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Research article

Attractive solutions for Hilfer fractional neutral stochastic integro-differential equations with almost sectorial operators

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Abstract: This paper studies the integro-differential equations of Hilfer fractional (HF) neutral stochastic evolution on an infinite interval with almost sectorial operators and their attractive solutions. We use semigroup theory, stochastic analysis, compactness methods, and the measure of noncompactness (MNC) as the foundation for our methodologies. We establish the existence and attractivity theorems for mild solutions by considering the fact that the almost sectorial operator is both compact and noncompact. Example that highlight the key findings are also provided.

Keywords: Hilfer fractional derivative; stochastic evolution equations; neutral systems; infinite interval; attractivity
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1. Introduction

The scientific world is now paying more and more attention to fractional calculus, which has an expanding variety of applications in fields including astronomy, electricity, life sciences, viscosity, medical science, control theory, data processing, etc. Due to the vast range of domains that fractional concepts are applied to, including physics, mechanics, chemistry, and engineering, fractional differential equations (FDEs), have become incredibly important. The study of ordinary and partial differential equations containing fractional derivatives has advanced significantly in recent years. We recommend the reader to read some books and articles published by Kilbas et al. [1], Diethelm [2], Zhou [3], Podulbny [4], Miller and Ross [5], Lakshmikantham et al. [6], and a series of papers [7–10] and the references cited therein.

The Caputo and Riemann-Liouville (R-L) fractional derivatives were among the initial fractional order derivatives that Hilfer [11] presented in his new operator named the Hilfer fractional derivative (HFD). Additionally, conceptual simulations of dielectric relaxation in glass components, polymers, rheological permanent simulation, and other domains have revealed the significance and usefulness of the HFD. To study the existence of an integral solution related to an evolution boundary value problem (BVP) equipped with the HFD, Gu and Trujillo [12], recently, employed the measure of noncompactness technique. Along with this article, other numerous academic articles have addressed the HFD in their theorems; see [13–16]. According to the methods used in [17–21], some researchers used almost sectorial operators to find a mild solution for some BVPs in the framework of the HFD systems.

Neutral differential equations have gained a lot of attention lately because of their many applications in a variety of domains, such as biological models, chemical kinetics, electronics, and fluid dynamics. We refer to the works on the theory and applications of neutral partial differential equations (PDEs) with non-local and classical situations as [21–26] and the references therein. We observe that there has been a recent surge in interest in neutral structures due to their prevalence in many applications of applied mathematics.

Since the above differential equations were originally used to numerically mimic a variety of occurrences in the humanities and natural sciences [27], stochastic PDEs have also attracted a lot of interest. Rather than focusing on deterministic models, more research should be done on stochastic models, as unpredictability and uncontrollable fluctuations are intrinsic to both manmade and natural systems. Stochastic differential equations (SDEs) represent a specific event mathematically by including irrationality. The research community has recently shown a great deal of interest in the use of SDEs in finite and infinite dimensions to represent a variety of processes in population fluctuations, mathematics, mechanical engineering, physical location, behavioral science, life sciences, and several other science and technology domains. See [13, 23, 28, 29] for a comprehensive introduction to SDEs and their applications.

Almost sectorial operators are being used by researchers to advance the existence concepts in fractional calculus. In this direction, for a system under study, researchers have created a unique way of identifying mild solutions. In addition, a theory has been developed to predict different requirements of linked semigroups formed by almost sectorial operators using multivalued maps, the Wright function, fractional calculus, semigroup operators, the MNC, and the fixed-point approach. For more details, refer to [18, 25, 30–33]. Some researchers in [17–19] analyzed their results via the almost sectorial operators by employing the Schauder's fixed-point theorem. The authors in [34–37] conducted an analysis of fractional evolution equations (FEEs) via a similar method with the sectorial operators. Further, Zhou [38] established the attractivity for FEEs with almost sectorial operators by using the Ascoli-Arzela theorem. Later, Zhou et al. [39] discussed the existence theorems related to the attractive solutions of the Hilfer FEEs with almost sectorial operators. Very recently, Yang et al. [40] established the HF stochastic evolution equations on infinite intervals via the fixed-point method.

To our knowledge, no work has been reported on the attractive solution for HF neutral stochastic evolution integro-differential equations on an infinite interval via almost sectorial operators. To fill this gap, by taking inspiration from the previous studies, this research intends to address this subject completely. In other words, the goal of this publication is to prove an attractive solution using the almost sectorial operators in the following form for HF neutral stochastic evolution integro-differential equations on an infinite interval:

$$\begin{cases} {}^H D_{0+}^{\lambda,\mu}[\mathfrak{g}(\mathfrak{s}) - \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))] = \mathcal{A}[\mathfrak{g}(\mathfrak{s}) - \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))] + \mathcal{F}(\mathfrak{s}, \mathfrak{g}(\mathfrak{s})) + \int_0^{\mathfrak{s}} G(\mathfrak{l}, \mathfrak{g}(\mathfrak{l}))dW(\mathfrak{l}), \\ I_{0+}^{(1-\lambda)(1-\mu)}\mathfrak{g}(0) = \mathfrak{g}_0, \quad \mathfrak{s} \in (0, \infty), \end{cases} \tag{1.1}$$

where ${}^H D_{0+}^{\lambda,\mu}$ is the HFD of order $0 < \mu < 1$ and type $0 \leq \lambda \leq 1$, $I_{0+}^{(1-\lambda)(1-\mu)}$ is a R-L integral of the fractional order $(1 - \lambda)(1 - \mu)$, and $\mathcal{A}$ denotes an almost sectorial operator in the Hilbert space $\mathscr{Y}$. $\mathcal{F} : (0, \infty) \times \mathscr{Y} \to \mathscr{Y}$, $G : [0, \infty) \times \mathscr{Y} \to L_2^0(\mathbb{K}, \mathscr{Y})$, and $\varpi : (0, \infty) \times \mathscr{Y} \to \mathscr{Y}$ are the given functions. $\{W(\mathfrak{s})\}_{\mathfrak{s}\geq 0}$ specifies a one-dimensional $\mathbb{K}$-valued Wiener process along with a finite trace nuclear covariance operator $Q \geq 0$ formulated on a filtered complete probability space $(\Xi, \mathscr{E}, \{\mathscr{E}_{\mathfrak{s}}\}_{\mathfrak{s}\geq 0}, \mathscr{P})$, and $\mathfrak{g}_0 \in L_2^0(\Xi, \mathscr{Y})$.

The following is a summary of this article's primary contributions:

(1) In this work, we investigate the attractive solution for HF neutral stochastic evolution integro-differential equations on an infinite interval via almost sectorial operators.

(2) This work applies some concepts of functional analysis, like the Wright function, the Ascoli-Arzela theorem, Kuratowski's measure of noncompactness, and Schauder's fixed point theorem, to prove the main results.

(3) The Ascoli-Arzela theorem, which is effectively employed to establish the new results, is the foundation of our method in the present research.

(4) The proved theorems are validated via a theoretical example.

The structure of this manuscript is as follows: Section 2 covers fractional calculus, MNC, and semigroup operators as a reminder. In Section 3, we establish the global existence and attractivity results of mild solutions for HF neutral stochastic evolution integro-differential equation (1.1). We present conceptual applications in Section 4 to assist us in making our discussion more successful.

2. Preliminaries

We present a few foundational definitions in this section. We require certain fundamental notations of fractional calculus and measures of noncompactness as a reminder.

Denote by $L_2(\Xi, \mathscr{Y})$, the collection of all strongly measurable square-integrable $\mathscr{Y}$-valued random variables, which is a Banach space for the norm $\|\mathfrak{g}(\cdot)\|_{L_2(\Xi,\mathscr{Y})} = (E\|\mathfrak{g}(\cdot, W)\|^2)^{\frac{1}{2}}$ for each $\mathfrak{g} \in L_2(\Xi, \mathscr{Y})$. Moreover, $L_2^0(\Xi, \mathscr{Y}) = \{\mathfrak{g} \in L_2(\Xi, \mathscr{Y}) : \ \mathfrak{g}$ is an subspace of $L_2(\Xi, \mathscr{Y})$ and is $\mathscr{E}_0$-measurable$\}$.

Let $C((0, \infty), L_2(\Xi, \mathscr{Y})) : (0, \infty) \to L_2(\Xi, \mathscr{Y})$ be a Banach space of all continuous functions. For each $\mathfrak{g} \in C((0, \infty), L_2(\Xi, \mathscr{Y}))$, define

$$\|\mathfrak{g}\|_{C((0,\infty),L_2(\Xi,\mathscr{Y}))} = (\sup_{\mathfrak{s}\in(0,\infty)} E\|\mathfrak{g}(\mathfrak{s})\|^2)^{\frac{1}{2}} < \infty.$$

Suppose that $(\Xi, \mathscr{E}, \mathscr{P})$ denotes the complete probability space defined with a complete family of right continuous increasing sub-σ-algebras $\{\mathscr{E}_{\mathfrak{s}}, \mathfrak{s} \in (0, \infty)\}$ fulfilling $\mathscr{E}_{\mathfrak{s}} \subset \mathscr{E}$, so that $\mathscr{Y}, \mathbb{K}$ denote two real separable Hilbert spaces, and $\{W(\mathfrak{s})\}_{\mathfrak{s}\geq 0}$ denotes a Q-Wiener process defined on $(\Xi, \mathscr{E}, \mathscr{P})$ with values in $\mathbb{K}$. Let $L(\mathbb{K}, \mathscr{Y}) : \mathbb{K} \to \mathscr{Y}$ be the space of all operators with boundedness property, and $L_Q(\mathbb{K}, \mathscr{Y}) : \mathbb{K} \to \mathscr{Y}$ stands for the space of all Q-Hilbert-Schmidt operators.

Furthermore, we suppose that $\mathscr{O}(\mathfrak{s})$ is continuous in the uniform operator topology for $\mathfrak{s} > 0$, and also, $\mathscr{O}(\mathfrak{s})$ has uniform boundedness, i.e., there exists $\mathcal{K} > 1$ such that $\sup_{\mathfrak{s}\in(0,\infty)} \|\mathscr{O}(\mathfrak{s})\| < \mathcal{K}$, throughout this paper.

Definition 2.1. *[31] For* $0 < \kappa < 1, \quad 0 < \varphi < \frac{\pi}{2}$, *we define that* $\Psi_{\varphi}^{-\kappa}$ *is a family of all closed linear operators with the sector* $S_{\varphi} = \{v \in \mathbb{C}\backslash\{0\} : |arg\, v| \leq \varphi\}$ *and let* $\mathcal{A} : D(\mathcal{A}) \subset \mathscr{Y} \to \mathscr{Y}$ *be such that*

(a) $\sigma(\mathcal{A}) \subseteq S_{\varphi}$ *;*
(b) for all $\omega < \lambda < \pi$, *there exists a constant* $\mathbb{R}_{\lambda} > 0$ *such that* $\|(vI - \mathcal{A})^{-1}\| \leq \mathbb{R}_{\lambda}|v|^{-\kappa}$.

Then, $\mathcal{A} \in \Psi_{\varphi}^{-\kappa}$ *is called an almost sectorial operator on* $\mathscr{Y}$.

Define the semigroup operator $\{\mathscr{T}(\mathfrak{s})\}_{\mathfrak{s}\geq 0}$ as

$$\mathscr{T}(\mathfrak{s}) = e^{-\mathfrak{s}v}(\mathcal{A}) = \frac{1}{2\pi i}\int_{\Gamma_{\varrho}} e^{-\mathfrak{s}v} R(v; \mathcal{A}) dv, \quad \mathfrak{s} \in S^{0}_{\frac{\pi}{2}-\varphi},$$

where $\Gamma_{\varrho} = \{\mathbb{R}^{+}e^{i\varrho}\} \bigcup \{\mathbb{R}^{+}e^{-i\varrho}\}$ with $\varphi < \varrho < \delta < \frac{\pi}{2} - |\arg \mathfrak{s}|$ is oriented counter-clockwise.

Proposition 2.2. *[31] Let* $\mathscr{T}(\mathfrak{s})$ *be the compact semigroup and* $\mathcal{A} \in \Psi_{\varphi}^{-\kappa}$ *for* $0 < \kappa < 1$ *and* $0 < \varphi < \frac{\pi}{2}$. *Then, we have the following:*

(1) $\mathscr{T}(\mathfrak{s} + v) = \mathscr{T}(\mathfrak{s})\mathscr{T}(v)$, *for all* $v, \mathfrak{s} \in S_{\frac{\pi}{2}-\varphi}$.
(2) $\|\mathscr{T}(\mathfrak{s})\|_{L(\mathscr{Y})} \leq \mathcal{K}_0 \mathfrak{s}^{\kappa-1}$, $\mathfrak{s} > 0$ *(*$\mathcal{K}_0 > 0$ *is a constant).*
(3) $R(\mathscr{T}(\mathfrak{s}))$ *belongs to* $\mathscr{T}(\mathfrak{s})$ *for* $\mathfrak{s} \in S_{\frac{\pi}{2}-\varphi} \subseteq D(\mathcal{A}^{\infty})$, *where* $R(\mathscr{T})$ *is the range of* $\mathscr{T}$. *Also,* $R(\mathscr{T}(\mathfrak{s})) \subset D(\mathcal{A}^{\theta})$, *for any* $\theta \in \mathbb{C}$ *with* $Re(\theta) > 0$, *and*

$$\mathcal{A}^{\theta}\mathscr{T}(\mathfrak{s})\mathfrak{g} = \frac{1}{2\pi i}\int_{\Gamma_{\mu}} v^{\theta} e^{-\mathfrak{s}v} R(v; \mathcal{A})\mathfrak{g} dv, \textit{ for all } \mathfrak{g} \in \mathscr{Y}.$$

Hence, there exists a constant $C' = C'(\gamma, \theta) > 0$ *such that*

$$\|\mathcal{A}^{\theta}\mathscr{T}(\mathfrak{s})\|_{L(\mathscr{Y})} \leq C'\mathfrak{s}^{-\gamma-Re(\theta)-1}, \textit{ for all } \mathfrak{s} > 0.$$

(4) If $\Sigma_{\mathscr{T}} = \{\mathfrak{g} \in \mathscr{Y} : \lim_{\mathfrak{s}\to 0^{+}} \mathscr{T}(\mathfrak{s})\mathfrak{g} = \mathfrak{g}\}$, *then* $D(\mathcal{A}^{\theta}) \subset \Sigma_{\mathscr{T}}$ *for* $\theta > 1 + \kappa$.
(5) $(vI - \mathcal{A})^{-1} = \int_0^{\infty} e^{-vv}\mathscr{T}(v)dv$, $v \in \mathbb{C}$, *and* $Re(v) > 0$.

Definition 2.3. *[41] The fractional integral of order* μ *for the function* $G : [0, \infty) \to \mathbb{R}$ *is defined as*

$$I^{\mu}_{0^{+}}G(\mathfrak{s}) = \frac{1}{\Gamma(\mu)}\int_0^{\mathfrak{s}} \frac{G(\mathfrak{l})}{(\mathfrak{s} - \mathfrak{l})^{1-\mu}} d\mathfrak{l}, \quad \mathfrak{s} > 0; \mu > 0,$$

provided the R.H.S. is point-wise convergent.

Definition 2.4. *[11] Let* $0 < \mu < 1$ *and* $0 \leq \lambda \leq 1$. *The HFD of order* μ *and type* λ *for* $G : [0, \infty) \to \mathbb{R}$ *is*

$$^{H}D_{0+}^{\lambda,\mu}G(\mathfrak{s}) = [I_{0+}^{\lambda(1-\mu)}D(I_{0+}^{(1-\lambda)(1-\mu)}G)](\mathfrak{s}).$$

For a Banach space $\mathscr{Y}$, let P be a non-empty subset in $\mathscr{Y}$. The Kuratowski's MNC α is introduced as

$$\alpha(P) = \inf\Big\{c > 0 : P \subset \bigcup_{\iota=1}^{n} M_\iota, \;\; \mathrm{diam}(M_\iota) \leq c\Big\}.$$

Here, the diameter of M_ι is provided by $\mathrm{diam}(M_\iota) = \sup\{|\mathsf{x} - \mathsf{y}| : \; \mathsf{x}, \mathsf{y} \in M_\iota\}, \; \iota = 1, 2, \cdots, n$.

Lemma 2.5. *[42] Let* V_1 *and* V_2 *be two bounded sets in the Banach space* E. *Then, we have the follwoing*

(i) $\alpha(V_1) = 0$ *if and only if* V_1 *is relatively compact;*
(ii) $\alpha(V_1) \leq \alpha(V_2)$ *if* $V_1 \subseteq V_2$;
(iii) $\alpha(V_1 + V_2) \leq \alpha(V_1) + \alpha(V_2)$, *where* $V_1 + V_2 = \{\mathfrak{g} + v : \mathfrak{g} \in V_1, v \in V_2\}$;
(iv) $\alpha\{\{\mathfrak{g}\} \cup V\} = \alpha(V)$ *for all* $\mathfrak{g} \in E$ *and every non-empty subset* $V \in E$;
(v) $\alpha\{V_1 \cup V_2\} \leq \max\{\alpha(V_1), \alpha(V_2)\}$;
(vi) $\alpha(\gamma V) \leq |\gamma|\alpha(V)$.

Lemma 2.6. *[43] Assume that* $\mathscr{Y}$ *is a Hilbert space, and the sequence* $\mathfrak{g}_n(\mathfrak{s}) : [0, \infty) \to \mathscr{Y}$, $(n = 1, 2, \cdots)$ *includes all continuous functions. If there exists* $\varrho \in L^1[0, \infty)$ *such that*

$$\|\mathsf{x}_n(\mathfrak{s})\| \leq \varrho(\mathfrak{s}), \quad \mathfrak{s} \in [0, \infty), \; n = 1, 2, \ldots,$$

then $\alpha(\{\mathsf{x}_n\}_{n=1}^{\infty})$ *is integrable on* $[0, \infty)$, *and*

$$\alpha\Big(\Big\{\int_0^\infty \mathsf{x}_n(\mathfrak{s})d\mathfrak{s} : n = 1, 2, \ldots\Big\}\Big) \leq 2\int_0^\infty \alpha(\{\mathsf{x}_n(\mathfrak{s}) : n = 1, 2, \ldots\})d\mathfrak{s}.$$

Definition 2.7. *[44] The Wright function* $\mathscr{M}_\lambda(\vartheta)$ *is formulated as*

$$\mathscr{M}_\lambda(\vartheta) = \sum_{n\in\mathbb{N}} \frac{(-\vartheta)^{n-1}}{(n-1)!\Gamma(1 - \vartheta n)}, \; \vartheta \in \mathbb{C},$$

with

$$\int_0^\infty \vartheta^\iota \mathscr{M}_\lambda(\vartheta)d\vartheta = \frac{\Gamma(1 + \iota)}{\Gamma(1 + \lambda\iota)}, \qquad \text{for } \iota \geq 0.$$

Lemma 2.8. *The system* (1.1) *has a solution in the form of the integral equation*

$$\begin{aligned}
\mathfrak{g}(\mathfrak{s}) = & \frac{[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))]}{\Gamma(\lambda(1 - \mu) + \mu)}\mathfrak{s}^{(\lambda-1)(1-\mu)} + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s})) \\
& + \frac{1}{\Gamma(\mu)}\int_0^{\mathfrak{s}} (\mathfrak{s} - \mathfrak{l})^{\mu-1}\mathcal{A}\mathfrak{g}(\mathfrak{l})d\mathfrak{l} + \frac{1}{\Gamma(\mu)}\int_0^{\mathfrak{s}} (\mathfrak{s} - \mathfrak{l})^{\mu-1}\mathcal{F}(\mathfrak{l}, \mathfrak{g}(\mathfrak{l}))d\mathfrak{l} \\
& + \frac{1}{\Gamma(\mu)}\int_0^{\mathfrak{s}} (\mathfrak{s} - \mathfrak{l})^{\mu-1}\int_0^{\mathfrak{l}} G(\omega, \mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}, \; \mathfrak{s} \in (0, \infty).
\end{aligned} \tag{2.1}$$

Proof. This proof is similar to that of [14]; therefore, we do not repeat it. □

Lemma 2.9. *Suppose that* $\mathfrak{g}(\mathfrak{s})$ *fulfills the integral equation* (2.1). *Then,*

$$\mathfrak{g}(\mathfrak{s}) = \mathscr{O}_{\lambda,\mu}[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s})) + \int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s} - \mathfrak{l})\mathcal{F}(\mathfrak{l}, \mathfrak{g}(\mathfrak{l}))d\mathfrak{l}$$
$$+ \int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s} - \mathfrak{l}) \int_0^{\mathfrak{l}} G(\omega, \mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}, \quad \mathfrak{s} \in (0, \infty),$$

where $\mathscr{O}_{\lambda,\mu} = I_{0+}^{\lambda(1-\mu)} \mathscr{P}_\mu(\mathfrak{s})$, $\mathscr{P}_\mu(\mathfrak{s}) = \mathfrak{s}^{\mu-1}\mathscr{Q}_\mu(\mathfrak{s})$, *and* $\mathscr{Q}_\mu(\mathfrak{s}) = \int_0^\infty \mu\vartheta\mathscr{M}_\mu(\vartheta)\mathscr{T}(\mathfrak{s}^\mu\vartheta)d\vartheta$.

Proof. This proof is similar to that of [14]; therefore, we do not repeat it. □

In relation to Lemma 2.8, we have a definition.

Definition 2.10. *An* $\mathscr{E}_{\mathfrak{s}}$*-adapted stochastic process* $\mathfrak{g}(\mathfrak{s}) : (0, \infty) \to \mathscr{Y}$ *is called a mild solution of the given system* (1.1)*, if* $I_{0+}^{(1-\lambda)(1-\mu)}\mathfrak{g}(0) = \mathfrak{g}_0$, $\mathfrak{g}_0 \in L_2^0(\Xi, \mathscr{Y})$*, and for each* $\mathfrak{s} \in (0, \infty)$*, the function* $G(\omega, \mathfrak{g}(\omega))$ *is integrable, and the stochastic integral equation*

$$\mathfrak{g}(\mathfrak{s}) = \mathscr{O}_{\lambda,\mu}[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s})) + \int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s} - \mathfrak{l})\mathcal{F}(\mathfrak{l}, \mathfrak{g}(\mathfrak{l}))d\mathfrak{l}$$
$$+ \int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s} - \mathfrak{l}) \int_0^{\mathfrak{l}} G(\omega, \mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}, \quad \mathfrak{s} \in (0, \infty),$$

holds.

Definition 2.11. *The mild solution* $\mathfrak{g}(\mathfrak{s})$ *of the system* (1.1) *is said to be attractive if* $\mathfrak{g}(\mathfrak{s}) \to 0$ *as* $\mathfrak{s} \to \infty$.

Lemma 2.12. *[18] For any fixed* $\mathfrak{s} > 0$, $\{\mathscr{Q}_\mu(\mathfrak{s})\}_{\mathfrak{s}>0}$, $\{\mathscr{P}_\mu(\mathfrak{s})\}_{\mathfrak{s}>0}$, *and* $\{\mathscr{O}_{\lambda,\mu}(\mathfrak{s})\}_{\mathfrak{s}>0}$ *are linear operators, and for every* $\mathfrak{g} \in \mathscr{Y}$,

$$\|\mathscr{Q}_\mu(\mathfrak{s})\mathfrak{g}\| \leq \mathcal{K}_1\mathfrak{s}^{\mu(\kappa-1)}\|\mathfrak{g}\|,\ \|\mathscr{P}_\mu(\mathfrak{s})\mathfrak{g}\| \leq \mathcal{K}_1\mathfrak{s}^{-1+\mu\kappa}\|\mathfrak{g}\|,\ \textit{and}\ \|\mathscr{O}_{\lambda,\mu}(\mathfrak{s})\mathfrak{g}\| \leq \mathcal{K}_2\mathfrak{s}^{-1+\lambda-\lambda\mu+\mu\kappa}\|\mathfrak{g}\|,$$

where

$$\mathcal{K}_1 = \mu\mathcal{K}_0\frac{\Gamma(1+\kappa)}{\Gamma(1+\mu\kappa)} \ \textit{and}\ \mathcal{K}_2 = \frac{\mathcal{K}_1\Gamma(\mu\kappa)}{\Gamma(\lambda(1-\mu)+\mu\kappa)}.$$

Lemma 2.13. *[18] Assume that* $\mathscr{O}(\mathfrak{s})$ *is equicontinuous for* $\mathfrak{s} > 0$. *Then,* $\{\mathscr{Q}_\mu(\mathfrak{s})\}_{\mathfrak{s}>0}$, $\{\mathscr{P}_\mu(\mathfrak{s})\}_{\mathfrak{s}>0}$ *and* $\{\mathscr{O}_{\lambda,\mu}(\mathfrak{s})\}_{\mathfrak{s}>0}$ *are strongly continuous, i.e., for any* $\mathfrak{g} \in \mathscr{Y}$ *and* $\mathfrak{s}'' > \mathfrak{s}' > 0$, *we have*

$$\|\mathscr{Q}_\mu(\mathfrak{s}')\mathfrak{g} - \mathscr{Q}_\mu(\mathfrak{s}'')\mathfrak{g}\| \to 0,\ \|\mathscr{P}_\mu(\mathfrak{s}')\mathfrak{g} - \mathscr{P}_\mu(\mathfrak{s}'')\mathfrak{g}\| \to 0,\ \textit{and}\ \|\mathscr{O}_{\lambda,\mu}(\mathfrak{s}')\mathfrak{g} - \mathscr{O}_{\lambda,\mu}(\mathfrak{s}'')\mathfrak{g}\| \to 0,$$

as $\mathfrak{s}'' \to \mathfrak{s}'$.

Let

$$C([0,\infty), L_2(\Xi, \mathscr{Y})) = \left\{\mathsf{x} : \mathsf{x} \in C([0,\infty), L_2(\Xi, \mathscr{Y})) : \lim_{\mathfrak{s}\to\infty} E\left\|\frac{\mathsf{x}(\mathfrak{s})}{1+\mathfrak{s}}\right\|^2 = 0\right\}.$$

Clearly, $(C([0,\infty), L_2(\Xi, \mathscr{Y})), \|\cdot\|)$ is a Banach space with

$$\|\mathsf{x}\|_\infty = \left(\sup_{\mathfrak{s}\in[0,\infty)} E\left\|\frac{\mathsf{x}(\mathfrak{s})}{1+\mathfrak{s}}\right\|^2\right)^{\frac{1}{2}} < \infty, \quad \text{for any } \mathsf{x} \in C([0,\infty), (\Xi, \mathscr{Y})).$$

We provide the generalized Ascoli-Arzela theorem below.

Lemma 2.14. *[45] The set $\Upsilon \subset C([0,\infty), L_2(\Xi, \mathscr{Y}))$ is relatively compact iff:*

(i) *for any $f > 0$, the set $\mathcal{I} = \{\mathsf{u} : \mathsf{u}(\mathfrak{s}) = \frac{\mathsf{y}(\mathfrak{s})}{1+\mathfrak{s}},\ \mathsf{y} \in \Upsilon\}$ is equicontinuous on $[0, f]$;*

(ii) $\lim_{\mathfrak{s}\to\infty} E\left\|\frac{\mathsf{y}(\mathfrak{s})}{1+\mathfrak{s}}\right\|^2 = 0$ *uniformly for $\mathsf{y} \in \Upsilon$;*

(iii) *for all $\mathfrak{s} \in [0,\infty)$, $\mathcal{I}(\mathfrak{s}) = \{\mathsf{u} : \mathsf{u}(\mathfrak{s}) = \frac{\mathsf{y}(\mathfrak{s})}{1+\mathfrak{s}},\ \mathsf{y} \in \Upsilon\}$ is relatively compact in $L_2(\Xi, \mathscr{Y})$.*

3. Main results

Now, the main theorems will be proved in this section. Some assumptions are required to prove these theorems. We list them as follows:

(H_1) For any $\mathfrak{g} \in \mathscr{Y}$, $\mathcal{F}(\cdot, \mathfrak{g})$ is measurable on $(0,\infty)$, and for any $\mathfrak{s} \in (0,\infty)$, $\mathcal{F}(\mathfrak{s}, \cdot)$ is continuous.

(H_2) There exists a function $\mathfrak{p} : (0,\infty) \to (0,\infty)$ such that for all $\mathfrak{g} \in \mathscr{Y}$ and all $\mathfrak{s} \in (0,\infty)$,

$$(I_{0+}^{\mu}\mathfrak{p})(\mathfrak{s}) \in C((0,\infty),(0,\infty)), \quad E\|\mathcal{F}(\mathfrak{s},\mathfrak{g})\|^2 \le \mathfrak{p}(\mathfrak{s}),$$

and

$$\lim_{\mathfrak{s}\to 0} \mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)+\mu}(I_{0+}^{\mu}\mathfrak{p})(\mathfrak{s}) = 0, \quad \lim_{\mathfrak{s}\to\infty} \frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)+\mu}}{(1+\mathfrak{s}^2)}(I_{0+}^{\mu}\mathfrak{p})(\mathfrak{s}) = 0.$$

(H_3) For every $\mathfrak{g} \in \mathscr{Y}$, $G(\cdot, \mathfrak{g})$ is $\mathscr{E}_{\mathfrak{s}}$-measurable on $(0,\infty)$, and for all $\mathfrak{s} \in (0,\infty)$, $G(\mathfrak{s},\cdot)$ is continuous.

(H_4) There exists a function $\mathfrak{q} : (0,\infty) \to (0,\infty)$ such that for all $\mathfrak{g} \in \mathscr{Y}$ and all $\mathfrak{s} \in (0,\infty)$,

$$(I_{0+}^{2\mu-1}\mathfrak{q})(\mathfrak{s}) \in C((0,\infty),(0,\infty)), \quad E\|\int_0^{\mathfrak{s}} G(\mathfrak{l}, \mathfrak{g}(\mathfrak{l}))d\mathfrak{l}\|^2 \le \mathfrak{q}(\mathfrak{s}),$$

and

$$\lim_{\mathfrak{s}\to 0} \mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}(I_{0+}^{2\mu-1}\mathfrak{q})(\mathfrak{s}) = 0, \quad \lim_{\mathfrak{s}\to\infty} \frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s}^2)}(I_{0+}^{2\mu-1}\mathfrak{q})(\mathfrak{s}) = 0.$$

(H_5) $\varpi : (0,\infty)\times\mathscr{Y} \to \mathscr{Y}$ is a continuous function, and there exists $\mathcal{K}_{\varpi} > 0$ such that ϖ is a $\mathscr{Y}$-valued function and satisfies

$$E\|\varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))\|^2 \le \mathcal{K}_{\varpi}\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}(1+\|\mathfrak{g}\|^2),\ \mathfrak{g} \in \mathscr{Y},\ \mathfrak{s} \in (0,\infty).$$

Define $C_{\mu}((0,\infty), L_2(\Xi,\mathscr{Y})) = \Big\{\mathfrak{g} \in C((0,\infty), L_2(\Xi,\mathscr{Y})) : \lim_{\mathfrak{s}\to 0+} \mathfrak{s}^{(1-\lambda)(1-\mu)}\mathfrak{g}(\mathfrak{s})$ exists and is finite, $\lim_{\mathfrak{s}\to\infty} E\left\|\frac{\mathfrak{s}^{(1-\lambda)(1-\mu)}\mathfrak{g}(\mathfrak{s})}{(1+\mathfrak{s})}\right\|^2 = 0\Big\}$, equipped with norm

$$\|\mathfrak{g}(\mathfrak{s})\|_{\mu}^2 = \left(\sup_{\mathfrak{s}\in[0,\infty)} E\left\|\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}\mathfrak{g}(\mathfrak{s})}{1+\mathfrak{s}}\right\|^2\right)^{\frac{1}{2}}.$$

Thus, $(C_{\mu}((0,\infty), L_2(\Xi,\mathscr{Y})), \|\cdot\|_{\mu}^2)$ is a Hilbert space. For each $\mathfrak{g} \in C_{\mu}((0,\infty), L_2(\Xi,\mathscr{Y}))$ and for any $\mathfrak{s} \in (0,\infty)$, define the operator Σ by

$$(\Sigma\mathfrak{g})(\mathfrak{s}) = (\Sigma_1\mathfrak{g})(\mathfrak{s}) + (\Sigma_2\mathfrak{g})(\mathfrak{s}),$$

where

$$(\Sigma_1 \mathfrak{g})(\mathfrak{s}) = \mathscr{O}_{\lambda,\mu}[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s})),$$

$$(\Sigma_2 \mathfrak{g})(\mathfrak{s}) = \int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s} - \mathfrak{l})\mathcal{F}(\mathfrak{l}, \mathfrak{g}(\mathfrak{l}))d\mathfrak{l} + \int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s} - \mathfrak{l}) \int_0^{\mathfrak{l}} G(\omega, \mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}.$$

Clearly, the neutral stochastic HF-system (1.1) has a mild solution $\mathfrak{g}^* \in C_\mu((0, \infty), L_2(\Xi, \mathscr{Y}))$ if and only if Σ has a fixed-point $\mathfrak{g}^* \in C_\mu((0, \infty), L_2(\Xi, \mathscr{Y}))$.

For each $\mathsf{x} \in C_\mu((0, \infty), L_2(\Xi, \mathscr{Y}))$, we set

$$\mathfrak{g}(\mathfrak{s}) = \mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}\mathsf{x}(\mathfrak{s}), \quad \mathfrak{s} \in (0, \infty).$$

Clearly, $\mathfrak{g} \in C_\mu((0, \infty), L_2(\Xi, \mathscr{Y}))$.

We now define the operator $\mho$ by

$$(\mho \mathsf{x})(\mathfrak{s}) = (\mho_1 \mathsf{x})(\mathfrak{s}) + (\mho_2 \mathsf{x})(\mathfrak{s}),$$

where

$$(\mho_1 \mathsf{x})(\mathfrak{s}) = \begin{cases} \mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}(\Sigma_1 \mathfrak{g})(\mathfrak{s}), & \text{for } \mathfrak{s} \in (0, \infty), \\ 0, & \mathfrak{s} = 0, \end{cases}$$

and

$$(\mho_2 \mathsf{x})(\mathfrak{s}) = \begin{cases} \mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}(\Sigma_2 \mathfrak{g})(\mathfrak{s}), & \text{for } \mathfrak{s} \in (0, \infty), \\ 0, & \mathfrak{s} = 0. \end{cases}$$

By using (H_2) and (H_4), we claim that there exists $r > 0$ such that the inequality

$$\sup_{\mathfrak{s}\in(0,\infty)} \left\{ \frac{8\mathcal{K}_2^2}{(1 + \mathfrak{s})^2}[E\|\mathfrak{g}_0\|^2 + \mathcal{K}_\varpi^2(1 + \|\mathfrak{g}_0\|^2)] + \frac{4\mathcal{K}_\varpi^2}{(1 + \mathfrak{s})^2}\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}(1 + \|\mathfrak{g}\|^2) \right.$$
$$+ \frac{4\mathcal{K}_1^2}{\mu\kappa(1 + \mathfrak{s})^2}\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)+\mu\kappa} \int_0^{\mathfrak{s}} (\mathfrak{s} - \mathfrak{l})^{\mu\kappa-1}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}$$
$$\left. + 4Tr(Q)\mathcal{K}_1^2 \frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1 + \mathfrak{s})^2} \int_0^{\mathfrak{s}} (\mathfrak{s} - \mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{q}(\mathfrak{l})d\mathfrak{l} \right\} \leq r$$

holds.

Let $\mathfrak{g}(\mathfrak{s}) = \mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}\mathsf{x}(\mathfrak{s})$. Define

$$\Phi_1 = \{\mathsf{x} : \mathsf{x} \in C([0, \infty), L_2(\Xi, \mathscr{Y})),\ E\|\mathsf{x}\|^2 \leq r\},$$
$$\widehat{\Phi}_1 = \{\mathfrak{g} : \mathfrak{g} \in C_\mu((0, \infty), L_2(\Xi, \mathscr{Y})),\ E\|\mathfrak{g}\|^2 \leq r\}.$$

It is clear that Φ_1 is a non-empty, closed, and convex subset of $C([0, \infty), L_2(\Xi, \mathscr{Y}))$. $\widehat{\Phi}_1$ is a closed, convex and non-empty set of $C_\mu((0, \infty), L_2(\Xi, \mathscr{Y})$, and $\mathfrak{g} \in \widehat{\Phi}_1$ whenever $\mathsf{x} \in \Phi_1$.

Let

$$\mathcal{D} := \left\{ \mathsf{z} : \mathsf{z}(\mathfrak{s}) = \frac{(\mho \mathsf{x})(\mathfrak{s})}{1 + \mathfrak{s}},\ \mathsf{x} \in \Phi_1 \right\}.$$

We must establish the next lemmas in order to establish the main theorems of this paper.

Lemma 3.1. *If (H_1)–(H_5) are satisfied, then, $\mathcal{D}$ is equicontinuous.*

Proof. We follow some steps.

Step 1: We prove $\mathcal{D}_1 := \left\{ \mathsf{z} : \mathsf{z}(\mathfrak{s}) = \frac{(\mho_1 \mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}},\ \mathsf{x} \in \Phi_1 \right\}$ is equicontinuous.

We have,

$$
\begin{aligned}
&\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s})[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))\\
=&\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{\Gamma(\lambda(1-\mu))}\int_0^{\mathfrak{s}} (\mathfrak{s} - \mathfrak{l})^{\lambda(1-\mu)-1}\mathfrak{l}^{\mu-1}\mathscr{Q}_\mu(\mathfrak{l})[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))]d\mathfrak{l} + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))\\
=&\int_0^1 (1-v)^{\lambda(1-\mu)-1}v^{\mu-1}\mathfrak{s}^{\mu(1-\kappa)}\mathscr{Q}_\mu(\mathfrak{s}v)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))]dv + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s})).
\end{aligned}
$$

Noting that $\lim_{\mathfrak{s}\to 0+} \mathfrak{s}^{\mu(1-\kappa)}\mathscr{Q}_\mu(\mathfrak{s}v)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))$ and $\int_0^1 (1-v)^{\lambda(1-\mu)-1}v^{\mu-1}dv$ are finite, we have

$$
\lim_{\mathfrak{s}\to 0+} \mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s})[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s})) = 0.
$$

Thus, from the aforesaid equality, when $\mathfrak{s}_1 = 0$, $\mathfrak{s}_2 \in (0, \infty)$, it follows that

$$
E\left\|\frac{(\mho_1\mathsf{x})(\mathfrak{s}_2)}{1+\mathfrak{s}_2} - (\mho_1\mathsf{x})(0)\right\|^2 \le E\left\|\frac{1}{1+\mathfrak{s}_2}\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_2)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s})) - 0\right\|^2 \to 0,
$$

as $\mathfrak{s}_2 \to 0$.

Furthermore, for any $0 < \mathfrak{s}_1 < \mathfrak{s}_2 < \infty$, using the elementary inequality, we get

$$
\begin{aligned}
E\left\|\frac{(\mho_1\mathsf{x})(\mathfrak{s}_2)}{1+\mathfrak{s}_2} - \frac{(\mho_1\mathsf{x})(\mathfrak{s}_1)}{1+\mathfrak{s}_1}\right\|^2 \le\ & E\left\|\frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_2)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))}{1+\mathfrak{s}_2}\right.\\
&\left. - \frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_1)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))}{1+\mathfrak{s}_1}\right\|^2\\
\le\ & 2E\left\|\frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_2)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))}{1+\mathfrak{s}_2}\right.\\
&\left. - \frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_2)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))}{1+\mathfrak{s}_1}\right\|^2\\
&+ 2E\left\|\frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_2)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))}{1+\mathfrak{s}_1}\right.\\
&\left. - \frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_2)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))}{1+\mathfrak{s}_1}\right\|^2\\
\le\ & 2E\|\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_1)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))]
\end{aligned}
$$

$$
\begin{aligned}
&+ \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))\|^2\Big(\frac{\mathfrak{s}_2 - \mathfrak{s}_1}{(1+\mathfrak{s}_2)(1+\mathfrak{s}_1)}\Big)^2 \\
&+ 2E\|\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_2)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s})) \\
&- \mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_1)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))\|^2\Big(\frac{1}{1+\mathfrak{s}_1}\Big)^2 \\
\leq\ & 2E\|\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_2)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] \\
&+ \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))\|^2\Big(\frac{\mathfrak{s}_2 - \mathfrak{s}_1}{(1+\mathfrak{s}_2)(1+\mathfrak{s}_1)}\Big)^2 \\
&+ 4E\|\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}[\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_2)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s})) \\
&- \mathscr{O}_{\lambda,\mu}(\mathfrak{s}_1)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] + \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))]\|^2\Big(\frac{\mathfrak{s}_2 - \mathfrak{s}_1}{(1+\mathfrak{s}_2)(1+\mathfrak{s}_1)}\Big)^2 \\
&+ 4E\|[\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa} - \mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}]\mathscr{O}_{\lambda,\mu}(\mathfrak{s}_2)[\mathfrak{g}_0 - \varpi(0, \mathfrak{g}(0))] \\
&+ \varpi(\mathfrak{s}, \mathfrak{g}(\mathfrak{s}))\|^2\Big(\frac{\mathfrak{s}_2 - \mathfrak{s}_1}{(1+\mathfrak{s}_2)(1+\mathfrak{s}_1)}\Big)^2 \\
\to\ & 0, \quad \text{as } \mathfrak{s}_2 \to \mathfrak{s}_1.
\end{aligned}
$$

Thus, $\mathcal{D}_1 := \Big\{\mathrm{z} : \mathrm{z}(\mathfrak{s}) = \frac{(\mho_1 \mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}},\ \mathsf{x} \in \Phi_1\Big\}$ is equicontinuous.

Step 2: Next we prove that $\mathcal{D}_2 := \Big\{\mathrm{z} : \mathrm{z}(\mathfrak{s}) = \frac{(\mho_2 \mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}},\ \mathsf{x} \in \Phi_1\Big\}$ is equicontinuous.

For every $\epsilon > 0$, one may write

$$
\begin{aligned}
E\Big\|\frac{(\mho_2\mathsf{x})(\mathfrak{s}_2)}{1+\mathfrak{s}_2} - \frac{(\mho_2\mathsf{x})(\mathfrak{s}_1)}{1+\mathfrak{s}_1}\Big\|^2 \leq\ & 4E\Big\|\frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}\int_0^{\mathfrak{s}_2}\mathscr{P}_\mu(\mathfrak{s}_2 - \mathfrak{l})\mathcal{F}(\mathfrak{l}, \mathfrak{g}(\mathfrak{l}))d\mathfrak{l}\Big\|^2 \\
&+ 4E\Big\|\frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}\int_0^{\mathfrak{s}_2}\mathscr{P}_\mu(\mathfrak{s}_2 - \mathfrak{l})\int_0^{\mathfrak{l}} G(\omega, \mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}\Big\|^2 \\
&+ 4E\Big\|\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\int_0^{\mathfrak{s}_1}\mathscr{P}_\mu(\mathfrak{s}_1 - \mathfrak{l})\mathcal{F}(\mathfrak{l}, \mathfrak{g}(\mathfrak{l}))d\mathfrak{l}\Big\|^2 \\
&+ 4E\Big\|\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\int_0^{\mathfrak{s}_1}\mathscr{P}_\mu(\mathfrak{s}_1 - \mathfrak{l})\int_0^{\mathfrak{l}} G(\omega, \mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}\Big\|^2 \\
\leq\ & 4\Big(\frac{\mathcal{K}_1\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}\Big)^2\int_0^{\mathfrak{s}_2}(\mathfrak{s}_2 - \mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{p}(\mathfrak{l})d\mathfrak{l} \\
&+ 4Tr(Q)\Big(\frac{\mathcal{K}_1\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}\Big)^2\int_0^{\mathfrak{s}_2}(\mathfrak{s}_2 - \mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{q}(\mathfrak{l})d\mathfrak{l}
\end{aligned}
$$

$$
\begin{aligned}
&+ 4\Big(\frac{\mathcal{K}_1 \mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big)^2 \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{p}(\mathfrak{l})d\mathfrak{l} \\
&+ 4Tr(Q)\Big(\frac{\mathcal{K}_1 \mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big)^2 \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{q}(\mathfrak{l})d\mathfrak{l} \\
< \epsilon.
\end{aligned}
$$

When $\mathfrak{s}_1 = 0,\ 0 < \mathfrak{s}_2 \leq T$, by using the hypotheses (H_2) and (H_4), we have

$$
\begin{aligned}
E\Big\|\frac{(\mho_2\mathsf{x})(\mathfrak{s}_2)}{1+\mathfrak{s}_2} - (\mho_2\mathsf{x})(0)\Big\|^2 \leq\ & 2E\Big\|\frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}\int_0^{\mathfrak{s}_2}\mathscr{P}_\mu(\mathfrak{s}_2-\mathfrak{l})\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\mathfrak{l}\Big\|^2 \\
&+ 2E\Big\|\frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}\int_0^{\mathfrak{s}_2}\mathscr{P}_\mu(\mathfrak{s}_2-\mathfrak{l})\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}\Big\|^2 \\
\leq\ & 4\Big(\frac{\mathcal{K}_1 \mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}\Big)^2 \int_0^{\mathfrak{s}_2} (\mathfrak{s}_2-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{p}(\mathfrak{l})d\mathfrak{l} \\
&+ 4Tr(Q)\Big(\frac{\mathcal{K}_1 \mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}\Big)^2 \int_0^{\mathfrak{s}_2} (\mathfrak{s}_2-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{q}(\mathfrak{l})d\mathfrak{l} \to 0, \text{ as } \mathfrak{s}_2 \to 0.
\end{aligned}
$$

When $0 < \mathfrak{s}_1 < \mathfrak{s}_2 \leq T$, we obtain

$$
\begin{aligned}
&E\Big\|\frac{(\mho_2\mathsf{x})(\mathfrak{s}_2)}{1+\mathfrak{s}_2} - \frac{(\mho_2\mathsf{x})(\mathfrak{s}_1)}{1+\mathfrak{s}_1}\Big\|^2 \\
\leq\ & 8E\Big\|\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}(\mathfrak{s}_2-\mathfrak{l})^{\mu-1}\mathscr{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\mathfrak{l}\Big\|^2 \\
&+ 8E\Big\|\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\int_0^{\mathfrak{s}_1}[(\mathfrak{s}_2-\mathfrak{l})^{\mu-1}-(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}]\mathscr{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\mathfrak{l}\Big\|^2 \\
&+ 8E\Big\|\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\int_0^{\mathfrak{s}_1}(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}[\mathscr{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})-\mathscr{Q}_\mu(\mathfrak{s}_1-\mathfrak{l})]\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\mathfrak{l}\Big\|^2 \\
&+ 8E\Big\|\Big[\frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}-\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big]\int_0^{\mathfrak{s}_2}(\mathfrak{s}_2-\mathfrak{l})^{\mu-1}\mathscr{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\mathfrak{l}\Big\|^2 \\
&+ 8E\Big\|\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}(\mathfrak{s}_2-\mathfrak{l})^{\mu-1}\mathscr{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}\Big\|^2 \\
&+ 8E\Big\|\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\int_0^{\mathfrak{s}_1}[(\mathfrak{s}_2-\mathfrak{l})^{\mu-1}-(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}]\mathscr{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}\Big\|^2 \\
&+ 8E\Big\|\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\int_0^{\mathfrak{s}_1}(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}[\mathscr{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})-\mathscr{Q}_\mu(\mathfrak{s}_1-\mathfrak{l})]\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}\Big\|^2
\end{aligned}
$$

$$+\,8E\Big\|\Big[\frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}-\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big]\int_0^{\mathfrak{s}_2}(\mathfrak{s}_2-\mathfrak{l})^{\mu-1}\mathscr{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})\int_0^{\mathfrak{l}}G(\omega,\mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}\Big\|^2$$

$$\leq 8\sum_{j=1}^{8}S_j,$$

where

$$S_1=\mathcal{K}_1^2\Big(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big)^2\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}(\mathfrak{s}_2-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{p}(\mathfrak{l})d\mathfrak{l},$$

$$S_2=\mathcal{K}_1^2\Big(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big)^2\int_0^{\mathfrak{s}_2}\|(\mathfrak{s}_2-\mathfrak{l})^{\mu-1}-(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}\|^2(\mathfrak{s}_2-\mathfrak{l})^{2\mu(\kappa-1)}\mathfrak{p}(\mathfrak{l})d\mathfrak{l},$$

$$S_3=\Big(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big)^2\int_0^{\mathfrak{s}_1}(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}\|\mathscr{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})-\mathscr{Q}_\mu(\mathfrak{s}_1-\mathfrak{l})\|^2E\|\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))\|^2d\mathfrak{l},$$

$$S_4=\mathcal{K}_1^2\Big[\frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}-\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big]^2\int_0^{\mathfrak{s}_2}(\mathfrak{s}_2-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{p}(\mathfrak{l})d\mathfrak{l},$$

$$S_5=Tr(Q)\mathcal{K}_1^2\Big(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big)^2\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}(\mathfrak{s}_2-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{q}(\mathfrak{l})d\mathfrak{l},$$

$$S_6=Tr(Q)\mathcal{K}_1^2\Big(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big)^2\int_0^{\mathfrak{s}_2}\|(\mathfrak{s}_2-\mathfrak{l})^{\mu-1}-(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}\|^2(\mathfrak{s}_2-\mathfrak{l})^{2\mu(\kappa-1)}\mathfrak{q}(\mathfrak{l})d\mathfrak{l},$$

$$S_7=Tr(Q)\Big(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big)^2\int_0^{\mathfrak{s}_1}(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}\|\mathscr{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})-\mathscr{Q}_\mu(\mathfrak{s}_1-\mathfrak{l})\|^2E\|\int_0^{\mathfrak{l}}G(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\omega\|^2d\mathfrak{l},$$

$$S_8=Tr(Q)\mathcal{K}_1^2\Big[\frac{\mathfrak{s}_2^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_2}-\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big]^2\int_0^{\mathfrak{s}_2}(\mathfrak{s}_2-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{q}(\mathfrak{l})d\mathfrak{l}.$$

By a straightforward calculation, we obtain

$$S_1\to 0\text{ as }\mathfrak{s}_2\to\mathfrak{s}_1.$$

Since, $\|(\mathfrak{s}_2-\mathfrak{l})^{\mu-1}-(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}\|^2(\mathfrak{s}_2-\mathfrak{l})^{2\mu(\kappa-1)}\leq(\mathfrak{s}_2-\mathfrak{l})^{2(\mu\kappa-1)}$, by using the Lebesgue dominated convergence theorem (LDCT), we obtain

$$\int_0^{\mathfrak{s}_2}\|(\mathfrak{s}_2-\mathfrak{l})^{\mu-1}-(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}\|^2\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\to 0\text{ as }\mathfrak{s}_2\to\mathfrak{s}_1.$$

Thus, $S_2\to 0$ as $\mathfrak{s}_2\to\mathfrak{s}_1$.

By (H_2), for $\epsilon>0$, we have

$$S_3\leq\Big(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\Big)^2\int_0^{\mathfrak{s}_1-\epsilon}(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}\|\mathscr{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})-\mathscr{Q}_\mu(\mathfrak{s}_1-\mathfrak{l})\|^2E\|\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))\|^2d\mathfrak{l}$$

$$
\begin{aligned}
&+\left(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\right)^2\frac{\epsilon^\mu}{\mu}\int_{\mathfrak{s}_1-\epsilon}^{\mathfrak{s}_1}(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}\|\mathcal{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})-\mathcal{Q}_\mu(\mathfrak{s}_1-\mathfrak{l})\|^2E\|\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))\|^2d\mathfrak{l}\\
\le&\left(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\right)^2\frac{\mathfrak{s}_1^\mu-\epsilon^\mu}{\mu}\int_0^{\mathfrak{s}_1-\epsilon}(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\sup_{\mathfrak{l}\in[0,\mathfrak{s}_1-\epsilon]}\|\mathcal{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})-\mathcal{Q}_\mu(\mathfrak{s}_1-\mathfrak{l})\|^2\\
&+2\mathcal{K}_1\left(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\right)^2\frac{\epsilon^\mu}{\mu}\int_{\mathfrak{s}_1-\epsilon}^{\mathfrak{s}_1}(\mathfrak{s}_1-\mathfrak{l})^{\mu\kappa-1}\mathfrak{p}(\mathfrak{l})d\mathfrak{l},\\
\le&\, S_{31}+S_{32}+S_{33},
\end{aligned}
$$

where

$$
\begin{aligned}
S_{31}&=\left(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\right)^2\frac{\mathfrak{s}_1^\mu-\epsilon^\mu}{\mu}\int_0^{\mathfrak{s}_1-\epsilon}(\mathfrak{s}_1-\mathfrak{l})^{\mu-1}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\sup_{\mathfrak{l}\in[0,\mathfrak{s}_1-\epsilon]}\|\mathcal{Q}_\mu(\mathfrak{s}_2-\mathfrak{l})-\mathcal{Q}_\mu(\mathfrak{s}_1-\mathfrak{l})\|^2\\
S_{32}&=2\mathcal{K}_1\left(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\right)^2\frac{\epsilon^\mu}{\mu}\left\|\int_0^{\mathfrak{s}_1}(\mathfrak{s}_1-\mathfrak{l})^{\mu\kappa-1}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}-\int_0^{\mathfrak{s}_1-\epsilon}(\mathfrak{s}_1-\epsilon-\mathfrak{l})^{\mu\kappa-1}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\right\|,\\
S_{33}&=2\mathcal{K}_1\left(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\right)^2\frac{\epsilon^\mu}{\mu}\int_0^{\mathfrak{s}_1-\epsilon}\|(\mathfrak{s}_1-\epsilon-\mathfrak{l})^{\mu\kappa-1}-(\mathfrak{s}_1-\mathfrak{l})^{\mu\kappa-1}\|\mathfrak{p}(\mathfrak{l})d\mathfrak{l}.
\end{aligned}
$$

From Lemma 2.13, we conclude that $S_{31}\to 0$ as $\mathfrak{s}_2\to\mathfrak{s}_1$. Using the corresponding deductions in relation to the proofs of $S_1,S_2\to 0$, we obtain $S_{32}\to 0$ and $S_{33}\to 0$ as $\epsilon\to 0$. Hence, $S_3\to 0$ as $\mathfrak{s}_2\to\mathfrak{s}_1$. We can also derive that $S_4\to 0$ as $\mathfrak{s}_2\to\mathfrak{s}_1$ by the continuity of $\left(\frac{\mathfrak{s}_1^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}_1}\right)^2$ with respect to $\mathfrak{s}$. For the terms $S_5,\cdots,S_8$, we can show that $S_5,\cdots,S_8\to 0$ as $\mathfrak{s}_2\to\mathfrak{s}_1$ by the similar proofs of $S_1,\cdots,S_4\to 0$ as $\mathfrak{s}_2\to\mathfrak{s}_1$, respectively.

Let $0\le\mathfrak{s}_1<T<\mathfrak{s}_2$. When $\mathfrak{s}_2\to\mathfrak{s}_1$, then $\mathfrak{s}_2\to T$ and $\mathfrak{s}_1\to T$ hold, simultaneously. So, for any $\mathsf{x}\in\Phi_1$,

$$
E\left\|\frac{(\mho_2\mathsf{x})(\mathfrak{s}_2)}{1+\mathfrak{s}_2}-\frac{(\mho_2\mathsf{x})(\mathfrak{s}_1)}{1+\mathfrak{s}_1}\right\|^2\le 2E\left\|\frac{(\mho_2\mathsf{x})(\mathfrak{s}_2)}{1+\mathfrak{s}_2}-\frac{(\mho_2\mathsf{x})(T)}{1+T}\right\|^2+E\left\|\frac{(\mho_2\mathsf{x})(T)}{1+T}-\frac{(\mho_2\mathsf{x})(\mathfrak{s}_1)}{1+\mathfrak{s}_1}\right\|^2,
$$

holds. So we have,

$$
E\left\|\frac{(\mho_2\mathsf{x})(\mathfrak{s}_2)}{1+\mathfrak{s}_2}-\frac{(\mho_2\mathsf{x})(\mathfrak{s}_1)}{1+\mathfrak{s}_1}\right\|^2\to 0,\ \text{as}\ \mathfrak{s}_2\to\mathfrak{s}_1.
$$

Hence, $\mathcal{D}_2:=\left\{\mathsf{z}:\mathsf{z}(\mathfrak{s})=\frac{(\mho_2\mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}},\ \mathsf{x}\in\Phi_1\right\}$ is equicontinuous. As a consequence, $\mathcal{D}=\mathcal{D}_1+\mathcal{D}_2$ is equicontinuous. Hence, the proof is ended. □

Lemma 3.2. *If* (H_1)–(H_5) *are satisfied, then, for all* $\mathsf{x}\in\Phi_1$, $\lim_{\mathfrak{s}\to\infty}E\left\|\frac{(\mho\mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}}\right\|^2=0$ *uniformly.*

Proof. Indeed, for any $\mathsf{x}\in\Phi_1$, by using Lemma 2.12 and the assumptions (H_2), (H_4), and (H_5), we obtain

$$E\|(\mho \mathsf{x})(\mathfrak{s})\|^2 \le 4E\|\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}\mathscr{O}_{\lambda,\mu}[\mathfrak{g}_0 - \varpi(0,\mathfrak{g}(0))]\|^2 + 4E\|\mathfrak{s}^{(1-\lambda)(1-\mu)}\varpi(\mathfrak{s},\mathfrak{g}(\mathfrak{s}))\|^2$$

$$+ 4E\|\mathfrak{s}^{(1-\lambda)(1-\mu)}\int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s}-\mathfrak{l})\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\mathfrak{l}\|^2$$

$$+ 4E\|\mathfrak{s}^{(1-\lambda)(1-\mu)}\int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s}-\mathfrak{l})\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}(\omega))dW(\omega)d\mathfrak{l}\|^2$$

$$\le 8\mathcal{K}_2^2[E\|\mathfrak{g}_0\|^2 + \mathcal{K}_\varpi^2(1+\|\mathfrak{g}_0\|^2)] + 4\mathcal{K}_\varpi^2\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}(1+\|\mathfrak{g}\|^2)$$

$$+ \frac{4\mathcal{K}_1^2}{\mu\kappa}\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)+\mu\kappa}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}\mathfrak{p}(\mathfrak{s})d\mathfrak{l}$$

$$+ 4Tr(Q)\mathcal{K}_1^2\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{q}(\mathfrak{s})d\mathfrak{l}.$$

Dividing both sides of the above inequalities by $(1+\mathfrak{s})^2$, we obtain

$$E\left\|\frac{(\mho \mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}}\right\|^2 \le \frac{8\mathcal{K}_2^2}{(1+\mathfrak{s})^2}[E\|\mathfrak{g}_0\|^2 + \mathcal{K}_\varpi^2(1+\|\mathfrak{g}_0\|^2)] + \frac{4\mathcal{K}_\varpi^2}{(1+\mathfrak{s})^2}\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}(1+\|\mathfrak{g}\|^2)$$

$$+ \frac{4\mathcal{K}_1^2}{\mu\kappa(1+\mathfrak{s})^2}\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)+\mu\kappa}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}\mathfrak{p}(\mathfrak{l})d\mathfrak{l} \tag{3.1}$$

$$+ 4Tr(Q)\mathcal{K}_1^2\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{q}(\mathfrak{l})d\mathfrak{l} \to 0,\ \text{as } \mathfrak{s}\to\infty,$$

which proves that for any $\mathsf{x} \in \Phi_1$, $\lim_{\mathfrak{s}\to\infty} E\left\|\frac{(\mho \mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}}\right\|^2 = 0$ holds uniformly. □

Lemma 3.3. *If* (H_1)–(H_5) *are satisfied, then* $\mho\Phi_1 \subset \Phi_1$.

Proof. For the case $\mathfrak{s} > 0$, by Eq (3.1), we have

$$E\left\|\frac{(\mho \mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}}\right\|^2 \le \frac{8\mathcal{K}_2^2}{(1+\mathfrak{s})^2}[E\|\mathfrak{g}_0\|^2 + \mathcal{K}_\varpi^2(1+\|\mathfrak{g}_0\|^2)] + \frac{4\mathcal{K}_\varpi^2}{(1+\mathfrak{s})^2}\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}(1+\|\mathfrak{g}\|^2)$$

$$+ \frac{4\mathcal{K}_1^2}{\mu\kappa(1+\mathfrak{s})^2}\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)+\mu\kappa}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}\mathfrak{p}(\mathfrak{s})d\mathfrak{l}$$

$$+ 4Tr(Q)\mathcal{K}_1^2\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{q}(\mathfrak{s})d\mathfrak{l} \le \mathfrak{r}.$$

For the case $\mathfrak{s} = 0$, we have

$$E\left\|\frac{(\mho \mathsf{x})(0)}{1+0}\right\|^2 = E\|(\mho \mathsf{x})(0)\|^2 \le 8\mathcal{K}_2^2[E\|\mathfrak{g}_0\|^2 + \mathcal{K}_\varpi^2(1+\|\mathfrak{g}_0\|^2)] \le \mathfrak{r}.$$

As a consequence, $\mho\Phi_1 \subset \Phi_1$. □

Lemma 3.4. *If (H_1)–(H_5) are satisfied, then $\mho$ is continuous.*

Proof. Let the sequence $\{\mathsf{x}_m\}_{m=1}^{\infty}$ be in Φ_1 and convergent to $\mathsf{x} \in \Phi_1$. In this case, it follows that $\lim_{m\to\infty} E\|\mathsf{x}_m(\mathfrak{s})\|^2 = E\|\mathsf{x}(\mathfrak{s})\|^2$ and $\lim_{m\to\infty} E\|\mathfrak{s}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathsf{x}_m(\mathfrak{s})\|^2 = E\|\mathfrak{s}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathsf{x}(\mathfrak{s})\|^2$, for $\mathfrak{s} \in (0,\infty)$.

We assume $\mathfrak{g}(\mathfrak{s}) = \mathfrak{s}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathsf{x}(\mathfrak{s})$, $\mathfrak{g}_m(\mathfrak{s}) = \mathfrak{s}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathsf{x}_m(\mathfrak{s})$, $\mathfrak{s} \in (0,\infty)$. Then, clearly $\mathfrak{g}, \mathfrak{g}_m \in \Phi_1$. According to (H_1) and (H_3), we get $\lim_{m\to\infty} E\|\mathcal{F}(\mathfrak{s},\mathfrak{g}_m(\mathfrak{s}))\|^2 = E\|\mathcal{F}(\mathfrak{s},\mathfrak{s}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathfrak{g}_m(\mathfrak{s}))\|^2 = E\|\mathcal{F}(\mathfrak{s},\mathfrak{s}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathfrak{g}(\mathfrak{s}))\|^2 = E\|\mathcal{F}(\mathfrak{s},\mathfrak{g}_m(\mathfrak{s}))\|^2$ and $\lim_{m\to\infty} E\|G(\mathfrak{s},\mathfrak{g}_m(\mathfrak{s}))\|^2 = E\|G(\mathfrak{s},\mathfrak{s}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathfrak{g}_m(\mathfrak{s}))\|^2 = E\|G(\mathfrak{s},\mathfrak{s}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathfrak{g}(\mathfrak{s}))\|^2 = E\|G(\mathfrak{s},\mathfrak{g}_m(\mathfrak{s}))\|^2$.

From (H_2), for all $\mathfrak{s} \in (0,\infty)$, we obtain

$$(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}E\|\mathcal{F}(\mathfrak{l},\mathfrak{g}_m(\mathfrak{l}))-\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))\|^2 \leq 2(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}\mathfrak{p}(\mathfrak{l}), \text{ a.e. in } [0,\mathfrak{s}).$$

Moreover, since $2(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}\mathfrak{p}(\mathfrak{l})$ is integrable for $\mathfrak{l} \in [0,\mathfrak{s})$ and $\mathfrak{s} \in [0,\infty)$, the LDCT enables us to claim that

$$\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}E\|\mathcal{F}(\mathfrak{l},\mathfrak{g}_m(\mathfrak{l}))-\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))\|^2 d\mathfrak{l} \to 0 \text{ as } m\to\infty.$$

Identically, by using (H_4) and LDCT, we obtain

$$\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}E\|[\int_0^{\mathfrak{l}}G(\omega,\mathfrak{g}_m(\omega))dW(\omega)-\int_0^{\mathfrak{l}}G(\omega,\mathfrak{g}(\omega))dW(\omega)]\|^2 d\mathfrak{l} \to 0 \text{ as } m\to\infty.$$

Thus, for $\mathfrak{s} \in [0,\infty)$, we have

$$\begin{aligned}
&E\left\|\frac{(\mho\mathsf{x}_m)(\mathfrak{s})}{1+\mathfrak{s}}-\frac{(\mho\mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}}\right\|^2\\
&\leq 2\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}E\left\|\int_0^{\mathfrak{s}}\mathscr{P}_\mu(\mathfrak{s}-\mathfrak{l})[\mathcal{F}(\mathfrak{l},\mathfrak{g}_m(\mathfrak{l}))-\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))]d\mathfrak{l}\right\|^2\\
&\quad+2\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}E\left\|\int_0^{\mathfrak{s}}\mathscr{P}_\mu(\mathfrak{s}-\mathfrak{l})\left[\int_0^{\mathfrak{l}}G(\omega,\mathfrak{g}_m(\omega))dW(\omega)-\int_0^{\mathfrak{l}}G(\omega,\mathfrak{g}(\omega))dW(\omega)\right]d\mathfrak{l}\right\|^2\\
&\leq 2\mathcal{K}_1\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}d\mathfrak{l}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}E\|\mathcal{F}(\mathfrak{l},\mathfrak{g}_m(\mathfrak{l}))-\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))\|^2 d\mathfrak{l}\\
&\quad+2\mathcal{K}_1 Tr(Q)\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}E\|\int_0^{\mathfrak{l}}G(\omega,\mathfrak{g}_m(\omega))d\omega-\int_0^{\mathfrak{l}}G(\omega,\mathfrak{g}(\omega))d\omega\|^2 d\mathfrak{l}\\
&\to 0 \text{ as } m\to\infty.
\end{aligned}$$

Hence, $\|\mho\mathsf{x}_m-\mho\mathsf{x}\| \to 0$ as $m\to\infty$; i.e., $\mho$ is continuous. □

We are now prepared to present and support our first theorem about the mild solutions of the neutral stochastic HF-system (1.1).

Theorem 3.5. *Assume that the semigroup operator $\mathscr{O}(\mathfrak{s})$ is compact, for every $\mathfrak{s} > 0$. If (H_1)–(H_5) are satisfied, then (i) there exist some mild solutions in $\widehat{\Phi}_1$ for the given neutral stochastic HF-system* (1.1)*; (ii) all mild solutions of* (1.1) *are attractive.*

Proof. (i) According to the properties of $\mho$ and Σ, we know that the neutral stochastic HF-system (1.1) possesses a mild solution $\mathfrak{g} \in \widehat{\Phi}_1$ if $\mho$ has a fixed-point $\mathsf{x} \in \Phi_1$, where $\mathsf{x}(\mathfrak{s}) = \mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}\mathfrak{g}(\mathfrak{s})$. We have to prove that $\mho$ has a fixed-point in Φ_1. In fact, from Lemmas 3.3 and 3.4, we already have that $\mho$ maps Φ_1 into itself and $\mho$ is continuous on Φ_1. To demonstrate that $\mho$ is completely continuous, we have to show that the set $\mho\Phi_1$ is relatively compact. According to Lemmas 3.1 and 3.2, the set $\mathcal{D} := \{\mathsf{z} : \mathsf{z}(\mathfrak{s}) = \frac{(\mho\mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}},\ \mathsf{x} \in \Phi_1\}$ is equicontinuous, and for any $\mathsf{x} \in \Phi_1$, $\lim_{\mathfrak{s}\to\infty} E\|\frac{(\mho\mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}}\|^2 = 0$ satisfies uniformly. From Lemma 2.14, for each $\mathfrak{s} \in [0,\infty)$, we prove $\mathcal{D} := \{\mathsf{z} : \mathsf{z}(\mathfrak{s}) = \frac{(\mho\mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}},\ \mathsf{x} \in \Phi_1\}$ is relatively compact in $L_2(\Xi, \mathscr{Y})$. It is obvious that $\mathcal{D}(0)$ is relatively compact in $L_2(\Xi, \mathscr{Y})$. Therefore, we just need to investigate the case $\mathfrak{s} \in (0,\infty)$. For any $\epsilon \in (0,\mathfrak{s})$ and $\gamma > 0$, we consider $\mho_{\epsilon,\gamma}$ on Φ_1 in the form:

$$
\begin{aligned}
(\mho_{\epsilon,\gamma}\mathsf{x})(\mathfrak{s}) :&= \mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}(\Sigma_{\epsilon,\gamma}\mathfrak{g})(\mathfrak{s}) \\
&= \mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}\Big\{\mathscr{O}_{\lambda,\mu}[\mathfrak{g}_0 - \varpi(0,\mathfrak{g}(0))] + \varpi(\mathfrak{s},\mathfrak{g}(\mathfrak{s})) \\
&\quad + \int_0^{\mathfrak{s}-\epsilon}\int_0^\infty \mu\vartheta(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathscr{M}_\mu(\vartheta)\mathscr{T}((\mathfrak{s}-\mathfrak{l})^\mu\vartheta)\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\vartheta d\mathfrak{l} \\
&\quad + \int_0^{\mathfrak{s}-\epsilon}\int_0^\infty \mu\vartheta(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathscr{M}_\mu(\vartheta)\mathscr{T}((\mathfrak{s}-\mathfrak{l})^\mu\vartheta)\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}(\omega))dW(\omega)d\vartheta d\mathfrak{l}\Big\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{(\mho_{\epsilon,\gamma}\mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}} &= \frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\Big\{\mathscr{O}_{\lambda,\mu}[\mathfrak{g}_0 - \varpi(0,\mathfrak{g}(0))] + \varpi(\mathfrak{s},\mathfrak{g}(\mathfrak{s})) \\
&\quad + \mathscr{T}(\epsilon^\mu\gamma)\int_0^{\mathfrak{s}-\epsilon}\int_0^\infty \mu\vartheta(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathscr{M}_\mu(\vartheta)\mathscr{T}((\mathfrak{s}-\mathfrak{l})^\mu\vartheta - \epsilon^\mu\gamma)\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\vartheta d\mathfrak{l} \\
&\quad + \mathscr{T}(\epsilon^\mu\gamma)\int_0^{\mathfrak{s}-\epsilon}\int_0^\infty \mu\vartheta(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathscr{M}_\mu(\vartheta)\mathscr{T}((\mathfrak{s}-\mathfrak{l})^\mu\vartheta - \epsilon^\mu\gamma)\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}(\omega))dW(\omega)d\vartheta d\mathfrak{l}\Big\}.
\end{aligned}
$$

Since the semigroup $\mathscr{O}(\mathfrak{s})$ is compact for any $\mathfrak{s} > 0$, so $\mathscr{O}_{\lambda,\mu}(\mathfrak{s})$ is also compact. Furthermore, $\mathscr{T}(\epsilon^\mu\gamma)$ is compact. Then for all $\epsilon \in (0,\mathfrak{s})$ and for any $\gamma > 0$, the set $\{\frac{(\mho_{\epsilon,\gamma}\mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}},\ \mathsf{x} \in \Phi_1\}$ is relatively compact in $L_2(\Xi, \mathscr{Y})$. From (H_2) and (H_4) and Lemma 2.12, for each $\mathsf{x} \in \Phi_1$, we derive that

$$
\begin{aligned}
&E\left\|\frac{(\mho\mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}} - \frac{(\mho_{\epsilon,\gamma}\mathsf{x})(\mathfrak{s})}{1+\mathfrak{s}}\right\|^2 \\
&\le 4E\left\|\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_0^{\mathfrak{s}}\int_0^\gamma \mu\vartheta(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathscr{M}_\mu(\vartheta)\mathscr{T}((\mathfrak{s}-\mathfrak{l})^\mu\vartheta)\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\vartheta d\mathfrak{l}\right\|^2 \\
&\quad + 4E\left\|\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_0^{\mathfrak{s}}\int_0^\gamma \mu\vartheta(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathscr{M}_\mu(\vartheta)\mathscr{T}((\mathfrak{s}-\mathfrak{l})^\mu\vartheta)\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}(\omega))dW(\omega)d\vartheta d\mathfrak{l}\right\|^2
\end{aligned}
$$

$$+ 4E\left\|\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_{\mathfrak{s}-\epsilon}^{\mathfrak{s}}\int_{\gamma}^{\infty}\mu\vartheta(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathscr{M}_{\mu}(\vartheta)\mathscr{T}((\mathfrak{s}-\mathfrak{l})^{\mu}\vartheta)\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\vartheta d\mathfrak{l}\right\|^2$$

$$+ 4E\left\|\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_{\mathfrak{s}-\epsilon}^{\mathfrak{s}}\int_{\gamma}^{\infty}\mu\vartheta(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathscr{M}_{\mu}(\vartheta)\mathscr{T}((\mathfrak{s}-\mathfrak{l})^{\mu}\vartheta)\int_0^{\mathfrak{l}}G(\omega,\mathfrak{g}(\omega))dW(\omega)d\vartheta d\mathfrak{l}\right\|^2$$

$$\leq 4(\mu\mathcal{K})^2\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu-1}d\mathfrak{l}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\Big(\int_0^{\gamma}\vartheta\mathscr{M}_{\mu}(\vartheta)d\vartheta\Big)^2$$

$$+ 4(\mu\mathcal{K})^2Tr(Q)\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu-1)}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\Big(\int_0^{\gamma}\vartheta\mathscr{M}_{\mu}(\vartheta)d\vartheta\Big)^2$$

$$+ 4(\mu\mathcal{K})^2\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_{\mathfrak{s}-\epsilon}^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu-1}d\mathfrak{l}\int_{\mathfrak{s}-\epsilon}^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\Big(\int_0^{\infty}\vartheta\mathscr{M}_{\mu}(\vartheta)d\vartheta\Big)^2$$

$$+ 4(\mu\mathcal{K})^2Tr(Q)\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_{\mathfrak{s}-\epsilon}^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu-1)}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\Big(\int_0^{\infty}\vartheta\mathscr{M}_{\mu}(\vartheta)d\vartheta\Big)^2$$

$$\leq 4\mu\mathcal{K}^2\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\Big(\int_0^{\gamma}\vartheta\mathscr{M}_{\mu}(\vartheta)d\vartheta\Big)^2$$

$$+ 4(\mu\mathcal{K})^2Tr(Q)\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu-1)}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\Big(\int_0^{\gamma}\vartheta\mathscr{M}_{\mu}(\vartheta)d\vartheta\Big)^2$$

$$+ 4\mu\mathcal{K}^2\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_{\mathfrak{s}-\epsilon}^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu-1}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\Big(\frac{1}{\Gamma(\mu+1)}\Big)^2$$

$$+ 4(\mu\mathcal{K})^2Tr(Q)\frac{\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}}{(1+\mathfrak{s})^2}\int_{\mathfrak{s}-\epsilon}^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu-1)}\mathfrak{p}(\mathfrak{l})d\mathfrak{l}\Big(\frac{1}{\Gamma(\mu+1)}\Big)^2$$

$$\to 0 \text{ as } \epsilon\to 0, \gamma\to 0.$$

Therefore, $\mathcal{D}(\mathfrak{s})$ is also a relatively compact set in $L_2(\Xi,\mathscr{Y})$ for $\mathfrak{s}\in[0,\infty)$. Now, the Schauder's fixed point theorem implies that $\mho$ has at least a fixed-point $\mathsf{x}^*\in\Phi_1$. Let $\mathfrak{g}^*(\mathfrak{s})=\mathfrak{s}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathsf{x}^*(\mathfrak{s})$. From the relationship between Σ and $\mho$, we have

$$\mathfrak{g}^*(\mathfrak{s})=\mathscr{O}_{\lambda,\mu}[\mathfrak{g}_0-\varpi(0,\mathfrak{g}(0))]+\varpi(\mathfrak{s},\mathfrak{g}^*(\mathfrak{s}))+\int_0^{\mathfrak{s}}\mathscr{P}_{\mu}(\mathfrak{s}-\mathfrak{l})\mathcal{F}(\mathfrak{l},\mathfrak{g}^*(\mathfrak{l}))d\mathfrak{l}$$

$$+\int_0^{\mathfrak{s}}\mathscr{P}_{\mu}(\mathfrak{s}-\mathfrak{l})\int_0^{\mathfrak{l}}G(\omega,\mathfrak{g}^*(\omega))dW(\omega)d\mathfrak{l},\quad \mathfrak{s}\in[0,\infty),$$

which shows that $\mathfrak{g}^*$ is a mild solution of the neutral stochastic HF-system (1.1).

(ii) If $\mathfrak{g}(\mathfrak{s})$ is a mild solution of the neutral stochastic HF-system (1.1), then

$$\mathfrak{g}(\mathfrak{s})=\mathscr{O}_{\lambda,\mu}[\mathfrak{g}_0-\varpi(0,\mathfrak{g}(0))]+\varpi(\mathfrak{s},\mathfrak{g}(\mathfrak{s}))+\int_0^{\mathfrak{s}}\mathscr{P}_{\mu}(\mathfrak{s}-\mathfrak{l})\mathcal{F}(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\mathfrak{l}$$

$$+\int_0^{\mathfrak{s}}\mathscr{P}_{\mu}(\mathfrak{s}-\mathfrak{l})\int_0^{\mathfrak{l}}G(\omega,\mathfrak{g}(\omega))dW(\omega)d\mathfrak{l},\quad \mathfrak{s}\in[0,\infty).$$

By (H_2), (H_4), and (H_5), noting that $-1+\lambda-\lambda\mu+\mu\kappa<0$, we obtain

$$\begin{aligned}E\|\mathfrak{g}(\mathfrak{s})\|^2 \leq\ & 8\mathcal{K}_2^2[E\|\mathfrak{g}_0\|^2+\mathcal{K}_\varpi^2(1+\|\mathfrak{g}_0\|^2)]+4\mathcal{K}_\varpi^2\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}(1+\|\mathfrak{g}\|^2)\\ &+\frac{4\mathcal{K}_1^2}{\mu\kappa}\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)+\mu\kappa}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}\mathfrak{p}(\mathfrak{s})d\mathfrak{l}\\ &+4Tr(Q)\mathcal{K}_1^2\mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{q}(\mathfrak{s})d\mathfrak{l}\to 0,\ \text{as } \mathfrak{s}\to\infty.\end{aligned}$$

Immediately, we can conclude that $\mathfrak{g}(\mathfrak{s})$ is an attractive solution which completes the proof. □

We assume that the subsequent hypothesis is true to demonstrate the existence results when the semigroup operator $\{\mathcal{O}(\mathfrak{s})\}_{\mathfrak{s}>0}$ is noncompact.

(H_6) There exists a constant $\mathscr{L}>0$ such that for every bounded set $D\subset\mathscr{Y}$, $\alpha(\mathcal{F}(\mathfrak{s},D))\vee\alpha(\int_0^{\mathfrak{l}}G(\mathfrak{l},D))\leq\mathscr{L}\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}\alpha(D)$, for a.e. $\mathfrak{s}\in[0,\infty)$.

Theorem 3.6. *Assume the semigroup operator $\mathcal{O}(\mathfrak{s})$ is noncompact for any $\mathfrak{s}>0$. If (H_1)–(H_6) are satisfied, then*
(i) there exists at least one mild solution in $\widehat{\Phi}_1$ for the neutral stochastic HF-system (1.1)*;*
(ii) all these mild solutions are attractive.

Proof. (i) We set $\mathsf{x}_0(\mathfrak{s})=\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}\mathcal{O}_{\lambda,\mu}(\mathfrak{s})\mathfrak{g}_0,\ \mathfrak{s}\in[0,\infty)$ and $\mathsf{x}_{m+1}=\mho\mathsf{x}_m,\ m=0,1,2,\cdots$. From Lemma 3.3, $\mho\mathsf{x}_m\subset\Phi_1$ whenever $\mathsf{x}_m\in\Phi_1,\ m=0,1,2,\cdots$. Define $\widehat{\mathcal{D}}=\{\mathsf{z}_m:\mathsf{z}_m(\mathfrak{s})=\dfrac{(\mho\mathsf{x}_m)(\mathfrak{s})}{1+\mathfrak{s}},\ \mathsf{x}_m\in\Phi_1\}_{m=0}^\infty$. We have to show that set $\widehat{\mathcal{D}}$ is relatively compact.

According to Lemmas 3.1 and 3.2, we already know that $\widehat{\mathcal{D}}$ is equicontinuous, and for $\mathsf{x}_m\in\Phi_1,\ \lim_{\mathfrak{s}\to\infty}E\|\dfrac{(\mho\mathsf{x}_m)(\mathfrak{s})}{1+\mathfrak{s}}\|^2=0$ uniformly. From Lemma 2.14, we have to show

$$\widehat{\mathcal{D}}=\{\mathsf{z}_m:\mathsf{z}_m(\mathfrak{s})=\frac{(\mho\mathsf{x})_m(\mathfrak{s})}{1+\mathfrak{s}},\ \mathsf{x}_m\in\Phi_1\}_{m=0}^\infty$$

is relatively compact in $L^2(\Xi,\mathscr{Y})$.

By Lemmas 2.6 and 2.12, along with the condition (H_6), we obtain

$$\begin{aligned}&\alpha\Big(\Big\{\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\int_0^{\mathfrak{s}}\mathscr{P}_\mu(\mathfrak{s}-\mathfrak{l})\mathcal{F}(\mathfrak{l},\mathfrak{g}_m(\mathfrak{l}))d\mathfrak{l}\Big\}_{m=0}^\infty\Big)\\ &\leq 2\mathcal{K}_1\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}\alpha\Big(\mathcal{F}\Big(\mathfrak{l},\Big\{\mathfrak{l}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathsf{x}_m(\mathfrak{l})\Big\}_{m=0}^\infty\Big)\Big)d\mathfrak{l}\\ &\leq 2\mathscr{L}\mathcal{K}_1\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}\mathfrak{l}^{1-\lambda+\lambda\mu-\mu\kappa}\alpha\Big(\Big\{\mathfrak{l}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathsf{x}_m(\mathfrak{l})\Big\}_{m=0}^\infty\Big)d\mathfrak{l}\\ &\leq 2\mathscr{L}\mathcal{K}_1\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}(1+\mathfrak{l})\alpha\Big(\Big\{\frac{\mathsf{x}_m(\mathfrak{l})}{1+\mathfrak{l}}\Big\}_{m=0}^\infty\Big)d\mathfrak{l}.\end{aligned}$$

On the other side, for all $\mathfrak{g}, v \in \mathscr{Y}$, from Lemmas 2.6 and 2.12, we obtain

$$\Big\| \int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s}-\mathfrak{l})\Big[\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}(\omega)) - \int_0^{\mathfrak{l}} G(\omega, v(\omega))\Big] dW(\omega)\Big\|$$

$$\leq \mathcal{K}_1\Big(\Big\| \int_0^{\mathfrak{s}} (\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}[\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}(\omega)) - \int_0^{\mathfrak{l}} G(\omega, v(\omega))] dW(\omega)\Big\|^2\Big)^{\frac{1}{2}}$$

$$\leq \mathcal{K}_1 Tr(Q)\Big(\int_0^{\mathfrak{s}} (\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}\mathcal{K}_1\Big\| \int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}(\omega)) - \int_0^{\mathfrak{l}} G(\omega, v(\omega))\mathcal{K}_1\Big\|^2 d\omega\Big)^{\frac{1}{2}}.$$

Thus, one has

$$\alpha\Big(\Big\{\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s}-\mathfrak{l})\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}_m(\omega))dW(\omega)\Big\}_{m=0}^{\infty}\Big)$$

$$\leq \mathcal{K}_1\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\Big[2Tr(Q)\int_0^{\mathfrak{s}} (\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}\Big[\alpha\Big(G\Big(\mathfrak{l},\Big\{\mathfrak{l}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathsf{x}_m(\mathfrak{l})\Big\}_{m=0}^{\infty}\Big)\Big)\Big]^2 d\mathfrak{l}\Big]^{\frac{1}{2}}$$

$$\leq \mathscr{L}\mathcal{K}_1\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\Big[2Tr(Q)\int_0^{\mathfrak{s}} (\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}\mathfrak{l}^{2(1-\lambda+\lambda\mu-\mu\kappa)}\Big[\alpha\Big(\Big\{\mathfrak{l}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathsf{x}_m(\mathfrak{l})\Big\}_{m=0}^{\infty}\Big)\Big]^2 d\mathfrak{l}\Big]^{\frac{1}{2}}$$

$$\leq \mathscr{L}\mathcal{K}_1\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\Big[2Tr(Q)\int_0^{\mathfrak{s}} (\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}(1+\mathfrak{l})^2\Big[\alpha\Big(\Big\{\frac{\mathsf{x}_m(\mathfrak{l})}{1+\mathfrak{l}}\Big\}_{m=0}^{\infty}\Big)\Big]^2 d\mathfrak{l}\Big]^{\frac{1}{2}}.$$

The above estimates yield that

$$\begin{aligned}
\alpha(\widehat{\mathcal{D}}(\mathfrak{s})) &= \alpha\Big(\Big\{\frac{(\mho\mathsf{x})_m(\mathfrak{s})}{1+\mathfrak{s}}\Big\}_{m=0}^{\infty}\Big)\\
&= \alpha\Big(\Big\{\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\mathscr{O}\lambda,\mu[\mathfrak{g}_0 - \varpi(0,\mathfrak{g}(0))] + \varpi(\mathfrak{s},\mathfrak{g}_m(\mathfrak{s}))\\
&\quad + \frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s}-\mathfrak{l})\mathcal{F}(\mathfrak{l},\mathfrak{g}_m(\mathfrak{l}))d\mathfrak{l}\\
&\quad + \frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s}-\mathfrak{l})\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}_m(\omega))dW(\omega)d\mathfrak{l}\Big\}_{m=0}^{\infty}\Big)\\
&= \alpha\Big(\Big\{\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s}-\mathfrak{l})\mathcal{F}(\mathfrak{l},\mathfrak{g}_m(\mathfrak{l}))d\mathfrak{l}\\
&\quad + \frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\int_0^{\mathfrak{s}} \mathscr{P}_\mu(\mathfrak{s}-\mathfrak{l})\int_0^{\mathfrak{l}} G(\omega,\mathfrak{g}_m(\omega))dW(\omega)d\mathfrak{l}\Big\}_{m=0}^{\infty}\Big)\\
&= 2\mathscr{L}\mathcal{K}_1\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\int_0^{\mathfrak{s}} (\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}(1+\mathfrak{l})\alpha\Big(\Big\{\frac{\mathsf{x}_m(\mathfrak{l})}{1+\mathfrak{l}}\Big\}_{m=0}^{\infty}\Big)d\mathfrak{l}\\
&\quad + \mathscr{L}\mathcal{K}_1\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\Big[2Tr(Q)\int_0^{\mathfrak{s}} (\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}(1+\mathfrak{l})^2\Big[\alpha\Big(\Big\{\frac{\mathsf{x}_m(\mathfrak{l})}{1+\mathfrak{l}}\Big\}_{m=0}^{\infty}\Big)\Big]^2 d\mathfrak{l}\Big]^{\frac{1}{2}}.
\end{aligned}$$

For any $\mathfrak{s} \in [0, \infty)$, from Lemma 2.5, one can derive that

$$\alpha\left(\left\{\frac{\mathsf{x}_m(\mathfrak{s})}{1+\mathfrak{s}}\right\}_{m=0}^{\infty}\right) = \alpha\left(\left\{\frac{\mathsf{x}_0(\mathfrak{s})}{1+\mathfrak{s}}\right\} \cup \left\{\frac{\mathsf{x}_m(\mathfrak{s})}{1+\mathfrak{s}}\right\}_{m=1}^{\infty}\right) = \alpha\left(\left\{\frac{\mathsf{x}_m(\mathfrak{s})}{1+\mathfrak{s}}\right\}_{m=1}^{\infty}\right) = \alpha(\widehat{\mathcal{D}}(\mathfrak{s})).$$

Hence, we deduce that

$$\begin{aligned}\alpha(\widehat{\mathcal{D}}(\mathfrak{s})) &\leq 2\mathscr{L}\mathcal{K}_1\mathcal{M}^* \int_0^{\mathfrak{s}} (\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}(1+\mathfrak{l})\alpha(\widehat{\mathcal{D}}(\mathfrak{l}))d\mathfrak{l} \\ &\quad + \mathscr{L}\mathcal{K}_1\mathcal{M}^*\left[2Tr(Q)\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}(1+\mathfrak{l})^2[\alpha(\widehat{\mathcal{D}}(\mathfrak{l}))]^2 d\mathfrak{l}\right]^{\frac{1}{2}} \\ &= M_1 + M_2,\end{aligned}$$

where $\mathcal{M}^* = \max\limits_{\mathfrak{s}\in[0,\infty)}\left\{\frac{\mathfrak{s}^{1-\lambda+\lambda\mu-\mu\kappa}}{1+\mathfrak{s}}\right\}$.

If $M_1 > M_2$, from the estimates above, we have

$$\alpha(\widehat{\mathcal{D}}(\mathfrak{s})) \leq 4\mathscr{L}\mathcal{K}_1\mathcal{M}^* \int_0^{\mathfrak{s}} (\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}(1+\mathfrak{l})\alpha(\widehat{\mathcal{D}}(\mathfrak{l}))d\mathfrak{l}.$$

Therefore, by a similar estimation, one of the inequalities

$$\alpha(\widehat{\mathcal{D}}(\mathfrak{s})) \leq 8\mathscr{L}\mathcal{K}_1\mathcal{M}^* \int_0^{\mathfrak{s}} (\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}\alpha(\widehat{\mathcal{D}}(\mathfrak{l}))d\mathfrak{l},$$

or

$$\alpha(\widehat{\mathcal{D}}(\mathfrak{s})) \leq 8\mathscr{L}\mathcal{K}_1\mathcal{M}^* \int_0^{\mathfrak{s}} (\mathfrak{s}-\mathfrak{l})^{\mu\kappa-1}\mathfrak{l}\alpha(\widehat{\mathcal{D}}(\mathfrak{l}))d\mathfrak{l}$$

holds. As a result, the inequality, in ([46], p. 188), enables us to claim that $\alpha(\widehat{\mathcal{D}}(\mathfrak{s})) = 0$.

If $M_1 < M_2$, a standard calculation yields that

$$(\alpha(\widehat{\mathcal{D}}(\mathfrak{s})))^2 \leq (2\mathscr{L}\mathcal{K}_1\mathcal{M}^*)^2\left(2Tr(Q)\int_0^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{l})^{2(\mu\kappa-1)}(1+\mathfrak{l})^2[\alpha(\widehat{\mathcal{D}}(\mathfrak{l}))]^2 d\mathfrak{l}\right).$$

We may also conclude that $\alpha(\widehat{\mathcal{D}}(\mathfrak{s})) = 0$ by using an analogous argument to the first scenario. Therefore, $\widehat{\mathcal{D}}(\mathfrak{s})$ is relatively compact. Lemma 2.14, finally, gives this fact that the set $\widehat{\mathcal{D}}$ is relatively compact. A subsequence of $\{\mathsf{x}_m\}_{m=0}^{\infty}$ exists so that it is convergent to, say, x^*, i.e., $\lim_{m\to\infty}\mathsf{x}_m = \mathsf{x}^* \in \Phi_1$. Thus, the continuity of the operator $\mho$ enables us to declare that

$$\mathsf{x}^* = \lim_{m\to\infty}\mathsf{x}_m = \lim_{m\to\infty}\mho\mathsf{x}_{m-1} = \mho(\lim_{m\to\infty}\mathsf{x}_{m-1}) = \mho\mathsf{x}^*.$$

Let $\mathfrak{g}^*(\mathfrak{s}) = \mathfrak{s}^{-1+\lambda-\lambda\mu+\mu\kappa}\mathsf{x}^*(\mathfrak{s})$. Thus, $\mathfrak{g}^*$ is a fixed-point of Σ, which will be the mild solution of the neutral stochastic HF-system (1.1).

(ii) This proof is similar to (ii) in Theorem 3.5. □

By Theorems 3.5 and 3.6, we have a corollary.

Corollary 3.7. *Assume that the semigroup operator $\mathscr{O}(\mathfrak{s})$ is compact for any $\mathfrak{s} > 0$ and assumptions (H_1) and (H_3) are fulfilled.*

(H_7) *There exist a function $\mathfrak{p} : (0,\infty) \to (0,\infty)$ and constants $\chi \in (0,1)$, $\mathscr{N} > 0$ such that for any $\mathfrak{g} \in \mathscr{Y}$, $\mathfrak{s} \in (0,\infty)$,*

$$(I_{0+}^{\mu}\mathfrak{p})(\mathfrak{s}) \in C((0,\infty),(0,\infty)),\ \mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)+\mu}(I_{0+}^{\mu}\mathfrak{p})(\mathfrak{s}) \le \mathscr{N}\mathfrak{s}^{2\chi},$$

and

$$E\|\mathcal{F}(\mathfrak{s},\mathfrak{g})\|^2 \le \mathfrak{p}(\mathfrak{s}).$$

(H_8) *There exist a function $\mathfrak{q} : (0,\infty) \to (0,\infty)$ and constants $\widehat{\chi} \in (0,1)$, $\widehat{\mathscr{N}} > 0$ such that for any $\mathfrak{g} \in \mathscr{Y}$, $\mathfrak{s} \in (0,\infty)$,*

$$(I_{0+}^{2\mu-1}\mathfrak{q})(\mathfrak{s}) \in C((0,\infty),(0,\infty)),\ \mathfrak{s}^{2(1-\lambda+\lambda\mu-\mu\kappa)}(I_{0+}^{2\mu-1}\mathfrak{q})(\mathfrak{s}) \le \widehat{\mathscr{N}}\mathfrak{s}^{2\widehat{\chi}},$$

and

$$E\|\int_0^{\mathfrak{s}} G(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\mathfrak{l}\|^2 \le \mathfrak{q}(\mathfrak{s}).$$

Then, there exists at least one mild solution in $\widehat{\Phi}_1$ for the neutral stochastic HF-system (1.1).

Corollary 3.8. *Suppose that the semigroup operator $\mathscr{O}(\mathfrak{s})$ is noncompact for all $\mathfrak{s} > 0$. If (H_1), (H_3), (H_7), (H_8), and (H_6) are hold, then one can find at least one mild solution in $\widehat{\Phi}_1$ to the neutral stochastic HF-system* (1.1).

4. Example

Consider the following HF neutral stochastic evolution integro-differential system on an infinite interval:

$$\begin{cases} {}^{H}D_{0+}^{\mu}[\mathfrak{g}(\mathfrak{s}) - w(\mathfrak{s},\mathfrak{g}(\mathfrak{s}))] = \mathscr{A}[\mathfrak{g}(\mathfrak{s}) - w(\mathfrak{s},\mathfrak{g}(\mathfrak{s}))] + f(\mathfrak{s},\mathfrak{g}(\mathfrak{s})) + \int_0^{\mathfrak{s}} g(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))dW(\mathfrak{l}), \\ I_{0+}^{1-\mu}\mathfrak{g}(0) = \mathfrak{g}_0, \qquad \mathfrak{s} \in (0,\infty), \end{cases} \tag{4.1}$$

where $f(\mathfrak{s},\mathfrak{g}(\mathfrak{s}))$ and $\int_0^{\mathfrak{s}} g(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))dW(\mathfrak{l})$ fulfill (H_1) and (H_3), respectively, and the constants $\zeta,\beta > 0$ exist such that $E\|f(\mathfrak{s},\mathfrak{g}(\mathfrak{s}))\|^2 \le \mathfrak{s}^{-\zeta}$, $E\|\int_0^{\mathfrak{s}} g(\mathfrak{l},\mathfrak{g}(\mathfrak{l}))d\mathfrak{l}\|^2 \le \mathfrak{s}^{-\beta}$ for $\zeta \in (\mu,1)$, $\beta \in (2\mu-1,1)$, and for $\mathfrak{s} \in (0,\infty)$, $\{\mathscr{O}(\mathfrak{s})\}_{\mathfrak{s}\ge 0}$ is compact.

Let $\mathfrak{p}(\mathfrak{s}) = \mathfrak{s}^{-\zeta}$, $\mathfrak{q}(\mathfrak{s}) = \mathfrak{s}^{-\beta}$, for $\mathfrak{s} > 0$. Then, it is easy to verify that

$$(I_{0+}^{\mu}\mathfrak{p})(\mathfrak{s}) = \frac{\Gamma(1-\zeta)}{\Gamma(1+\mu-\zeta)}\mathfrak{s}^{\mu-\zeta} \in C((0,\infty),(0,\infty)),\ \mathfrak{s}^{2(1-\mu)+\mu}(I_{0+}^{\mu}\mathfrak{p})(\mathfrak{s}) \le \mathscr{N}\mathfrak{s}^{2\chi},$$

$$(I_{0+}^{2\mu-1}\mathfrak{q})(\mathfrak{s}) = \frac{\Gamma(1-\beta)}{\Gamma(2\mu-\beta)}\mathfrak{s}^{2\mu-\beta-1} \in C((0,\infty),(0,\infty)),\ \mathfrak{s}^{2(1-\mu)}(I_{0+}^{2\mu-1}\mathfrak{q})(\mathfrak{s}) \le \widehat{\mathscr{N}}\mathfrak{s}^{2\widehat{\chi}},$$

where $\chi = \frac{1}{2}(2-\zeta) \in (0,1)$, $\widehat{\chi} = \frac{1}{2}(1-\beta) \in (0,1)$, $\mathscr{N} \geq \frac{\Gamma(1-\zeta)}{\Gamma(1+\mu-\zeta)}\mathfrak{s}^{\mu-\zeta}$, $\widehat{\mathscr{N}} \geq \frac{\Gamma(1-\beta)}{\Gamma(2\mu-\beta)}\mathfrak{s}^{2\mu-\beta-1}$, which means that the conditions (H_7) and (H_8) are fulfilled. Further, it is easy to prove that $1-\frac{\mu}{2} > \frac{1}{2}(2-\zeta) = \chi$ and $1-\mu > \frac{1}{2}(1-\beta) = \widehat{\chi}$. By Corollary 3.7, the neutral stochastic HF-system (4.1) has at least a mild solution and also an attractive solution.

Remark 4.1. *This result may also extend to the attractive solution for Hilfer fractional neutral stochastic differential equations with Poisson jump.*

5. Conclusions

In this paper, we proved that Hilfer fractional neutral stochastic integro-differential equations on an infinite interval with almost sectorial operators have global mild and attractive solutions, and that the corresponding semigroup is either compact or noncompact. We determined the Wright function, the measure of noncompactness, and several alternative criteria to ensure the worldwide existence of mild solutions to the HF-system (1.1) by using the generalized Ascoli-Arzela theorem. To demonstrate the acquired theoretical findings, an example was given. This result may also be used to study Hilfer fractional neutral stochastic integro-differential equations with impulses on an infinite interval and their approximate controllability.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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