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Research article

(ϵ, δ)-complex anti fuzzy subgroups and their applications

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Abstract: The complex anti-fuzzy set (CAFS) is an extension of the traditional anti-fuzzy set with a wider range for membership function beyond real numbers to complex numbers with unit disc aims to address the uncertainty of data. The complex anti-fuzzy set is more significant because it provides two dimensional information and versatile representation of vagueness and ambiguity of data. In terms of the characteristics of complex anti-fuzzy sets, we proposed the concept of (ϵ, δ)-CAFSs that offer a more comprehensive representation of the uncertainty of data than CAFSs by considering both the magnitude and phase of the membership functions and explain the (ϵ, δ)-complex anti fuzzy subgroups (CAFSG) in the context of CAFSs. Moreover, we showed that every CAFSG is a (ϵ, δ)-CAFSG. Also, we used this approach to define (ϵ, δ)-complex anti-fuzzy(CAF) cosets and (ϵ, δ)-CAF normal subgroups of a certain group as well as to investigate some of their algebraic properties. We elaborated the (ϵ, δ)-CAFSG of the classical quotient group and demonstrated that the set of all (ϵ, δ)-CAF cosets of such a particular CAFs normal subgroup formed a group. Furthermore, the index of (ϵ, δ)-CAFSG was demonstrated and (ϵ, δ)-complex anti fuzzification of Lagrange theorem corresponding to the Lagrange theorem of classical group theory was briefly examined.

Keywords: complex anti-fuzzy set; (ϵ, δ)-complex anti-fuzzy set; (ϵ, δ)-complex anti-fuzzy subgroup; (ϵ, δ)-complex anti-fuzzy normal subgroup
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1. Introduction

Numerous applications of algebraic theory can be found not only in theoretic and practical mathematics such as game theory, algebraic geometry, etc. but also in other scientific disciplines like physics, genetics, and engineering. Group theory is a fundamental branch of algebra that investigates the properties and structures of various groups. It plays a central role in various areas of mathematics, such as physics, chemistry, and computer science, including cryptography, algebraic geometry, algebraic number theory, harmonic analysis, etc. [1–7]. Life is filled with unpredictability, which is impossible to avoid. This universe is also not built on accurate measurements or suppositions. Sometimes, the classical mathematical framework of probability is unable to handle every situation. The novel idea of fuzzy sets introduced by Zadeh [8], is briefly explained by the uncertainty, vagueness, and ambiguity of data. A wide range of academics from other disciplines have used this idea because it was so inspirational. By taking fuzzy sets and logic into consideration, a number of novel theories are developed in parallel with traditional approaches. In 1970, Rosenfeld [9] proposed the fuzzy concepts into group theory, and classified the outcomes as fuzzy subgroup. The discussion of the fuzzy subgroups, fuzzy quotient groups, and fuzzy normal subgroups are also done in this research work. Ray [10] pioneered the idea of cartesian product of a the fuzzy subgroups. In 1986, Atanassov [11] published his first article on intuitionistic fuzzy (IF) sets, which is an extension of fuzzy sets, and introduced certain operations, like subtraction, addition, composition union, and intersection under the influence of the intuitionistic fuzzy set. Biswas introducced the IF subgroup with basic findings [12], and Sharma investigated some fundamental results of the IF subgroup. Also, IF homomorphism is under the influence of group theory [13, 14].

Gulzar et al. [15] established a new category of t-IF-subgroups. The explanation of the t-IF centralizer, normalizer, and t-intuitionistic Abelian subgroups are also discussed. Intuitionistic fuzzy set techniques have acquired importance over fuzzy set techniques in recent years throughout a number of technical fields. The distance measurements approach is used in a variety of applications of IF sets. Researcher have used IF sets in a variety of situations in clinical diagnosis, medical application, etc. It plays a very important role in engineering issues, professional selection, real-life issues, and education. In 2001, Supriya et al. [16–18] studied the Sanchez's approach for medical diagnosis and extended this theory with the notion of the IF set theory.

Biswas [19] presented the principle of anti-fuzzy subgroups and initiated the fundamental algebraic structures. The fundamental results of anti-fuzzy subgroup are discussed and the relationships between complements of fuzzy subgroup and anti-fuzzy subgroup are also addressed [20]. In 2013, Azam et al. [21] introduced a few basic operations and structures of anti-fuzzy ideals of ring. Gang [22] introduced the factor rings and investigated some results. In 1999, Kim and Jun [23] developed the novel idea of anti fuzzy R-subgroups of near rings, and Kim et al. [24] initiated the anti-fuzzy ideals in near rings, discussed basic algebraic properties, and established the relation between the near rings and anti-fuzzy sets. Sharma [25] developed the definition of α-anti fuzzy subgroup and explored the fundamental algebraic structure of the α-anti-fuzzy subgroup. In addition, the techniques of the α-anti-fuzzy normal subgroups and quotient group of α-anti-fuzzy cosets are also explained. In 2022, Razaq [26] introduced the concept of Pythagorean fuzzy normal subgroups, Pythagorean fuzzy isomorphism, and developed the basic characteristics of Pythagorean fuzzy normal subgroups and proved the fascinating results of Pythagorean fuzzy isomorphism. Moreover, they looked at the concept of Pythagorean fuzzy ideas and investigated some results [27]. Xiao et. al [28] presented the q-ROFDM model with new score function, and the best-worst methods for manufacturer selection also discussed the fuzzy criteria weights, and several comparisons are conducted to illustrate the developed model.

Sharma [29] applied the fundamental properties of group theory to the (α,β)-anti fuzzy set and introduced the (α,β)-anti-fuzzy subgroup, which is an extension of the (α,β)-anti fuzzy set. They also, demonstrated the basics of the result of the (α,β)-anti-fuzzy subgroup and certain features of this ideology are discussed. Moreover, they investigated the homomorphic images and pre-images of certain group. Wan et al. [30] presented the method for interactive and complementary feature selection via fuzzy multigranularity uncertainty measures and compared them with the benchmark approaches on several datasets.

Further, changes in the process (periodicity) of the data overlap with uncertainty in our daily lives and ambiguity in the data. Due to the insufficiency of current hypotheses that provide explanations for the information, data is lost during the process. Ramot et al. [31, 32] initiated a complex fuzzy set (CFS) to deal with the problem by extending the range of the membership function from real numbers to complex numbers with the unit disc. Because the CFS considers only the degree of membership than the non-membering part of data entities, which also play an equal role in the decision-making process for evaluating the system, it only gives weight to the degree of membership. However, it is frequently difficult to describe membership degree estimation by a fuzzy set's accurate value in the real world. This may reflect using two-dimensional information than one in these circumstances, when it may be simpler to reflect the vagueness and ambiguity that exist in the real world. Given that uncertainties are uneasy to be evaluated in the complex problem of decision-making, an expansion of the existing theories may therefore be very helpful for explaining uncertainties. To address this, Alkouri and Salleh [33, 34] examined the fundamental features of complex intuitionistic fuzzy sets and extended the definition of CFSs to consist of complex degrees of non-membership functions.

Furthermore, Gulzar et al. [35] introduced the idea of Q-complex fuzzy subrings and covered some of their basic algebraic features. Additionally, the examine the homomorphic image and invert image of Q-complex fuzzy subrings, and enlarge this concept to develop the concept of the direct product of two Q-complex fuzzy subrings. Hanan et al. [36] started the abstraction of (α,β)-CFSs and defined (α,β)-complex fuzzy subgroups (CFSG). After that, they established that each CFSG is a (α,β)-CFSG and defined (α,β)-complex fuzzy normal subgroups of a given group. This concept is expanded to define (α,β)-complex fuzzy cosets, and some of their algebraic properties are examined.
The following are the motivation of this novel work.

1) Biswas [19] presented the principle of anti-fuzzy subgroups and initiated the fundamental algebraic structures. Sharma [25] developed the definition of α-anti fuzzy subgroup and explored the fundamental algebraic structure of α-anti-fuzzy subgroup. Sharma [29] applied the fundamental properties of group theory to the (α,β)-anti fuzzy set and introduced (α,β)-anti-fuzzy subgroup, which is an extension of the (α,β)-anti fuzzy set.

2) Ramot et al. [31, 32] initiated a CFS to deal with the problem by extending the range of the membership function from real numbers to complex numbers with the unit disc. Because the CFS considers only the degree of membership than the non-membering part of data entities, which also play an equal role in the decision-making process for evaluating the system, it gives weight only to the degree of membership.

3) The proppsed method is (ϵ, δ)-CAFSG. is a generalized form of CAFSG. The motivation for the recommended concept is expressed as follows: (1) To communicate a general concept such as the (ϵ, δ)-CAFSG; (2) For $\epsilon = 1$ and $\delta = 2\pi$, the idea that we propose can be convert into a classical CAFS. As a effective generalization of fuzzy subgroups, the (ϵ, δ)-CAFSGs are the subject of this article investigation.

1.1. Objectives

1) To propose the concept of (ϵ, δ)-CAFSs, examine the (ϵ, δ)-CAFSG in the context of CAFSs and prove that every complex fuzzy subgroup is a (ϵ, δ)-CAFSG.

2) To define (ϵ, δ)-CAF cosets and (ϵ, δ)-CAFNSGs of a certain group, as well as to investigate some algebraic properties under the (ϵ, δ)-CAFSG. We elaborate the (ϵ, δ)-CAFSG of the classical quotient group.

3) To demonstrate the index of (ϵ, δ)-CAFSG and (ϵ, δ)-complex anti-fuzzification of the Lagrange theorem corresponding to the Lagrange theorem of classical group theory.

This paper is organized as follows: Section 1 introduces the fundamental concepts of complex anti fuzzy sets, complex anti fuzzy subgroups, and related features. In Section 2, we construct (ϵ, δ)-CAFS and (ϵ, δ)-CAFSG as generalizations of CAFSG. We show that any complex anti fuzzy subgroup is also a (ϵ, δ)-CAFSG, and examined some of the essential aspects of these newly define CAFSGs. In Section 3, the (ϵ, δ)-CAF cosets and (ϵ, δ)-CAFNSGs are describe and various algebraic properties of these particular groups are investigate. Furthermore, we discuss (ϵ, δ)-complex anti fuzzy quotient groups (CAFQG) and establish the quotient group with regard to (ϵ, δ)-CAF cosets. The indices of the (ϵ, δ)-CAFSG is define and the (ϵ, δ)-complex anti fuzzification of Lagrange's theorem is develop.

2. Preliminaries

We start by analyzing the fundamental idea of $CAFS$s and $CAFSG$s, both are essential for study.

Definition 2.1. [8] If H is universal set and x is an arbitrary element of H then an anti-fuzzy set φ is define as $\varphi = \{(x, \ \lambda), x \in H\}$, where λ is a non membership function and $\lambda \in [0, 1]$.

Definition 2.2. [37] A $CFS\ S$ of a universe set H, characterized by the degree of membership $\theta_S(l) = \nu_S(l)e^{i\eta_S(l)}$ and is defined as $\theta_S : l \to \{l \in H : |l| \leq 1\}$, H is complex plain. Whose range is not limited to [0 1] but extens to unit circle in complex plane, where $i = \sqrt{-1}$, $\nu_S(l)$ and $\eta_S(l)$ are both real valued including $\nu_S(l) \in [0, 1]$ and $\eta_S(l) \in [0, 2\pi]$. As for purpose of simplicity, we will employ $\nu_S(l)e^{i\eta_S(l)}$ membership function for complex fuzzy set S.

Definition 2.3. [11] Assume that $S = \{(l, \ \rho_S(l)) : \ l \in H\}$ be a anti fuzzy subset where H is a universal set. Now the set

$$S_\pi = \{(l, \vartheta_{S_\pi}(l)) : \vartheta_{S_\pi}(l) = 2\pi\rho_S(l), l \in G\}$$

is called π-anti fuzzy subset.

Definition 2.4. [11] A π-anti fuzzy set S_π of group G is known as π-anti fuzzy subgroup of G if the following conditions are satisfied

(i) $S_\pi(lm) \le \max\{S_\pi(l), S_\pi(m)\}$, $\forall\, l, m \in G$,

(ii) $S_\pi(l^{-1}) \le S_\pi(l)$, $\forall\, l, m \in G$.

Definition 2.5. [11] Assume $S = \left\{(l, \nu_S(l)\, e^{i\eta_S(l)}) : l \in G\right\}$ and $T = \left\{(l, \nu_T(l)\, e^{i\eta_T(l)}) : l \in G\right\}$ are both $CAFS$s of G. Then

(i) A $CAFS$ S is homogeneous $CAFS$, if $\forall\, l, m \in G$, we have $\nu_S(l) \le \nu_S(m)$ if and only if $\eta_S(l) \le \eta_S(m)$.

(ii) A $CAFS$ A is homogeneous complex anti fuzzy set with B, if $\forall\, p, q \in G$, we have $\nu_A(p) \le \nu_B(p)$ if and only if $\eta_A(p) \le \eta_B(p)$.

Definition 2.6. [35] Let $S = \left\{(l, \nu_S(l)\, e^{i\eta_S(l)}) : l \in G\right\}$ and $T = \left\{(l, \nu_T(l)\, e^{i\eta_T(l)}) : l \in G\right\}$ be a CAF subsets of set G. Then intersection and union of S and T is examined as:

(i) $(S \cap T)(l) = \nu_{S\cap T}(l)\, e^{i\eta_{S\cap T}(l)}$
$= \max\left\{\nu_S(l)\, e^{i\eta_S(l)}, \nu_S(l)\, e^{i\eta_S(l)}\right\}$, $\forall\, l \in L$.

(ii) $(S \cup T)(l) = \nu_{S\cup T}(l)\, e^{i\eta_{S\cup T}(l)}$
$= \min\left\{\nu_S(l)\, e^{i\eta_S(l)}, \nu_S(l)\, e^{i\eta_S(l)}\right\}$, $\forall\, l \in L$.

Definition 2.7. [11] Let S be a$CAFS$ of group G. Then S is know as $CAFSG$ of group G, if the following criteria are fulfilled.

(i) $\nu_S(lm)\, e^{i\eta_S(lm)} \le \max\left\{\nu_S(l)\, e^{i\eta_S(l)},\ \nu_S(m)\, e^{i\eta_S(q)}\right\}$,

(ii) $\nu_S\left(l^{-1}\right) e^{i\eta_S\left(l^{-1}\right)} \le \nu_S(l)\, e^{i\eta_S(l)}$ for all $l, m \in G$.

Definition 2.8. [11] A complex anti fuzzy set S of group G is said to be $CAFNSG$ of group G, if: $\nu_S(lm)\, e^{i\eta_S(lm)} = \nu_S(ml)\, e^{i\eta_S(ml)}$, for all $l, m \in G$.

Definition 2.9. [25] Let S be a anti fuzzy subset of a group G. Then anti fuzzy set S_ϵ of G is known as ϵ-anti fuzzy subset of G, where $\epsilon \in [0, 1]$ and define as $S_\epsilon(p) = \max\{S(p), 1 - \epsilon\}$ for all $p \in G$.
Some results:

(i) (i) Let S and T be two anti fuzzy subsets of X. Then
$(S \cup T)_\epsilon = S_\epsilon \cup T_\epsilon$.

(ii) (ii) Suppose g : $L \to M$ be a mapping and S and T be two anti fuzzy subsets of L and M sequentially, then
(a) $g^{-1}(T_\epsilon) = (g^{-1}(T))_\epsilon$,
(b) $g(T)_\epsilon = (g(T))_\epsilon$.

Definition 2.10. [38] Suppose S^ϵ and S_δ respectively indicate, the ϵ- fuzzy set and δ-anti fuzzy set of L, where L is universal set. Then the anti fuzzy set $S_{(\epsilon,\delta)}$ is define by
$S_{(\epsilon,\delta)}(u)$=min$\{u,\ (S^\epsilon)^c(u), S_\delta(u)\}\forall u \in L$ and is called $S_{(\epsilon,\delta)}$-anti fuzzy set of L due respect the fuzzy set S, where $\epsilon, \delta \in [0, 1]$ such that $\epsilon + \delta \le 1$.

Remark 2.11.

(i) $S_{(0,1)}(u)$=min$\{(S^1)^c(u), S_0(u)\}$=min$\{S^c(u), 1\} = 1$,

(ii) $S_{(0,1)}(u)$=min$\{(S^0)^c(u), S_1(u)\}$=min$\{1, S^c(u)\} = 1$.

3. Algebraic attributes of (ϵ, δ)-complex anti fuzzy subgroups

Now this section introduces the (ϵ, δ)-$CAFS$ and (ϵ, δ)-$CAFSG$s methodology. We establish that any complex fuzzy subgroup is also a (ϵ, δ)-$CAFSG$ but the converse does not hold and we explore certain fundamentals categorization of this phenomena.

Definition 3.1. Let $S = \{(l, \mu_S(l)\, e^{i\eta_S(l)}) : l \in G\}$ be $CAFS$ of group G, for any $\epsilon \in [0, 1]$ and $\delta \in [0, 2\pi]$, such that $\mu_S(l) \geq \epsilon$ and $\eta_S(l) \geq \delta$ or ($\nu_S(l) \leq \epsilon$ and $\eta_S(l) \leq \delta$). Then, the set $S_{(\epsilon,\delta)}$ is called (ϵ, δ)-$CAFS$t and defined as: $\nu_{S_\epsilon}(l)e^{i\eta_{S_\delta}(l)} = \max\{\nu_S(l)e^{i\eta_{S_\delta}(l)},\ \epsilon e^{i\delta}\} = \max\{\nu_S(l),\ \epsilon\}e^{i\max\{\eta_S(l),\ \delta\}}$, where $\nu_{S_\epsilon}(l) = \max\{\nu_S(l), \epsilon\}$ and $\eta_{S_\delta}(l) = \max\{\eta_S(l), \delta\}$.

Throughout manuscript, we will focused on the non-membership function of (ϵ, δ)-$CAFS$s $S_{(\epsilon,\delta)}$ and $T_{(\epsilon,\delta)}$ such as $\nu_{S_\epsilon}(l)e^{i\eta_{S_\delta}(l)}$ and $\nu_{T_\epsilon}(l)e^{i\eta_{T_\delta}(l)}$, respectively.

Definition 3.2. Let $S_{(\epsilon,\delta)}$ and $T_{(\epsilon,\delta)}$ be a two (ϵ, δ)-$CAFS$s of G. Then

(i) A (ϵ, δ)-$CAFS$ $S_{(\epsilon,\delta)}$ is homogeneous (ϵ, δ)-$CAFS$, for all $l, m \in G$, we have $\nu_{S_\epsilon}(l) \geq \nu_{S_\epsilon}(m)$ if and only if $\eta_{S_\delta}(l) \geq \eta_{S_\delta}(m)$.

(ii) A (ϵ, δ)-$CAFS$ $S_{(\epsilon,\delta)}$ is homogeneous (ϵ, δ)-$CAFS$ with $T_{(\epsilon,\delta)}$, for all $l, m \in G$, such that $\nu_{S_\epsilon}(l) \geq \nu_{T_\epsilon}(l)$ if and only if $\eta_{S_\delta}(l) \geq \eta_{T_\delta}(l)$.

In this research article, we use (ϵ, δ)-$CAFS$ as homogeneous (ϵ, δ)-complex anti fuzzy set.

Remark 3.3. By taking the values of $\epsilon = 1$ and $\delta = 2\pi$ in the given definition, we obtain the classical $CAFS$ S.

Remark 3.4. Let $S_{(\epsilon,\delta)}$ and $T_{(\epsilon,\delta)}$ be two (ϵ, δ)-$CAFS$s of group G. Then $(S\cap M)_{(\epsilon,\delta)} = S_{(\epsilon,\delta)} \cap T_{(\epsilon,\delta)}$.

Definition 3.5. Let $S_{(\epsilon,\delta)}$ be an (ϵ, δ)-$CAFS$ of group G for $\epsilon \in [0, 1]$ and $\delta \in [0, 2\pi]$. Then $S_{(\epsilon,\delta)}$ is known as (ϵ, δ)-$CAFS$G of group G, if it satisfy the following conditions:

(i) $\nu_{S_\epsilon}(lq)e^{i\eta_{S_\delta}(lq)} \geq \max\{\ \nu_{S_\epsilon}(l)e^{i\eta_{S_\delta}(l)},\ \nu_{S_\epsilon}(q)e^{i\eta_{S_\delta}(q)}\}$,

(ii) $\nu_{S_\epsilon}(l^{-1})e^{i\eta_{S_\delta}(l^{-1})} \leq \nu_{S_\epsilon}(l)e^{i\eta_{S_\delta}(l)}$ for all $l, m \in G$.

Theorem 3.6. *If $S_{(\epsilon,\delta)}$ is an (ϵ, δ)-$CAFSG$ of group G, for all $l, m \in G$. Then*

(i) $\nu_{S_\epsilon}(l)e^{i\eta_{S_\delta}(l)} \geq \nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}$,

(ii) $\nu_{S_\epsilon}(lm^{-1})e^{i\eta_{S_\delta}(lm^{-1})} = \nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}$.

It suggests that $\nu_{S_\epsilon}(l)e^{i\eta_{S_\delta}(l)} = \nu_{S_\epsilon}(m)e^{i\eta_{S_\delta}(m)}$.

The proof of this theorem is straightforward.

Now, in this theorem we show that $CAFNSG$ is a spacial case of (ϵ, δ)- $CAFNSG$.

Theorem 3.7. *Every $CAFSG$ of the group G is also a (ϵ, δ)-$CAFSG$ of G.*

Proof. Assume that S be $CAF\mathbb{S}$G of group G, for every $l, m \in G$. Suppose that

$$\nu_{S_\epsilon}(lm)e^{i\epsilon_{S_\delta}(lm)} = \max\{\nu_S(lm)e^{i\epsilon_S(lm)},\ \epsilon e^{i\delta}\}$$

$$
\begin{aligned}
&\leq \max\{\max\{\nu_S(l)e^{i\epsilon_S(l)},\ \nu_S(m)e^{i\epsilon_S(m)}\}\ ,\ \epsilon e^{i\delta}\} \\
&= \max\{\max\{\nu_S(l)e^{i\epsilon_S(l)}, \epsilon e^{i\delta}\}\ , \\
&\quad \max\{\nu_S(m)e^{i\epsilon_S(m)},\ \epsilon e^{i\delta}\}\ \} \\
&= \max\{\nu_{S_\epsilon}(l)e^{i\epsilon_{S_\delta}(l)}, \nu_{S_\epsilon}(m)e^{i\epsilon_{S_\delta}(m)}\}\ .
\end{aligned}
$$

Further, we assume that

$$
\begin{aligned}
\nu_{S_\epsilon}(l^{-1})e^{i\epsilon_{S_\delta}(l^{-1})} &= \max\{\ \nu_S(l^{-1})e^{i\epsilon_S(l^{-1})}, \epsilon e^{i\delta}\ \} \\
&\leq \max\{\ \nu_S(l)e^{i\epsilon_S(l)}, \epsilon e^{i\delta}\ \} \\
&= \nu_{S_\epsilon}(l)e^{i\epsilon_{S_\delta}(l)}.
\end{aligned}
$$

This established the proof.

Remark 3.8. If $S_{(\epsilon,\delta)}$-$CAFSG$ then it is not essential S is $CAFSG$.

Example 3.9. The Klein four group is referred by $G = \{e, l, m, lm\}$. It can be written as $S = \{< e,\ 0.2e^{i\frac{\pi}{12}} >, < l, 0.4e^{i\frac{\pi}{6}} >, < m, 0.4e^{i\frac{\pi}{6}} >,\ < lm,\ 0.3e^{i\frac{\pi}{7}} >\}$ is not $CAFSG$ of G. Take $\epsilon = 0.2$ and $\delta = \frac{\pi}{6}$. Then, it's simple to see $\nu_S(l)e^{i\eta_S(l)} > \epsilon e^{i\delta}$, for all $l \in G$. Moreover, we have $\nu_{S_\epsilon}(l)e^{i\eta_{S_\delta}(l)} = \epsilon e^{i\delta},\ \forall\ l \in G$. Therefore, $\nu_{S_\epsilon}(lm)e^{i\eta_{S_\delta}(lm)} \leq \max\{\nu_{S_\epsilon}(l)e^{i\eta_{S_\delta}(l)},\ \nu_{S_\epsilon}(m)e^{i\eta_{S_\delta}(m)}\}$, $\forall l, m \in G$. Furthermore, $l^{-1} = l,\ m^{-1} = m,\ (lm)^{-1} = lm$. So, $\nu_{S_\epsilon}(l^{-1})e^{i\eta_{S_\delta}(l^{-1})} \geq\ \nu_{S_\epsilon}(l)e^{i\eta_{S_\delta}(l)}$. Hence, $S_{(\epsilon,\delta)}$ is (ϵ, δ)-$CAFSG$.

Theorem 3.10. *Let S be a complex anti fuzzy set of group G such that $\nu_S(l^{-1})e^{i\epsilon_S(l^{-1})} = \nu_S(l)e^{i\epsilon_S(l)},\ \forall\ l \in G$. Let $\epsilon e^{i\delta} \geq re^{i\theta}$ such that $\epsilon \geq r$ and $\delta \geq \theta$, where $re^{i\theta} = \max\{\nu_S(l)e^{i\epsilon_S(l)} : l \in G\}$ and $\epsilon, r \in [0, 1]$ and $\delta, \theta \in [0, 2\pi]$. Then $S_{(\epsilon,\delta)}$ is an (ϵ, δ)-$CAFSG$ of G.*

Proof. Note that $\epsilon e^{i\delta} \geq re^{i\theta}$. Implies that $\max\{\nu_S(l)e^{i\epsilon_S(l)} : l \in G\} \leq \epsilon e^{i\delta}$. This indicates $\max\{\nu_S(l)e^{i\epsilon_S(l)}, \epsilon e^{i\delta}\} = \epsilon e^{i\delta}$, for all $l \in G$. Implies that $\nu_{S_\epsilon}(l)e^{i\epsilon_{S_\delta}(l)} = \epsilon e^{i\delta}$.

$$
\begin{aligned}
\nu_{S_\epsilon}(lm)e^{i\epsilon_{S_\delta}(lm)} &\leq \max\{\nu_{S_\epsilon}(l)e^{i\epsilon_{S_\delta}(l)}, \\
&\qquad \nu_{S_\epsilon}(m)e^{i\epsilon_{S_\delta}(m)}\}. \\
\text{Moreover,}\qquad \nu_S(l^{-1})e^{i\epsilon_S(l^{-1})} &= \nu_S(l)e^{i\epsilon_S(l)}, \forall\ l \in G. \\
\text{Implies that,}\qquad \nu_{S_\epsilon}(l^{-1})e^{i\epsilon_{S_\delta}(l^{-1})} &= \nu_{S_\epsilon}(l)e^{i\epsilon_{S_\delta}(l)}.
\end{aligned}
$$

Hence, $S_{(\epsilon,\delta)}$ is (ϵ, δ)-$CAFSG$ of G.

Theorem 3.11. *Intersection of two (ϵ, δ)-$CAFSG$s of G is also (ϵ, δ)-$CAFSG$ of G.*

Proof. Let $S_{(\epsilon,\delta)}$ and $T_{(\epsilon,\delta)}$ be two (ϵ, δ)-$CAFSG$s of G, for any $l, m \in G$.

Consider,

$\nu_{(S\cap T)_\epsilon}(lm)e^{\epsilon_{(S\cap T)_\delta}(lm)} = \nu_{(S_\epsilon \cap T_\epsilon)}(lm)e^{i\epsilon_{S_\delta \cap T_\delta}(lm)}$

$$
\begin{aligned}
&= \max\{\nu_{S_\epsilon}(lm)e^{i\epsilon_{S_\delta}(lm)},\ \nu_{T_\epsilon}(lm)e^{i\epsilon_{T_\delta}(lm)}\} \\
&\leq \max\left\{\begin{array}{l} \max\{\nu_{S_\epsilon}(l)e^{i\epsilon_{S_\delta}(l)},\ \nu_{S_\epsilon}(m)e^{i\epsilon_{S_\delta}(m)}\}\ , \\ \max\{\nu_{T_\epsilon}(l\ )e^{i\epsilon_{T_\delta}(l)}\ \nu_{T_\epsilon}(m)e^{i\epsilon_{T_\delta}(m)}\}\ . \end{array}\right\} \\
&= \max\left\{\begin{array}{l} \max\{\nu_{S_\epsilon}(l)e^{i\epsilon_{S_\delta}(l)}, \nu_{T_\epsilon}(l)e^{i\epsilon_{T_\delta}(l)}\}\ , \\ \max\{\nu_{S_\epsilon}(m)e^{i\epsilon_{T_\delta}(m)}, \nu_{T_\epsilon}(m)e^{i\epsilon_{T_\delta}(m)}\}. \end{array}\right\}
\end{aligned}
$$

$$= \max\{\nu_{(S_\epsilon \cap T_\delta)}(l)e^{i\epsilon_{(S_\delta \cap T_\delta)}(l)},\ \nu_{(S_\epsilon \cap T_\delta)}(m)e^{i\epsilon_{(S_\delta \cap T_\delta)}(m)}\}$$
$$= \max\{\nu_{(S\cap T)_\epsilon}(l)e^{i\epsilon_{(S\cap T)_\delta}(l)}, \nu_{(S\cap T)_\epsilon}(m)e^{i\epsilon_{(S\cap T)_\delta}(m)}\}\ .$$

Further,

$$\nu_{(S\cap T)_\epsilon}(l^{-1})e^{\epsilon_{(S\cap T)_\delta}(l^{-1})} = \nu_{S_\epsilon \cap T_\epsilon}(l^{-1})e^{i\epsilon_{(S_\delta \cap T_\delta)}(l^{-1})}$$
$$= \max\{\nu_{S_\epsilon}(l^{-1})e^{i\epsilon_{S_\delta}(l^{-1})},\ \nu_{T_\epsilon}(l^{-1})e^{i\epsilon_{T_\delta}(l^{-1})}\}$$
$$\leq \max\{\nu_{S_\epsilon}(l)e^{i\epsilon_{S_\delta}(l)},\ \nu_{T_\epsilon}(l)e^{i\epsilon_{T_\delta}(l)}\}$$
$$= \nu_{(S\cap T)_\epsilon}(l)e^{\epsilon_{(S\cap T)_\delta}(l)}.$$

Consequently, $S_{(\epsilon,\delta)} \cap T_{(\epsilon,\delta)}$ is (ϵ, δ)-$CAFSG$ of G.

Corollary 3.12. *Intersection of a family of (ϵ, δ)-$CAFSG$s of G is also (ϵ, δ)-$CAFSG$.*

Remark 3.13. Union of two (ϵ, δ)-$CAFSG$s may not be a (ϵ, δ)-complex anti fuzzy subgroup.

Example 3.14. Assume that a symmetric group S_4 with permutation of four elements$\{(1), (2\ 3), (2\ 3\ 4),$ $(2\ 4\ 3), (3\ 4), (2\ 4), (1\ 2), (1\ 2\ 4), (1\ 2\ 3), (1\ 2\ 3\ 4), (1\ 2)(3\ 4), (1\ 2\ 4), (1\ 3\ 2), (1\ 3\ 4\ 2), (1\ 3), (1\ 3\ 4), (1\ 3\ 2\ 4),$ $(1\ 3)(2\ 4), (1\ 4\ 3\ 2), (1\ 4\ 2), (1\ 4\ 3), (1\ 4), (1\ 4\ 2\ 3), (1\ 4)(2\ 3)\}$. Define two (ϵ, δ)-$CAFSG$s $S_{(0.9,\pi/2)}$ and $T_{(0.6,\pi/2)}$ of S_4 for value $\epsilon e^{i\delta} = 0.9e^{\pi}$ are delivered as:

$$S_{(0.9,\pi/2)}(l) = \begin{cases} 0.8e^{\pi/4}, & \text{if } l \in <(1\ 3)> \\ 0.7e^{\pi/6}, & \text{otherwise} \end{cases} \quad \text{and}$$
$$T_{(0.9,\pi/2)}(l) = \begin{cases} 0.9e^{\pi/2}, & \text{if } l \in <(1\ 3\ 2\ 4)> \\ 0.6e^{\pi/7}, & \text{otherwise} \end{cases}$$

indicates that $$(S_{(0.9,\pi/2)} \cup T_{(0.9,\pi/2)})(l) = \begin{cases} 0.8e^{\pi/4}, & \text{if } l \in <(1\ 3\ 2\ 4)> \cap <(1\ 3)> \\ 0.7e^{\pi/6}, & \text{if } l \in <(1\ 3\ 2\ 4)> -e \\ 0.6e^{\pi/7}, & \text{if } l \in <(1\ 3)> -e \end{cases}$$

Take $l = (1\ 2)(3\ 4)$, $m = (1\ 3)$ and $lm = (1\ 2\ 3\ 4)$. Moreover, $(S_{(0.9,\pi/2)} \cup T_{(0.9,\pi/2)})(l) = 0.7e^{\pi/6}$. $(S_{(0.9,\pi/2)} \cup T_{(0.9,\pi/2)})(l) = 0.6e^{\pi/7}$ and $(S_{(0.9,\pi/2)} \cup T_{(0.9,\pi/2)})(lm) = 0.6e^{\pi/7}$.

We can clearly observe that $(S_{(0.9,\pi/2)} \cup T_{(0.9,\pi/2)})(lm) \nleq \max\{(S_{(0.9,\pi/2)} \cup T_{t(0.9,\pi/2)})(l), (S_{(0.9,\pi/2)} \cup T_{(0.9,\pi/2)})(m)\}$. So, this establishes the assertion.

4. (ϵ, δ)-complex anti fuzzification of Lagrange's theorem

The algebraic features of (ϵ, δ)-$CAFNSG$s are explore in this section. We investigate (ϵ, δ)-CAF cosets of (ϵ, δ)-$CAFSG$s and create a quotient framework that focuses on these $CAFNSG$s. The (ϵ, δ)-$CAFSG$ of the classical quotient group is also discussed and several key characteristics of these $CAFNSG$s are illustrated.

Definition 4.1. Suppose that $S_{(\epsilon,\delta)}$ be an (ϵ, δ)-$CAFS$G of group G, as $\epsilon \in [0, 1]$ and $\eta \in [0, 2\pi]$. Then (ϵ, δ)-$CAFS$ $lS_{(\epsilon,\delta)}(w) = \{(w, \nu_{lS_\epsilon}(w)e^{i\eta_{lS_\eta}(w)}),\ w \in G\}$ of G is known as a(ϵ, δ)-CAF left coset of G examine by $S_{(\epsilon,\delta)}$ and is define as:

$$\nu_{lS_\epsilon}(w)e^{i\eta_{lS_\eta}(w)} = \nu_{S_\epsilon}(l^{-1}w)e^{i\eta_{S_\eta}(l^{-1}w)}$$

$$= \max\{\nu_S(l^{-1}w)e^{i\eta_S(l^{-1}w)}, \epsilon e^{i\delta}\}, \forall w, l \in G.$$

In same way we explain (ϵ, δ)-CAF right coset $S_{(\epsilon,\delta)}w = \{(w, \nu_{S_\epsilon l}(w)e^{i\eta_{S_\delta l}(w)}),\ w \in G\}$ of of G determine by $S_{(\epsilon,\delta)}$ and l also define as : $\nu_{S_\epsilon l}(w)e^{i\eta_{S_\delta l}(w)} = \nu_{S_\epsilon}(wl^{-1})e^{i\eta_{S_\delta}(wl^{-1})} = \max\{\nu_S(wl^{-1})e^{i\eta_S(wl^{-1})},\ \epsilon e^{i\delta}\}$, for all w , $l \in G$.

The next given example demonstrates the concept of (ϵ, δ)-CAF cosets of $S_{(\epsilon,\delta)}$.

Example 4.2. Take $G = \{(1), (1\ 3), (1\ 2)(3\ 4), (2\ 4), (1\ 4)(2\ 3),$
$(1\ 4\ 3\ 2), (1\ 3)(2\ 4), (1\ 2\ 3\ 4)\}$ a symmetric group with 8 elements represent (ϵ, δ)-$CAFSG$ of G only when $\epsilon = 0.4$ and $\delta = \pi/6$ as follows: $S_{(0.4,\pi/6)}(w)$

$$= \begin{cases} 0.9e^{\pi} & \text{if } w \in \{(1\ 3)(2\ 4), (1)\} \\ 0.8e^{\pi/3}, & \text{if } w \in \{(1\ 2)(3\ 4), (1\ 4)(2\ 3)\}, \\ 0.7e^{\pi/5}, & \text{if } w \in \{(2\ 4), (1\ 3), (1\ 2\ 3\ 4), (1\ 4\ 3\ 2)\} \end{cases}$$

From the definition of cosets we have
$\nu_{lS_{(0.4,\pi/6)}}(w)e^{\eta_{lS_{(0.4,\pi/6)}}(w)} = \nu_{S_{(0.4,\pi/6)}}(l^{-1}w)e^{\eta_{S_{(0.4,\pi/6)}}(l^{-1}w)}$.
Thus, $(0.4, \pi/6)$-CAF left coset of $S_{(0.4,\pi/6)}(w)$ in G for $l = (2\ 4)$ as seen below: $lS_{(0.4,\pi/6)}(w)$

$$= \begin{cases} 0.9e^{\pi} & \text{if } w \in \{(1\ 3)(2\ 4), (1)\} \\ 0.8e^{\pi/3}, & \text{if } w \in \{(1\ 4)(2\ 3), (1\ 2)(3\ 4)\} \\ 0.6e^{\pi/5}, & \text{if } w \in \{(2\ 4), (1\ 4\ 3\ 2), (1\ 3), (1\ 2\ 3\ 4)\} \end{cases} .$$

In same way, $(0.4, \pi/6)$-CF right coset of $S_{(0.4,\pi/6)}(w)$ is find, for every $l \in G$.

Definition 4.3. Let $S_{(\epsilon,\delta)}$ be an (ϵ, δ)-$CAFSG$ of group G, where $\epsilon \in [0, 1]$ and $\delta \in [0, 2\pi]$.Therefore $S_{(\epsilon,\delta)}$ is known as (ϵ, δ)-CAFNSG of G if $S_{(\epsilon,\delta)}(lm) = S_{(\epsilon,\delta)}(ml)$. Equivalently, (ϵ, δ)-$CAFSG$ $S_{(\epsilon,\delta)}$ is (ϵ, δ)-CAFNSG of group G if: $S_{(\epsilon,\delta)}l(m) = lS_{(\epsilon,\delta)}(m)$, for all $l,\ m \in G$.

Note that each $(1, 2\pi)$-$CAFNSG$ is a classical $CAFNSG$ of G.

Remark 4.4. Let $S_{(\epsilon,\delta)}$ be an (ϵ, δ)-CAFNSG of the group G. Then $S_{(\epsilon,\delta)}(m^{-1}lm) = S_{(\epsilon,\delta)}(l)$, for all $l, m \in G$.

Theorem 4.5. *If S is $CAFNSG$ of group G. Then $S_{(\epsilon,\delta)}$ is (ϵ, δ)- $CAFNSG$ of G.*

Proof. Assume that w, l arbitrary of elements of G. Consequently, we have $\nu_S(l^{-1}w)e^{i\eta_S(l^{-1}w)} = \nu_S(xl^{-1})e^{i\eta_S(wl^{-1})}$, This implies that, $\{\nu_S(l^{-1}w)e^{i\eta_S(l^{-1}w)}, \epsilon e^{i\delta}\} = \max\{\nu_S(wl^{-1})e^{i\eta_S(wl^{-1})}, \epsilon e^{i\delta}\}$
we obtain, $\nu_{lS_\epsilon}(w)e^{i\eta_{lS_\delta}(w)} = \nu_{S_\epsilon l}(w)e^{i\eta_{S_\delta l}(w)}$. we get $lS_{(\epsilon,\delta)}(w) = S_{(\epsilon,\delta)}l(w)$. Consequently, $S_{(\epsilon,\delta)}$ is (ϵ, δ)-$CAFNSG$ of G. In most circumstances, the converse of the following outcome is not valid. This fact is discuss in given bellow example.

Example 4.6. Suppose $G = D_3 \ =< l, m :\ l^3 = m^2 = e,\ ml = l^2m >$ be the Dihedral group. Suppose that S be a complex anti fuzzy set of G and described as:

$$S = \begin{cases} 0.5e^{\pi/4} & \text{if } w \in < m >, \\ 0.3e^{\pi/8} & \text{if } w \notin < m > . \end{cases}$$

Note that S is not a complex anti fuzzy normal subgroup of group G. For $\nu_S(l^2(lm))e^{i\eta_S(l^2(lm))} = 0.5e^{\pi/4} \neq 0.3e^{\pi/8} = \nu_S((lm)l^2)e^{i\eta_S((lm)l^2)}$. Now we take $\epsilon e^{i\delta} = 0.6e^{i\pi/3}$, we get $\nu_{lS_{0.6}}(w)e^{i\eta_{wS_{\pi/3}}} = \max\{\nu_S(l^{-1}w)e^{i\eta_{S(l^{-1}w)}}, 0.6e^{i\pi/3}\} = 0.6e^{\frac{i\pi}{3}} = \max\{\nu_S(wl^{-1})e^{i\eta_{S(wl^{-1})}}, 0.6e^{i\pi/3}\}$
$= \nu_{S_{0.6}l}(w)e^{i\eta_{S\pi/3}}(w)$.

Next, we show that each (ϵ, δ)- $CAFSG$ of group G will be (ϵ, δ)- $CAFNSG$ of group G, include some particular values of ϵ and δ. The following outcomes are illustrate in this direction.

Theorem 4.7. *Let $S_{(\epsilon,\delta)}$ be (ϵ, δ)-$CAFSG$ of group G as a result $\epsilon e^{i\delta} > re^{i\theta}$, $\epsilon \geq r$ and $\delta \geq \theta$, where $re^{i\theta} = \max\{\hat{A}\mu_S(w)e^{i\eta_S(w)}, \forall\, w \in G\}$ and r, $\epsilon \in [0, 1]$ and $\delta, \theta \in [0, 2\pi]$. So $S_{(\epsilon,\delta)}$ be $(\epsilon, \delta)-$ $CAFNSG$ of the group G.*

Proof. Given that $\epsilon e^{i\delta} \geq re^{i\theta}$. This implies $\max\{\nu_S(w)e^{i\eta_S(w)} :$ for all $w \in G\} \leq \epsilon e^{i\delta}$. This shows $\nu_S(w)e^{i\eta_S(w)} \leq \epsilon e^{i\delta}$, for all $w \in G$. So, $\nu_{lS_\epsilon}(w)e^{i\eta_{lS_\delta}(w)} = \max\{\nu_S(l^{-1}w)e^{i\eta_S(l^{-1}w)}, \epsilon e^{i\delta}\} = \epsilon e^{i\delta}$, for any $w \in G$. Similarly, $\nu_{S_\epsilon l}(w)e^{i\eta_{S_\delta l}(w)} = \max\{\nu_S(wl^{-1})e^{i\eta_S(wl^{-1})}, \epsilon e^{i\delta}\} = \epsilon e^{i\delta}$. Implies that $\nu_{lS_\epsilon}(w)e^{i\eta_{lS_\delta}(w)} = \nu_{S_\epsilon l}(w)e^{i\eta_{S_\delta l}(w)}$. Hence, it proved the result.

Theorem 4.8. *Let $S_{(\epsilon,\delta)}$ be (ϵ, δ)- $CAFNSG$ of group G. Then the set $S^e_{(\epsilon,\delta)} = \{w \in G : S_{(\epsilon,\delta)}(w^{-1}) = S_{(\epsilon,\delta)}(e)\}$ is normal subgroup of group G.*

Proof. Obviously $S^e_{(\epsilon,\delta)} \neq \eta$ because $e \in G$. Let $w, v \in S^e_{(\epsilon,\delta)}$ be any elements. Consider,
$\nu_{S_\epsilon}(wv)e^{i\eta_{S_\delta}(wv)} \leq \max\{\nu_{S_\epsilon}(w)e^{i\eta_{S_\delta}(w)}, \nu_{S_\epsilon}(v)e^{i\eta_{S_\delta}(v)}\} = \max\{\nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}, \nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}\}$. Implies that $\nu_{S_\epsilon}(wv)e^{i\eta_{S_\delta}(wv)} \leq \nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}$. However, $\nu_{S_\epsilon}(wv)e^{i\eta_{S_\delta}(wv)} \geq \nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}$. Therefore,
$\nu_{S_\epsilon}(wv)e^{i\eta_{S_\delta}(wv)} = \nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}$. It implies that $S_{(\epsilon,\delta)}(w^{-1}) = S_{(\epsilon,\delta)}(e)$. It implies that $wv \in S^e_{(\epsilon,\delta)}$. Further, $\nu_{S_\epsilon}(v^{-1})e^{i\eta_{S_\delta}(v^{-1})} \leq \nu_{S_\epsilon}(v)e^{i\eta_{S_\delta}(v)} = \nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}$. But $\nu_{S_\epsilon}(w)e^{i\eta_{S_\delta}(w)} \geq \nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}$. Thus $S^e_{(\epsilon,\delta)}$ is subgroup of group G. Moreover, let $w \in S^e_{(\epsilon,\delta)}$and $\in G$. We have $\nu_{S_{(\epsilon,\delta)}}(v^{-1}wv)e^{i\eta_{S_{(\epsilon,\delta)}}(v^{-1}wv)}$
$= \nu_{S_{(\epsilon,\delta)}}(w)e^{i\eta_{S_{(\epsilon,\delta)}}(w)}$. It implies that $y^{-1}wv \in S^e_{(\epsilon,\delta)}$. Hence, $S^e_{(\epsilon,\delta)}$ is a normal subgroup.

Theorem 4.9. *Assume that $S_{(\epsilon,\delta)}$ be an (ϵ, δ)- $CAFNSG$ of group G. Then*

(i) $lS_{(\epsilon,\delta)} = mS_{(\epsilon,\delta)}$ if and only if $l^{-1}m \in S^e_{(\epsilon,\delta)}$,

(ii) $S_{(\epsilon,\delta)}l = S_{(\epsilon,\delta)}m$ if and only if $lm^{-1} \in S^e_{(\epsilon,\delta)}$.

Proof. For any $l, m \in G$, we have $lS_{(\epsilon,\delta)} = mS_{(\epsilon,\delta)}$. Assume that,
$\nu_{S_\epsilon}(l^{-1}m)e^{i\eta_{S_\delta}(l^{-1}m)} = \max\{\nu_S(l^{-1}m)e^{i\eta_S(l^{-1}m)}, \epsilon e^{i\delta}\}$

$$\begin{aligned}
&= \max\{\nu_{lS}(m)e^{i\eta_{lS}(m)}, \epsilon e^{i\delta}\} \\
&= \nu_{lS_\epsilon}(m)e^{i\eta_{lS_\delta}(m)} \\
&= \nu_{mS_\epsilon}(m)e^{i\eta_{mS_\delta}(m)} \\
&= \max\{\nu_S(m^{-1}m)e^{i\eta_\delta(m^{-1}m)}, \epsilon e^{i\delta}\} \\
&= \max\{\nu_S(e)e^{i\eta_S(e)}, \epsilon e^{i\delta}\} \\
&= \nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}.
\end{aligned}$$

Therefore, $l^{-1}m \in S^e_{(\epsilon,\delta)}$.

Conversely, let $l^{-1}m \in S^e_{(\epsilon,\delta)}$ then $\nu_{S_\epsilon}(l^{-1}m)e^{i\eta_{S_\delta}(l^{-1}m)} = \nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}$.

Consider
$\nu_{lS_\epsilon}(a)e^{i\eta_{lS_\delta}(a)} = \max\{\nu_S(l^{-1}a)e^{i\eta_S(l^{-1}a)}, \epsilon e^{i\delta}\}$

$$\begin{aligned} &= \nu_{S_\epsilon}(l^{-1}a)e^{i\eta_S(l^{-1}a)} \\ &= \nu_{S_\epsilon}(l^{-1}m)(m^{-1}a)e^{i\eta_{S_\delta}(l^{-1}m)(m^{-1}a)} \\ &\leq \max\{\nu_{S_\epsilon}(l^{-1}m)e^{i\eta_{S_\delta}(l^{-1}m)}, \nu_{S_\epsilon}(m^{-1}a)e^{i\eta_{S_\delta}(m^{-1}a)}\} \\ &= \max\{\nu_{S_\epsilon}(e)e^{i\eta_{S_\delta}(e)}, \nu_{S_\epsilon}(m^{-1}a)e^{i\eta_{S_\delta}(m^{-1}a)}\} \\ &= \nu_{S_\epsilon}(m^{-1}a)e^{i\eta_{S_\delta}(m^{-1}a)} \\ &= \nu_{mS_\epsilon}(a)e^{i\eta_{mS_\delta}(a)}. \end{aligned}$$

Interchange the role of l and we get
$\nu_{mS_\epsilon}(a)e^{i\eta_{mS_\delta}(a)} \leq \nu_{lS_\epsilon}(a)e^{i\eta_{lS_\delta}(a)}$. Thus, $\nu_{lS_\epsilon}(a)e^{i\eta_{lS_\delta}(a)} = \nu_{mS_\epsilon}(a)e^{i\eta_{mS_\delta}(a)}$.
(ii) In similar way, this can be present as part (i).

Theorem 4.10. *Let $S_{(\epsilon,\delta)}$ be an (ϵ,δ)-CAFNSG of group G and l, m, a, b arbitrary elements of G. If $lS_{(\epsilon,\delta)} = aS_{(\epsilon,\delta)}$ and $mS_{(\epsilon,\delta)} = bS_{(\epsilon,\delta)}$, then $lmS_{(\epsilon,\delta)} = abS_{(\epsilon,\delta)}$.*

Proof. Given that $lS_{(\epsilon,\delta)} = aS_{(\epsilon,\delta)}$ and $mS_{(\epsilon,\delta)} = bS_{(\epsilon,\delta)}$. Implies that $l^{-1}a, m^{-1}b \in S^e_{(\epsilon,\delta)}$.

Consider, $(lm)^{-1}(ab) = m^{-1}(l^{-1}a)b = m^{-1}(l^{-1}a)(lm^{-1})b = [m^{-1}(l^{-1}a)(m)](m^{-1}b)$. As $S^e_{(\epsilon,\delta)}$ is normal subgroup of G. Thus, $(lm)^{-1}(ab) \in S^e_{(\epsilon,\delta)}$. Similarly, $lmS_{(\epsilon,\delta)} = abS_{(\epsilon,\delta)}$. As a result of this, we can say that (ϵ,δ)-$CAFQG$ along to classical quotient group.

Theorem 4.11. *Assume that $G/S_{(\epsilon,\delta)} = \{lS_{(\epsilon,\delta)} : l \in G\}$ be the collection of all (ϵ,δ)-CF cosets of (ϵ,δ)-CAFNSG $S_{(\epsilon,\delta)}$ of G. Consequently, the set action of the binary operator is well define $G/S_{(\epsilon,\delta)}$ and is present as $lS_{(\epsilon,\delta)} * mS_{(\epsilon,\delta)} = lmS_{(\epsilon,\delta)}$* for all $l, m \in G$.

Proof. We have $lS_{(\epsilon,\delta)} = mS_{(\epsilon,\delta)}$ *and* $aS_{(\epsilon,\delta)} = bS_{(\epsilon,\delta)}$, for arbitrary $a, b, l, m \in G$. Assume that $g \in G$ be arbitrary element, so

$$[lS_{(\epsilon,\delta)}aS_{(\epsilon,\delta)}]\,(g) = (laS_{(\epsilon,\delta)}(g)) = \nu_{laS_\epsilon}(g)e^{i\eta_{laS_\delta}(g)}.$$

Consider,
$\nu_{laS_\epsilon}(g)e^{i\eta_{laS_\delta}(g)} = \max\{\nu_{laS}(g)e^{i\eta_{laS}(g)}, \epsilon e^{i\delta}\}$

$$\begin{aligned} &= \max\{\nu_S((la)^{-1}g)e^{i\eta_S((la)^{-1}g)},\ \epsilon e^{i\delta}\} \\ &= \max\{\nu_S(a^{-1}(l^{-1}g))e^{i\eta_S(a^{-1}(l^{-1}g))},\ \epsilon e^{i\delta}\} \\ &= \nu_{aS_\epsilon}(l^{-1}g)e^{i\eta_{aS_\delta}(l^{-1}g)} \\ &= \nu_{bS_\epsilon}(l^{-1}g)e^{i\eta_{bS_\delta}(l^{-1}g)} \\ &= \max\{\nu_S(b^{-1}(l^{-1}g))e^{I\eta_S(b^{-1}(l^{-1}g))}, \epsilon e^{i\delta}\} \\ &= \max\{\nu_S(l^{-1}(gb^{-1})), \epsilon e^{i\delta}\} \\ &= \nu_{lS_\epsilon}(gb^{-1})e^{i\eta_{lS_\delta}(gb^{-1})} \\ &= \nu_{lS_\epsilon}(gb^{-1})e^{i\eta_{mS_\delta}(gb^{-1})} \\ &= \max\{\nu_S(m^{-1}(gb^{-1}))e^{i\eta_S(m^{-1}(gb^{-1}))}, \epsilon e^{i\delta}\} \end{aligned}$$

$$\begin{aligned} &= \max\{\nu_S(m^{-1}g)b^{-1}e^{i\eta_S(m^{-1}g)b^{-1}}, \epsilon e^{i\delta}\} \\ &= \max\{\nu_S(b^{-1}m^{-1}(g))e^{I\eta_S(b^{-1}m^{-1}(g))}, \epsilon e^{i\delta}\} \\ &= \max\{\nu_S((mb)^{-1}(g))e^{I\eta_S((mb)^{-1}(g))}, \epsilon e^{i\delta}\} \\ &= \nu_{qbS_\epsilon}(g)e^{i\eta_{qbS_\delta}(g)}. \end{aligned}$$

Hence, the operation $*$ on $G/S_{(\epsilon,\delta)}$ is well defined. It can be observed that $*$ operation is a closed and associative on set $G/S_{(\epsilon,\delta)}$. Moreover, $\nu_{S_\epsilon}e^{i\eta_{S_\delta}} * \nu_{lS_\epsilon}e^{i\eta_{lS_\delta}} = \nu_{eS_\epsilon}e^{i\eta_{eS_\delta}} * \nu_{lS_\epsilon}e^{i\eta_{lS_\delta}} = \nu_{lS_\epsilon}e^{i\eta_{lS_\delta}} = \nu_{lS_\epsilon}e^{i\eta_{lS_\delta}} \implies \nu_{S_\epsilon}e^{i\eta_S}$ is neutral element of $G/S_{(\epsilon,\delta)}$. Obviously, the inverse of every entity of $G/S_{(\epsilon,\delta)}$ exist if $\nu_{lS_\epsilon}e^{i\eta_{lS_\delta}} \in G/S_{(\epsilon,\delta)}$, so there is a element, $\nu_{l^{-1}S_\epsilon}e^{i\eta_{l^{-1}S_\delta}} \in G/S_{(\epsilon,\delta)}$ such that $\nu_{l^{-1}pS_\epsilon}e^{i\eta_{l^{-1}lS_\delta}} = \nu_{S_\epsilon}e^{i\eta_{S_\delta}}$. As a consequence, $G/S_{(\epsilon,\delta)}$ is a group. The group $G/S_{(\epsilon,\delta)}$ is known as quotient group of the G by $S_{(\epsilon,\delta)}$.

Lemma 4.12. *Assume that a natural homomorphism from group G onto $G/S_{(\epsilon,\delta)}$ is $f : G$ to $G/S_{(\epsilon,\delta)}$ and the rule specifies, $f(l) = lS_{(\epsilon,\delta)}$ with kernel $f = S^e_{(\epsilon,\delta)}$.*

Proof. Suppose an arbitrary elements l, m taken from group G, then $f(lm) = lmS_{(\epsilon,\delta)} = \nu_{lmS_\epsilon}e^{i\eta_{lmS_\delta}} = \nu_{lS_\epsilon}e^{i\eta_{lS_\delta}} * \nu_{mS_\epsilon}e^{i\eta_{mS_\delta}} = lS_{(\epsilon,\delta)} * mS_{(\epsilon,\delta)} = f(l) * s(m)$. Hence f is a homomorphism and f is an onto mapping.

$$\begin{aligned} \text{Then, } Kerf &= \{l \in G : f(l) = eS_{(\epsilon,\delta)}\} \\ &= \{l \in G : lS_{(\epsilon,\delta)} = eS_{(\epsilon,\delta)}\} \\ &= \{l \in G : le^{-1} \in S^e_{(\epsilon,\delta)}\} \\ &= \{l \in G : l \in S^e_{(\epsilon,\delta)}\} \\ &= S^e_{(\epsilon,\delta)}. \end{aligned}$$

As a result of this, we introduce (ϵ,δ)-$CAFG$ of quotient group generates by normal subgroup $S^e_{\epsilon,\delta}$.

Theorem 4.13. *Let $S^e_{\epsilon,\delta}$ be normal subgroup of G. If $S_{(\epsilon,\delta)} = \{(l, \nu_{S_\epsilon}(l)e^{i\eta_{S_\delta}(l)}) : l \in G\}$ is (ϵ,δ)-$CAFSG$. Then the (ϵ,δ)-complex anti fuzzy set $\overline{S}_{(\epsilon,\delta)} = \{(lS^e_{(\epsilon,\delta)}, \overline{\nu}_{S_\epsilon}(lS^e_{(\epsilon,\delta)})e^{i\eta_{S_\delta}(lS^e_{(\epsilon,\delta)})}) : l \in G\}$ of $G/S^e_{(\epsilon,\delta)}$ is also a (ϵ,δ)-$CAFSG$ of $G/S^e_{\epsilon,\delta}$. Where $\overline{\nu}_{S_\epsilon}(lS^e_{(\epsilon,\delta)})e^{i\overline{\eta}_{S_\delta}(lS^e_{(\epsilon,\delta)})} = \min\{\nu_{S_\epsilon}(la)e^{i\eta_{S_\delta}(la)} : a \in S^e_{(\epsilon,\delta)}\}$.*

Proof. First we shall prove that $\overline{\nu}_{S_\epsilon}(lS^e_{(\epsilon,\delta)})e^{i\overline{\eta}_{S_\delta}(mS^e_{(\epsilon,\delta)})}$ is well defined. Let $lS^e_{\epsilon,\delta} = mS^e_{\epsilon,\delta}$ then $m = la$, for some $a \in S^e_{\epsilon,\delta}$. Now $\overline{\nu}_{S_\epsilon}(mS^e_{(\epsilon,\delta)})e^{i\overline{\eta}_{S_\delta}(mS^e_{\epsilon,\delta})} = \min\{\nu_{S_\epsilon}(mb)e^{i\eta_{S_\delta}(mb)} : b \in S^e_{(\epsilon,\delta)}\}$

$$\begin{aligned} &= \min\{\nu_{S_\epsilon}(lab)e^{i\eta_{S_\delta}(lab)} : c = ab \in S^e_{(\epsilon,\delta)}\} \\ &= \min\{\nu_{S_\epsilon}(lc)e^{i\eta_{S_\delta}(lc)} : c \in S^e_{(\epsilon,\delta)}\} \\ &= \overline{\nu}_{S_\epsilon}(lS^e_{(\epsilon,\delta)})\, e^{i\overline{\eta}_{S_\delta}(lS^e_{(\epsilon,\delta)})} \end{aligned}$$

Therefore, $\overline{\nu}_{S_\epsilon}(lS^e_{(\epsilon,\delta)})e^{i\overline{\eta}_{S_\delta}(lS^e_{(\epsilon,\delta)})}$ is well defined.
Consider, $\overline{\nu}_{S_\epsilon}\{(lS^e_{(\epsilon,\delta)})(mS^e_{(\epsilon,\delta)})\}e^{i\overline{\eta}_{S_\delta}\{(lS^e_{(\epsilon,\delta)})(mS^e_{(\epsilon,\delta)})\}}$

$$\begin{aligned} &= \overline{\nu}_{S_\epsilon}(lmS^e_{(\epsilon,\delta)})e^{i\overline{\eta}_{S_\delta}(lmS^e_{(\epsilon,\delta)})} \\ &= \min\{\nu_{S_\epsilon}(lma)e^{i\eta_{S_\delta}(lma)} : a \in S^e_{(\epsilon,\delta)}\} \end{aligned}$$

$$
\begin{aligned}
&\leq \min\{\max\{\nu_{S_\epsilon}(lb)e^{i\eta_{S_{(\epsilon,\delta)}}(lb)}, \\
&\quad \nu_{S_\epsilon}(mc)e^{i\eta_{S_\delta}(mc)}\} \ : b, c \in S^e_{\epsilon,\delta}\} \\
&\leq \max\{\min\{\nu_{S_\epsilon}(lb)e^{i\eta_{S_\delta}(lb)}\} \ : b \in S^e_{\epsilon,\delta}, \\
&\quad \min\{\nu_{S_\epsilon}(mc)e^{i\eta_{S_\delta}(mc)}\} \ : c \in S^e_{\epsilon,\delta} \ \} \\
&\leq \max\{\overline{\nu}_{S_\epsilon}(lS^e_{(\epsilon,\delta)})e^{i\overline{\eta}_{S_\delta}(lS^e_{(\epsilon,\delta)})}, \\
&\quad \overline{\nu}_{S_\epsilon}(mS^e_{(\epsilon,\delta)})e^{i\overline{\eta}_{S_\delta}(mS^e_{(\epsilon,\delta)})}\}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Also,} \qquad & \overline{\nu}_{S_\epsilon}((lS^e_{(\epsilon,\delta)})^{-1})e^{i\overline{\eta}_{S_\delta}((lS^e_{(\epsilon,\delta)})^{-1})} = \\
& \overline{\nu}_{S_\epsilon}(l^{-1}S^e_{\epsilon,\delta})e^{i\overline{\eta}_{S_\delta}(l^{-1}S^e_{\epsilon,\delta})} \\
&= \min\{\nu_{S_\epsilon}(l^{-1}a)e^{i\eta_{S_\delta}(l^{-1}a)} : a \in S^e_{\epsilon,\delta}\} \\
&\leq \min\{\nu_{S_\epsilon}(la)e^{i\eta_{S_\delta}(la)} : \ a \in S^e_{\epsilon,\delta}\} \\
&= \overline{\nu}_{S_\epsilon}(lS^e_{(\epsilon,\delta)})e^{i\overline{\eta}_{S_\delta}(lS^e_{(\epsilon,\delta)})}.
\end{aligned}
$$

This established the proof.

Definition 4.14. Let $S_{(\epsilon,\delta)}$ be a (ϵ,δ)-$CAFSG$ of finite the group G. Then the cardinality of the set $G/S_{(\epsilon,\delta)}$ for (ϵ,δ)-CAF left cosets of G by $S_{(\epsilon,\delta)}$ is known as the index of (ϵ,δ)-$CAFSG$ and is represent by $[G : l]$.

Theorem 4.15. *(ϵ,δ)-complex anti fuzzification of Lagrange's Theorem: Assume that G be finite group and $S_{(\epsilon,\delta)}$ be (ϵ,δ)-$CAFSG$ of G then G is divisible by the index of (ϵ,δ)-$CAFSG$ of G.*

Proof. By Lemma 4.13, natural homomorphism h introduced from G to $G/S_{(\epsilon,\delta)}$. A subgroup is introduced by $H = \{w \in G : wS_{(\epsilon,\delta)} = eS_{(\epsilon,\delta)}\}$. By attempting to make use of the definition $w \in H$ and $g \in G$, we have $wS_{(\epsilon,\delta)}(g) = eS_{(\epsilon,\delta)}(g)$. This indicates $S_{(\epsilon,\delta)}(w^{-1}g) = S_{(\epsilon,\delta)}(g)$. By Theorem 4.11, which shows that $w \in S^e_{(\epsilon,\delta)}$. As a result H is contain in $S^e_{(\epsilon,\delta)}$. Now, we can take arbitrary element $w \in S^e_{(\epsilon,\delta)}$ and applying knowledge $S^e_{(\epsilon,\delta)}$ is subgroup of G, we have $S_{(\epsilon,\delta)}(w^{-1}) = S_{(\epsilon,\delta)}(e)$. From Theorem 4.11 , the elements $w^{-1}, \ g \in S^e_{(\epsilon,\delta)}$, this mean $wS_{(\epsilon,\delta)} = eS_{(\epsilon,\delta)}$, implies that $w \in H$. Hence $S^e_{(\epsilon,\delta)}$ is contain in H. We can conclude this the discussion that $H = S^e_{(\epsilon,\delta)}$.

Unions of disjoint of right cosets is establish the partition of group G and is defined as $G = z_1G \cup z_2H \cup \cdots \cup z_lH$. Where $z_1H = H$. There is a (ϵ,δ)-CAF cosets $z_iS_{(\epsilon,\delta)}$ in $G/S^e_{(\epsilon,\delta)}$ and also is a differentiable.

Consider any coset $z_iS^e_{(\epsilon,\delta)}$. Let $w \in S^e_{(\epsilon,\delta)}$, then

$$
\begin{aligned}
h(z_iw) = z_iwS_{(\epsilon,\delta)} &= z_iS_{(\epsilon,\delta)}wS_{(\epsilon,\delta)} \\
&= z_iS_{(\epsilon,\delta)}eS_{(\epsilon,\delta)} \\
&= z_iS_{(\epsilon,\delta)}.
\end{aligned}
$$

Hence, h maps every entity of $z_iS^e_{(\epsilon,\delta)}$ into the (ϵ,δ)-CAF cosets $z_iS_{(\epsilon,\delta)}$.

Currently, we can establish a basic association. h among the set $\{z_iS^e_{(\epsilon,\delta)} : 1 \leq i \leq l\}$ and the set $G/S^e_{(\epsilon,\delta)}$ defined by

$$h(z_iS^e_{(\epsilon,\delta)}) = z_iS_{(\epsilon,\delta)}, \ 1 \leq i \leq l.$$

The h is injective.

As a result, suppose $z_i S_{(\epsilon,\delta)} = z_l S_{(\epsilon,\delta)}$, then $z_l^{-1} z_i S_{(\epsilon,\delta)} = eS_{(\epsilon,\delta)}$. Using (S), we have $z_l^{-1} z_i \in H$, this means that $z_i S^e_{(\epsilon,\delta)} = z_i S^e_{(\epsilon,\delta)}$ and thus h is injective. It is evident from the preceding discussion that $[G : S^e_{(\epsilon,\delta)}]$ *and* $[G : S_{(\epsilon,\delta)}]$ are equal. Since $[G : S^e_{(\epsilon,\delta)}]$ divides $O(G)$.

This algebraic concept is shown in example.

Example 4.16. Assume $G = \{< l, m : l^3 = m^2 = e,\ lm = ml^2\}$ be a group of order 6 finite permutations. The (ϵ, δ)-$CAFS$G $S_{(\epsilon,\delta)}$ of G according to the value $\epsilon = 0.2$ and $\delta = \frac{\pi}{4}$ is discuss.

$$S_{(\epsilon,\delta)}(\omega) = \begin{cases} 0.3e^{\frac{\pi i}{3}} & \text{if } \omega = e, \\ 0.4e^{\frac{\pi i}{2}}, & \text{if } \omega = l, l^2, \\ 0.6e^{\pi i}, & \text{otherwise.} \end{cases}$$

The set of all (ϵ, δ)-CAF left cosets of G by $S_{(\epsilon,\delta)}$ is given by:

$$G/S_{(\epsilon,\delta)} = \{eS_{(\epsilon,\delta)},\ lS_{(\epsilon,\delta)},\ mS_{(\epsilon,\delta)}\}.$$

It represents that $[G : S_{(\epsilon,\delta)}] = Card(G/S_{(\epsilon,\delta)}) = 3$.

5. Conclusions

In this article, we defined the concept of (ϵ, δ)-$CAFS$ as a useful modification of classical $CAFS$. We established (ϵ, δ)-$CAFSG$s and presented certain fundamental algebraic characterizations of this novel framework. In addition, we developed the (ϵ, δ)-CAF cosets and analyzed some of their algebraic characteristics. Furthermore, we investigated the (ϵ, δ)-$CAFNSG$ that generates the (ϵ, δ)-$CAFQG$. As for the future works, we will extend the novel approach to the different algebraic models and then apply on the extension of group theory, and introduce (ϵ, δ)-CAF subrings. Furthermore, we will work on its applications. Moreover, the proposed method can be applied to other areas, such as design concept evaluation, and the assessment of a method for complex products based on cloud rough numbers [39]. This assessment can be regarded as multi-attribute group decision-making.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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