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Research article

Further improvements of the Jensen inequality in the integral sense by virtue of 6-convexity along with applications

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Abstract: The Jensen inequality is of fundamental importance because of its influential and interesting consequences. In recent years, the Jensen inequality has been supposed to be the most engaging source for research. We present interesting improvements to the continuous version of Jensen's inequality through the application of the concept of 6-convexity. For real visualization and comparison to other results, some numerical experiments were provided. With the aid of the acquired results, improvements for the Hermite-Hadamard and Hölder inequalities were presented. Some relationships between the means were granted as applications of established improvements. In addition, some estimations of the Csiszár divergence and its associated cases were received as further applications of the obtained results. The major techniques employed in formulating the proposed improvements included the Jensen inequality and the concept of convexity.

Keywords: information theory; convex function; Jensen's inequality; 6-convex function; power means; Hölder inequality; quasi-arithmetic means
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1. Introduction and preliminaries

Jensen's inequality stands out as a fundamental concept that is crucial across a multitude of scientific and technological fields, with particular significance in the realms of statistics and mathematics (see the works [1, 2]). In statistical estimation, this inequality is used to set constraints on the biases and variances of estimators [3, 4]. It assists in comprehending the connection among the expected value of a function applied to an estimator, and the function of the expected value of the estimator [5]. Jensen's inequality is a fundamental concept in the study of convex functions in mathematical analysis [6]. It serves as a bridge between convex and linear functions, making it easier to analyze inequalities and the behavior of functions over intervals of time. In the study of economics, risk assessment and utility theory depend heavily on Jensen's inequality [7]. It functions as a basic building block for financial models like the capital asset pricing model, which facilitates understanding of the risk and return characteristics of assets [8–10]. Jensen's inequality holds important significance for optimization because it gives a tool to evaluate the convexity of objective functions, which helps formulate and solve optimization issues. In information theory, Jensen's inequality is a key idea because it sets boundaries, analyzes entropy and expectations, and explains the manner in which various information measures respond [11]. Furthermore, this inequality is essential to engineering since it helps to manage and quantify uncertainty and connects expected values and convex functions to promote optimal decision-making [12]. Convex functions play an important role in different branches of mathematics, optimization, and applied sciences due to their inherent properties and significance [13–16]. Convex functions have geometric properties that simplify their analysis, and their derivatives provide valuable information about the function's behavior [14, 17, 18]. This makes them amenable to both geometric intuition and rigorous mathematical analysis [19–21].

The convex function can be summarized as the following:

Definition 1.1. *Let the function ϑ: $[\beta_1, \beta_2] \to \mathbb{R}$ be a convex, if for $\varrho_1, \varrho_2 \in [\beta_1, \beta_2]$ and $\upsilon \in [0, 1]$, the inequality*

$$\vartheta(\upsilon\varrho_1 + (1-\upsilon)\varrho_2) \leq \upsilon\vartheta(\varrho_1) + (1-\upsilon)\vartheta(\varrho_2) \tag{1.1}$$

is valid.

If the inequality in (1.1) *flips its direction, then ϑ is regarded as concave.*

To define n-convex function, first, we give the definition of the n-divided difference.

For distinct points $\delta_1, \delta_2, \delta_3, \ldots, \delta_n$, the divided difference of a function $\mathcal{F}$ is defined recursively as follows [22]:

$$[\delta_n]\mathcal{F} = \mathcal{F}(\delta_n),$$

$$[\delta_1, \delta_2, \ldots, \delta_n]\mathcal{F} = \frac{[\delta_2, \delta_3, \ldots, \delta_n]\mathcal{F} - [\delta_1, \delta_2, \ldots, \delta_{n-1}]\mathcal{F}}{\delta_n - \delta_1}. \tag{1.2}$$

The n-convex(n-concave) function is defined as [22]:

Definition 1.2. *Any real valued function $\mathcal{F}$ is said to be n-convex (n-concave) on I, if for any distinct points $\delta_1, \delta_2, \delta_3, \ldots, \delta_{n+1} \in I$, the relation*

$$[\delta_1, \delta_2, \ldots, \delta_{n+1}]\mathcal{F} \geq (\leq) 0 \tag{1.3}$$

holds.

In the following theorem, a very useful tool has been presented for examining the n-convexity (n-concavity) of a function [22].

Theorem 1.3. *Consider the n times differentiable function $\mathcal{F}: I \to \mathbb{R}$. Then $\mathcal{F}$ is n-convex (n-concave), if and only if*

$$\mathcal{F}^{(n)}(\delta) \geq (\leq) 0, \qquad \delta \in I. \tag{1.4}$$

The following is the discrete classical version of Jensen's inequality:

Theorem 1.4. *Let the function $\mathcal{F}$: $[\gamma_1, \gamma_2] \to \mathbb{R}$ be convex and $\delta_i \in [\gamma_1, \gamma_2]$, $\varrho_i \geq 0$, with $\varrho^* := \sum_{i=1}^n \varrho_i > 0$. Then*

$$\mathcal{F}\left(\frac{1}{\varrho^*}\sum_{i=1}^{n}\varrho_i\delta_i\right) \leq \frac{1}{\varrho^*}\sum_{i=1}^{n}\varrho_i\mathcal{F}(\delta_i). \tag{1.5}$$

The reverse inequality in (1.5) *holds, if $\mathcal{F}$ is concave.*

Due to the broad applicability and significance of Jensen's inequality, it has undergone improvements, generalizations, and modifications to explore its diverse behaviors and characteristics [23]. The integral version of this renowned inequality is presented in the following theorem.

Theorem 1.5. *Let the functions f: $[\beta_1, \beta_2] \to I$, w: $[\beta_1, \beta_2] \to [0, \infty)$ be integrable with $\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)d\delta > 0$. If the function $\mathcal{F}$ is convex over I and $\mathcal{F} \circ f$ is an integrable on I, then*

$$\mathcal{F}\left(\frac{1}{\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)d(\delta)}\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)f(\delta)d(\delta)\right) \leq \frac{1}{\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)d(\delta)}\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)\mathcal{F}(f(\delta))d(\delta). \tag{1.6}$$

The inequality (1.6) *reverses for the concave function $\mathcal{F}$.*

The Hermite-Hadamard inequality is a powerful tool that is often used to understand the behavior of convex functions in a variety of mathematical contexts [24–27]. Its geometric interpretation and various generalizations highlight its significance in mathematical analysis. The Hermite-Hadamard inequality is also notable for its purpose in establishing a relationship between the values of a convex function and the corresponding average value of the function. In [28], Miftah et al. present new integral inequalities of the Hermite-Hadamard type for functions that are twice differentiable and s-convex. These inequalities extend the classical Hermite-Hadamard inequalities, providing bounds for the integral of such functions in terms of their derivatives and s-convexity properties. These results contribute to the development of mathematical inequalities theory, particularly in the context of s-convex functions. A novel type of generalised convex function called extended (s, m)-prequasiinvex functions on coordinates is introduced in this article [29]. The mathematical expression of this inequality can be articulated as follows:

Theorem 1.6. *If the function $\mathcal{F}$ is a convex function over $[\beta_1, \beta_2]$, then*

$$\mathcal{F}\left(\frac{\beta_1+\beta_2}{2}\right) \leq \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\mathcal{F}(\varrho)d\varrho \leq \frac{\mathcal{F}(\beta_1)+\mathcal{F}(\beta_2)}{2}. \tag{1.7}$$

For the concave function $\mathcal{F}$, the inequality (1.7) *flips.*

In the recent wave of research activity, researchers increasingly recognize the Jensen inequality in providing insights and establishing connections across different domains, contributing to the ongoing advancement of knowledge and understanding in diverse disciplines. In 2000, Dragomir and Fitzpatrick [30] utilized s-convexity in the sense of Breckner defined on linear space to develop some

Jensen's type inequalities, and they also demonstrated how these inequalities were applied to means. In 2010, Dragomir [31] discussed a refinement of the prominent Jensen's inequality by taking the convex function over linear space. Zabandan and Kiliçman [32] generalized the Jensen inequality for convex functions of two variables and established the Hermite-Hadamard inequality lower bound for the convex function within a bounded region on the region. In 2012, Horváth [33] provided improvements of the classical Jensen's inequality in discrete form by introducing a novel approach, and additionally obtained integral analogs of these discrete inequalities. Also, they gave new refinements to the Hermite-Hadamard inequality. In 2020, Ullah et al. [34] discovered soft margin estimator bounds by applying the Jensen inequality, and further generalized the obtained bounds for the soft margin estimator by taking functions defined on rectangles and utilizing the properties of the Jaccard similarity function. Khan et al. [35] utilized the Jensen inequality to derive bounds for the Slater gap and then applied those results to means and divergences. You et al. [36] utilized integral identity and presented improvements of companion inequality to the Jensen inequality. Further, they discussed the consequences of the main finding in information theory. Basir et al. [23] acquired different improved majorization inequality through the application of celebrated Jensen's inequality and also discussed several cases through which the main findings gave bitter results for the majorization difference. Ullah et al. [37] examined improvements to Jensen's inequality by incorporating generalized convexity and also explored its applications in various domains. Khan et al. [38] applied the convexity of the second order derivative absolute function and established some interesting bounds for the Jensen gap. In 2023, Khan et al. [39] improved the Jensen inequality by taking differentiable functions and performed experiments by taking particular functions to make comparisons with some other earlier results to highlight its bitter performance.

We introduce captivating enhancements to the continuous version of Jensen's inequality by utilizing the notion of 6-convexity. To provide a concrete perspective and facilitate comparison with recent findings, several numerical experiments are conducted. Utilizing the outcomes obtained, enhancements to the Hölder and Hermite-Hadamard inequalities are elucidated. In addition, we will explore the implications of these improvements for relationships between different means. Additionally, we investigate estimations of the Csiszár divergence and its related cases, employing the established results. The principal methodologies employed in devising these enhancements encompass the utilization of the Jensen inequality and the exploration of convexity.

2. Improvements of Jensen's inequality

In this part, we closely examine improvements to the Jensen inequality using the innovative concept of 6-convexity. We start to present an improvement of the Jensen inequality that gives an estimate of the Jensen difference derived through the application of the 6-convexity principle.

Theorem 2.1. *Let $\mathcal{F}\colon I \to \mathbb{R}$ be a four times differentiable function such that $\mathcal{F}$ is 6-convex and $\mathcal{F}''''$ is integrable. Also, let*

$$f : [\beta_1, \beta_2] \to I, \quad \mathrm{w} : [\beta_1, \beta_2] \to [0, \infty)$$

be any integrable functions such that

$$\overline{\mathrm{w}} := \int_{\beta_1}^{\beta_2} \mathrm{w}(\delta) d\delta > 0$$

and

$$\overline{f} := \frac{1}{\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta) f(\delta) d\delta.$$

Then

$$\begin{aligned}\frac{1}{\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)\mathcal{F}(f(\delta))d\delta - \mathcal{F}(\overline{f}) \leq & \frac{1}{120\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(f(\delta)-\overline{f})^4\left(4\mathcal{F}^{''''}(\overline{f}) + \mathcal{F}^{''''}(f(\delta))\right)d\delta \\ & - \frac{1}{6\overline{\mathrm{w}}}\mathcal{F}^{'''}(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^3 d\delta + \frac{1}{2\overline{\mathrm{w}}}\mathcal{F}^{''}(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(f(\delta)-\overline{f})^2 d\delta.\end{aligned} \tag{2.1}$$

The inequality (2.1) *holds in the reverse direction, if the function* $\mathcal{F}$ *is* 6*-concave.*

Proof. Without the misfortune of a sweeping statement, consider that, $\overline{f} \neq f(\delta)$ for all $\delta \in [\beta_1, \beta_2]$. Using the rule of integration by parts, we achieve

$$\begin{aligned}&\frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4 \int_0^1 t^3\mathcal{F}^{''''}(t\overline{f} + (1-t)f(\delta))dtd\delta \\ &= \frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\bigg(\frac{t^3}{\overline{f} - f(\delta)}\mathcal{F}^{'''}(t\overline{f} + (1-t)f(\delta))\Big|_0^1 \\ &- \frac{3}{\overline{f} - f(\delta)}\int_0^1 t^2\mathcal{F}^{'''}(t\overline{f} + (1-t)f(\delta))dt\bigg)d\delta \\ &= \frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\bigg(\frac{\mathcal{F}^{'''}(\overline{f})}{\overline{f} - f(\delta)} - \frac{3}{\overline{f} - f(\delta)}\bigg(t^2\frac{\mathcal{F}^{''}(t\overline{f} + (1-t)f(\delta))\Big|_0^1}{\overline{f} - f(\delta)} \\ &- \frac{2}{\overline{f} - f(\delta)}\int_0^1 t\mathcal{F}^{''}(t\overline{f} + (1-t)f(\delta))dt\bigg)\bigg)d\delta \\ &= \frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\bigg(\frac{\mathcal{F}^{'''}(\overline{f})}{\overline{f} - f(\delta)} - 3\frac{\mathcal{F}^{''}(\overline{f})}{(\overline{f} - f(\delta))^2} \\ &+ \frac{6}{(\overline{f} - f(\delta))^2}\int_0^1 t\mathcal{F}^{''}(t\overline{f} + (1-t)f(\delta))dt\bigg)d\delta \\ &= \frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\bigg(\frac{\mathcal{F}^{'''}(\overline{f})}{\overline{f} - f(\delta)} - 3\frac{\mathcal{F}^{''}(\overline{f})}{(\overline{f} - f(\delta))^2} + \frac{6}{(\overline{f} - f(\delta))^2} \\ &\bigg(t\frac{\mathcal{F}^{'}(t\overline{f} + (1-t)f(\delta))\Big|_0^1}{\overline{f} - f(\delta)} - \frac{\int_0^1\mathcal{F}^{'}(t\overline{f} + (1-t)f(\delta))dt}{\overline{f} - f(\delta)}\bigg)\bigg)d\delta \\ &= \frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\bigg(\frac{\mathcal{F}^{'''}(\overline{f})}{\overline{f} - f(\delta)} - 3\frac{\mathcal{F}^{''}(\overline{f})}{(\overline{f} - f(\delta))^2} + 6\frac{\mathcal{F}^{'}(\overline{f})}{(\overline{f} - f(\delta))^3} \\ &- \frac{6}{(\overline{f} - f(\delta))^3}\int_0^1 \mathcal{F}^{'}(t\overline{f} + (1-t)f(\delta))dt\bigg)d\delta \\ &= \frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\bigg(\frac{\mathcal{F}^{'''}(\overline{f})}{\overline{f} - f(\delta)} - 3\frac{\mathcal{F}^{''}(\overline{f})}{(\overline{f} - f(\delta))^2} + 6\frac{\mathcal{F}^{'}(\overline{f})}{(\overline{f} - f(\delta))^3}\end{aligned}$$

$$- \frac{6}{(\overline{f} - f(\delta))^4}\mathcal{F}(t\overline{f} + (1-t)f(\delta))\Big|_0^1\Big)d\delta$$

$$= \frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\Big(\frac{\mathcal{F}'''(\overline{f})}{\overline{f} - f(\delta)} - 3\frac{\mathcal{F}''(\overline{f})}{(\overline{f} - f(\delta))^2} + 6\frac{\mathcal{F}'(\overline{f})}{(\overline{f} - f(\delta))^3}$$

$$- \frac{6}{(\overline{f} - f(\delta))^4}(\mathcal{F}(\overline{f}) - \mathcal{F}(f(\delta)))\Big)d\delta$$

$$= \frac{1}{6\overline{\mathrm{w}}}\mathcal{F}'''(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^3 - \frac{1}{2\overline{\mathrm{w}}}\mathcal{F}''(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^2$$

$$+ \frac{1}{\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)\mathcal{F}(f(\delta)) - \mathcal{F}(\overline{f})d\delta.$$

This implies that

$$\begin{aligned}\frac{1}{\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)\mathcal{F}(f(\delta))d\delta - \mathcal{F}(\overline{f}) =& \frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\int_0^1 t^3\mathcal{F}''''(t\overline{f} + (1-t)f(\delta))dtd\delta \\ &- \frac{1}{6\overline{\mathrm{w}}}\mathcal{F}'''(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^3 d\delta + \frac{1}{2\overline{\mathrm{w}}}\mathcal{F}''(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^2 d\delta.\end{aligned} \tag{2.2}$$

Since $\mathcal{F}''''$ is convex in the interval I. As a result, using the definition of convexity on the right-hand side of the Eq (2.2), we will be able to obtain

$$\begin{aligned}\frac{1}{\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)\mathcal{F}(f(\delta)) - \mathcal{F}(\overline{f}) \leq& \frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\int_0^1 t^3(t\mathcal{F}''''(\overline{f}) + (1-t)\mathcal{F}''''(f(\delta)))dtd\delta \\ &- \frac{1}{6\overline{\mathrm{w}}}\mathcal{F}'''(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^3 d\delta + \frac{1}{2\overline{\mathrm{w}}}\mathcal{F}''(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^2 d\delta \\ =& \frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\Big(\mathcal{F}''''(\overline{f})\int_0^1 t^4 dt + \mathcal{F}''''(f(\delta))\int_0^1 (t^3 - t^4)dt\Big)d\delta \\ &- \frac{1}{6\overline{\mathrm{w}}}\mathcal{F}'''(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^3 d\delta + \frac{1}{2\overline{\mathrm{w}}}\mathcal{F}''(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^2 d\delta \\ =& \frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\Big(\mathcal{F}''''(\overline{f})\frac{1}{5} + \mathcal{F}''''(f(\delta))\frac{1}{20}\Big)d\delta \\ &- \frac{1}{6\overline{\mathrm{w}}}\mathcal{F}'''(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^3 d\delta + \frac{1}{2\overline{\mathrm{w}}}\mathcal{F}''(\overline{f})\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^2 d\delta.\end{aligned}$$

This will lead to the main result (2.1) after a bit of simplification. □

In the preceding theorem, we derive a bound for the Jensen gap through utilization the Jensen inequality.

Theorem 2.2. *Presume that Theorem* 2.1 *assumptions are fulfilled. Then*

$$\begin{aligned}\frac{1}{\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)\mathcal{F}(f(\delta))d\delta - \mathcal{F}(\overline{f}) \geq& \frac{1}{24\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^4\mathcal{F}''''\Big(\frac{4\overline{f} + f(\delta)}{5}\Big)d\delta \\ &- \frac{\mathcal{F}'''(\overline{f})}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^3 d\delta + \frac{\mathcal{F}''(\overline{f})}{2\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2} \mathrm{w}(\delta)(\overline{f} - f(\delta))^2 d\delta.\end{aligned} \tag{2.3}$$

The inequality (2.3) *reverses, if* $\mathcal{F}$ *is* 6*-concave.*

Proof. By consuming Jensen's inequality on the right side of (2.2), we arrive to

$$\begin{aligned}
&\frac{1}{\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)\mathcal{F}(f(\delta))d\delta-\mathcal{F}(\overline{f})\\
&=\frac{1}{6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2}\left(\mathrm{w}(\delta)(\overline{f}-f(\delta))^4\int_0^1 t^3\mathcal{F}''''(t\overline{f}+(1-t)f(\delta))dt\right)d\delta\\
&-\frac{1}{6\overline{\mathrm{w}}}\mathcal{F}'''(\overline{f})\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^3d\delta+\frac{1}{2\overline{\mathrm{w}}}\mathcal{F}''(\overline{f})\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^2d\delta\\
&=\frac{1}{4\times 6\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2}\left(\mathrm{w}(\delta)(\overline{f}-f(\delta))^4\left(\frac{\int_0^1 t^3\mathcal{F}''''(t\overline{f}+(1-t)f(\delta))dt}{\int_0^1 t^3dt}\right)\right)d\delta\\
&-\frac{1}{6\overline{\mathrm{w}}}\mathcal{F}'''(\overline{f})\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^3d\delta+\frac{1}{2\overline{\mathrm{w}}}\mathcal{F}''(\overline{f})\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^2d\delta\\
&\geq\frac{1}{24\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^4\mathcal{F}''''\left(\frac{\int_0^1 t^3(t\overline{f}+(1-t)f(\delta))dt}{\int_0^1 t^3dt}\right)d\delta \qquad (2.4)\\
&-\frac{1}{6\overline{\mathrm{w}}}\mathcal{F}'''(\overline{f})\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^3d\delta+\frac{1}{2\overline{\mathrm{w}}}\mathcal{F}''(\overline{f})\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^2d\delta\\
&=\frac{1}{24\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^4\mathcal{F}''''\left(\frac{\overline{f}\int_0^1 t^4dt+f(\delta)\int_0^1(t^3-t^4)dt}{\int_0^1 t^3dt}\right)d\delta\\
&-\frac{1}{6\overline{\mathrm{w}}}\mathcal{F}'''(\overline{f})\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^3d\delta+\frac{1}{2\overline{\mathrm{w}}}\mathcal{F}''(\overline{f})\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^2d\delta\\
&=\frac{1}{24\overline{\mathrm{w}}}\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^4\mathcal{F}''''\left(\frac{4\overline{f}+f(\delta)}{5}\right)d\delta\\
&-\frac{1}{6\overline{\mathrm{w}}}\mathcal{F}'''(\overline{f})\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^3d\delta+\frac{1}{2\overline{\mathrm{w}}}\mathcal{F}''(\overline{f})\int_{\beta_1}^{\beta_2}\mathrm{w}(\delta)(\overline{f}-f(\delta))^2d\delta.
\end{aligned}$$

The inequality (2.4) confirms the required inequality (2.3). □

Remark 2.3. *The inequality* (2.3) *can also be obtained using weighted Hermite-Hadamard inequaliy.*

3. Significance of acquired improvements

This section provides the importance of the obtained improvements in the sense of its bitterness as compared to other established results.

3.1. Functions meet the criteria

It is crucial to emphasize that the obtained results demonstrate strong performance in meeting the convexity requirement for the function $\mathcal{F}''''$. This is particularly noteworthy because there are specific functions $\mathcal{F}$ for which $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$, and $\mathcal{F}'''$ are not convex, while $\mathcal{F}''''$ is convex. Therefore, there are many improvements to the Jensen inequality which are not applicable to functions that are not 6-convex. Examples of such functions are provided below:

- $\mathcal{F}(x) = \exp x - x^5$, $x \in [-4, 4]$.
- $\mathcal{F}(x) = -x^3 - 2x^5 - \log x$, $x > 0$.
- $\mathcal{F}(x) = x^6 - 100x^5$, $x \in [0, 1]$.

3.2. Numerical experiments

In this part of the manuscript, we focus to give some examples to know the betterness of our results.

Example 3.1. *Let us take the functions* $\mathcal{F}(\delta) = (1-\delta)^7$, $f(\delta) = \delta$ *and* $\mathrm{w}(\delta) = 1$ *for all* $\delta \in [0, 1]$. *Then* $\mathcal{F}''(\delta) = 42(1-\delta)^5$ *and* $\mathcal{F}''''''(\delta) = 5040(1-\delta)$. *Clearly both* $\mathcal{F}''(\delta)$ *and* $\mathcal{F}''''''(\delta)$ *are positive on* $[0, 1]$. *This shows that, the function* $\mathcal{F}(\delta) = (1-\delta)^7$ *is* 6*-convex as well as convex. Utilizing* $\mathcal{F}(\delta) = (1-\delta)^7$, $f(\delta) = \delta$ *and* $\mathrm{w}(\delta) = 1$ *in* (2.1), *we will get*

$$0.1172 < 0.1328. \tag{3.1}$$

If we use the same functions in the inequality (6) *of [37], we get*

$$0.1172 < 0.2666. \tag{3.2}$$

From (3.1) *and* (3.2) *it is obvious that the inequality* (2.1) *gives better and efficient estimate as compared to the inequality* (6) *in [37].*

Example 3.2. *Consider the function* $f(\delta) = \delta^6$ *defined on* $(-\infty, \infty)$. *Then* $f''(\delta) = 30\delta^4$ *and* $f''''''(\delta) = 720$, *clearly* $f''(\delta) = 30\delta^4$ *and* $f''''''(\delta) = 720$ *are positive on* $(-\infty, \infty)$. *As a result, this confirms the* 6*-convexity of the given function. Therefore, use these information in inequality* (2.1), *by substituting* $f(\delta) = \delta^6$, $f(\delta) = \delta$ *and* $\mathrm{w}(\delta) = 1$, *we get*

$$0.1273 < 0.1317. \tag{3.3}$$

Now, utilize the inequality (6) *in [37] for the above hypotheses, we obtain*

$$0.1273 < 0.1830. \tag{3.4}$$

From the inequalities (3.3) *and* (3.4), *we concluded that the Jensen gap estimates in* (2.1) *are better than those in* (6) *in [37].*

4. Applications of the obtained results

This section is assigned to give applications of the obtained results in various domains. They are presented in the subsequent subsections.

4.1. Applications to the Hölder inequality

The study on the Hölder inequality is an interesting topic because it has turned out to be a very useful tool to deal with complicated issues arising in a variety of domains [40]. In recent years, a large number of implications, expansions, and generalizations have been investigated [41]. In 2015, Chen and Wei [42] examined reversed Hölder inequality under $\alpha,\beta-$symmetric integral, and further discussed some associated inequalities. In 2019, Yan and Gao [43] gave extensions of the renowned Hölder inequality, and granted inequalities based on their primary results. This section pertains the applications of the major findings for Hölder inequality. We begin with the below result, in which an improvement for the Hölder inequality is presented as an application of Theorem 2.1.

Proposition 4.1. *Let a, b, ω: $[\beta_1, \beta_2] \to (0, \infty)$ be functions such that ωa^q, ωb^p, ωab and $ba^{\frac{-q}{p}}$ are integrable and $q, p > 1$ with $q^{-1} + p^{-1} = 1$.*

(i) *If $p \in [3, 4] \cup (1, 2] \cup [5, \infty)$, then*

$$
\begin{aligned}
&\Big(\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta\Big)^{\frac{1}{q}}\Big(\int_{\beta_1}^{\beta_2} \omega(\delta)b^p(\delta)d\delta\Big)^{\frac{1}{p}} - \int_{\beta_1}^{\beta_2} \omega(\delta)a(\delta)b(\delta)d\delta \\
&\leq \Bigg[\frac{p(p-1)(p-2)(p-3)}{120}\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)\Bigg(\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta} - b(\delta)a^{-\frac{q}{p}}(\delta)\Bigg)^4 \\
&\times\Bigg(4\Bigg(\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta}\Bigg)^{p-4} + \Big(b(\delta)a^{-\frac{q}{p}}(\delta)\Big)^{p-4}\Bigg)d\delta - \frac{p(p-1)(p-2)}{6} \\
&\times\Bigg(\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta}\Bigg)^{p-3}\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)\Bigg(\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta} - b(\delta)a^{-\frac{q}{p}}(\delta)\Bigg)^3 d\delta \\
&+ \frac{p(p-1)}{2}\Bigg(\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta}\Bigg)^{p-2}\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta) \\
&\times\Bigg(\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta} - b(\delta)a^{-\frac{q}{p}}(\delta)\Bigg)^2\Bigg]^{\frac{1}{p}} \times \Bigg(\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta\Bigg)^{\frac{1}{q}} d\delta.
\end{aligned} \tag{4.1}
$$

(ii) *If $p \in (4, 5) \cup (2, 3)$, then* (4.1) *holds in the contrary sense.*

Proof. (i) The function $\mathcal{F}(\delta) = \delta^p$ is convex as well as 6-convex on $(0, \infty)$ for all

$$p \in [3, 4] \cup (1, 2] \cup [5, \infty).$$

Therefore, utilizing (2.1), by choosing $\mathcal{F}(\delta) = \delta^p$ and

$$\mathrm{w}(\delta) = \omega(\delta)a^q(\delta),\ f(\delta) = b(\delta)a^{-\frac{q}{p}}(\delta),$$

and then taking power $\frac{1}{p}$, will get

$$\begin{aligned}
&\Big(\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta\Big)^{(p-1)}\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta\Big)-\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)a(\delta)b(\delta)d\delta\Big)^p\Big)^{\frac{1}{p}}\\
&\leq\Bigg[\frac{p(p-1)(p-2)(p-3)}{120}\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)\\
&\times\Bigg(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}-b(\delta)a^{-\frac{q}{p}}(\delta)\Bigg)^4\times\Bigg(4\Bigg(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}\Bigg)^{p-4}+\Big(b(\delta)a^{-\frac{q}{p}}(\delta)\Big)^{p-4}\Bigg)d\delta\\
&-\frac{p(p-1)(p-2)}{6}\Bigg(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}\Bigg)^{p-3}\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)\\
&\times\Bigg(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}-b(\delta)a^{-\frac{q}{p}}(\delta)\Bigg)^3d\delta+\frac{p(p-1)}{2}\Bigg(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}\Bigg)^{p-2}\\
&\times\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)\Bigg(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}-b(\delta)a^{-\frac{q}{p}}(\delta)\Bigg)^2\Bigg]^{\frac{1}{p}}\times\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta\Big)^{\frac{1}{q}}d\delta.
\end{aligned}\tag{4.2}$$

As the inequality

$$x^r-y^r\leq(x-y)^r\tag{4.3}$$

holds for all $x,y\geq 0$ and $r\in[0,1]$. Thus, using (4.3), for

$$x=\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta\Big)^{p-1}\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta\Big),\quad y=\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)a(\delta)b(\delta)\Big)^p$$

and $r=\frac{1}{p}$, we get

$$\begin{aligned}
&\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta\Big)^{\frac{1}{q}}\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta\Big)^{\frac{1}{p}}-\int_{\beta_1}^{\beta_2}\omega(\delta)a(\delta)b(\delta)d\delta\\
&\leq\Big(\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta\Big)^{p-1}\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta\Big)-\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)a(\delta)b(\delta)d\delta\Big)^p\Big)^{\frac{1}{p}}.
\end{aligned}\tag{4.4}$$

In the end, we compare both inequalities, (4.2) and (4.4) and finally, we get the result (4.1).

(ii) The function $\mathcal{F}(\delta)=\delta^p$ is 6-concave in the open interval $(0,\infty)$ for all $p>1$ such that $p\in(2,3)\cup(4,5)$. Therefore, we can get the reverse inequity in (4.2) by adopting the procedure of (i) . □

Another outcome of the main Theorem 2.1 regarding Hölder inequality is given below:

Corollary 4.2. *Assume that* a,b,ω: $[\beta_1,\beta_2]\to(0,\infty)$ *are functions such that* ωa^q, ωb^p, ωab *and* b^pa^{-q} *are integrable and* p *lies in* $(0,1)$ *with*

$$q=\frac{p}{p-1}.$$

The following statements are true as a result:

(i) *If* $\frac{1}{p} \in [3,4] \cup (1,2] \cup [5,\infty)$, *then*

$$\int_{\beta_1}^{\beta_2} \omega(\delta)a(\delta)b(\delta)d\delta - \Big(\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta\Big)^{\frac{1}{q}}\Big(\int_{\beta_1}^{\beta_2} \omega(\delta)b^p(\delta)d\delta\Big)^{\frac{1}{p}}$$
$$\leq \frac{(1-p)(1-2p)(1-3p)}{120p^4}\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta} - a^{-q}(\delta)b^p(\delta)\right)^4$$
$$\times\left(4\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}\right)^{\frac{1}{p}-4} + \left(b(\delta)a(\delta)^{-\frac{q}{p}}\right)^{\frac{1}{p}-4}\right)d\delta - \frac{(1-p)(1-2p)}{6p^3} \tag{4.5}$$
$$\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}\right)^{\frac{1}{p}-3}\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta} - a^{-q}(\delta)b^p(\delta)\right)^3 d\delta$$
$$+\frac{(1-p)}{2p^2}\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}\right)^{\frac{1}{p}-2}\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta} - a^{-q}(\delta)b^p(\delta)\right)^2 d\delta.$$

(ii) *If the values of p lies in* $(0,1)$ *such that* $\frac{1}{p} \in (2,3) \cup (4,5)$, *then the inequality* (4.5), *is reversed.*

Proof. (i) For $p \in (0,1)$, such that $\frac{1}{p} \in (1,2] \cup [3,4] \cup [5,\infty)$, the function $\mathcal{F}(\delta) = \delta^{\frac{1}{p}}$ is convex as well as 6-convex on $(0,\infty)$. Therefore, using (2.1) by substituting $\mathcal{F}(\delta) = \delta^{\frac{1}{p}}$, $f(\delta) = a^{-q}(\delta)b^p(\delta)$, and $\mathrm{w}(\delta) = \omega a^q(\delta)$, we get

$$\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)a(\delta)b(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta} - \frac{\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta\Big)^{\frac{1}{p}}}{\Big(\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta\Big)^{\frac{1}{p}}} \leq \frac{(1-p)(1-2p)(1-3p)}{120p^4\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}$$
$$\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta} - a^{-q}(\delta)b^p(\delta)\right)^4\left(4\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}\right)^{\frac{1}{p}-4}\right.$$
$$\left.+\,(b(\delta)a(\delta)^{-\frac{q}{p}})^{\frac{1}{p}-4}\right)d\delta - \frac{(1-p)(1-2p)}{6p^3\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}\right)^{\frac{1}{p}-3}$$
$$\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta} - a^{-q}(\delta)b^p(\delta)\right)^3 d\delta \tag{4.6}$$
$$+\frac{(1-p)}{2p^2\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta}\right)^{\frac{1}{p}-2}\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)$$
$$\times\left(\frac{\int_{\beta_1}^{\beta_2}\omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta} - a^{-q}(\delta)b^p(\delta)\right)^2 d\delta.$$

After multiplying $\int_{\beta_1}^{\beta_2}\omega(\delta)a^q(\delta)d\delta$ on both hand sides, will get inequality (4.5).

(ii) The function $\mathcal{F}(\delta) = \delta^p$ is 6-concave on $(0, \infty)$ for all p in $(0, 1)$ such that $\frac{1}{p} \in (2, 3) \cup (4, 5)$. Therefore, we can get the reverse inequity in (4.5). Using the procedure described in (i). □

In the following corollary an improvement of the Hölder inequality obtained through Theorem 2.2.

Corollary 4.3. *Assume that all the assumptions of Corollary* 4.2 *exist.*

(i) *If* $\frac{1}{p} \in (1, 2] \cup [3, 4] \cup [5, \infty)$, *then*

$$\begin{aligned}
&\int_{\beta_1}^{\beta_2} \omega(\delta)a(\delta)b(\delta)d\delta - \Big(\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta\Big)^{\frac{1}{q}}\Big(\int_{\beta_1}^{\beta_2} \omega(\delta)b^p(\delta)d\delta\Big)^{\frac{1}{p}} \\
&\geq \frac{(1-p)(1-2p)(1-3p)}{24p^4}\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)\left(\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta} - a^{-q}(\delta)b^p(\delta)\right)^4 \\
&\times\left(4\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)b^p(\delta)d\delta + a^{-q}(\delta)b^p(\delta)\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta}{5\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta}\right)^{\frac{1-4p}{p}} d\delta - \frac{(1-p)(1-2p)}{6p^3} \\
&\left(\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta}\right)^{\frac{1-3p}{p}}\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)\left(\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta} - a^{-q}(\delta)b^p(\delta)\right)^3 d\delta \\
&+ \frac{(1-p)}{2p^2}\left(\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta}\right)^{\frac{1-2p}{p}} \times \int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)\left(\frac{\int_{\beta_1}^{\beta_2} \omega(\delta)b^p(\delta)d\delta}{\int_{\beta_1}^{\beta_2} \omega(\delta)a^q(\delta)d\delta} - a^{-q}(\delta)b^p(\delta)\right)^2 d\delta.
\end{aligned} \tag{4.7}$$

(ii) *If* $\frac{1}{p} \in (2, 3) \cup (4, 5)$, *then the inequality* (4.7) *will be reverse.*

Proof. (i) Consider the function $\mathcal{F}(\delta) = \delta^{\frac{1}{p}}$. The function $\mathcal{F}$ is convex and 6-convex on $(0, \infty)$, $0 < p < 1$ such that

$$\frac{1}{p} \in [3, 4] \cup (1, 2] \cup [5, \infty).$$

So, using (2.3) by putting $\mathcal{F}(\delta) = \delta^{\frac{1}{p}}$, $\mathrm{w}(\delta) = \omega(\delta)a^q(\delta)$ and $f(\delta) = a^{-q}(\delta)b^p(\delta)$, will get (4.7).

(ii) For the given assumptions, the function $\mathcal{F}(\delta) = \delta^{\frac{1}{p}}$ is 6-concave. Therefore, applying the method of (i), we get the reverse inequality given in (4.7). □

4.2. Applications to the Hermite-Hadamard inequality

The Hermite-Hadamard inequality can be used to establish bounds on the average value of a convex function, which is often used to model the preferences of consumers in economics. The Hermite-Hadamard inequality plays a role in optimization problems involving convex functions. It can be useful to establish bounds on the objective function in optimization algorithms, helping to guide the search for optimal solutions efficiently. The Hermite-Hadamard inequality can be used in various physical and engineering contexts where convex functions arise, such as modeling material properties, optimizing engineering designs, and analyzing physical systems with constraints [12].

Our major results are supported by some corollaries presenting some improvements to the Hermite-Hadamard inequality.

Corollary 4.4. *Let $\mathcal{F}$: $[\beta_1,\beta_2] \to \mathbb{R}$ be four times differentiable function such that $\mathcal{F}^{''''}$ is integrable and $\mathcal{F}$ is* 6*-convex function. Then*

$$\begin{aligned}
&\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\mathcal{F}(\delta)d\delta-\mathcal{F}\Big(\frac{\beta_1+\beta_2}{2}\Big)\\
&\leq\frac{1}{120(\beta_2-\beta_1)}\int_{\beta_1}^{\beta_2}\Big(\frac{\beta_1+\beta_2}{2}-\delta\Big)^4\Big(4\mathcal{F}^{''''}\Big(\frac{\beta_1+\beta_2}{2}\Big)+\mathcal{F}^{''''}(\delta)\Big)d\delta\\
&-\frac{1}{6(\beta_2-\beta_1)}\mathcal{F}^{'''}\Big(\frac{\beta_1+\beta_2}{2}\Big)\int_{\beta_1}^{\beta_2}\Big(\frac{\beta_1+\beta_2}{2}-\delta\Big)^3d\delta\\
&+\frac{1}{6(\beta_2-\beta_1)}\mathcal{F}^{''}\Big(\frac{\beta_1+\beta_2}{2}\Big)\int_{\beta_1}^{\beta_2}\Big(\frac{\beta_1+\beta_2}{2}-\delta\Big)^2d\delta.
\end{aligned}\tag{4.8}$$

If the function $\mathcal{F}$ is 6*-concave, then the inequality* (4.8) *holds in the reverse direction.*

Proof. As the function $\mathcal{F}$ is 6-convex on $[\beta_1,\beta_2]$. Therefore, using (2.1) for $f(\delta)=\delta$ and $\mathrm{w}(\delta)=1$, we get (4.8). □

Corollary 4.5. *Assume that the suppositions of the Corollary* 4.4 *are true. Then*

$$\begin{aligned}
&\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\mathcal{F}(\delta)d\delta-\mathcal{F}\Big(\frac{\beta_1+\beta_2}{2}\Big)\\
&\geq\frac{1}{24(\beta_2-\beta_1)}\int_{\beta_1}^{\beta_2}\Big(\frac{\beta_1+\beta_2}{2}-x\Big)^4\mathcal{F}^{''''}\Big(\frac{2(\beta_1+\beta_2)+\delta}{5}\Big)d\delta\\
&-\frac{1}{6(\beta_2-\beta_1)}\mathcal{F}^{'''}\Big(\frac{\beta_1+\beta_2}{2}\Big)\int_{\beta_1}^{\beta_2}\Big(\frac{\beta_1+\beta_2}{2}-\delta\Big)^3d\delta\\
&+\frac{1}{6(\beta_2-\beta_1)}\mathcal{F}^{''}\Big(\frac{\beta_1+\beta_2}{2}\Big)\int_{\beta_1}^{\beta_2}\Big(\frac{\beta_1+\beta_2}{2}-\delta\Big)^2d\delta.
\end{aligned}\tag{4.9}$$

If the function $\mathcal{F}$ is 6*-concave. Then the inequality* (4.9) *will be reverse.*

Proof. Inequality (4.9) can easily be deduced by choosing $f(\delta)=\delta$ and $\mathrm{w}(\delta)=1$ in (2.3). □

4.3. Applications to the means

Since 1930, many people have recognized the importance of means, and multiple studies have concentrated on the characteristics and uses of means [44]. Several articles have been written on the means by which they are thoroughly researched from all angles [45]. Recently, numerous mathematical inequalities for diverse means have been published, and these inequalities have since been developed, expanded, and improved in numerous ways through a wide range of techniques and methodologies [46]. The primary topics to be covered in this section are power and quasi-arithmetic means. Based on our major findings, we shall establish several inequalities for the means. By incorporating a few specific functions into the special results, we will be able to achieve the required inequalities.

This section begins with a definition of power means.

Definition 4.6. *(Power mean) Let a, b: $[\beta_1, \beta_2] \to (0, \infty)$ be integrable functions. Then, the power of mean of order $\vartheta \in \mathbb{R}$ is defined dy:*

$$\mathcal{P}_\vartheta(a, b) = \begin{cases} \left(\frac{1}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta} \int_{\beta_1}^{\beta_2} a(\delta)b^\vartheta(\delta)d\delta\right)^{\frac{1}{\vartheta}}, & \vartheta \neq 0, \\ \exp\left(\frac{\int_{\beta_1}^{\beta_2} a(\delta)\log b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta}\right), & \vartheta = 0. \end{cases}$$

In the first instance, we prove some inequalities for the power mean based on Theorem 2.1.

Corollary 4.7. *Let a and b be two positive integrable functions on $[\beta_1, \beta_2]$ with $\ddot{a} = \int_{\beta_1}^{\beta_2} a(\delta)d\delta$. Also assume that $\mho, \jmath \in \mathbb{R} - \{0\}$.*

(i) *If $\jmath > 0$ with $\jmath \leq \mho \leq 2\jmath$ or $3\jmath \leq \mho \leq 4\jmath$ or $\mho \geq 5\jmath$ or $\mho<0$, then*

$$\begin{aligned} \mathcal{P}_\mho^\mho(a, b) - \mathcal{P}_\jmath^\mho(a, b) \leq & \frac{\mho(\mho - \jmath)(\mho - 2\jmath)(\mho - 3\jmath)}{120\jmath^4\ddot{a}} \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_\jmath^\jmath(a, b) - b(\delta)^\jmath)^4 \\ & \times (\mathcal{P}_\jmath^{\mho-4\jmath}(a, b) - b^{\mho-4\jmath}(\delta))d\delta - \frac{\mho(\mho - \jmath)(\mho - 2\jmath)\mathcal{P}_\jmath^{\mho-3\jmath}(a, b)}{6\jmath^3\ddot{a}} \\ & \times \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_\jmath^\jmath(a, b) - b(\delta)^\jmath)^3 d\delta + \frac{\mho(\mho - \jmath)\mathcal{P}_\jmath^{\mho-2\jmath}(a, b)}{2\jmath^2\ddot{a}} \\ & \times \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_\jmath^\jmath(a, b) - b(\delta)^\jmath)^2 d\delta. \end{aligned} \tag{4.10}$$

(ii) *If $\jmath < 0$ with $2\jmath \leq \mho \leq \jmath$ or $4\jmath \leq \mho \leq 3\jmath$ or $\mho \leq 5\jmath$ or $\mho>0$, then* (4.10) *holds.*

(iii) *If $\jmath > 0$ with $0 < \mho < \jmath$, $2\jmath < \mho < 3\jmath$ or $4\jmath < \mho < 5\jmath$, then* (4.10) *holds in the reverse direction.*

(iv) *If $\jmath < 0$ with $\jmath < \mho < 0$, $3\jmath < \mho < 2\jmath$ or $5\jmath < \mho < 4\jmath$, then* (4.10) *holds in the reverse direction.*

Proof. (i) Consider the function $\mathcal{F}(\delta) = \delta^{\frac{\mho}{\jmath}}$ for $\delta > 0$, then the function is 6-convex with the given intervals. In order to get (4.10), use the inequality (2.1) by substituting $\mathcal{F}(\delta) = \delta^{\frac{\mho}{\jmath}}$, $\mathrm{w}(\delta) = a(\delta)$ and $f(\delta) = b^\jmath(\delta)$.

(ii) There is another possibility that the function $\mathcal{F}(\delta) = \delta^{\frac{\mho}{\jmath}}$, $\delta > 0$ will be 6-convex. If the given conditions are satisfied for $\mho$ and $\jmath$, then the procedure for obtaining (4.10), can easily be followed by adopting the procedure used for obtaining (i).

(iii) For such values of $\mho$, $\jmath$ the function $\mathcal{F}(\delta) = \delta^{\frac{\mho}{\jmath}}$, $\delta > 0$ is 6-concave. Therefore, we can get the inequality (4.10), by adopting the procedure of (i) in the reverse direction.

(iv) In such cases of $\mho$, $\jmath$ the function $\mathcal{F}(\delta) = \delta^{\frac{\mho}{\jmath}}$, $\delta > 0$ is 6-concave. Thus using the method of proof of part (i) but for $\mathcal{F}$ as a 6-concave function, we get the inequality (4.10) in the reverse direction. □

It will be interesting to see an application based on Theorem 2.2 in our next corollary.

Corollary 4.8. *Let a and b are two positive integrable functions defined on $[\beta_1, \beta_2]$ with*

$$\ddot{a} = \int_{\beta_1}^{\beta_2} a(\delta)d\delta$$

and $\mho, \jmath \in \mathbb{R} - \{0\}$.

(i) *If* $\jmath > 0$ *with* $\jmath \leq \mho \leq 2\jmath$ *or* $3\jmath \leq \mho \leq 4\jmath$ *or* $\mho \geq 5\jmath$ *or* $\mho<0$*, then*

$$\begin{aligned} \mathcal{P}_{\mho}^{\mho}(a,b) - \mathcal{P}_{\jmath}^{\mho}(a,b) \geq & \frac{\mho(\mho-\jmath)(\mho-2\jmath)(\mho-3\jmath)}{24\jmath^4 \ddot{a}} \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_{\jmath}^{\jmath}(a,b) - b^{\jmath}(\delta))^4 \\ & \times \left(\frac{4\mathcal{P}_{\jmath}^{\jmath}(a,b) + b^{\jmath}(\delta)}{5}\right)^{\frac{\mho}{\jmath}-4} d\delta - \frac{\mho(\mho-\jmath)(\mho-2\jmath)\mathcal{P}_{\jmath}^{\mho-3\jmath}(a,b)}{6\jmath^3 \ddot{a}} \\ & \times \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_{\jmath}^{\jmath}(a,b) - b^{\jmath}(\delta))^3 d\delta + \frac{\mho(\mho-\jmath)\mathcal{P}_{\jmath}^{\mho-2\jmath}(a,b)}{2\jmath^2 \ddot{a}} \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_{\jmath}^{\jmath}(a,b) - b^{\jmath}(\delta))^2 d\delta. \end{aligned} \tag{4.11}$$

(ii) *If* $\jmath < 0$ *with* $2\jmath \leq \mho \leq \jmath$ *or* $4\jmath \leq \mho \leq 3\jmath$ *or* $\mho \leq 5\jmath$ *or* $\mho>0$*, then* (4.11) *holds.*

(iii) *If* $\jmath > 0$ *with* $0 < \mho < \jmath$, $2\jmath < \mho < 3\jmath$ *or* $4\jmath < \mho < 5\jmath$*, then* (4.11) *holds in the reverse direction.*

(iv) *If* $\jmath < 0$ *with* $\jmath < \mho < 0$, $3\jmath < \mho < 2\jmath$ *or* $5\jmath < \mho < 4\jmath$*, then* (4.11) *holds in the reverse direction.*

Proof. As the function $\mathcal{F}(\delta) = \delta^{\frac{\mho}{\jmath}}$, $\delta > 0$, is 6-convex. If the conditions given in (i) and (ii) of Corollary 4.8 are true. Then, in order to get (4.11) use the inequality (2.3) by putting $\mathcal{F}(\delta) = \delta^{\frac{\mho}{\jmath}}$, $w(\delta) = a(\delta)$ and $f(\delta) = b^{\jmath}(\delta)$. Moreover, the conditions mentioned in (iii) and (iv) on $\mho$, $\jmath$ of Corollary 4.10 are true, then the function $\mathcal{F}(\delta) = \delta^{\frac{\mho}{\jmath}}$, $\delta > 0$ will be 6-concave. Therefore, utilizing (2.3) when put $w(\delta) = a(\delta)$, $f(\delta) = b^{\jmath}(\delta)$ and $\mathcal{F}(\delta) = \delta^{\frac{\mho}{\jmath}}$, we get the reverse inequality in (4.11). □

Theorem 2.1 leads to a corollary which gives a further interesting relationship for different means.

Corollary 4.9. *If a and b are two positive integrable functions defined on* $[\beta_1, \beta_2]$ *with* $\ddot{a} = \int_{\beta_1}^{\beta_2} a(\delta) d\delta$*, then*

$$\begin{aligned} \frac{\mathcal{P}_1(a,b)}{\mathcal{P}_0(a,b)} \leq \exp\Bigg[& \frac{1}{120\ddot{a}} \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_1(a,b) - b(\delta))^4 \times (24\mathcal{P}_1^{-4}(a,b) + 6b(\delta)^{-4}) d\delta \\ & + \frac{\mathcal{P}_1^{-3}(a,b)}{3\ddot{a}} \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_1(a,b) - b(\delta))^3 d\delta + \frac{\mathcal{P}_1^{-2}(a,b)}{2\ddot{a}} \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_1(a,b) - b(\delta))^2 d\delta \Bigg]. \end{aligned} \tag{4.12}$$

Proof. Consider the function $\mathcal{F}(\delta) = -\ln \delta$, $\delta \in (0, \infty)$. It is evident that the functions $\mathcal{F}''(\delta) = \frac{1}{\delta^2}$ and $\mathcal{F}''''''(\delta) = \frac{120}{\delta^6}$ are positive. So, the function $\mathcal{F}$ is 6-convex as well as convex. Therefore, using (2.1) by putting $\mathcal{F}(\delta) = -\ln \delta$, $w(\delta) = a(\delta)$ and $f(\delta) = b(\delta)$, will get (4.12). □

In the next corollary we will see another application based on the Theorem 2.2.

Corollary 4.10. *Suppose the statement of Corollary* 4.9 *is true. Then*

$$\begin{aligned} \frac{\mathcal{P}_1(a,b)}{\mathcal{P}_0(a,b)} \geq \exp\Bigg[& \frac{625}{4\ddot{a}} \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_1(a,b) - b(\delta))^4 \times (4\mathcal{P}_1(a,b) + b(\delta))^{-4} d\delta \\ & + \frac{\mathcal{P}_1^{-3}(a,b)}{3\ddot{a}} \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_1(a,b) - b(\delta))^3 d\delta + \frac{\mathcal{P}_1^{-2}(a,b)}{2\ddot{a}} \int_{\beta_1}^{\beta_2} a(\delta)(\mathcal{P}_1(a,b) - b(\delta))^2 d\delta \Bigg]. \end{aligned} \tag{4.13}$$

Proof. As we accept the hypothesis of Corollary 4.9, therefore, treating inequality (2.3) by putting $\mathcal{F}(\delta) = -\ln \delta$, $w(\delta) = a(\delta)$ and $f(\delta) = b(\delta)$, will get (4.13). □

We obtain a relation for power mean using Theorem 2.1 in the following corollary.

Corollary 4.11. *It is assumed that the assumptions of Corollary* 4.9 *are true. Then*

$$\begin{aligned}\mathcal{P}_1(a,b)-\mathcal{P}_0(a,b)\leq&\frac{1}{120\ddot{a}}\int_{\beta_1}^{\beta_2}a(\delta)\big(\ln\mathcal{P}_0(a,b)-\ln b(\delta)\big)^4\\&\times(4\mathcal{P}_0(a,b)+b(\delta))d\delta-\frac{\mathcal{P}_0(a,b)}{6\ddot{a}}\int_{\beta_1}^{\beta_2}a(\delta)\big(\ln\mathcal{P}_1(a,b)-\ln b(\delta)\big)^3d\delta\\&+\frac{\mathcal{P}_0(a,b)}{2\ddot{a}}\int_{\beta_1}^{\beta_2}a(\delta)\big(\ln\mathcal{P}_1(a,b)-\ln b(\delta)\big)^2d\delta.\end{aligned}\tag{4.14}$$

Proof. Consider the function $\mathcal{F}(\delta)=\exp(\delta)$. Then it is clear that the functions $\mathcal{F}''(\delta)=\exp(\delta)$ and $\mathcal{F}''''''(\delta)=\exp(\delta)$ are always non negative. Therefore, applying (2.1) by putting $f(\delta)=\ln b(\delta)$, $\mathrm{w}(\delta)=a(\delta)$ and $\mathcal{F}(\delta)=\exp(\delta)$. So we can easily obtain the inequality (4.14). □

As a result of Theorem 2.2, another relationship for power mean exists.

Corollary 4.12. *Let the hypotheses of the Corollary* 4.9 *are true. Then*

$$\begin{aligned}\mathcal{P}_1(a,b)-\mathcal{P}_0(a,b)\geq&\frac{1}{24\ddot{a}}\int_{\beta_1}^{\beta_2}a(\delta)\big(\ln\mathcal{P}_0(a,b)-\ln b(\delta)\big)^4\\&\times\mathcal{P}_0^{\frac{4}{5}}(a,b)d\delta-\frac{\mathcal{P}_0(a,b)}{6\ddot{a}}\int_{\beta_1}^{\beta_2}a(\delta)\big(\ln\mathcal{P}_1(a,b)-\ln b(\delta)\big)^3d\delta\\&+\frac{\mathcal{P}_0(a,b)}{2\ddot{a}}\int_{\beta_1}^{\beta_2}a(\delta)\big(\ln\mathcal{P}_1(a,b)-\ln b(\delta)\big)^2d\delta.\end{aligned}\tag{4.15}$$

Proof. As accepted by the statement of the Corollary 4.9. Therefore, using inequality (2.3) by substituting $\mathcal{F}(\delta)=\exp\delta$, $\mathrm{w}(\delta)=a(\delta)$ and $f(\delta)=\ln b(\delta)$, will get (4.15). □

The quasi-arithmetic mean can be defined as follows.

Definition 4.13. *(Quasi-arithmetic mean) Let a and b be any two positive integrable functions defined on* $[\beta_1,\beta_2]$ *and* $\mathfrak{g}$ *be strictly monotonic continuous function on* $[\beta_1,\beta_2]$. *Then*

$$Q_{\mathfrak{g}}(a,b)=\mathfrak{g}^{-1}\left(\frac{1}{\ddot{a}}\int_{\beta_1}^{\beta_2}a(\delta)\mathfrak{g}(b(\delta))d\delta\right),$$

where

$$\ddot{a}=\int_{\beta_1}^{\beta_2}a(\delta)d\delta.$$

In addition, the following corollary provides a relationship for the quasi-arithmetic mean by applying Theorem 2.1.

Corollary 4.14. *Let a, b and* $\mathcal{F}$ *be positive integrable functions defined on* $[\beta_1,\beta_2]$ *with*

$$\ddot{a}=\int_{\beta_1}^{\beta_2}a(\delta)d\delta.$$

Also, let g *be strictly monotonic continuous function and* $\mathcal{F}\circ \mathrm{g}^{-1}$ *be* 6*-convex on* $(0,\infty)$*. Then*

$$\begin{aligned}\frac{1}{\ddot{a}}\int_{\beta_1}^{\beta_2} a(\delta)\mathcal{F}(b(\delta))d\delta-\mathcal{F}(Q_{\mathrm{g}}(a,b)) \leq &\frac{1}{120\ddot{a}}\int_{\beta_1}^{\beta_2} a(\delta)\left(\frac{1}{\ddot{a}}\int_{\beta_1}^{\beta_2} a(\delta)\mathrm{g}(b(\delta))d\delta-\mathrm{g}(b(\delta))\right)^4\\ &\times\left(4\left(\mathcal{F}\circ \mathrm{g}^{-1}\right)^{''''} Q_{\mathrm{g}}(a,b)-(\mathcal{F}\circ \mathrm{g}^{-1})^{''''}\mathrm{g}(b(\delta))\right)d\delta\\ &-\frac{(\mathcal{F}\circ \mathrm{g}^{-1})^{'''}(\mathrm{g}(Q_{\mathrm{g}}(a,b)))}{6\ddot{a}}\int_{\beta_1}^{\beta_2} a(\delta)(\mathrm{g}(Q_{\mathrm{g}}(a,b))-\mathrm{g}(b(\delta))))^3 d\delta\\ &+\frac{(\mathcal{F}\circ \mathrm{g}^{-1})^{''}(\mathrm{g}(Q_{\mathrm{g}}(a,b)))}{2\ddot{a}}\int_{\beta_1}^{\beta_2} a(\delta)(\mathrm{g}(Q_{\mathrm{g}}(a,b))-\mathrm{g}(b(\delta))))^2 d\delta.\end{aligned}\tag{4.16}$$

Proof. As the function $\mathcal{F}\circ \mathrm{g}^{-1}$ is 6-convex on $(0,\infty)$. In order to obtain an inequality (4.16), we must choose inequality (2.1) by putting $\mathcal{F}=\mathcal{F}\circ \mathrm{g}^{-1}$, $\mathrm{w}(\delta)=a(\delta)$ and $f(\delta)=\mathrm{g}(b)$. □

As an application of Theorem 2.2, the following corollary provides a relation for the Quasi-arithmetic mean.

Corollary 4.15. *Let us assume that the statement of Corollary* 4.14 *is true. Then*

$$\begin{aligned}\frac{1}{\ddot{a}}\int_{\beta_1}^{\beta_2} a(\delta)\mathcal{F}(b(\delta))d\delta-\mathcal{F}(Q_{\mathrm{g}}(a,b)) \geq &\frac{1}{24\ddot{a}}\int_{\beta_1}^{\beta_2} a(\delta)(\mathrm{g}(Q_{\mathrm{g}}(a,b)))-\mathrm{g}(b(\delta)))^4\\ &\times((\mathcal{F}\circ \mathrm{g}^{-1})^{''''}\left(\frac{4\left(\mathrm{g}(Q_{\mathrm{g}}(a,b)))+\mathrm{g}(b(\delta))\right)^4}{5}\right)d\delta\\ &-\frac{(\mathcal{F}\circ \mathrm{g}^{-1})^{'''}(\mathrm{g}(Q_{\mathrm{g}}(a,b)))}{6\ddot{a}}\int_{\beta_1}^{\beta_2} a(\delta)(\mathrm{g}(Q_{\mathrm{g}}(a,b))-\mathrm{g}(b(\delta))))^3 d\delta\\ &+\frac{(\mathcal{F}\circ \mathrm{g}^{-1})^{''}(\mathrm{g}(Q_{\mathrm{g}}(a,b)))}{2\ddot{a}}\int_{\beta_1}^{\beta_2} a(\delta)(\mathrm{g}(Q_{\mathrm{g}}(a,b))-\mathrm{g}(b(\delta))))^2 d\delta.\end{aligned}\tag{4.17}$$

Proof. As we accept the statement of the Corollary 4.14. Therefore, using inequality (2.3) by substituting $\mathcal{F}=\mathcal{F}\circ \mathrm{g}^{-1}$, $\mathrm{w}(\delta)=a(\delta)$ and $f(\delta)=g(b)$, will get (4.17). □

4.4. Information theory applications

In this part, we will discuss a few applications of certain inequalities in the field of information theory. There are a few bounds associated with Renyi divergence, Csiszár divergence, Shannon entropy , Kullback-Leibler divergence, and Bhattacharyya coefficient. Csisár divergence is defined at the beginning of this section.

Definition 4.16. *(Csiszár divergence) Let* $\varrho\colon I\to\mathbb{R}$ *be a convex function and integrable. Also let*

$$b:[\beta_1,\beta_2]\to\mathbb{R},\quad a:[\beta_1,\beta_2]\to(0,\infty)$$

be integrable functions such that $\varrho\circ\frac{b}{a}$ *is integrable and* $\frac{b(\delta)}{a(\delta)}\in I$ *for all* $\delta\in[\beta_1,\beta_2]$*. Then the Csiszár divergence is defined as:*

$$C_d(a,b)=\int_{\beta_1}^{\beta_2} a(\delta)\varrho\left(\frac{b(\delta)}{a(\delta)}\right)d\delta.$$

Corollary 4.17. *Let ϱ: $I \to \mathbb{R}$ be a four times differentiable function such that ϱ'''' integrable. Also, let*

$$b : [\beta_1,\beta_2] \to \mathbb{R} \quad \text{and} \quad a : [\beta_1,\beta_2] \to (0,\infty)$$

be integrable functions and

$$\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta}, \quad \frac{b(\delta)}{a(\delta)} \in I$$

for all $\delta \in [\beta_1,\beta_2]$. If ϱ is 6-convex, then

$$\begin{aligned} C_d(a,b) - \varrho\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta}\right)\int_{\beta_1}^{\beta_2} a(\delta)d\delta \leq & \frac{1}{120}\int_{\beta_1}^{\beta_2} a(\delta)\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta} - \frac{b(\delta)}{a(\delta)}\right)^4 \\ & \times\left(4\varrho''''\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta}\right) + \varrho''''\left(\frac{b(\delta)}{a(\delta)}\right)\right)d\delta \\ & - \frac{1}{6}\varrho'''\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta}\right)\int_{\beta_1}^{\beta_2} a(\delta)\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta} - \frac{b(\delta)}{a(\delta)}\right)^3 d\delta \\ & + \frac{1}{2}\varrho''\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta}\right)\int_{\beta_1}^{\beta_2} a(\delta)\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta} - \frac{b(\delta)}{a(\delta)}\right)^2 d\delta. \end{aligned} \tag{4.18}$$

Proof. To obtain an inequality (4.18), apply (2.1) by putting

$$\mathcal{F} = \varrho, \quad \mathrm{w}(\delta) = \frac{a(\delta)}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta} \quad \text{and} \quad f(\delta) = \frac{b(\delta)}{a(\delta)}.$$

□

Next, we look at the corollary, based on Theorem 2.2 we get a relation for the Csiszár divergence.

Corollary 4.18. *Let us suppose that all the assumptions of Corollary 4.17 are true. Then*

$$\begin{aligned} C_d(a,b) - \varrho\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta}\right)\int_{\beta_1}^{\beta_2} a(\delta)d\delta \geq & \frac{1}{24}\int_{\beta_1}^{\beta_2} a(\delta)\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta} - \frac{b(\delta)}{a(\delta)}\right)^4 \\ & \times\left(\varrho''''\left(\frac{4}{5}\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta} + \frac{b(\delta)}{a(\delta)}\right)\right)d\delta \\ & - \frac{1}{6}\varrho'''\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta}\right)\int_{\beta_1}^{\beta_2} a(\delta)\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta} - \frac{b(\delta)}{a(\delta)}\right)^3 d\delta \\ & + \frac{1}{2}\varrho''\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta}\right)\int_{\beta_1}^{\beta_2} a(\delta)\left(\frac{\int_{\beta_1}^{\beta_2} b(\delta)d\delta}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta} - \frac{b(\delta)}{a(\delta)}\right)^2 d\delta. \end{aligned} \tag{4.19}$$

Proof. Utilizing (2.3) for

$$\mathrm{w}(\delta) = \frac{a(\delta)}{\int_{\beta_1}^{\beta_2} a(\delta)d\delta}, \quad \mathcal{F} = \varrho \quad \text{and} \quad f(\delta) = \frac{b(\delta)}{a(\delta)},$$

we get (4.19). □

The Rényi divergence can be defined as follows:

Definition 4.19. *(Rényi divergence) Let a and b be positive probability density functions on* $[\beta_1, \beta_2]$ *such that* $\varsigma \in [0, \infty)$ *and* $\varsigma \neq 1$. *Then the Rényi divergence is defined by*

$$R_d(a, b) = \frac{1}{\varsigma - 1} \log\left(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta) b^{1-\varsigma}(\delta) d\delta \right).$$

Corollary 4.20. *Assume that a and b are positive probability density functions and* $\varsigma \in [0, \infty)$ *with* $\varsigma \neq 1$. *Then*

$$\begin{aligned} R_d(a, b) - \int_{\beta_1}^{\beta_2} a(\delta) \log\left(\frac{a(\delta)}{b(\delta)}\right) d\delta \leq & \frac{1}{20} \int_{\beta_1}^{\beta_2} a(\delta) \Big(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta) b^{1-\varsigma}(\delta) d\delta \\ & - \left(\frac{a(\delta)}{b(\delta)}\right)^{\varsigma - 1} \Big)^4 \Big(\frac{4}{\varsigma - 1} \left(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta) b^{1-\varsigma}(\delta) d\delta \right)^{-4} + \frac{1}{\varsigma - 1} \left(\frac{a(\delta)}{b(\delta)}\right)^{4(1-\varsigma)} \Big) d\delta \\ & + \frac{1}{3(\varsigma - 1)} \left(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta) b^{1-\varsigma}(\delta) d\delta \right)^{-3} \int_{\beta_1}^{\beta_2} a(\delta) \\ & \times \Big(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta) b^{1-\varsigma}(\delta) d\delta - \left(\frac{a(\delta)}{b(\delta)}\right)^{\varsigma - 1} \Big)^3 d\delta + \frac{1}{2(\varsigma - 1)} \left(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta) b^{1-\varsigma}(\delta) d\delta \right)^{-2} \\ & \times \int_{\beta_1}^{\beta_2} a(\delta) \Big(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta) b^{1-\varsigma}(\delta) - \left(\frac{a(\delta)}{b(\delta)}\right)^{\varsigma - 1} \Big)^2 d\delta. \end{aligned} \tag{4.20}$$

Proof. If

$$\mathcal{F}(\delta) = -\frac{1}{\varsigma - 1} \log \delta, \quad \delta > 0,$$

then

$$\mathcal{F}''(\delta) = \frac{1}{(\varsigma - 1)\delta^2}$$

and

$$\mathcal{F}''''''(\delta) = \frac{120}{(\varsigma - 1)\delta^6}.$$

It is clear that the double and sixth derivative of the function $\mathcal{F}$ is positive for all $\delta \in (0, \infty)$. This indicates that the function $\mathcal{F}(\delta) = -\frac{1}{\varsigma - 1} \log \delta$ is both convex and 6-convex on $(0, \infty)$. Therefore, substituting

$$\mathcal{F}(\delta) = -\frac{1}{\varsigma - 1} \log \delta, \quad \mathrm{w}(\delta) = a(\delta) \quad \text{and} \quad f(\delta) = \left(\frac{a(\delta)}{b(\delta)}\right)^{\varsigma - 1}$$

in (2.1), we get (4.20). □

Corollary 4.21. *Assume that the hypotheses of Corollary* 4.20 *are true. Then*

$$
\begin{aligned}
R_d(a,b) - \int_{\beta_1}^{\beta_2} a(\delta)\log\left(\frac{a(\delta)}{b(\delta)}\right)d\delta \geq & \frac{1}{4(\varsigma-1)}\int_{\beta_1}^{\beta_2} a(\delta)\Big(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta)b^{1-\varsigma}(\delta)d\delta - \left(\frac{a(\delta)}{b(\delta)}\right)^{\varsigma-1}\Big)^4 \\
& \times\left(\frac{4\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta)b^{1-\varsigma}(\delta)d\delta + \left(\frac{a(\delta)}{b(\delta)}\right)^{(\varsigma-1)}}{5}\right)^{-4} d\delta \\
& + \frac{1}{3(\varsigma-1)}\left(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta)b^{1-\varsigma}(\delta)d\delta\right)^{-3} \\
& \times\int_{\beta_1}^{\beta_2} a(\delta)\Big(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta)b^{1-\varsigma}(\delta)d\delta - \left(\frac{a(\delta)}{b(\delta)}\right)^{\varsigma-1}\Big)^3 d\delta \\
& + \frac{1}{2(\varsigma-1)}\left(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta)b^{1-\varsigma}(\delta)d\delta\right)^{-2}\int_{\beta_1}^{\beta_2} a(\delta) \\
& \times\Big(\int_{\beta_1}^{\beta_2} a^{\varsigma}(\delta)b^{1-\varsigma}(\delta)d\delta - \left(\frac{a(\delta)}{b(\delta)}\right)^{\varsigma-1}\Big)^2 d\delta.
\end{aligned}
\tag{4.21}
$$

Proof. By using inequality (2.3) for

$$
\mathcal{F}(\delta) = -\frac{1}{\varsigma-1}\log\delta, \quad \mathrm{w}(\delta) = a(\delta) \quad \text{and} \quad f(\delta) = \left(\frac{a(\delta)}{b(\delta)}\right)^{\varsigma-1},
$$

we receive (4.21). □

The Shannon entropy is defined by:

Definition 4.22. *(Shannon entropy) Assume that a is a positive probability density function on* $[\beta_1, \beta_2]$. *Then the Shannon entropy is defined by:*

$$
S_e(a) = -\int_{\beta_1}^{\beta_2} a(\delta)\log a(\delta)d\delta.
$$

Corollary 4.23. *Let a be any positive probability density function and b is an integrable function on* $[\beta_1, \beta_2]$. *Then*

$$
\begin{aligned}
& \int_{\beta_1}^{\beta_2} b(\delta)d\delta - \int_{\beta_1}^{\beta_2} a(\delta)\log b(\delta)d\delta - S_e(a) \\
& \leq \frac{1}{20}\int_{\beta_1}^{\beta_2} a(\delta)\Big(\int_{\beta_1}^{\beta_2} b(\delta)d\delta - \frac{b(\delta)}{a(\delta)}\Big)^4\left(4\left(\int_{\beta_1}^{\beta_2} b(\delta)d\delta\right)^{-4} + \left(\frac{a(\delta)}{b(\delta)}\right)^4\right)d\delta \\
& + \frac{1}{3}\left(\int_{\beta_1}^{\beta_2} b(\delta)d\delta\right)^{-3}\Big(\int_{\beta_1}^{\beta_2} b(\delta)d\delta - \frac{b(\delta)}{a(\delta)}\Big)^3 d\delta \\
& + \frac{1}{3}\left(\int_{\beta_1}^{\beta_2} b(\delta)d\delta\right)^{-2}\Big(\int_{\beta_1}^{\beta_2} b(\delta)d\delta - \frac{b(\delta)}{a(\delta)}\Big)^2 d\delta.
\end{aligned}
\tag{4.22}
$$

Proof. As the function $\varrho(\delta) = -\log\delta,\ \ \delta > 0$ is convex as well as 6-convex on $(0,\infty)$. Therefore, use (4.18) for $\varrho(\delta) = -\log\delta$, we acquire (4.22). □

Corollary 4.24. *Assume that the assumptions of Corollary* 4.23 *are valid. Then*

$$\begin{aligned}
&\int_{\beta_1}^{\beta_2} b(\delta)d\delta - \int_{\beta_1}^{\beta_2} a(\delta)\log b(\delta)d\delta - S_e(a)\\
&\geq \frac{1}{4}\int_{\beta_1}^{\beta_2} a(\delta)\left(\int_{\beta_1}^{\beta_2} b(\delta)d\delta - \frac{b(\delta)}{a(\delta)}\right)^4\left(\frac{4}{5}\int_{\beta_1}^{\beta_2} b(\delta)d\delta + \frac{b(\delta)}{a(\delta)}\right)^{-4} d\delta\\
&+\frac{1}{3}\left(\int_{\beta_1}^{\beta_2} b(\delta)d\delta\right)^{-3}\left(\int_{\beta_1}^{\beta_2} b(\delta)d\delta - \frac{b(\delta)}{a(\delta)}\right)^3 d\delta\\
&+\frac{1}{2}\left(\int_{\beta_1}^{\beta_2} b(\delta)d\delta\right)^{-2}\left(\int_{\beta_1}^{\beta_2} b(\delta)d\delta - \frac{b(\delta)}{a(\delta)}\right)^2 d\delta.
\end{aligned} \tag{4.23}$$

Proof. By substituting $g(\delta) = -\log\delta$ in (4.19), we receive (4.23). □

The definition of the Kullback-Leibler divergence is stated below:

Definition 4.25. *(Kullback-Leibler divergence) Let a and b be positive probability density function on* $[\beta_1, \beta_2]$. *Then*

$$K_{bl}(a,b) = \int_{\beta_1}^{\beta_2} b(\delta)\log\left(\frac{b(\delta)}{a(\delta)}\right)d\delta.$$

Corollary 4.26. *Let a and b be probability density functions with* $a(\delta), b(\delta) > 0$. *Then*

$$\begin{aligned}
D_{kl}(a,b) \leq & \frac{1}{60}\int_{\beta_1}^{\beta_2} a(\delta)\left(1 - \frac{b(\delta)}{a(\delta)}\right)^4\left(4 + \left(\frac{a(\delta)}{b(\delta)}\right)^3\right)d\delta\\
&+\frac{1}{6}\int_{\beta_1}^{\beta_2} a(\delta)\left(1 - \frac{b(\delta)}{a(\delta)}\right)^3 d\delta + \frac{1}{2}\int_{\beta_1}^{\beta_2} a(\delta)\left(1 - \frac{b(\delta)}{a(\delta)}\right)^2 d\delta.
\end{aligned} \tag{4.24}$$

Proof. Consider the function $\varrho(\delta) = \delta\log\delta,\ \ \delta > 0$ is 6-convex. Then, clearly $\varrho''''''(\delta) = 24\delta^{-5}\ \forall\ \delta > 0$, which confirms that ϱ is convex as well as 6-convex. Therefore, using (4.18) by taking $\varrho(\delta) = \delta\log\delta$, we get (4.24). □

Corollary 4.27. *Presume that the postulates of Corollary* 4.26 *are true, then*

$$\begin{aligned}
D_{kl}(a,b) \geq & \frac{1}{60}\int_{\beta_1}^{\beta_2} a(\delta)\left(1 - \frac{b(\delta)}{a(\delta)}\right)^4\left(4 + \frac{b(\delta)}{a(\delta)}\right)^{-3} d\delta\\
&+\frac{1}{6}\int_{\beta_1}^{\beta_2} a(\delta)\left(1 - \frac{b(\delta)}{a(\delta)}\right)^3 d\delta + \frac{1}{2}\int_{\beta_1}^{\beta_2} a(\delta)\left(1 - \frac{b(\delta)}{a(\delta)}\right)^2 d\delta.
\end{aligned} \tag{4.25}$$

Proof. Applying inequality (4.19) for $\varrho(\delta) = \delta\log\delta,\ \ \delta > 0$, we obtain (4.25). □

The Bhattacharyya coefficient can be defined as follows:

Definition 4.28. *(Bhattacharyya coefficient) Let a and b be any positive probability density functions. Then*

$$B_c(a,b) = \int_{\beta_1}^{\beta_2} \sqrt{a(\delta)b(\delta)}d\delta.$$

Corollary 4.29. *Assume that a and b be arbitrary probability density functions with $a(\delta), b(\delta) > 0$. Then*

$$\begin{aligned}1 - B_c(a,b) \leq & \frac{15}{480}\int_{\beta_1}^{\beta_2} a(\delta)\left(1-\frac{b(\delta)}{a(\delta)}\right)^4\left(1+\frac{1}{4}\left(\frac{a(\delta)}{b(\delta)}\right)^{\frac{7}{2}}\right)d\delta \\ & + \frac{1}{16}\int_{\beta_1}^{\beta_2} a(\delta)\left(1-\frac{b(\delta)}{a(\delta)}\right)^3 d\delta + \frac{1}{8}\int_{\beta_1}^{\beta_2} a(\delta)\left(1-\frac{b(\delta)}{a(\delta)}\right)^2 d\delta.\end{aligned} \tag{4.26}$$

Proof. Let us consider $\varrho(\delta) = -\sqrt{\delta}$, $\delta > 0$. Then certainly, ϱ is both convex and 6-convex over $(0,\infty)$. Therefore, utilizing (4.18) for $\varrho(\delta) = -\sqrt{\delta}$, we get (4.26). □

Corollary 4.30. *Suppose that the assumptions of Corollary* 4.29 *hold. Then*

$$\begin{aligned}1 - B_c(a,b) \geq & \frac{15}{384}\int_{\beta_1}^{\beta_2} a(\delta)\left(1-\frac{b(\delta)}{a(\delta)}\right)^4\left(\frac{4}{5}+\frac{b(\delta)}{5a(\delta)}\right)^{-\frac{7}{2}} d\delta \\ & + \frac{1}{16}\int_{\beta_1}^{\beta_2} a(\delta)\left(1-\frac{b(\delta)}{a(\delta)}\right)^3 d\delta + \frac{1}{8}\int_{\beta_1}^{\beta_2} a(\delta)\left(1-\frac{b(\delta)}{a(\delta)}\right)^2 d\delta.\end{aligned} \tag{4.27}$$

Proof. Apply inequality (4.19) for $\varrho(\delta) = -\sqrt{\delta}$, $\delta > 0$, we deduce (4.27). □

5. Conclusions and future research

The Jensen inequality is a fundamental concept in mathematical analysis that is essential to many areas of applied sciences and mathematics. Its importance stems from offering crucial insights into the behavior of convex functions and serving as a potent tool for estimating expectations and averages in the realms of probability theory and optimization problems. In the article, we introduced new improvements to the Jensen inequality in the context of the Riemann integral. The established improvements provide estimates for the Jensen difference. The significance of the achieved improvements is demonstrated by comparing them with other results and further emphasizing the circumstances under which these improvements become essential. The applications of the key discovery are demonstrated in the context of improvements for the Hermite-Hadamard and Hölder inequalities. Connections between means are achieved emerging from the obtained improvements. In addition, we have provided estimates for the Csiszár divergence, Shannon entropy, Kullback-Libeler divergence, and Bhattacharyya coefficient as direct consequences of the obtained improvements. The Jensen inequality for the integral version and the concept of convexity are the base concepts that have been utilized in the acquirement of the proposed improvements. The ideas and concepts discussed in this acquisition may simulate further research in this direction.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no conflicts of interest regarding the publication of this article, according to the authors.

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