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Research article

Well-posedness and order preservation for neutral type stochastic differential equations of infinite delay with jumps

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Abstract: In this work, we are concerned with the order preservation problem for multidimensional neutral type stochastic differential equations of infinite delay with jumps under non-Lipschitz conditions. By using a truncated Euler-Maruyama scheme and adopting an approximation argument, we have developed the well-posedness of solutions for a class of stochastic functional differential equations which allow the length of memory to be infinite, and the coefficients to be non-Lipschitz and even unbounded. Moreover, we have extended some existing conclusions on order preservation for stochastic systems to a more general case. A pair of examples have been constructed to demonstrate that the order preservation need not hold whenever the diffusion term contains a delay term, although the jump-diffusion coefficient could contain a delay term.

Keywords: order preservation; infinite memory; neutral type stochastic differential equation; jump
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1. Introduction

In [1], Asker studied well-posedness for a class of neutral type stochastic differential equations driven by Brownian motions with infinite delay; Bao et al. [2] also investigated the exponential ergodicity, weak convergence, and asymptotic Log-Harnack inequality for several kinds of models with infinite memory. So far, there is no order preservation available for stochastic differential equations with infinite memory. Moreover, the order preservation theorems play an essential role in the theory of stochastic systems and their applications because, in many fields of analysis, they constitute an effective way to control a complicated stochastic system by using a simpler one. These types of theorems are used in a wide range of practical problems in fields such as finance, economics, biology, and mathematics; see also [3–8]. Consequently, we focus on establishing order preservation for neutral-type stochastic differential equations of infinite memory with jumps and obtaining the well-posedness for these stochastic systems under non-Lipschitz conditions.

The pioneering work on order preservation for stochastic differential equations is detailed in [9], and was later generalized in [10]. Since their works, the order preservation for two stochastic differential equations driven by continuous noise processes has been investigated extensively. With regard to the order preservation under various settings, we can refer to, for example, [11] for one-dimensional stochastic differential equations, [12] for one-dimensional stochastic hybrid delay systems, and [13] for multidimensional stochastic functional differential equations.

Meanwhile, the order preservation for two stochastic differential equations subject to the discontinuous case has also garnered much attention. For example, applying criteria of a "viability condition", the authors of [14] showed a comparison theorem of stochastic differential equations with jumps under Lipschitz and linear growth conditions; using a Tanaka-type formula, [15] further established a comparison theorem for one-dimensional stochastic differential delay equations with jumps, where the coefficients satisfy local Lipschitz and linear growth conditions; adopting an approximation argument, the work in [16] extends the results on one-dimensional equations to multidimensional stochastic functional differential equations with jumps, where the coefficients satisfy a non-Lipschitz condition.

It is worth pointing out that [13, 15, 17, 18] focus on order preservation for stochastic functional differential equations with Lipschitz coefficients, which rules out the case of non-Lipschitz conditions. On the other hand, few studies have focused on stochastic functional differential equations with non-Lipschitz coefficients, and, in the existing literature, most have focused on stochastic functional differential equations of finite delay. Yet, the corresponding issue for stochastic functional differential equations with infinite memory is rarely addressed in the literature. Moreover, the multidimensional order preservation theorem affords a further widening of the field of application, especially for those processes whose dynamics are influenced by each other. Based on the above motivations, in this work, we aimed to develop an approximation method to investigate order preservation for multidimensional neutral type stochastic functional differential equations, which allow the coefficients to be non-Lipschitz and depend on the whole history of the system. Compared to the existing results on order preservation, the innovations of our work can be described as follows:

(i) We introduce the truncated Euler-Maruyama scheme method into the analysis of the well-posedness problem of neutral-type stochastic differential equations of infinite delay with jumps, and we establish the existence of the solutions;

(ii) Our model is more applicable and practical, as we deal with neutral-type stochastic differential equations under non-Lipschitz conditions.

The rest of the paper is arranged as follows. In Section 2, we introduce some notations and present the framework of our paper; Section 3 is devoted to the existence and uniqueness of solutions for a class of neutral stochastic functional differential equations of infinite delay for pure jumps; Section 4 focuses on the order preservation for this system.

2. Preliminaries

For $d, m \in \mathbb{N}$, i.e., the set of all positive integers, let $(\mathbb{R}^d, \langle\cdot,\cdot\rangle, |\cdot|)$ be the d-dimensional Euclidean space with the inner product $\langle\cdot,\cdot\rangle$ inducing the norm $|\cdot|$ and $\mathbb{R}^d \otimes \mathbb{R}^m$ denote the collection of all $d \times m$

matrixes with real entries, which is endowed with the Hilbert-Schmidt norm $\|\cdot\|$. $\mathscr{D} := D([-\infty, 0]; \mathbb{R}^d)$ denotes the family of all càdlàg functions $f : [-\infty, 0] \to \mathbb{R}^d$. For a càdlàg map $f : [-\infty, \infty) \to \mathbb{R}^d$ and $t \geq 0$, let $f_{t-} \in \mathscr{D}$ be such that $f_{t-}(\theta) = f((t+\theta)-) = \lim_{s\uparrow t+\theta} f(s)$ for $\theta \in [-\infty, 0]$, and $(f_t)_{t\geq 0}$ is called the segment process of $(f(t))_{t>-\infty}$.

For a fixed number $r > 0$, set

$$\mathscr{D}_r := \Big\{\phi \in \mathscr{D} : \|\phi\|_r := \sup_{-\infty<\theta<0} (e^{r\theta}|\phi(\theta)|) < \infty\Big\}.$$

Then, $(\mathscr{D}_r, \|\cdot\|_r)$ is a Banach space. Under the uniform norm $\|\cdot\|_r$, the space $\mathscr{D}_r$ is complete but not separable. Let $(W(t))_{t\geq 0}$ be an m-dimensional Brownian motion and $N(\mathrm{d}t, \mathrm{d}u)$ a Poisson counting process with characteristic measure λ on a measurable subset $\mathbb{Y}$ defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfying the usual condition (i.e., $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets and $\mathscr{F}_t = \mathscr{F}_{t+} := \bigcap_{s>t} \mathscr{F}_s$). We assume that $W(t)$ and $N(\mathrm{d}t, \mathrm{d}u)$ are independent.

Consider the following neutral-type stochastic differential equations of infinite delay on $(\mathbb{R}^d, \langle\cdot,\cdot\rangle, |\cdot|)$:

$$\begin{cases} \mathrm{d}\{X(t) - G(X_t)\} = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W(t) + \int_{\mathbb{Y}} \gamma(t, X_t, u)N(\mathrm{d}t, \mathrm{d}u), \quad t \geq 0, \\ X_0 = \xi \in \mathscr{D}_r, \end{cases} \tag{2.1}$$

$$\begin{cases} \mathrm{d}\{\bar{X}(t) - G(\bar{X}_t)\} = \bar{b}(t, \bar{X}_t)\mathrm{d}t + \bar{\sigma}(t, \bar{X}_t)\mathrm{d}W(t) + \int_{\mathbb{Y}} \bar{\gamma}(t, \bar{X}_t, u)N(\mathrm{d}t, \mathrm{d}u), \quad t \geq 0, \\ \bar{X}_0 = \bar{\xi} \in \mathscr{D}_r, \end{cases} \tag{2.2}$$

where $G : \mathscr{D}_r \to \mathbb{R}^d$, $b, \bar{b} : \mathbb{R} \times \mathscr{D}_r \to \mathbb{R}^d$, $\sigma, \bar{\sigma} : \mathbb{R} \times \mathscr{D}_r \to \mathbb{R}^d \times \mathbb{R}^d$, and $\gamma, \bar{\gamma} : \mathbb{R} \times \mathscr{D}_r \times \mathbb{Y} \to \mathbb{R}^d$ are progressively measurable.

Set $\Lambda(t) := X(t) - G(X_t)$, $\bar{\Lambda}(t) := \bar{X}(t) - G(\bar{X}_t)$, $Z(t) := X(t) - \bar{X}(t)$, and $\tilde{Z}(t) := \Lambda(t) - \bar{\Lambda}(t)$. In order to derive the well-posedness of solutions, we assume the following

(A1) $G(0) = 0$ and there exists a constant $\alpha \in (0, \frac{1}{2})$ such that $|G(\xi) - G(\eta)| \leq \alpha\|\xi - \eta\|_r$ for $\xi, \eta \in \mathscr{D}_r$.

(A2) There exist some functions $K \in ([0, \infty])$ and $u \in \mathscr{U}$ such that $\mathbb{P}$-a.s.

$$\begin{aligned} &|b(t,\xi) - b(t,\eta)|^2 + |\bar{b}(t,\xi) - \bar{b}(t,\eta)|^2 + \|\sigma(t,\xi) - \sigma(t,\eta)\|^2 + \|\bar{\sigma}(t,\xi) - \bar{\sigma}(t,\eta)\|^2 \\ &\quad + \Big(\int_{\mathbb{Y}} (|\gamma(t,\xi,u) - \gamma(t,\eta,u)| + |\bar{\gamma}(t,\xi,u) - \bar{\gamma}(t,\eta,u)|)\lambda(\mathrm{d}u)\Big)^2 \\ &\quad + \int_{\mathbb{Y}} (|\gamma(t,\xi,u) - \gamma(t,\eta,u)|^2 + |\bar{\gamma}(t,\xi,u) - \bar{\gamma}(t,\eta,u)|^2)\lambda(\mathrm{d}u) \\ &\leq K(t)\|\xi - \eta\|_r^2 u(\|\xi - \eta\|_r^2), \quad \xi, \eta \in \mathscr{D}_r, \quad t \geq 0, \end{aligned}$$

where $\mathscr{U}$ is a class of control functions and

$$\mathscr{U} = \Big\{u \in C^1((0,\infty); [1,\infty)) : \int_0^1 \frac{\mathrm{d}s}{su(s)} = \infty, s \mapsto su(s) \text{ is increasing and concave}\Big\}.$$

(A3) For any $T > 0$, there exists a constant $C(T)$ such that $\mathbb{P}$-a.s.

$$\begin{aligned} &\sup_{t\in[0,T]} (|b(t,0)|^2 + |\bar{b}(t,0)|^2 + \|\sigma(t,0)\|^2 + \|\bar{\sigma}(t,0)\|^2) \\ &\quad + \int_0^T \int_{\mathbb{Y}} (|\gamma(t,0,u)|^2 + |\bar{\gamma}(t,0,u)|^2)\mathrm{d}t\lambda(\mathrm{d}u) \leq C(T). \end{aligned}$$

(**A4**) $G(\xi) \le G(\eta)$ for $\xi \le \eta$ and there exists a constant $\alpha \in (0, \frac{1}{2})$ such that

$$|G(\xi) - G(\eta)| \le \alpha \max_{1\le i\le d} \|\xi^i - \eta^i\|_r, \quad \xi, \eta \in \mathscr{D}_r.$$

Under (**A1**)–(**A3**), (2.1) admits a unique strong solution $(X(t))_{t\ge 0}$; see Theorem 3.1 below for more details. For the existence and uniqueness of strong solutions to stochastic functional differential equations with infinite delay, we refer the reader to [1, 19, 20] and the references therein. In particular, using the Picard approximation, Ren and Chen [19] studied the existence and uniqueness for a class of neutral-type stochastic differential equations of infinite delay with Poisson jumps in an abstract space under non-Lipschitz. We remark that we provide an alternative method to establishing the well-posedness of neutral type stochastic differential equations of infinite delay with jumps. The Lipschitz coefficient α in (**A1**) is set to less than one-half rather than $\frac{1}{40}$, as detailed in [19]. So, in some sense, our result is more general. Assumption (**A4**) is just imposed for the sake of the monotonicity principle of the solution process; see Theorem 4.1 below for more details.

Meanwhile, to establish the order preservation for multidimensional neutral-type stochastic differential equations of infinite delay, in view of [21], we introduce the partial orders on $\mathbb{R}^d$ and $\mathscr{C}_r$ as follows: for $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \mathbb{R}^d$,

$$x \le y \Leftrightarrow x_i \le y_i, \quad i = 1, 2, \cdots, d,$$

$$x < y \Leftrightarrow x \le y \text{ and } x \ne y,$$

$$x \ll y \Leftrightarrow x_i < y_i, \quad i = 1, 2, \cdots, d,$$

and, for $\xi = (\xi_1, \cdots, \xi_n), \eta = (\eta_1, \cdots, \eta_n) \in \mathscr{D}_r$,

$$\xi \le \eta \Leftrightarrow \xi(\theta) \le \eta(\theta), \ \ \theta \in (-\infty, 0],$$

$$\xi < \eta \Leftrightarrow \xi \le \eta \text{ and } \xi \ne \eta,$$

$$\xi \ll \eta \Leftrightarrow \xi(\theta) < \eta(\theta), \ \ \theta \in (-\infty, 0],$$

$$\xi \le_D \eta \Leftrightarrow \xi \le \eta \text{ and } \xi(0) - G(\xi) \le \eta(0) - G(\eta),$$

$$\xi <_D \eta \Leftrightarrow \xi \le_D \eta \text{ and } \xi \ne \eta.$$

$$\xi \wedge \eta := (\xi_1 \wedge \eta_1, \cdots, \xi_d \wedge \eta_d).$$

In this section, we finally recall the definition of D-order preservation (see, e.g., [21, Definition 4.1]).

Definition 2.1. *Equations* (2.1) *and* (2.2) *represent D-order preservation, if, for any* $\xi, \bar{\xi} \in \mathscr{D}_r$ *with* $\mathbb{P}(\xi \le_D \bar{\xi}) = 1$, *one has*

$$\mathbb{P}(X_t^{\xi} \le_D \bar{X}_t^{\bar{\xi}}) = 1, \quad t \ge 0.$$

3. Existence and uniqueness of solutions

In the case that $G = 0$ and $N = 0$, the existence and uniqueness of solutions to (2.1) with weak one-sided local Lipschitz conditions has been studied in [2]. On the other hand, under the same conditions the authors of [1] has extended the result to neutral-type stochastic differential equations of infinite delay. Compared with these, we point out that the following result is included in [1, 2]. In contrast to the assumptions put forward in [1, 2], the assumptions (**A1**)–(**A3**) are more general. Moreover, in [16], where order preservation of a stochastic functional differential equation with non-Lipschitz coefficients is given, a tried-and-true method shows that we can approximate the non-Lipschitz stochastic functional differential equations by using those with Lipschitz coefficients to prove the existence of solutions. It is worth pointing out that the Bismut formula for stochastic functional differential equations of finite delay plays a crucial role in the analysis of the existence of those with non-Lipschitz coefficients. Alternatively, for the neutral-type stochastic differential equations of infinite delay, this method is no longer valid. To prove the well-posedness of solutions, we adopt a truncated Euler-Maruyama approximation argument (see, e.g., [1,2]), where the essential ingredient is to construct the associated segment process and introduce an approximate function in a good way.

For any $k \geq 1$, let $\psi_k : \mathbb{R} \to [0, \infty)$ such that $\psi_k(s) = \psi_k'(s) = 0$ for $s \in (-\infty, 0]$ and

$$\psi_k''(s) = \begin{cases} 4k^2 s, & s \in \Big[0, \dfrac{1}{2k}\Big], \\ -4k^2\Big(s - \dfrac{1}{k}\Big), & s \in \Big[\dfrac{1}{2k}, \dfrac{1}{k}\Big], \\ 0, & \textit{otherwise}. \end{cases}$$

Then, one has

$$0 \leq \psi_k' \leq I_{(0,\infty)} \text{ and } 0 \leq \psi_k(s) \uparrow s^+, \ \ s\psi_k''(s) \leq I_{(0,\frac{1}{k})}(s) \downarrow 0, \ \text{as } k \uparrow \infty. \tag{3.1}$$

Theorem 3.1. Let (**A1**)–(**A3**) hold with $\bar{b} = 0$, $\bar{\sigma} = 0$, and $\bar{\gamma} = 0$. Then, for any $t \geq 0$ and $\xi \in \mathscr{D}_r$, (2.1) has a unique solution such that

$$\mathbb{E}\|X_t^\xi\|_r^2 \leq C < \infty, \ \ t \geq 0.$$

Proof. In what follows, we write X_t in lieu of X_t^ξ for brevity.

(a) First, we shall show that $\mathbb{E}\|X_t\|_r^2 \leq C\mathrm{e}^{-2rt} < \infty, \ \ t \geq 0$. Let $X(t)$ be a solution to (2.1). Define

$$\tau_n = \inf\{t \geq 0, \|X_t\|_r \geq \|\xi\|_r + n\}, \ n \geq 1.$$

Then, by (**A1**), one infers that

$$\mathrm{e}^{2rt}\|X_t\|_r^2 \leq \frac{1}{1-\alpha}\|\xi\|_r^2 + \frac{1}{(1-\alpha)^2}\sup_{0\leq s\leq t}(\mathrm{e}^{2rs}|\Lambda(s)|^2). \tag{3.2}$$

Combining the Itô formula with the assumption (**A1**), one has, for any $0 \le t \le T$,

$$\begin{aligned}
e^{2rt}|\Lambda(t)|^2 \le & 2(1+\alpha^2)\|\xi\|_r^2 + 2\int_0^t r e^{2rs}|\Lambda(s)|^2 ds + 2\int_0^t e^{2rs}\langle \Lambda(s), b(s,X_s)\rangle ds \\
& + 2\int_0^t e^{2rs}\langle \Lambda(s), \sigma(s,X_s)\rangle dW(s) \\
& + \int_0^t e^{2rs}\|\sigma(s,X_s)\|^2 ds + \int_0^t\int_{\mathbb{Y}} e^{2rs}(|\Lambda(s)+\gamma(s,X_{s^-},u)|^2 - |\Lambda(s)|^2)N(ds,du) \\
=: & \sum_{i=1}^6 I_i(t).
\end{aligned}$$

By taking the Young inequality into consideration, one gets

$$\begin{aligned}
I_3(t) \le & 8T\int_0^t e^{2rs}\Big(2|b(s,X_s)-b(s,0)|^2 + 2|b(s,0)|^2\Big)ds + \frac{1}{8T}\int_0^t e^{2rs}|\Lambda(s)|^2 ds \\
\le & 16T\int_0^t \Big(e^{2rs}K(s)\|X_s\|_r^2 u(\|X_s\|_r^2) + e^{2rs}|b(s,0)|^2\Big)ds + \frac{1}{8}\sup_{0\le s\le t}\Big(e^{2rs}|\Lambda(s)|^2\Big) \\
\le & C(T)\int_0^t e^{2rs}\Big(1+\|X_s\|_r^2 u(\|X_s\|_r^2)\Big)ds + \frac{1}{8}\Big(\sup_{0\le s\le t} e^{2rs}|\Lambda(s)|^2\Big).
\end{aligned}$$

The Burkholder-Davis-Gundy inequality, together with the assumptions (**A2**) and (**A3**), implies that

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} I_4(s)\Big) \le & \mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n}\int_0^s e^{2ru}\langle\Lambda(u), \sigma(u,X_u)dW(u)\rangle\Big) \\
\le & \frac{1}{8}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}|\Lambda(s)|^2\Big) + C(T)\mathbb{E}\int_0^{t\wedge\tau_n} e^{2rs}\|\sigma(s,X_s)\|^2 ds \\
\le & \frac{1}{8}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}|\Lambda(s)|^2\Big) + C(T)\mathbb{E}\int_0^{t\wedge\tau_n} e^{2rs}\Big(1+\|X_s\|_r^2 u(\|X_s\|_r^2)\Big)ds.
\end{aligned}$$

It follows from the assumptions (**A2**) and (**A3**) that

$$\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} I_5(s)\Big) \le C(T)\mathbb{E}\int_0^{t\wedge\tau_n} e^{2rs}\Big(1+\|X_s\|_r^2 u(\|X_s\|_r^2)\Big)ds.$$

The Young inequality implies that

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} I_6(s)\Big) \le & \mathbb{E}\int_0^{t\wedge\tau_n}\int_Y e^{2rs}(2|\gamma(s,X_{s^-},u)||\Lambda(s)| + |\gamma(s,X_{s^-},u)|^2)^+ N(ds,du) \\
\le & \frac{1}{4}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}|\Lambda(s)|^2\Big) + C\mathbb{E}\int_0^{t\wedge\tau_n}\int_{\mathbb{Y}} e^{2rs}(|\gamma(s,X_{s^-},u)-\gamma(s,0,u)|^2 \\
& + |\gamma(s,0,u)|^2)\lambda(du)ds \\
\le & \frac{1}{4}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}|\Lambda(s)|^2\Big) + C(T)\mathbb{E}\int_0^{t\wedge\tau_n} e^{2rs}\Big(1+\|X_s\|_r^2 u(\|X_s\|_r^2)\Big)ds.
\end{aligned}$$

Therefore, from the above inequalities, we obtain

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}|\Lambda(s)|^2\Big) \le & 8\|\xi\|_r^2 + C(T)\int_0^t e^{2r(s\wedge\tau_n)}\Big(1+\mathbb{E}\|X_{s\wedge\tau_n}\|_r^2 u(\|X_{s\wedge\tau_n}\|_r^2)\Big)\mathrm{d}s \\ & + 4r\int_0^t \mathbb{E}\Big(\sup_{0\le u\le s\wedge\tau_n} e^{2ru}|\Lambda(u)|^2\Big)\mathrm{d}s.\end{aligned} \tag{3.3}$$

Applying the Gronwall inequality leads to

$$\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}|\Lambda(s)|^2\Big) \le C(T)\Big(\|\xi\|_r^2 + \int_0^t e^{2r(s\wedge\tau_n)}\Big(1+\mathbb{E}\|X_{s\wedge\tau_n}\|_r^2 u(\|X_{s\wedge\tau_n}\|_r^2)\Big)\mathrm{d}s\Big),$$

which, together with (3.2), implies that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}\|X_s\|_r^2\Big) \le & \frac{1}{1-\alpha}\|\xi\|_r^2 + \frac{1}{(1-\alpha)^2}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}|\Lambda(s)|^2\Big) \\ \le & C(T)\|\xi\|_r^2 + \frac{C(T)}{(1-\alpha)^2}\int_0^t e^{2r(s\wedge\tau_n)}\Big(1+\mathbb{E}\|X_{s\wedge\tau_n}\|_r^2 u(\|X_{s\wedge\tau_n}\|_r^2)\Big)\mathrm{d}s \\ \le & C(T)\|\xi\|_r^2 + \frac{C(T)}{(1-\alpha)^2} \\ & + \frac{C(T)}{(1-\alpha)^2}\int_0^t \mathbb{E}\Big(\sup_{0\le v\le s\wedge\tau_n} e^{2rv}\|X_v\|_r^2\Big)u(\sup_{0\le v\le s\wedge\tau_n} e^{2rv}\|X_v\|_r^2)\mathrm{d}s.\end{aligned}$$

Let $G(s)=\int_1^s \frac{1}{ru(r)}\mathrm{d}r$, $s>0$. Then, by the Bihari inequality, we have $\mathbb{P}$-a.s.

$$\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}\|X_s\|_r^2\Big) \le G^{-1}\Big\{G\Big(C(T)\|\xi\|_r^2 + \frac{C(T)}{(1-\alpha)^2}\Big) + \frac{C(T)}{(1-\alpha)^2}t\Big\} < \infty, \quad t\in[0,T],$$

where G^{-1} is the inverse function of G. Let $n\uparrow\infty$; then, $\tau_n\uparrow\infty$. Therefore, we obtain

$$\mathbb{E}\|X_t\|_r^2 \le C < \infty, \quad t\ge 0$$

due to the arbitrariness of T.

(b) Second, we aim to derive the uniqueness of the solution. Let $X(t)$ and $Y(t)$ be two solutions to (2.1) with the same initial value X_0. Set

$$\varphi_n(t) := \sup_{0\le s\le t\wedge\tau_n} e^{2rs}|X(s)-Y(s)|^2 = e^{2r(t\wedge\tau_n)}\|X_{t\wedge\tau_n}-Y_{t\wedge\tau_n}\|_r^2 \le \frac{1}{(1-\alpha)^2}\sup_{0\le s\le t\wedge\tau_n}\Big(e^{2rs}|\Lambda^{X,Y}(s)|^2\Big),$$

where $\Lambda^{X,Y}(t) = X(t)-Y(t)-(G(X_t)-G(Y_t))$, and, in the last step we apply the assumption (**A1**). Then, carrying out the same technique to deduce (3.3), one has

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}|\Lambda^{X,Y}(s)|^2\Big) \le & 4r\int_0^t \mathbb{E}\Big(\sup_{0\le u\le s\wedge\tau_n} e^{2ru}|\Lambda^{X,Y}(u)|^2\Big)\mathrm{d}s \\ & + C(T)\mathbb{E}\int_0^{t\wedge\tau_n} e^{2rs}\|X_s-Y_s\|_r^2 u(\|X_s-Y_s\|_r^2)\mathrm{d}s.\end{aligned}$$

Due to the fact that the function $su(s)$ is increasing, and by the Gronwall inequality, we have

$$\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}|\Lambda^{X,Y}(s)|^2\Big) \le C(T)\mathbb{E}\int_0^{t\wedge\tau_n} e^{2rs}\|X_s - Y_s\|_r^2 u(\|X_s - Y_s\|_r^2)\mathrm{d}s.$$

Furthermore, using Jensen's inequality, we get

$$\begin{aligned}\mathbb{E}\varphi_n(t) \le&\frac{1}{(1-\alpha)^2}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_n} e^{2rs}|\Lambda^{X,Y}(s)|^2\Big)\\ \le&\frac{C(T)}{(1-\alpha)^2}\mathbb{E}\int_0^t e^{2r(s\wedge\tau_n)}\|X_{s\wedge\tau_n} - Y_{s\wedge\tau_n}\|_r^2 u(e^{2r(s\wedge\tau_n)}\|X_{s\wedge\tau_n} - Y_{s\wedge\tau_n}\|_r^2)\mathrm{d}s\\ \le&\frac{C(T)}{(1-\alpha)^2}\int_0^t (\mathbb{E}\varphi_n(s))u(\mathbb{E}\varphi_n(s))\mathrm{d}s, \ \ t\in[0,T],\ n\ge 1.\end{aligned}$$

Since $\int_0^1 \frac{1}{ru(r)}\mathrm{d}r = \infty$, $s > 0$. By the Bihari inequality, we have that $\mathbb{P}$-a.s. $\mathbb{E}\varphi_n(T) = 0$, $t\in[0,T]$, $n\ge 1$. Let $n\uparrow\infty$; then, $\mathbb{E}\Big(\sup_{0\le s\le T} e^{2rs}|X(s) - Y(s)|^2\Big) = 0$, which implies that $X(s) = Y(s)$ for any $t\ge 0$ $\mathbb{P}$-a.s.

(c) Finally, we shall divide two cases to show the existence of the solution to (2.1). We shall adopt the truncated Euler-Maruyama scheme approach (see, e.g., [1, 2]), where the essential ingredient is to construct an approximation of the segment process in a good way.

Case 1. In this part, we shall show existence of the solution for bounded b, σ and $\beta := \int_{\mathbb{Y}}(|\gamma(\cdot,\cdot,u)|^2 + |\gamma(\cdot,\cdot,u)|)\lambda(\mathrm{d}u)$. Define

$$\Psi_k(x) = \psi_k(|x|), \qquad x\in\mathbb{R}^d.$$

By the definition of ψ_k, it is easy to see that $\Psi_k\in C^2(\mathbb{R}^d;\mathbb{R}_+)$. Let

$$(\Psi_k)_x(x) = \Big(\frac{\partial\Psi_k(x)}{\partial x_1},\cdots,\frac{\partial\Psi_k(x)}{\partial x_d}\Big) \text{ and } (\Psi_k)_{xx}(x) = \Big(\frac{\partial^2\Psi_k(x)}{\partial x_i\partial x_j}\Big)_{d\times d}, \quad x\in\mathbb{R}^d.$$

A straightforward calculation leads to the following for $x\in\mathbb{R}^d$ and $i = 1, 2, \cdots, d$:

$$\frac{\partial\Psi_k(x)}{\partial x_i} = \psi_k'(|x|)\frac{x_i}{|x|} \text{ and } \frac{\partial^2\Psi_k(x)}{\partial x_i\partial x_j} = \psi_k'(|x|)(\delta_{ij}|x|^2 - x_ix_j)|x|^{-3} + \psi_k''(|x|)x_ix_j|x|^{-2},$$

where $\delta_{ij} = 1$ if $i = j$, or 0 otherwise. Then, it follows from (3.1) that, for $x\in\mathbb{R}^d$,

$$0\le|(\Psi_k)_x(x)|\le 1, \text{ and } 0\le\Psi_k(x)^2\le|x|^2, \ |x|\cdot\|(\Psi_k)_{xx}(x)\|\le 2I_{(0,\frac{1}{k})}(|x|)\downarrow 0, \text{ as } k\uparrow\infty. \tag{3.4}$$

Set $N_0 := \{n\in\mathbb{N} : n\ge\frac{r}{\log 2}\}$ and $\lfloor s\rfloor := \sup\{k\in\mathbb{Z}; k\le s\}$, i.e., the integer part of $s > 0$. For any $n\in N_0$, consider a stochastic differential equation:

$$\begin{cases}\mathrm{d}\{X^n(t) - G(\hat{X}_t^n)\} = b(t,\hat{X}_t^n)\mathrm{d}t + \sigma(t,\hat{X}_t^n)\mathrm{d}W(t) + \int_{\mathbb{Y}}\gamma(t,\hat{X}_{t-}^n,u)N(\mathrm{d}t,\mathrm{d}u), \ \ t\ge 0,\\ \hat{X}_0^n = X_0^n = X_0 = \xi\in\mathscr{D}_r,\end{cases} \tag{3.5}$$

where $\hat{X}_t^n(\theta) := X^n((t+\theta)\wedge t_n)$, $\theta\in(-\infty, 0]$, $t_n := \frac{\lfloor nt\rfloor}{n}$. In view of a similar technique as in the proof of the uniqueness in (b), (3.5) has a unique solution by piecewise solving piece-wisely using the time step length $\frac{1}{n}$. And, beyond that, we can find an $n\in N_0$ satisfying that $e^{r/n}\le 2$; then,

$$\|\hat{X}_t^n\|_r\le\|X_t^n\|_r\vee|X^n(t_n)|\le e^{r(t-t_n)}\|X_t^n\|_r\le 2\|X_t^n\|_r. \tag{3.6}$$

Let

$$\tau_R^n = \inf\{t \geq 0 : |X^n(t)| \geq R\} = \inf\{t \geq 0 : \|X_t^n\|_r \geq R\}, \quad \|\xi\|_r < R, \quad n \in N_0,$$

$Z^{n,m}(t) = X^n(t) - X^m(t)$, $Z_t^{n,m} = X_t^n - X_t^m$, $\Lambda^n(t) = X^n(t) - G(\hat{X}_t^n)$, $\Lambda^{n,m}(t) = \Lambda^n(t) - \Lambda^m(t)$, and $H_t^n = X_t^n - \hat{X}_t^n$. Using (3.5) and the assumption (**A1**), the Young inequality leads to the following for any $\varepsilon > 0$:

$$\begin{aligned}
\|Z_t^{n,m}\|_r^2 &= \sup_{-\infty \leq \theta \leq 0} \mathrm{e}^{2r\theta} |Z^{n,m}(t+\theta)|^2 = \sup_{0 \leq s \leq t} \mathrm{e}^{2r(s-t)} |Z^{n,m}(s)|^2 \\
&\leq \frac{1}{1-\alpha} \sup_{0 \leq s \leq t} |\Lambda^{n,m}(s)|^2 + \alpha \sup_{0 \leq s \leq t} \|\hat{X}_s^n - \hat{X}_s^m\|_r^2 \\
&\leq \frac{1}{1-\alpha} \sup_{0 \leq s \leq t} |\Lambda^{n,m}(s)|^2 + \alpha(1+\varepsilon) \sup_{0 \leq s \leq t} \|H_s^n - H_s^m\|_r^2 + \alpha(1 + \frac{1}{\varepsilon}) \sup_{0 \leq s \leq t} \|Z_s^{n,m}\|_r^2.
\end{aligned}$$

Set $\varepsilon > \frac{\alpha}{1-\alpha}$; then, $\delta := \alpha(1 + \frac{1}{\varepsilon}) < 1$. It is easy to see that

$$\sup_{0 \leq s \leq t} \|Z_s^{n,m}\|_r^2 \leq \kappa_1 \sup_{0 \leq s \leq t} |\Lambda^{n,m}(s)|^2 + \kappa_2 \sup_{0 \leq s \leq t} \|H_s^n - H_s^m\|_r^2, \tag{3.7}$$

where

$$\kappa_1 = \frac{1}{(1-\alpha)(1-\delta)}, \quad \text{and} \quad \kappa_2 = \frac{\alpha\delta}{(\delta-\alpha)(1-\delta)}.$$

Moreover, since b and σ are bounded on bounded subsets of $[0, \infty) \times \mathscr{D}_r$, then

$$|b(t, X_t^n)| \leq C(R) := \sup_{\|\zeta\|_r \leq R} |b(t, \zeta)| < \infty, \quad R \in (\|\xi\|_r, \infty), \quad t \in [0, \tau_R^n] \tag{3.8}$$

and

$$|\sigma(t, X_t^n)| \leq C(R) := \sup_{\|\zeta\|_r \leq R} |\sigma(t, \zeta)| < \infty, \quad R \in (\|\xi\|_r, \infty), \quad t \in [0, \tau_R^n]. \tag{3.9}$$

It follows from the definition of τ_R^n and (3.6) that, for $t \leq \tau_R^n$,

$$\|H_t^n\|_r = \|X_t^n - \hat{X}_t^n\|_r \leq \|X_t^n\|_r + \|\hat{X}_t^n\|_r \leq 3\|X_t^n\|_r \leq 3R.$$

In addition, it is easy to see from (3.5) and (**A1**) that

$$\begin{aligned}
\|H_t^n\|_r &= \sup_{t_n < s < t} (\mathrm{e}^{r(s-t)} |X^n(s) - X^n(s \wedge t_n)|) \\
&\leq \int_{t_n}^t |b(s, \hat{X}_s^n)| \mathrm{d}s + \sup_{t_n \leq s \leq t} \Big| \int_{t_n}^s \sigma(r, \hat{X}_r^n) \mathrm{d}W(r) \Big| \\
&\quad + \sup_{t_n \leq s \leq t} \Big| \int_{t_n}^s \int_{\mathbb{Y}} \gamma(r, \hat{X}_{r-}^n, u) N(\mathrm{d}r, \mathrm{d}u) \Big| + \alpha \sup_{t_n \leq s \leq t} \|\hat{X}_s^n - \hat{X}_{s \wedge t_n}^n\|_r \\
&\leq \int_{t_n}^t |b(s, \hat{X}_s^n)| \mathrm{d}s + \sup_{t_n \leq s \leq t} \Big| \int_{t_n}^s \sigma(r, \hat{X}_r^n) \mathrm{d}W(r) \Big| \\
&\quad + \sup_{t_n \leq s \leq t} \Big| \int_{t_n}^s \int_{\mathbb{Y}} \gamma(r, \hat{X}_{r-}^n, u) N(\mathrm{d}r, \mathrm{d}u) \Big|,
\end{aligned} \tag{3.10}$$

where, in the last step, we have used the fact that

$$\|\hat{X}_s^n - \hat{X}_{s \wedge t_n}^n\|_r = \sup_{-\infty < \theta < 0} (\mathrm{e}^{r\theta} |\hat{X}^n(s+\theta) - \hat{X}^n(s+\theta)|) \leq \sup_{t_n \leq u \leq s} (\mathrm{e}^{r(u-s)} |\hat{X}^n(u) - \hat{X}^n(t_n)|) = 0.$$

In view of (3.8) and (3.9), besides the Burkholder-Davis-Gundy inequality, we get

$$\lim_{n\to\infty}\mathbb{E}\Big(\int_{t_n}^{t\wedge\tau_R^n}|b(s,\hat{X}_s^n)|\mathrm{d}s\Big)^2\leq\lim_{n\to\infty}\frac{1}{n^2}C(R)=0,$$

$$\lim_{n\to\infty}\mathbb{E}\Big(\sup_{t_n\leq s\leq t\wedge\tau_R^n}\Big|\int_{t_n}^{s}\sigma(r,\hat{X}_r^n)\mathrm{d}W(r)\Big|^2\Big)\leq C\lim_{n\to\infty}\mathbb{E}\Big(\int_{t_n}^{t\wedge\tau_R^n}|\sigma(s,\hat{X}_s^n)|^2\mathrm{d}s\Big)\leq\lim_{n\to\infty}\frac{C_R}{n}=0$$

and

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}\Big(\sup_{t_n\leq s\leq t\wedge\tau_R^n}\Big|\int_{t_n}^{s}\int_{\mathbb{Y}}\gamma(r,\hat{X}_{r-}^n,u)N(\mathrm{d}r,\mathrm{d}u)\Big|^2\Big)\\&\leq\lim_{n\to\infty}\mathbb{E}\Big|\int_{t_n}^{t\wedge\tau_R^n}\int_{\mathbb{Y}}\gamma(s,\hat{X}_{s-}^n,u)\lambda(\mathrm{d}u)\mathrm{d}s\Big|^2+\lim_{n\to\infty}\mathbb{E}\int_{t_n}^{t\wedge\tau_R^n}\int_{\mathbb{Y}}|\gamma(s,\hat{X}_{s-}^n,u)|^2\lambda(\mathrm{d}u)\mathrm{d}s\\&\leq\lim_{n\to\infty}\Big(\frac{\beta^2}{n^2}+\frac{\beta}{n}\Big)=0.\end{aligned}$$

Combining these inequalities with (3.10), one has

$$\sup_{t\in[0,T]}\lim_{n\to\infty}\mathbb{E}(\|H_t^n\|_r^2 I_{\{t\leq\tau_R^n\}})=0. \tag{3.11}$$

In what follows, we shall prove that $X^n(\cdot)$ is a Cauchy sequence. Fix $T>0$ and set

$$\mathscr{D}_r^2=\Big\{(v(t))_{t\in(-\infty,T]}\text{ is a adapted process on }\mathscr{D}\text{ with }v_0=\xi\text{ and }\mathbb{E}\Big(\sup_{t\in(-\infty,T]}(\mathrm{e}^{2rt}|v(t)|^2)\Big)<\infty\Big\}.$$

Then, $\mathscr{D}_r^2$ is a complete metric space with

$$\rho(u,v):=\|u-v\|_{\mathscr{D}_r^2}:=\Big(\mathbb{E}\Big(\sup_{t\in[0,T]}(\mathrm{e}^{2rt}|u(t)-v(t)|^2)\Big)\Big)^{\frac{1}{2}}.$$

By the Itô formula, one has the following for any $m,n\in N_0$:

$$\begin{aligned}&\Psi_k(\Lambda^{n,m}(t))^2\\=&2\int_0^t\langle\{\Psi_k\cdot(\Psi_k)_x\}(\Lambda^{n,m}(s)),b(t,\hat{X}_s^n)-b(s,\hat{X}_s^m)\rangle\mathrm{d}s\\&+\int_0^t\mathrm{trace}\{(\sigma(s,\hat{X}_s^n)-\sigma(s,\hat{X}_s^m))^*\{(\Psi_k)_x^2+\Psi_k(\Psi_k)_{xx}\}(\Lambda^{n,m}(s))\\&\times(\sigma(s,\hat{X}_s^n)-\sigma(s,\hat{X}_s^m))\}\mathrm{d}s\\&+2\int_0^t\langle\{\Psi_k(\Psi_k)_x\}(\Lambda^{n,m}(s)),(\sigma(s,\hat{X}_s^n)-\sigma(s,\hat{X}_s^m))\mathrm{d}W(s)\rangle\\&+\int_0^t\int_{\mathbb{Y}}\Big(\Psi_k(\Lambda^{n,m}(s)+\gamma(s,\hat{X}_{s-}^n,u)-\gamma(s,\hat{X}_{s-}^m,u))^2-\Psi_k(\Lambda^{n,m}(s))^2\Big)N(\mathrm{d}s,\mathrm{d}u).\end{aligned} \tag{3.12}$$

By applying the elementary inequality, the assumption (**A2**) and (3.4) yield that

$$\begin{aligned}
&2\int_0^{T\wedge\tau_R^n\wedge\tau_R^m}\langle\{\Psi_k\cdot(\Psi_k)_x\}(\Lambda^{n,m}(t)), b(t,\hat{X}_t^n)-b(t,\hat{X}_t^m)\rangle\mathrm{d}t\\
\leq&2\int_0^{T\wedge\tau_R^n\wedge\tau_R^m}\Big(|\{\Psi_k\cdot(\Psi_k)_x\}(\Lambda^{n,m}(t))|\cdot|b(t,\hat{X}_t^n)-b(t,\hat{X}_t^m)|\Big)\mathrm{d}t\\
\leq&\frac{1}{8}\sup_{0\leq t\leq T\wedge\tau_R^n\wedge\tau_R^m}\Psi_k(\Lambda^{n,m}(t))^2+C(T)\int_0^{T\wedge\tau_R^n\wedge\tau_R^m}\Big(\|\hat{X}_t^n-X_t^n\|_r^2\cdot u(\|\hat{X}_t^n-X_t^n\|_r^2)\\
&+\|X_t^n-X_t^m\|_r^2\cdot u(\|X_t^n-X_t^m\|_r^2)+\|X_t^m-\hat{X}_t^m\|_r^2\cdot u(\|X_t^m-\hat{X}_t^m\|_r^2)\Big)\mathrm{d}t
\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}
&\int_0^{T\wedge\tau_R^n\wedge\tau_R^m}\mathrm{trace}\{(\sigma(s,\hat{X}_s^n)-\sigma(s,\hat{X}_s^m))^*\{(\Psi_k)_x^2+\Psi_k(\Psi_k)_{xx}\}(\Lambda^{n,m}(t))\\
&\quad\times(\sigma(s,\hat{X}_s^n)-\sigma(s,\hat{X}_s^m))\}\mathrm{d}s\\
\leq&C(T)\int_0^{T\wedge\tau_R^n\wedge\tau_R^m}\Big(\|\hat{X}_t^n-X_t^n\|_r^2\cdot u(\|\hat{X}_t^n-X_t^n\|_r^2)\\
&+\|X_t^n-X_t^m\|_r^2\cdot u(\|X_t^n-X_t^m\|_r^2)+\|X_t^m-\hat{X}_t^m\|_r^2\cdot u(\|X_t^m-\hat{X}_t^m\|_r^2)\Big)\mathrm{d}t\\
&+K(T)\int_0^{T\wedge\tau_R^n\wedge\tau_R^m}2I_{(0,\frac{1}{k})}(|\Lambda^{n,m}(t)|)\|\hat{X}_t^n-\hat{X}_t^m\|_r^2\cdot u(\|\hat{X}_t^n-\hat{X}_t^m\|_r^2)\mathrm{d}t\\
\leq&C(T)\int_0^{T\wedge\tau_R^n\wedge\tau_R^m}\Big(\|\hat{X}_t^n-X_t^n\|_r^2\cdot u(\|\hat{X}_t^n-X_t^n\|_r^2)\\
&+\|X_t^n-X_t^m\|_r^2\cdot u(\|X_t^n-X_t^m\|_r^2)+\|X_t^m-\hat{X}_t^m\|_r^2\cdot u(\|X_t^m-\hat{X}_t^m\|_r^2)\Big)\mathrm{d}t\\
&+C(T)\int_0^{T\wedge\tau_R^n\wedge\tau_R^m}I_{(0,\frac{1}{k(1-2\alpha)})}(|X^n(t)-X^m(t)|)2\sup_{0\leq s\leq t}\mathrm{e}^{2rs}|X^n(s)-X^m(s)|^2\\
&\cdot u(2\sup_{0\leq s\leq t}\mathrm{e}^{2rs}|X^n(s)-X^m(s)|^2)\mathrm{d}t\\
\leq&C(T)\int_0^{T\wedge\tau_R^n\wedge\tau_R^m}\Big(\|\hat{X}_t^n-X_t^n\|_r^2\cdot u(\|\hat{X}_t^n-X_t^n\|_r^2)\\
&+\|X_t^n-X_t^m\|_r^2\cdot u(\|X_t^n-X_t^m\|_r^2)+\|X_t^m-\hat{X}_t^m\|_r^2\cdot u(\|X_t^m-\hat{X}_t^m\|_r^2)\Big)\mathrm{d}t+\epsilon(k),
\end{aligned}\tag{3.14}$$

where, in the penultimate inequality, we have used the fact that, for any $0\leq t\leq T\wedge\tau_R^n\wedge\tau_R^m$,

$$\begin{aligned}
\|\hat{X}_t^n-\hat{X}_t^m\|_r^2\leq&\sup_{-\infty\leq\theta\leq0}\mathrm{e}^{2r\theta}|X^n(t+\theta)-X^m(t+\theta)|^2I_{\{t+\theta\leq t_n\}}\\
&+\sup_{-\infty\leq\theta\leq0}\mathrm{e}^{2r\theta}|X^n(t_n)-X^m(t_n)|^2I_{\{t+\theta\geq t_n\}}\\
\leq&2\sup_{0\leq s\leq t}\mathrm{e}^{2rs}|X^n(s)-X^m(s)|^2
\end{aligned}$$

and

$$|X^n(s)-X^m(s)|\leq|\Lambda^{n,m}(s)|+\alpha\|\hat{X}_s^n-\hat{X}_s^m\|_r\leq|\Lambda^{n,m}(s)|+2\alpha\sup_{0\leq u\leq s}|X^n(u)-X^m(u)|,\quad 0\leq s\leq t,$$

by aid of the definition of $\hat{X}_t^n$ and (**A1**). Here,

$$\epsilon(k) := C(T) \sup_{0<s<s_0} su(s) \downarrow 0 \ \text{ as } \ k \uparrow \infty, \ s_0 := \frac{2e^{2rT}}{(k(1-2\alpha))^2},$$

because $u \in \mathscr{U}$. By the Burkholder-Davis-Gundy inequality, (**A2**), and (3.4), one gets

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R^n\wedge\tau_R^m} 2\int_0^t \langle\{\Psi_k(\Psi_k)_x\}(\Lambda^{n,m}(s)), (\sigma(s,\hat{X}_s^n)-\sigma(s,\hat{X}_s^m))\mathrm{d}W(s)\rangle \\
\le&\frac{1}{8}\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R^n\wedge\tau_R^m} |\Psi_k(\Lambda^{n,m}(t))|^2\Big) + C(T)\int_0^{T\wedge\tau_R^n\wedge\tau_R^m} \Big(\|\hat{X}_t^n - X_t^n\|_r^2 \cdot u(\|\hat{X}_t^n - X_t^n\|_r^2) \\
&+ \|X_t^n - X_t^m\|_r^2 \cdot u(\|X_t^n - X_t^m\|_r^2) + \|X_t^m - \hat{X}_t^m\|_r^2 \cdot u(\|X_t^m - \hat{X}_t^m\|_r^2)\Big)\mathrm{d}t.
\end{aligned} \tag{3.15}$$

By virtue of (3.4) and a Taylor expansion, one infers that, for $t \in [0,T]$,

$$\begin{aligned}
&\Psi_k(\Lambda^{n,m}(t) + \gamma(s,\hat{X}_{t-}^n,u) - \gamma(s,\hat{X}_{t-}^m,u))^2 - \Psi_k(\Lambda^{n,m}(t))^2 \\
\le&|\gamma(t,\hat{X}_{t-}^n,u) - \gamma(t,\hat{X}_{t-}^m,u)| \int_0^1 \{2\Psi_k(\Psi_k)_x\}(\Lambda^{n,m}(s) + \theta(\gamma(t,\hat{X}_{t-}^n,u) - \gamma(t,\hat{X}_{t-}^m,u)))\mathrm{d}\theta \\
\le&2|\gamma(t,\hat{X}_{t-}^n,u) - \gamma(t,\hat{X}_{t-}^m,u)| \int_0^1 \Psi_k(\Lambda^{n,m}(s) + \theta(\gamma(t,\hat{X}_{t-}^n,u) - \gamma(t,\hat{X}_{t-}^m,u)))\mathrm{d}\theta \\
\le&2|\gamma(t,\hat{X}_{t-}^n,u) - \gamma(t,\hat{X}_{t-}^m,u)|^2 + 2\Psi_k(\Lambda^{n,m}(s))|\gamma(t,\hat{X}_{t-}^n,u) - \gamma(t,\hat{X}_{t-}^m,u)|.
\end{aligned}$$

This and (**A2**) imply that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R^n\wedge\tau_R^m} \int_0^t \int_{\mathbb{Y}} \Big(\Psi_k(\Lambda^{n,m}(s) + \gamma(s,\hat{X}_{s-}^n,u) - \gamma(s,\hat{X}_{s-}^m,u))^2 - \Psi_k(\Lambda^{n,m}(s))^2\Big)N(\mathrm{d}s,\mathrm{d}u)\Big) \\
\le&\frac{1}{4}\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R^n\wedge\tau_R^m} |\Psi_k(\Lambda^{n,m}(t))|^2\Big) + 3\mathbb{E}\int_0^{T\wedge\tau_R^n\wedge\tau_R^m} \int_{\mathbb{Y}} |\gamma(t,\hat{X}_{t-}^n,u) - \gamma(t,\hat{X}_{t-}^m,u)|^2 \lambda(\mathrm{d}u)\mathrm{d}t \\
\le&\frac{1}{4}\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R^n\wedge\tau_R^m} |\Psi_k(\Lambda^{n,m}(t))|^2\Big) + C(T)\mathbb{E}\int_0^{T\wedge\tau_R^n\wedge\tau_R^m} \Big(\|\hat{X}_t^n - X_t^n\|_r^2 \cdot u(\|\hat{X}_t^n - X_t^n\|_r^2) \\
&+ \|X_t^n - X_t^m\|_r^2 \cdot u(\|X_t^n - X_t^m\|_r^2) + \|X_t^m - \hat{X}_t^m\|_r^2 \cdot u(\|X_t^m - \hat{X}_t^m\|_r^2)\Big)\mathrm{d}t.
\end{aligned} \tag{3.16}$$

Substituting (3.13)–(3.16) into (3.12), we infer that, for any $k \ge 1$ and $m,n \in N_0$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R^n\wedge\tau_R^m} \Psi_k(\Lambda^{n,m}(t))^2\Big) \\
\le&C(T)\int_0^T \Big(\mathbb{E}\|H_t^n\|_r^2 \cdot u(\|H_t^n\|_r^2)I_{\{t\le\tau_R^n\}} + \mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_R^n\wedge\tau_R^m} \|Z_s^{n,m}\|_r^2 \cdot u(\|Z_s^{n,m}\|_r^2)\Big) \\
&+ \mathbb{E}\|H_t^m\|_r^2 \cdot u(\|H_t^m\|_r^2)I_{\{t\le\tau_R^m\}}\Big)\mathrm{d}t + C(T)\epsilon(k).
\end{aligned}$$

Let $k \uparrow \infty$; using Jensen's inequality and noting (3.7), we get the following for any $n, m \in N_0$:

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R^n\wedge\tau_R^m}\|Z^{n,m}(t)\|_r^2\Big)\\
\le& C(T)\int_0^T\Big(\mathbb{E}\|H_t^n\|_r^2\cdot u(\mathbb{E}\|H_t^n\|_r^2)I_{\{t\le\tau_R^n\}}+\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_R^n\wedge\tau_R^m}\|Z_s^{n,m}\|_r^2\Big)\cdot u(\mathbb{E}\sup_{0\le s\le t\wedge\tau_R^n\wedge\tau_R^m}\|Z_s^{n,m}\|_r^2)\\
&+2\kappa_2\mathbb{E}\|H_t^m\|_r^2\cdot u(\mathbb{E}\|H_t^m\|_r^2)I_{\{t\le\tau_R^m\}}\Big)\mathrm{d}t+\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R^n\wedge\tau_R^m}(\|H^n(t)\|_r^2+\|H^m(t)\|_r^2)\Big)\\
\le& C(T,n,m)+C(T)\int_0^T\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_R^n\wedge\tau_R^m}\|Z_s^{n,m}\|_r^2\Big)\cdot u\Big(\mathbb{E}\sup_{0\le s\le t\wedge\tau_R^n\wedge\tau_R^m}\|Z_s^{n,m}\|_r^2\Big)\mathrm{d}t,
\end{aligned}$$

where

$$\begin{aligned}
C(T,n,m):=&2\kappa_2\mathbb{E}\Big(\sup_{0\le t\le T}\mathrm{e}^{2rt}(\|H^n(t)\|_r^2I_{\{t\le\tau_R^n\}}+\|H^m(t)\|_r^2I_{\{t\le\tau_R^m\}})\Big)\\
&+C(T)\int_0^T\Big(\mathbb{E}\|H_t^n\|_r^2\cdot u(\mathbb{E}\|H_t^n\|_r^2)I_{\{t\le\tau_R^n\}}\\
&+\mathbb{E}\|H_t^m\|_r^2\cdot u(\mathbb{E}\|H_t^m\|_r^2)I_{\{t\le\tau_R^m\}}\Big)\mathrm{d}t\to 0,\ \text{ as }\ n,m\to\infty.
\end{aligned}$$

Let $G(s)=\int_1^s\frac{1}{ru(r)}\mathrm{d}r,\ \ s>0$. Since $\int_0^1\frac{1}{ru(r)}\mathrm{d}r=\infty$, by the Bihari inequality, we have

$$\lim_{n,m\to\infty}\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R^n\wedge\tau_R^m}\|Z^{n,m}(t)\|_r^2\Big)\le G^{-1}(-\infty)=0, \tag{3.17}$$

because of (3.11). Thus, to prove that $X^n_\cdot$ converges in probability to a solution of (2.1), it is sufficient to show that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\mathbb{P}(\tau_R^n\le T)=0. \tag{3.18}$$

Indeed, this and (3.17) yield that, for any $\varepsilon>0$,

$$\lim_{n,m\to\infty}\rho(X^n,X^m)=\lim_{n,m\to\infty}\Big(\mathbb{E}\Big(\sup_{t\in[0,T]}(\mathrm{e}^{2rt}|X^n(t)-X^m(t)|^2)\Big)\Big)^{\frac12}=0.$$

This implies that $X^n(t)$ is a Cauchy sequence in $\mathscr{D}_r^2$ with the norm ρ and has a unique limit $X(t)$ on $\mathscr{D}_r^2$ due to the completeness of $(\mathscr{D}_r^2,\rho)$. Then, by using a standard argument, we can show that $(X(t))_{t\in[0,T]}$ is the unique solution to (2.1). So, to achieve the desired assertion, it is sufficient to show that (3.18) holds true. By a simple calculation, and using the assumption (**A1**), we have

$$\mathrm{e}^{2rt}\|X_t^n\|_r^2\le\frac{1}{1-\alpha}\|\xi\|_r^2+\frac{1}{(1-\alpha)^2}\sup_{0\le s\le t}(\mathrm{e}^{2rs}|\Lambda^n(s)|^2). \tag{3.19}$$

By Itô's formula, the Burkholder-Davis-Gundy inequality, and Gronwall's inequality, together with (**A2**), (3.8), and (3.9), one has

$$\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_R^n}\mathrm{e}^{2rs}|\Lambda^n(s)|^2\Big)\le C\|\xi\|_r^2+C(R)t+C\int_0^t\mathrm{e}^{2rs}\mathbb{E}\|H_s^n\|_r^2I_{\{s\le\tau_R^n\}}\mathrm{d}s,\ \ t\ge0. \tag{3.20}$$

Noting that for the definition τ_R^n, one has $\{\tau_R^n \le T, \sup_{0\le t\le T\wedge\tau_R^n} |X^n(t)| < \frac{R}{4}\} = \emptyset$, it follows from (3.11), (3.19), and (3.20) that

$$\begin{aligned}\lim_{R\to\infty}\lim_{n\to\infty}\mathbb{P}(\tau_R^n \le T) &= \lim_{R\to\infty}\lim_{n\to\infty}\mathbb{P}\Big(\tau_R^n \le T, \sup_{0\le t\le T\wedge\tau_R^n}|X^n(t)| \ge \frac{R}{4}\Big)\\ &\le \lim_{R\to\infty}\lim_{n\to\infty}\mathbb{P}\Big(\sup_{0\le t\le T\wedge\tau_R^n}|X^n(t)| \ge \frac{R}{4}\Big)\\ &\le \lim_{R\to\infty}\lim_{n\to\infty}\frac{16}{R^2}\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R^n} e^{2rt}|X^n(t)|^2\Big) = 0,\end{aligned}$$

where, in the second step, we have used the Chebyshev inequality. Therefore, (3.18) holds.

Case 2. Next, we present the existence of the solution for unbounded b, σ and β. For any $n \ge 1$, let $\vec{n} = (n, n, \cdots, n) \in \mathbb{R}^d$. Set

$$\rho_n(\xi) := (\xi \wedge \vec{n}) \vee (-\vec{n}), \quad n \ge 1,\ \xi \in \mathscr{D}_r,$$

$$b_n(t,\xi) := b(t\wedge n, \rho_n(\xi)), \quad \sigma_n(t,\xi) := \sigma(t\wedge n, \rho_n(\xi)),$$

and

$$\gamma_n(t,\xi,u) := \gamma(t\wedge n, \rho_n(\xi), u).$$

Then, b_n, σ_n, and $\beta_n := \int_{\mathbb{Y}}(|\gamma_n(\cdot,\cdot,u)|^2 + |\gamma_n(\cdot,\cdot,u)|)\lambda(\mathrm{d}u)$ are locally bounded. Therefore, according to (a), (b), and Case 1,

$$\begin{cases} \mathrm{d}\{X^n(t) - G(X_t^n)\} = b_n(t, X_t^n)\mathrm{d}t + \sigma_n(t, X_t^n)\mathrm{d}W(t) + \displaystyle\int_{\mathbb{Y}}\gamma_n(t, X_{t-}^n, u)N(\mathrm{d}t, \mathrm{d}u), \quad t \ge 0,\\ X_0^n = X_0 = \xi \in \mathscr{D}_r,\end{cases}$$

has a unique solution $X^n(t)$, $t \ge 0$. For any $m \ge n \ge 1$ and $\xi \in \mathscr{D}_r$ with $\|\xi\|_r \le n$, one has

$$b_n(t,\xi) = b_m(t,\xi), \ \ \sigma_n(t,\xi) = \sigma_m(t,\xi), \ \ \gamma_n(t,\xi,u) = \gamma_m(t,\xi,u). \ \ t \in [0,n],$$

Then it follows that $X^n(t) = X^m(t)$ for $t \le \bar{\tau}_n$, where $\bar{\tau}_n := n \wedge \inf\{t \ge 0 : \|X_t^n\|_r \ge n\}$. Let $n \uparrow \infty$; then, $\bar{\tau}_n \uparrow \infty$. So, for $t < \bar{\tau}_n$, $X(t) := X^n(t)$ is a solution of (2.1). □

4. Order preservation for neutral-type stochastic differential equations of infinite delay with jumps

In this section, we first show the D-order preservation problems for a class of neutral-type stochastic differential equations of infinite delay with pure jump processes.

Theorem 4.1. Let (**A1**)–(**A4**) hold. The solutions of (2.1) and (2.2) are D-order-preserving if the following conditions are satisfied:
(i) The drift $b = (b_1, b_2, \cdots, b_d)$ and $\bar{b} = (\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_d)$ satisfy that $b_i(t,\xi) \le \bar{b}_i(t,\bar{\xi})$ for any $1 \le i \le d$ if $\xi, \bar{\xi} \in \mathscr{D}_r$ with $\xi \le_D \bar{\xi}$ and $\xi_i(0) - G_i(\xi) = \bar{\xi}_i(0) - G_i(\bar{\xi})$.

(ii) The diffusion $\sigma = (\sigma_{ij})$, and $\bar{\sigma} = (\bar{\sigma}_{ij})$ satisfy that $\sigma = \bar{\sigma}$ for any $1 \le i \le d$, $1 \le j \le m$ and $\xi, \bar{\xi} \in \mathscr{D}_r$. Moreover,

$$\begin{aligned}&\sum_{j=1}^{m}(|\sigma_{ij}(t,\xi) - \sigma_{ij}(t,\eta)|^2 + |\bar{\sigma}_{ij}(t,\xi) - \bar{\sigma}_{ij}(t,\eta)|^2)\\ &\le K(t)|\xi_i(0) - G_i(\xi) - \bar{\xi}_i(0) + G_i(\bar{\xi})|^2 u(|\xi_i(0) - G_i(\xi) - \bar{\xi}_i(0) + G_i(\bar{\xi})|^2), \ \ t \ge 0, \ \xi, \bar{\xi} \in \mathscr{D}_r.\end{aligned}$$

In other words, $\sigma_{ij}(t,\xi)$ only depends on t and $\xi_i(0) - G_i(\xi)$.
(iii) The jump diffusion term $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_d)$ satisfies that $\xi^i(0) - G_i(\xi) + \gamma_i(t,\xi,\cdot) \le \bar{\xi}_i(0) - G_i(\bar{\xi}) + \gamma_i(t,\bar{\xi},\cdot)$ for any $1 \le i \le d$ if $\xi, \bar{\xi} \in \mathscr{D}_r$ with $\xi \le_D \bar{\xi}$.

Proof. For any $T > 0$ and the initial values $\xi, \bar{\xi} \in \mathscr{D}_r$ with $\xi \le_D \bar{\xi}$, we first seek to prove that

$$\mathbb{E} \sup_{0 \le t \le T} (\Lambda^i(t) - \bar{\Lambda}^i(t))^+ = 0, \quad 1 \le i \le d, \tag{4.1}$$

where $\Lambda^i(t) = X^i(t) - G^i(X_t)$ and $\bar{\Lambda}^i(t) = \bar{X}^i(t) - G^i(\bar{X}_t)$. Define a stopping time

$$\tau_k = \inf\{t \ge 0 : |\Lambda(t) - \Lambda(t) \wedge \bar{\Lambda}(t)| \ge k\}. \quad k \ge 1.$$

By the definition of ψ_n and $\xi \le_D \bar{\xi}$, it follows that

$$\psi_n(\Lambda^i(0) - \bar{\Lambda}^i(0)) = \psi_n(\xi^i(0) - G^i(\xi) - \bar{\xi}^i(0) - G^i(\bar{\xi})) = 0.$$

Then, it follows from the Itô formula and the condition (ii) that

$$\begin{aligned}&e^{2r(t\wedge\tau_k)}\psi_n(\Lambda^i(t\wedge\tau_k) - \bar{\Lambda}^i(t\wedge\tau_k))^2\\ \le& 2r\int_0^{t\wedge\tau_k} e^{2rs}\psi_n(\Lambda^i(s) - \bar{\Lambda}^i(s))^2 ds\\ &+ 2\int_0^{t\wedge\tau_k} e^{2rs}(b^i(s,X_s) - \bar{b}^i(s,\bar{X}_s))\{\psi_n\psi_n^{'}\}(\Lambda^i(s) - \bar{\Lambda}^i(s))ds\\ &+ \sum_{j=1}^{m}\int_0^{t\wedge\tau_k} e^{2rs}(\sigma^{ij}(s,X_s) - \sigma^{ij}(s,\bar{X}_s))^2\{\psi_n\psi_n^{''} + \psi_n^{'2}\}(\Lambda^i(s) - \bar{\Lambda}^i(s))ds\\ &+ 2\sum_{j=1}^{m}\int_0^{t\wedge\tau_k} e^{2rs}(\sigma^{ij}(s,X_s) - \sigma^{ij}(s,\bar{X}_s))\{\psi_n\psi_n^{'}\}(\Lambda^i(s) - \bar{\Lambda}^i(s))dB^j(s)\\ &+ \int_0^{t\wedge\tau_k}\int_{\mathbb{Y}} e^{2rs}\{\psi_n(\Lambda^i(s-) - \bar{\Lambda}^i(s-) + \gamma^i(s,X_s,u)\\ &- \bar{\gamma}^i(s,\bar{X}_s,u))^2 - \psi_n(\Lambda^i(s-) - \bar{\Lambda}^i(s-))^2\}N(ds,du).\end{aligned} \tag{4.2}$$

Set $\Lambda^i(t) \wedge \bar{\Lambda}^i(t) = (X^i(t) - G^i(X_t)) \wedge (\bar{X}^i(t) - G^i(\bar{X}_t)) =: Y^i(t) - D^i(Y_t)$, $i = 1, \cdots, d$. Then, it is easy to see that $Y_t \le_D \bar{X}_t$ because $Y^i(t) - D^i(Y_t) \le \bar{X}^i(t) - G^i(\bar{X}_t)$, $i = 1, \cdots, d$. Due to the fact that $0 \le \psi_n^{'}(\Lambda^i(s) - \bar{\Lambda}^i(s)) \le I_{\{\Lambda^i(s) - \bar{\Lambda}^i(s)\}}$, and when $\Lambda^i(s) > \bar{\Lambda}^i(s)$, one has that $\Lambda^i(s) \wedge \bar{\Lambda}^i(s) = \bar{\Lambda}^i(s)$, that is, $Y^i(s) - D^i(Y_s) = \bar{X}^i(s) - G^i(\bar{X}_s)$. It follows from the condition (ii) that

$$(b^i(s,Y_s) - \bar{b}^i(s,\bar{X}_s))\{\psi_n\psi_n^{'}\}(\Lambda^i(s) - \bar{\Lambda}^i(s)) \le 0, \ n \ge 1, \ \ s \in [0,T].$$

This and the assumption (**A2**), for $t \in [0,T]$, $n,k \geq 1$, imply that

$$
\begin{aligned}
&2\int_0^{t\wedge\tau_k} \mathrm{e}^{2rs}(b^i(s,X_s) - \bar{b}^i(s,\bar{X}_s))\{\psi_n\psi_n^{'}\}(\Lambda^i(s) - \bar{\Lambda}^i(s))\mathrm{d}s \\
&= 2\int_0^{t\wedge\tau_k} \mathrm{e}^{2rs}(b^i(s,X_s) - b^i(s,Y_s) + b^i(s,Y_s) - \bar{b}^i(s,\bar{X}_s))\{\psi_n\psi_n^{'}\}(\Lambda^i(s) - \bar{\Lambda}^i(s))\mathrm{d}s \\
&\leq 2\int_0^{t\wedge\tau_k} \mathrm{e}^{2rs}(b^i(s,X_s) - b^i(s,Y_s))\{\psi_n\psi_n^{'}\}(\Lambda^i(s) - \bar{\Lambda}^i(s))\mathrm{d}s \\
&\leq \int_0^{t\wedge\tau_k} \mathrm{e}^{2rs}\Big(8T|b^i(s,X_s) - b^i(s,Y_s)|^2 + \frac{1}{8T}\psi_n(\Lambda^i(s) - \bar{\Lambda}^i(s))^2\Big)\mathrm{d}s \\
&\leq C(T)\int_0^{t\wedge\tau_k} \mathrm{e}^{2rs}\|X_s - Y_s\|_r^2 \cdot u(\|X_s - Y_s\|_r^2)\mathrm{d}s \\
&\quad + \frac{1}{8(1-\alpha)}\sup_{0\leq s\leq t\wedge\tau_k} \mathrm{e}^{2rs}\psi_n(\Lambda^i(s) - \bar{\Lambda}^i(s))^2,
\end{aligned} \tag{4.3}
$$

where, in the third step, we have used the fact that $0 \leq \psi_n^{'}(\Lambda^i(s) - \bar{\Lambda}^i(s)) \leq I_{\{\Lambda^i(s)>\bar{\Lambda}^i(s)\}}$. Meanwhile, the condition (ii) implies that $\sigma^{ij}(s,X_s) = \sigma^{ij}(s,X_s)$ depends only on $X^i(s) - G^i(X_s)$. Then, it follows from (**A2**) and (3.1) that, for some constant $C(T) > 0$,

$$
\begin{aligned}
&\sum_{j=1}^m \mathrm{e}^{2rs}(\sigma^{ij}(s,X_s) - \sigma^{ij}(s,\bar{X}_s))^2\{\psi_n\psi_n^{''} + \psi_n^{'2}\}(\Lambda^i(s) - \bar{\Lambda}^i(s)) \\
&= \sum_{j=1}^m \mathrm{e}^{2rs}(\sigma^{ij}(s,X_s) - \sigma^{ij}(s,\bar{X}_s))^2\psi_n\psi_n^{''}(\Lambda^i(s) - \bar{\Lambda}^i(s)) \\
&\quad + \sum_{j=1}^m \mathrm{e}^{2rs}(\sigma^{ij}(s,X_s) - \sigma^{ij}(s,\bar{X}_s))^2\psi_n^{'2}(\Lambda^i(s) - \bar{\Lambda}^i(s)) \\
&\leq \sum_{j=1}^m \mathrm{e}^{2rs}(\sigma^{ij}(s,X_s) - \sigma^{ij}(s,\bar{X}_s))^2 I_{\{\Lambda^i(s)-\bar{\Lambda}^i(s)\in(0,\frac{1}{n})\}} \\
&\quad + \sum_{j=1}^m \mathrm{e}^{2rs}(\sigma^{ij}(s,X_s) - \sigma^{ij}(s,Y_s) + \sigma^{ij}(s,Y_s) - \sigma^{ij}(s,\bar{X}_s))^2 I_{\{\Lambda^i(s)>\bar{\Lambda}^i(s)\}} \\
&\leq C(T)I_{\{\Lambda^i(s)-\bar{\Lambda}^i(s)\in(0,\frac{1}{n})\}}\mathrm{e}^{2rs}|\Lambda^i(s) - \bar{\Lambda}^i(s)|^2 \cdot u(|\Lambda^i(s) - \bar{\Lambda}^i(s)|^2) \\
&\quad + C(T)\mathrm{e}^{2rs}\|X_s - Y_s\|_r^2 \cdot u(\|X_s - Y_s\|_r^2) \\
&\leq \epsilon(n) + C(T)\mathrm{e}^{2rs}\|X_s - Y_s\|_r^2 \cdot u(\|X_s - Y_s\|_r^2),
\end{aligned} \tag{4.4}
$$

where

$$
\epsilon(n) := C(T)\sup_{s\in(0,s_1)} s\cdot u(s) \downarrow 0, \text{ as } n\uparrow\infty,\ s_1 := \frac{\mathrm{e}^{2rT}}{n^2}
$$

because $u \in \mathcal{U}$. Moreover, the assumption (**A2**), (3.1), and the condition (ii) imply that, for any $n \geq 1$

and $0 \le s \le T$,

$$
\begin{aligned}
&\sum_{j=1}^{m} \mathrm{e}^{4rs}(\sigma^{ij}(s,X_s)-\sigma^{ij}(s,\bar{X}_s))^2\{\psi_n\psi_n^{'}\}^2(\Lambda^i(s)-\bar{\Lambda}^i(s))\\
&\le \sum_{j=1}^{m} \mathrm{e}^{4rs}(\sigma^{ij}(s,X_s)-\sigma^{ij}(s,\bar{X}_s))^2\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2\\
&\le C(T)\mathrm{e}^{4rs}\|X_s-Y_s\|_r^2\cdot u(\|X_s-Y_s\|_r^2)\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2,
\end{aligned}
$$

which, together with the Burkholder-Davis-Gundy inequality, leads to

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t} 2\sum_{j=1}^{m}\int_0^{t\wedge\tau_k} \mathrm{e}^{2rs}(\sigma^{ij}(s,X_s)-\sigma^{ij}(s,\bar{X}_s))\{\psi_n\psi_n^{'}\}(\Lambda^i(s)-\bar{\Lambda}^i(s))\mathrm{d}B^j(s)\\
&\le C(T)\mathbb{E}\Big(\int_0^{t\wedge\tau_k} \mathrm{e}^{2rs}\|X_s-Y_s\|_r^2\cdot u(\|X_s-Y_s\|_r^2)\mathrm{e}^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2\mathrm{d}s\Big)^{\frac{1}{2}}\\
&\le C(T)\mathbb{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{2rs}\|X_s-Y_s\|_r^2\cdot u(\|X_s-Y_s\|_r^2)\mathrm{d}s\\
&\quad+\frac{1}{8}\mathbb{E}\sup_{0\le s\le t\wedge\tau_k} \mathrm{e}^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2.
\end{aligned}
\tag{4.5}
$$

At last, the condition (iii) implies that

$$
Y^i(s)-G^i(Y_s)+\gamma^i(s,Y_s,\cdot)\le\bar{\Lambda}^i(s)+\gamma^i(s,\bar{X}_s,\cdot),\quad \lambda\times\mathbb{P}\text{-a.e.} \tag{4.6}
$$

If $\Lambda^i(s)\le\bar{\Lambda}^i(s)$, then (4.6) becomes

$$
\Lambda^i(s)+\gamma^i(t,Y_s,\cdot)\le\bar{\Lambda}^i(s)+\gamma^i(t,\bar{X}_s,\cdot),\quad \lambda\times\mathbb{P}\text{-a.e.}
$$

This, by the notion of ψ_n, and (3.1) leads to

$$
\begin{aligned}
&\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s)+\gamma^i(s,X_s,\cdot)-\bar{\gamma}^i(s,\bar{X}_s,\cdot))^2-\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2\\
&\quad=\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s)+\gamma^i(s,X_s,\cdot)-\bar{\gamma}^i(s,\bar{X}_s,\cdot))^2\\
&\quad=\psi_n(\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)+(\Lambda^i(s)+\gamma^i(s,Y_s,\cdot))-(\bar{\Lambda}^i(s)+\bar{\gamma}^i(s,\bar{X}_s,\cdot)))^2\\
&\quad\le\psi_n(\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot))^2\\
&\quad\le|\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)|^2,\quad \lambda\times\mathbb{P}\text{-a.e.}
\end{aligned}
$$

If $\Lambda^i(s)\ge\bar{\Lambda}^i(s)$, then (4.6) becomes

$$
\gamma^i(s,Y_s,\cdot)\le\gamma^i(s,\bar{X}_s,\cdot),\quad \lambda\times\mathbb{P}\text{-a.e.}
$$

This, by the notion of ψ_n and (3.1) leads to

$$\begin{aligned}
&\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s)+\gamma^i(s,X_s,\cdot)-\bar{\gamma}^i(s,\bar{X}_s,\cdot))^2-\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2\\
=&\psi_n(\Lambda^i(s)+\gamma^i(s,X_s,\cdot)-\bar{\Lambda}^i(s)-\gamma^i(s,Y_s,\cdot)+\gamma^i(s,Y_s,\cdot)-\bar{\gamma}^i(s,\bar{X}_s,\cdot))^2\\
&-\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2\\
=&\psi_n(\Lambda^i(s)+\gamma^i(s,X_s,\cdot)-\bar{\Lambda}^i(s)-\gamma^i(s,Y_s,\cdot))^2-\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2\\
\leq&2\psi_n\psi_n^{'}(\Lambda^i(s)-\bar{\Lambda}^i(s)+\theta(\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)))|\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)|\\
\leq&2\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s)+\theta(\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)))|\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)|\\
\leq&2|\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)|\Big(\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))+|\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)|\Big)\\
\leq&2|\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)|^2+2\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))|\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)|.
\end{aligned}$$

Therefore, it is easy to see that

$$\begin{aligned}
&\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s)+\gamma^i(s,X_s,\cdot)-\bar{\gamma}^i(s,\bar{X}_s,\cdot))^2-\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2\\
\leq&2|\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)|^2+2\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))|\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)|.
\end{aligned}$$

This, together with (**A2**) and (3.1), implies that

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq s\leq t}\int_0^{s\wedge\tau_k}\int_{\mathbb{Y}}\mathrm{e}^{2rv}\{\psi_n(\Lambda^i(v-)-\bar{\Lambda}^i(v-)+\gamma^i(v,X_v,u)-\bar{\gamma}^i(v,\bar{X}_v,u))^2\\
&-\psi_n(\Lambda^i(v-)-\bar{\Lambda}^i(v-))^2\}^+N(\mathrm{d}v,\mathrm{d}u)\\
=&\mathbb{E}\int_0^{t\wedge\tau_k}\int_{\mathbb{Y}}\mathrm{e}^{2rs}\{\psi_n(\Lambda^i(s-)-\bar{\Lambda}^i(s-)+\gamma^i(s,X_s,u)-\bar{\gamma}^i(s,\bar{X}_s,u))^2-\psi_n(\Lambda^i(s-)-\bar{\Lambda}^i(s-))^2\}^+N(\mathrm{d}s,\mathrm{d}u)\\
=&\mathbb{E}\int_0^{t\wedge\tau_k}\int_{\mathbb{Y}}\mathrm{e}^{2rs}\{\psi_n(\Lambda^i(s-)-\bar{\Lambda}^i(s-)+\gamma^i(s,X_s,u)-\bar{\gamma}^i(s,\bar{X}_s,u))^2\\
&-\psi_n(\Lambda^i(s-)-\bar{\Lambda}^i(s-))^2\}^+\lambda(\mathrm{d}u)\mathrm{d}s\\
\leq&2\mathbb{E}\int_0^{t\wedge\tau_k}\int_{\mathbb{Y}}\mathrm{e}^{2rs}\Big(|\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)|^2+2\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))|\gamma^i(s,X_s,\cdot)-\gamma^i(s,Y_s,\cdot)|\Big)\lambda(\mathrm{d}u)\mathrm{d}s\\
\leq&C(T)\mathbb{E}\int_0^{t\wedge\tau_k}\mathrm{e}^{2rs}\|X_s-Y_s\|_r^2\cdot u(\|X_s-Y_s\|_r^2)\mathrm{d}s+\frac{1}{4(1-\alpha)}\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_k}\mathrm{e}^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2.
\end{aligned} \tag{4.7}$$

Substituting (4.3)–(4.5) and (4.7) into (4.2), one has the following for $1 \leq i \leq d$:

$$
\begin{aligned}
&\mathbb{E} \sup_{-\infty<s\leq t\wedge\tau_k} e^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2 = \mathbb{E} \sup_{0\leq s\leq t\wedge\tau_k} e^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2 \\
\leq& 2r\mathbb{E}\int_0^{t\wedge\tau_k} e^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2 \mathrm{d}s + \frac{1}{2}\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_k} e^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2 \\
&+ \epsilon(n) + C(T)\mathbb{E}\int_0^{t\wedge\tau_k} e^{2rs}\|X_s-Y_s\|_r^2 \cdot u(\|X_s-Y_s\|_r^2)\mathrm{d}s \\
\leq& 2r\int_0^t \mathbb{E} \sup_{0<v\leq s\wedge\tau_k} e^{2rv}\psi_n(\Lambda^i(v)-\bar{\Lambda}^i(v))^2 \mathrm{d}s + \frac{1}{2}\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_k} e^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2 \\
&+ \epsilon(n) + C(T)\frac{1}{1-\alpha}\int_0^t \mathbb{E} \sup_{0\leq v\leq s\wedge\tau_k} e^{2rv}|\Lambda(v)-\Lambda(v)\wedge\bar{\Lambda}(v)|^2 \cdot u\Big(\frac{1}{1-\alpha}\sup_{0\leq v\leq s\wedge\tau_k} e^{2rv}|\Lambda(v)-\Lambda(v)\wedge\bar{\Lambda}(v)|^2\Big)\mathrm{d}s \\
\leq& 2r\int_0^t \mathbb{E} \sup_{0<v\leq s\wedge\tau_k} e^{2rv}\psi_n(\Lambda^i(v)-\bar{\Lambda}^i(v))^2 \mathrm{d}s + \frac{1}{2}\mathbb{E} \sup_{0\leq s\leq t\wedge\tau_k} e^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2 \\
&+ \epsilon(n) + C(T)\int_0^t \mathbb{E}\Big(\phi_k(s)\cdot u(\phi_k(s))\Big)\mathrm{d}s,
\end{aligned}
$$

where $\phi_k(s) = \frac{1}{1-\alpha}\sup_{-\infty<v\leq s\wedge\tau_k} e^{2rv}|\Lambda(v)-\Lambda(v)\wedge\bar{\Lambda}(v)|^2$, $s \geq 0$. This further implies that, for any $n, k \geq 1$ and $0 \leq t \leq T$,

$$
\begin{aligned}
\mathbb{E} \sup_{-\infty<s\leq t\wedge\tau_k} e^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2 \leq& 4r\int_0^t \mathbb{E} \sup_{0<v\leq s\wedge\tau_k} e^{2rv}\psi_n(\Lambda^i(v)-\bar{\Lambda}^i(v))^2 \mathrm{d}s \\
&+ 2C(T)\int_0^t \mathbb{E}\Big(\phi_k(s)\cdot u(\phi_k(s))\Big)\mathrm{d}s + 2\epsilon(n).
\end{aligned}
$$

It now follows by Gronwall's inequality that

$$
\mathbb{E} \sup_{-\infty<s\leq t\wedge\tau_k} e^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2 \leq 2C(T)\int_0^t \mathbb{E}\Big(\phi_k(s)\cdot u(\phi_k(s))\Big)\mathrm{d}s + 2C(T)\epsilon(n),
$$

which leads to

$$
\sum_{i=1}^d \mathbb{E} \sup_{-\infty<s\leq t\wedge\tau_k} e^{2rs}\psi_n(\Lambda^i(s)-\bar{\Lambda}^i(s))^2 \leq 2dC(T)\int_0^t \mathbb{E}\Big(\phi_k(s)\cdot u(\phi_k(s))\Big)\mathrm{d}s + 2dC(T)\epsilon(n).
$$

Let $n \uparrow \infty$, and, by Jensen's inequality, one has

$$
(1-\alpha)\mathbb{E}\phi_k(s) \leq 2dC(T)\int_0^t \Big(\mathbb{E}\phi_k(s)\Big)\cdot u\Big(\mathbb{E}\phi_k(s)\Big)\mathrm{d}s, \quad 0 \leq t \leq T, \quad k \geq 1.
$$

Let $G(s) = \int_1^s \frac{1}{ru(r)}\mathrm{d}r$, $s > 0$. Since $\int_0^1 \frac{1}{ru(r)}\mathrm{d}r = \infty$, by the Bihari inequality, we have

$$
\mathbb{E}\phi_k(t) \leq G^{-1}(G(0) + 2dC(T)) = G^{-1}(-\infty) = 0, \quad k \geq 1.
$$

Let $k \uparrow \infty$; then, (4.1) holds. Thus,

$$
X(t) - G(X_t) \leq \bar{X}(t) - G(\bar{X}_t).
$$

Moreover, under condition (**A4**) and Proposition 4.3 in [21], one has

$$\mathbb{P}(X_t \leq \bar{X}_t) = 1, \quad t \geq 0,$$

which yields the desired assertion. □

Next, we aim to study the order preservation for neutral-type stochastic differential equations with compensatory jump processes. Consider two multidimensional neutral-type stochastic differential equations of infinite delay with jumps for any $t \in [0, T]$:

$$\begin{cases} \mathrm{d}\{X(t) - G(X_t)\} = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W(t) + \int_{\mathbb{Y}} \gamma(t, X_{t-}, u)\tilde{N}(\mathrm{d}t, \mathrm{d}u), \\ X(t) = \xi(t), \quad t \in (-\infty, 0], \end{cases} \tag{4.8}$$

and

$$\begin{cases} \mathrm{d}\{\bar{X}(t) - G(\bar{X}_t)\} = \bar{b}(t, \bar{X}_t)\mathrm{d}t + \bar{\sigma}(t, \bar{X}_t)\mathrm{d}W(t) + \int_{\mathbb{Y}} \bar{\gamma}(t, \bar{X}_{t-}, u)\tilde{N}(\mathrm{d}t, \mathrm{d}u), \\ \bar{X}(t) = \bar{\xi}(t), \quad t \in (-\infty, 0]. \end{cases} \tag{4.9}$$

Then, (4.8) and (4.9) are respectively equivalent to

$$\begin{cases} \mathrm{d}\{X(t) - G(X_t)\} = \Big\{b(t, X_t) - \int_{\mathbb{Y}} \gamma(t, X_{t-}, u)\lambda(\mathrm{d}u)\Big\}\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W(t) + \int_{\mathbb{Y}} \gamma(t, X_{t-}, u)N(\mathrm{d}t, \mathrm{d}u), \\ X(t) = \xi(t), \quad t \in (-\infty, 0], \end{cases}$$

and

$$\begin{cases} \mathrm{d}\{\bar{X}(t) - G(\bar{X}_t)\} = \Big\{\bar{b}(t, \bar{X}_t) - \int_{\mathbb{Y}} \bar{\gamma}(t, \bar{X}_{t-}, u)\lambda(\mathrm{d}u)\Big\}\mathrm{d}t + \bar{\sigma}(t, \bar{X}_t)\mathrm{d}W(t) + \int_{\mathbb{Y}} \bar{\gamma}(t, \bar{X}_{t-}, u)N(\mathrm{d}t, \mathrm{d}u), \\ \bar{X}(t) = \bar{\xi}(t), \quad t \in (-\infty, 0]. \end{cases}$$

According to Theorem 4.1, and together with the method of [15, Theorem 3.1], we can infer the following D-order preservation result for multidimensional neutral-type stochastic differential equations of infinite delay with compensatory jump processes.

Theorem 4.2. Let (**A1**)–(**A4**) hold. The solutions of (4.8) and (4.9) are D-order-preserving if the following conditions are satisfied:
(i) For any $1 \leq i \leq d$ and $t \geq 0$, if $\xi, \bar{\xi} \in \mathscr{D}_r$ with $\xi \leq_D \bar{\xi}$ and $\xi_i(0) - G_i(\xi) = \bar{\xi}_i(0) - G_i(\bar{\xi})$, it holds that $b_i(t, \xi) - \int_{\mathbb{Y}} \gamma_i(t, \xi, u)\lambda(\mathrm{d}u) \leq \bar{b}_i(t, \bar{\xi}) - \int_{\mathbb{Y}} \bar{\gamma}_i(t, \bar{\xi}, u)\lambda(\mathrm{d}u)$.
(ii) The diffusion $\sigma = (\sigma_{ij})$ and $\bar{\sigma} = (\bar{\sigma}_{ij})$ satisfy that $\sigma = \bar{\sigma}$ for any $1 \leq i \leq d$, $1 \leq j \leq m$, and $\xi, \bar{\xi} \in \mathscr{D}_r$. Moreover,

$$\begin{aligned} &\sum_{i=1}^{m} (|\sigma_{ij}(t, \xi) - \sigma_{ij}(t, \eta)|^2 + |\bar{\sigma}_{ij}(t, \xi) - \bar{\sigma}_{ij}(t, \eta)|^2) \\ &\leq K(t)|\xi_i(0) - G_i(\xi) - \bar{\xi}_i(0) + G_i(\bar{\xi})|^2 u(|\xi_i(0) - G_i(\xi) - \bar{\xi}_i(0) + G_i(\bar{\xi})|^2), \quad t \geq 0, \ \xi, \bar{\xi} \in \mathscr{D}_r. \end{aligned}$$

(iii) The jump diffusion term $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_d)$ satisfies that $\xi^i(0) - G_i(\xi) + \gamma_i(t, \xi, \cdot) \leq \bar{\xi}_i(0) - G_i(\bar{\xi}) + \gamma_i(t, \bar{\xi}, \cdot)$ for any $1 \leq i \leq d$, $t \geq 0$ if $\xi, \bar{\xi} \in \mathscr{D}_r$ with $\xi \leq_D \bar{\xi}$.

Remark 4.1. In Theorems 4.1 and 4.2, it is easy to find that D-order preservation theorems hold in when the diffusion term does not contain a segment process. However, the jump-diffusion coefficient can contain a delay term. It is consistent with the result for one-dimensional stochastic differential delay equations in [15].

In the following part inspired by [15, Examples 2.2 and 2.3], we will also establish two examples to support the above two opinions respectively.

Example 4.3. *Consider the following two one-dimensional neutral-type stochastic differential equations only as a matter of convenience:*

$$\begin{cases} \mathrm{d}\{X(t) - G(X_t)\} = X_t dW(t) - X_{t-}N(\mathrm{d}t), & t \in [0, T], \\ X(\theta) = c, & \theta \in (-\infty, 0), \end{cases}$$

and

$$\begin{cases} \mathrm{d}\{Y(t) - G(Y_t)\} = Y_t dW(t) - Y_{t-}N(\mathrm{d}t), & t \in [0, T], \\ Y(\theta) = 0, & \theta \in (-\infty, 0), \end{cases}$$

where $c < 0$ is a constant, N is a Poisson process and there is independence of Brownian motion W. Due to the infiniteness of the length of memory, for any $t \in [0, T \wedge (-\theta)]$, $Y(t) \equiv 0$ while $X(t) = c(1 + W(t) - N(t))$. On the other side, the following relation is obvious:

$$\{\omega \in \Omega : W(t) < -1\} \subseteq \{\omega \in \Omega : 1 + W(t) - N(t) < 0\}.$$

Then,

$$0 \le \mathbb{P}\{\omega \in \Omega : W(t) < -1\} \le \mathbb{P}\{\omega \in \Omega : 1 + W(t) - N(t) < 0\}.$$

This implies that

$$\mathbb{P}\{\omega \in \Omega : X(t) > 0\} > 0.$$

Therefore, it holds that D-order preservation need not hold if the diffusion coefficient includes a delay term. However, the following example shows that the jump diffusion can conclude a delay function.

Example 4.4. *Consider a pair of neutral stochastic functional differential equations of infinite delay with jumps described by the following for fixed $T > 0$:*

$$\begin{cases} \mathrm{d}\{X(t) - G(X_t)\} = \int_0^\infty X_{t-}\gamma(u)\tilde{N}(\mathrm{d}t, \mathrm{d}u), & t \in [0, T], \\ X(\theta) = c, & \theta \in (-\infty, 0), \end{cases}$$

and

$$\begin{cases} \mathrm{d}\{Y(t) - G(Y_t)\} = \int_0^\infty Y_{t-}\gamma(u)\tilde{N}(\mathrm{d}t, \mathrm{d}u), & t \in [0, T], \\ Y(\theta) = 0, & \theta \in (-\infty, 0), \end{cases}$$

where $c < 0$ is a constant and $\tilde{N}$ is a compensated Poisson random measure on $[0, \infty]$ with parameter $\lambda(\mathrm{d}u)\mathrm{d}t$ such that $T \int_0^\infty \gamma(u)\lambda(\mathrm{d}u) < 1$ for $\gamma(u) > 0, u \in (0, \infty)$. Assume, moreover, that $\gamma(u) > 0, u \in (0, \infty)$. Then, for any $t \in [0, T \wedge (-\theta)]$,

$$\begin{aligned} X(t) =& c \int_0^t \int_0^\infty \gamma(u)N(\mathrm{d}t, \mathrm{d}u) + c\Big(1 - \int_0^t \int_0^\infty \gamma(u)\lambda(\mathrm{d}u)\mathrm{d}s\Big) \\ \le& c\Big(1 - \int_0^t \int_0^\infty \gamma(u)\lambda(\mathrm{d}u)\mathrm{d}s\Big) \\ \le& 0, \end{aligned}$$

while $Y(t) \equiv 0$.

5. Conclusions

In this work, we established the well-posedness of neutral-type stochastic differential equations of infinite delay with jumps under non-Lipschitz conditions. We presented the order preservation for this system. We also gave some examples to support our results. It would be interesting to continue the study of neutral-type stochastic differential equations of infinite delay with jumps. For instance, [1] studied the stability in distribution of numerical solutions of neutral stochastic functional differential equations with infinite delay. A natural question is to ask whether it is possible to extend these results in [1] to the model established in our work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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