrm{E}(A_{1T}^2) = \exp\{2\mu_1 + 2\delta_1^2\},\\
\mathrm{E}(e^{\ln A_{2T}}) &= \mathrm{E}(A_{2T}) = \exp\{\mu_2 + \frac{1}{2}\delta_2^2\},\\
\mathrm{E}(e^{2\ln A_{2T}}) &= \mathrm{E}(A_{2T}^2) = \exp\{2\mu_2 + 2\delta_2^2\},\\
\mathrm{E}(e^{\ln A_{1T} + \ln A_{2T}}) &= \mathrm{E}(A_{1T}A_{2T}) = \exp\{\mu_1 + \mu_2 + \frac{1}{2}(\delta_1^2 + 2\hat{\rho}\delta_1\delta_2 + \delta_2^2)\}.
\end{aligned}$$

Therefore, we have the five parameters as follows:

$$\begin{aligned}
\mu_i &= 2\ln \mathrm{E}(A_{iT}) - \frac{1}{2}\ln \mathrm{E}(A_{iT}^2),\\
\delta_i^2 &= \ln \mathrm{E}(A_{iT}^2) - 2\ln \mathrm{E}(A_{iT}), \quad i = 1, 2,\\
\hat{\rho} &= \frac{2[\ln \mathrm{E}(A_{1T}A_{2T}) - (\mu_1 + \mu_2)] - (\delta_1^2 + \delta_2^2)}{2\delta_1\delta_2},
\end{aligned}$$

where $\mathrm{E}(A_{iT})$ and $\mathrm{E}(A_{iT}^2)$ for $i = 1, 2$ are defined by (2.6) and (2.7). Now, it is key to compute the expectation $\mathrm{E}(A_{1T}A_{2T})$ in the third expression above.

Next, we use the theory of the polynomial diffusion process suggested by Willems [63] to compute such moment $\mathrm{E}(A_{1T}A_{2T})$. Since

$$\begin{aligned}
tA_{it} &= S_{i0}\int_0^t \mathrm{e}^{(r-\frac{1}{2}\sigma_i^2)u + \sigma_i W_{iu}}\mathrm{d}u\\
&\overset{\text{law}}{=} S_{i0}\int_0^t \mathrm{e}^{(r-\frac{1}{2}\sigma_i^2)(t-u) + \sigma_i(W_{it} - W_{iu})}\mathrm{d}u := S_{i0}\cdot Z_{it},
\end{aligned} \tag{A.7}$$

for $i = 1, 2$ and fixed $t > 0$. Using Ito's formula, we have

$$\mathrm{d}Z_{it} = (1 + rZ_{it})\mathrm{d}t + \sigma_i Z_{it}\mathrm{d}W_{it}, \quad i = 1, 2, \tag{A.8}$$

and $Z_{it},\ i = 1, 2$ are polynomial diffusion processes. So, we know that $\mathrm{E}(A_{1T}A_{2T}) = \frac{S_{10}S_{20}}{T^2}\mathrm{E}(Z_{1T}Z_{2T})$. On the other hand, we get

$$\mathrm{d}(Z_{1t}Z_{2t}) = Z_{1t}\mathrm{d}Z_{2t} + Z_{2t}\mathrm{d}Z_{1t} + \mathrm{d}Z_{1t}\mathrm{d}Z_{2t}$$

$$= (Z_{1t} + Z_{2t} + 2rZ_{1t}Z_{2t} + \rho\sigma_1\sigma_2 Z_{1t}Z_{2t})\mathrm{d}t + Z_{1t}Z_{2t}(\sigma_1 \mathrm{d}W_{1t} + \sigma_2 \mathrm{d}W_{2t}).$$

Hence, we have (using Fubini's theorem)

$$\mathrm{E}(Z_{1t}Z_{2t}) = \int_0^t \Big[\mathrm{E}(Z_{1u}) + \mathrm{E}(Z_{2u}) + (2r + \rho\sigma_1\sigma_2)\mathrm{E}(Z_{1u}Z_{2u})\Big]\mathrm{d}u, \tag{A.9}$$

where $\mathrm{E}(Z_{iu}) = \frac{\mathrm{e}^{rt}-1}{r}$, $i = 1, 2$. it shows that $y(t) = \mathrm{E}(Z_{1t}Z_{2t})$ is the solution of the following ordinary differential equation

$$\begin{cases} \frac{\mathrm{d}y(t)}{\mathrm{d}t} - (2r + \rho\sigma_1\sigma_2)y(t) = 2\frac{\mathrm{e}^{rt}-1}{r}, \\ y(0) = 0. \end{cases} \tag{A.10}$$

After solving the equation above, we obtain:

$$\mathrm{E}(Z_{1t}Z_{2t}) = \frac{2}{r}\Big[\frac{\mathrm{e}^{(2r+\rho\sigma_1\sigma_2)t} - \mathrm{e}^{rt}}{r + \rho\sigma_1\sigma_2} - \frac{\mathrm{e}^{(r+\rho\sigma_1\sigma_2)t} - 1}{2r + \rho\sigma_1\sigma_2}\Big]. \tag{A.11}$$

Finally, similar to the proof of Proposition 2, we calculate the price of the AASO as follows:

$$\begin{aligned} B &= \Big\{\omega : \frac{A_{1T}}{A_{2T}^{\beta}} > \frac{\mathrm{e}^{\hat{k}}}{\mathrm{E}(A_{2T}^{\beta})}\Big\} \\ &= \Big\{\omega : \ln A_{1T} - \beta \ln A_{2T} > \hat{k} - \beta\mu_2 - \frac{1}{2}\beta^2\delta_2^2\Big\}. \end{aligned} \tag{A.12}$$

Thus, the price of the AASO is given by

$$\begin{aligned} AASO(S_1, S_2, K) &= \mathrm{E}\{\mathrm{e}^{-rT}(A_{1T} - A_{2T} - K)\mathbf{1}_B\}, \\ &= \mathrm{e}^{-rT}(\hat{I}_1 - \hat{I}_2 - \hat{I}_3), \end{aligned}$$

where

$$\begin{aligned} \hat{I}_1 &= \mathrm{E}(A_{1T}\mathbf{1}_B) = \mathrm{E}\Big[e^{\ln A_{1T}}\mathbf{1}_{(\ln A_{1T} - \beta\ln A_{2T} > \ln D_2)}\Big] = e^{\mu_1 + \frac{1}{2}\delta_1^2}\mathrm{N}(\hat{d}_1), \\ \hat{I}_2 &= \mathrm{E}(A_{2T}\mathbf{1}_B) = \mathrm{E}\Big[e^{\ln A_{2T}}\mathbf{1}_{(\ln A_{1T} - \beta\ln A_{2T} > \ln D_2)}\Big] = e^{\mu_2 + \frac{1}{2}\delta_2^2}\mathrm{N}(\hat{d}_2), \\ \hat{I}_3 &= K\mathrm{E}(\mathbf{1}_B) = KQ\Big(\ln A_{1T} - \beta\ln A_{2T} > \ln D_2\Big) = K\mathrm{N}(\hat{d}_3). \end{aligned}$$

By combining the expressions above, we obtain the stated formulae.