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Research article

Analysis of stochastic delay differential equations in the framework of conformable fractional derivatives

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Abstract: In numerous domains, fractional stochastic delay differential equations are used to model various physical phenomena, and the study of well-posedness ensures that the mathematical models accurately represent physical systems, allowing for meaningful predictions and analysis. A fractional stochastic differential equation is considered well-posed if its solution satisfies the existence, uniqueness, and continuous dependency properties. We established the well-posedness and regularity of solutions of conformable fractional stochastic delay differential equations (CFrSDDEs) of order $\gamma \in (\frac{1}{2}, 1)$ in $\mathbb{L}^{\mathrm{p}}$ spaces with $\mathrm{p} \geq 2$, whose coefficients satisfied a standard Lipschitz condition. More specifically, we first demonstrated the existence and uniqueness of solutions; after that, we demonstrated the continuous dependency of solutions on both the initial values and fractional exponent γ. The second section was devoted to examining the regularity of time. As a result, we found that, for each $\Phi \in (0, \gamma - \frac{1}{2})$, the solution to the considered problem has a $\Phi-$Hölder continuous version. Lastly, two examples that highlighted our findings were provided. The two main elements of the proof were the Burkholder-Davis-Gundy inequality and the weighted norm.

Keywords: conformable fractional stochastic delay differential equations; well-posedness; regularity
Mathematics Subject Classification: 34A07, 34A08, 60G22

1. Introduction

The intriguing field of fractional calculus (FrC) expands the traditional concepts of differentiation and integration to non-integer orders in mathematics. Complex physical system behavior can be best described by the use of FrC. We examine a few fascinating uses for FrC [1–4]:

- Control theory makes use of FrC to construct and evaluate complicated systems with memory effects. FrC is useful in understanding the behavior and stability of electrical circuits.
- Fractional-order multipoles in electromagnetics improve our comprehension of intricate electromagnetic processes. The description of electrochemical processes relies heavily on fractional differential equations. They depict the nonlocal characteristics of diffusion and charge transport in porous media.
- FrC aids fluid mechanics in the modeling of flow in non-Newtonian fluids and porous materials.
- Understanding population dynamics, such as the spread of illnesses or ecological interactions, is aided by FrC.
- Fractional-order optical systems are employed in optics for imaging and aberration correction.
- Furthermore, FrC sheds light on neural networks and the actions of individual brain neurons.
- A useful tool for simulating viscoelastic systems, such as polymers and biological tissues, is FrC.

A large number of natural phenomena are nonlocal, which means that recent and distant past events have an impact on them now. These nonlocal relationships are expressed more precisely by fractional operators than by traditional integer-order. Different mathematical definitions and attributes are used for different sorts of fractional-order derivatives (FrOD). The kind of system being represented, the particular problem at hand, and the required mathematical qualities of the derivative all influence which fractional derivative should be used.

Compared to integer-order calculus, the FrOD is a better way to represent many real-world occurrences. Several definitions of FrOD exist, including Riemann-Liouville, Caputo-Fabrizio, Caputo-Grunuwald Letnikov, Atangana-Baleanu, and conformable. There are various definitions for FrOD, in contrast to integer-order derivatives [5–8]. In general, these definitions are not identical.

The conformable fractional derivative (CFrD) was established by Khalil et al. [9]. For $\eta(\mathrm{t}) : [0, \infty[\to \mathbb{R}$, the CFrD of order γ is specified via:

$$\mathcal{T}_{\mathrm{t}}^{\gamma}\eta(\mathrm{t}) = \lim_{\epsilon\to 0}\frac{\eta^{\lceil\gamma\rceil-1}(\mathrm{t}+\epsilon \mathrm{t}^{\lceil\gamma\rceil-\gamma}) - \eta^{\lceil\gamma\rceil-1}(\mathrm{t})}{\epsilon},$$

$\mu - 1 < \gamma \leq \mu,\ \mathrm{t} > 0, \mu \in \mathcal{N}$, and $\lceil\gamma\rceil$, with the smallest number greater than or equal to γ. Given a certain scenario, when $0 < \gamma \leq 1$, we get

$$\mathcal{T}_{\mathrm{t}}^{\gamma}\eta(\mathrm{t}) = \lim_{\epsilon\to 0}\frac{\eta(\mathrm{t}+\epsilon \mathrm{t}^{1-\gamma}) - \eta(\mathrm{t})}{\mathrm{t}},\ \mathrm{t} > 0.$$

When $\eta(\mathrm{t})$ is γ-differentiable in $(0, \mathbb{G})$, $\mathbb{G} > 0$, and $\lim_{\mathrm{t}\to 0^+}\eta^{\gamma}(\mathrm{t})$ exists, then $\eta^{(\gamma)}(0) = \lim_{\mathrm{t}\to 0^+}\eta^{(\gamma)}(\mathrm{t})$.

Formulated below, the conformable fractional integral of $\eta(\mathrm{t})$ beginning at $\tilde{\alpha} \geq 0$ is:

$$\mathcal{I}_{\gamma}^{\tilde{\alpha}}(\eta)(\mathrm{t}) = \int_{\tilde{\alpha}}^{\mathrm{t}}\frac{\eta(s)}{(s-\tilde{\alpha})^{1-\gamma}}ds,\ \gamma \in (0,\ 1].$$

Taylor series, Gronwall's inequality, exponential functions, Leibniz rule, chain rule, physical interpretation, and integration by parts were among the subjects discussed in [10–14]. Ma and colleagues [15] showed that the conformable derivative is acceptable and operates well in a gray system model. In addition, a lot of study has been done on Sturm's theorems, Ulam's stability, and the variational iteration method. Recent research has been conducted by the authors of [16–22] on conformable Itô stochastic differential equations, existence results for solutions, Lyapunov stability, virtually definitely exponential stability, and Ulam-type stability.

The 1940s saw the development of the mathematical theory of stochastic differential equations (SDEs), largely due to the groundbreaking research of Japanese mathematician Kiyosi Itô. Itô started the study of SDEs and proposed the idea of the stochastic integral. These formulas are essential for simulating a wide range of phenomena in disciplines including engineering, physics, and finance.

The far more established and ancient subjects of ordinary and partial differential equations are connected to the probability theory through SDEs. SDEs introduce unpredictability into the differential equations, allowing us to define systems affected by random fluctuations. SDEs contain stochastic terms that represent the inherent uncertainty in real-world processes, in contrast to classical differential equations, where the coefficients are fixed. Because of this, they are especially helpful for simulating phenomena that are affected by random variables. SDEs offer a potent framework for comprehending uncertainty and randomness in a wide range of natural and artificial phenomena [23–26].

- Stock price models are frequently created using SDEs. SDEs are used in the well-known Black-Scholes model for option pricing. SDEs are essential for modeling bond prices and interest rates. SDEs aid in the analysis of the best investment plans. Financial risk assessment is aided by SDEs.
- SDEs have their roots in the analysis of Brownian motion. They explain the haphazard motion of suspended particles in a liquid. SDEs simulate thermally fluctuating physical systems. Schrödinger's equation, which describes the time evolution of quantum wave functions, and SDEs are similar. Diffusion processes and chemical reactions can be represented using SDEs.
- SDEs aid in the understanding of ecological systems, species interactions, and population dynamics. The spread of infectious diseases can be modeled by using SDEs.
- Communication channel noise is modeled using SDEs. SDEs are useful for the analysis of wireless networks.
- The stochastic behavior of neurons is described by SDEs. Gene expression and regulatory interactions are modeled by SDEs.
- SDEs can be used to forecast extreme events and research climate variability.

Fractional stochastic delay differential equations (FrSDDEs) are a fascinating area of research that combines FrC, delay concepts, and stochastic processes. They simulate systems in which random fluctuations operate on the underlying dynamics and lead to fractional behavior. Financial time series, the spread of infectious illnesses, and other topics are modeled in applications.

In the context of FrSDDEs, the notion of well-posedness is essential to comprehending their behavior. FrSDDEs are used in physics and engineering to model a wide range of physical processes. The study of well-posedness guarantees that the mathematical models faithfully capture the physical systems under study, enabling significant analysis and prediction. If the existence, uniqueness, and continuous dependency conditions are satisfied by the solution to an FrSDDE, the equation is said to be well-posed.

The process of studying the existence of the solutions of various models tells us that, under appropriate conditions, solutions to differential equations can be found. Uniqueness is another important aspect of well-being. It ensures that, under certain conditions, the solution to the fractional stochastic differential equation (FSDE) is unique. The regularity property ensures that solution behavior changes continuously with variations in the initial conditions and fractional order value.

In this study, we proved the existence and uniqueness (EU) and continuous dependency (Con-D) of the solutions of the CFrSDDE on the fractional exponent γ and on the initial values, in addition to the regularity of the solutions. Specifically, we have demonstrated in this study the EU and Con-D of the solutions of CFrSDDEs in $\mathbb{L}^{\mathrm{p}}$ spaces via multiplicative noise drives combined with the generalized Lipschitz-type coefficients. Second, the Hölder continuity of the solutions to CFrSDDEs is of relevance in $\mathbb{L}^{\mathrm{p}}$ spaces. The three main elements used in the results demonstration are the Hölder inequality, the Burkholder-Davis-Gundy inequality, and the temporally weighted norm.

The CFrSDDE of order $\frac{1}{2} < \gamma < 1$ was analyzed in this study, which is a generalization of the traditional SDE driven by Brownian motions.

$$\begin{cases} \mathcal{T}_{\mathrm{t}}^{\gamma}\mathbb{U}(\mathrm{t}) = \Theta(\mathrm{t}, \mathbb{U}(\mathrm{t}), \mathbb{U}(\mathrm{t}-\upsilon)) + \Xi(\mathrm{t}, \mathbb{U}(\mathrm{t}), \mathbb{U}(\mathrm{t}-\upsilon))\frac{d\mathcal{W}_{\mathrm{t}}}{dt}, \\ \mathbb{U} = \lambda(\mathrm{t}), \end{cases} \tag{1.1}$$

where $\Theta : [0,\ \mathbb{G}] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $\Xi : [0,\ \mathbb{G}] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ are measured and based on a whole filtered probability space $(\mho, \widetilde{\mathbb{F}}_{\mathrm{t}}, \mathbb{P})$, and $(\mathcal{W}_{\mathrm{t}})_{\mathrm{t}\in[0,\ \infty)}$ is Brownian motion with $\{\widetilde{\mathcal{F}_{\mathrm{t}}}\}_{t\geq 0}$. The m-dimensional Euclidean space with norm $|.|$ can be described by $\mathbb{R}^m$, and the family of bounded continuous $\mathbb{R}^m$-valued functions Δ on $[-\upsilon, 0]$ with norm $\|\Delta\| = \sup_{\mathrm{t}\in[-\upsilon,0]} |\Delta|$ is indicated by $C([-\upsilon, 0]; \mathbb{R}^m)$. For every $\mathrm{t} \in [-\upsilon, 0]$, the history function is represented by $\lambda(\mathrm{t})$ and the delay period is denoted by the positive constant $\upsilon \in \mathbb{R}$.

The study follows this format: In the next section, we utilize certain significant concepts and theorems from the theory to provide an adequate foundation for the results we make about CFrSDDEs. We first demonstrate the well-posedness of the CFrSDDEs solution in the first subsection of Section 3, then we will demonstrate the regularity in the second section. In Section 4, examples are provided to emphasize our findings. In the last section, there are some final comments.

2. Preliminaries

Several fundamental ideas, definitions, and theorems that are relevant to this article are covered in this section.

Definition 2.1. *For* $\mathrm{p} \geq 2$, $\mathrm{t} \in [0, \infty)$, *assume* $\widetilde{\mathcal{M}}_{\mathrm{t}}^{\mathrm{p}} = \mathbb{L}^{\mathrm{p}}(\mho, \widetilde{\mathbb{F}}_{\mathrm{t}}, \mathbb{P})$ *denotes whole* $\widetilde{\mathbb{F}}_{\mathrm{t}}$*-measurable* p^{th} *functions that can be integrated by* $\aleph = (\aleph_1, \aleph_2, \cdots, \aleph_m)^T : \mho \to \mathbb{R}^m$ *with*

$$\|\aleph\|_{\mathrm{p}} = \left(\sum\nolimits_{\iota=1}^{m} \mathrm{E}(|\aleph_\iota|^{\mathrm{p}})\right)^{\frac{1}{\mathrm{p}}}. \tag{2.1}$$

Consider that the following $\mathbb{U}(\mathrm{t})$ is the solution of Eq (1.1) with the initial condition (In.C) $\mathbb{U}(0) = \lambda$ if $\mathbb{U}(0) = \lambda$: $\widetilde{\mathcal{M}}_{\mathrm{t}}^{\mathrm{p}}$ for $\mathrm{t} \in [0, \mathbb{G}]$:

$$\mathbb{U}(\mathrm{t}) = \lambda + \int_0^{\mathrm{t}} s^{\gamma-1}\Theta(s, \mathbb{U}(s), \mathbb{U}(s-\upsilon))ds + \int_0^{\mathrm{t}} s^{\gamma-1}\Xi(s, \mathbb{U}(s), \mathbb{U}(s-\upsilon))d\mathcal{W}_{\mathrm{t}}. \tag{2.2}$$

Definition 2.2. *We assume that the coefficients* Θ *and* Ξ *in Eq* (1.1) *satisfy the following for the intent of this work:*

(i) $(\widetilde{\mathbb{H}}_1)$ *For* $\forall \mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Y}_1, \mathbb{Y}_2 \in \mathbb{R}^m$*, there is* $\mathbb{L}$ *such that*

$$\|\Xi(\mathrm{t}, \mathbb{Q}_1, \mathbb{Q}_2) - \Xi(\mathrm{t}, \mathbb{Y}_1, \mathbb{Y}_2)\|_{\mathrm{p}} \leq \mathbb{L}(\|\mathbb{Q}_1 - \mathbb{Y}_1\|_{\mathrm{p}} + \|\mathbb{Q}_2 - \mathbb{Y}_2\|_{\mathrm{p}}).$$

$$\|\Theta(\mathrm{t}, \mathbb{Q}_1, \mathbb{Q}_2) - \Theta(\mathrm{t}, \mathbb{Y}_1, \mathbb{Y}_2)\|_{\mathrm{p}} \leq \mathbb{L}(\|\mathbb{Q}_1 - \mathbb{Y}_1\|_{\mathrm{p}} + \|\mathbb{Q}_2 - \mathbb{Y}_2\|_{\mathrm{p}}).$$

(ii) $(\widetilde{\mathbb{H}}_2)$ *The* $\Theta(\mathrm{t}, 0, 0)$ *and the* $\Xi(\mathrm{t}, 0, 0)$ *are generally restricted by time, i.e.,*

$$\underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup}\|\Theta(\mathrm{t}, 0, 0)\|_{\mathrm{p}} < \mathcal{Z}, \; \underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup}\|\Xi(\mathrm{t}, 0, 0)\|_{\mathrm{p}} < \mathcal{Z}. \tag{2.3}$$

Note that neither $(\widetilde{\mathbb{H}}_1)$ *nor* $(\widetilde{\mathbb{H}}_1)$ *is dependent on the norm chosen on* $\mathbb{R}^m$*. However, for simplicity in our next estimations, we give* $\mathbb{R}^m$ *with the* p *norm: For every vector* $\mathbb{Q} = (\mathbb{Q}_1, \mathbb{Q}_1, \mathbb{Q}_1, \cdots, \mathbb{Q}_m)^T \in \mathbb{R}^m$, $\|\mathbb{Q}\|_{\mathrm{p}} = \left(\sum_{\iota=1}^{m} |\mathbb{Q}_\iota|^{\mathrm{p}}\right)^{\frac{1}{\mathrm{p}}}$*, give* $\|\mathbb{Q}\|_{\mathrm{p}}$ *of* $\mathbb{Q}$.

Theorem 2.3. *Suppose that* $(\widetilde{\mathbb{H}}_1)$ *and* $(\widetilde{\mathbb{H}}_2)$ *are fulfilled, then a constant* $\mathcal{J} > 0$ *that relies on* $\gamma, \mathbb{L}, \mathcal{J}, \mathcal{Z}, \mathbb{G}$ *is there; therefore,*

$$\|\mathbb{V}_\gamma(\lambda, \mathrm{t}) - \mathbb{V}_\gamma(\lambda, \mathrm{f})\|_{\mathrm{p}} \leq \mathcal{J}|\mathrm{t} - \mathrm{f}|^{\gamma - \frac{1}{2}}, \; \forall \mathrm{t}, \mathrm{f} \in [0, \mathbb{G}]. \tag{2.4}$$

Corollary 2.4. *For* $\Phi \in (0, \gamma - \frac{1}{2})$*, a modification* $\mathcal{V}$ *of* $\mathcal{X}$ *when* $\Phi-$*Hölder continuous paths, i.e.,*

$$\mathbb{P}(\mathbb{U}_{\mathrm{t}} = \mathcal{V}_{\mathrm{t}}) = 1, \; \forall \mathrm{t} \in [0, \mathbb{G}]. \tag{2.5}$$

Proof. *From Eq* (2.4) *and Kolmogorov test [27],* $\mathcal{X}(\mathrm{t})$ *has* Φ*-Hölder continuous modification* $\forall \Phi \in (0, \gamma - \frac{1}{2})$. □

3. The main results

We demonstrate in this section that the CFrSDDE solutions are well-posed and regular.

3.1. The well-posedness of solutions to CFrSDDEs

To accomplish this, we need to show the EU and Con-D of the solutions on γ and the initial data to verify the well-posedness of solutions.

Assume that $\widetilde{\mathcal{H}}^{\mathrm{p}}(0, \mathbb{G})$ is made up of complete, measurable processes $\mathbb{U}(\mathrm{t})$, $\widetilde{\mathcal{F}}_{\mathbb{G}}-$adapted, with $\widetilde{\mathcal{F}}_{\mathbb{G}} = (\widetilde{\mathbb{F}}_{\mathrm{t}})_{\mathrm{t}\in[0,\mathbb{G}]}$, and satisfies the following:

$$\|\mathbb{U}(\mathrm{t})\|_{\widetilde{\mathcal{H}}^{\mathrm{p}}} = \underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup}\|\mathbb{U}(\mathrm{t})\|_{\mathrm{p}} < \infty. \tag{3.1}$$

$(\widetilde{\mathcal{H}}^{\mathrm{p}}(0, \mathbb{G}), \|.\|_{\widetilde{\mathcal{H}}^{\mathrm{p}}})$; undoubtedly, $(\widetilde{\mathcal{H}}^{\mathrm{p}}(0, \mathbb{G}), \|.\|_{\widetilde{\mathcal{H}}^{\mathrm{p}}})$ is a Banach space. We build an operator $F_\lambda : \widetilde{\mathcal{H}}^{\mathrm{p}}(0, \mathbb{G}) \to \widetilde{\mathcal{H}}^{\mathrm{p}}(0, \mathbb{G})$ by $F_\lambda(\mathbb{M}(0)) = \lambda$ for any $\lambda \in \widetilde{\mathcal{M}}_0^{\mathrm{p}}$, and for $\mathrm{t} \in [0, \mathbb{G}]$, the equality that follows is true.

$$F_\lambda(\mathbb{M}(\mathrm{t})) = \lambda + \int_0^{\mathrm{t}} s^{\gamma-1}\Theta(s, \mathbb{M}(s), \mathbb{M}(s - \upsilon))ds + \int_0^{\mathrm{t}} s^{\gamma-1}\Xi(s, \mathbb{M}(s), \mathbb{M}(s - \upsilon))d\mathcal{W}_s. \tag{3.2}$$

The following lemma illustrates this operator's well-defined property. The aforementioned finding, along with several others that follow, is proved using the elementary inequality below.

$$\left\|\mathbb{U}_1+\mathbb{U}_2\right\|_{\mathrm{p}}^{\mathrm{p}}+\left\|\mathbb{U}_1-\mathbb{U}_2\right\|_{\mathrm{p}}^{\mathrm{p}} \leq 2^{\mathrm{p}-1}(\|\mathbb{U}_1\|_{\mathrm{p}}^{\mathrm{p}}+(\|\mathbb{U}_2\|_{\mathrm{p}}^{\mathrm{p}}). \tag{3.3}$$

Lemma 3.1. *Let us suppose that $\widetilde{\mathbb{H}}_1$ and $\widetilde{\mathbb{H}}_2$ are satisfied. We then obtain a well-defined operator F_λ with the value $\lambda \in \widetilde{\mathcal{M}}_0^p$.*
Proof. *Suppose $\mathbb{M}(\mathrm{t}) \in \widetilde{\mathcal{H}}^{\mathrm{p}}[0,\mathbb{G}]$, and here $\mathbb{M}(\mathrm{t})$ is arbitrary. We possess the following $\forall \mathrm{t} \in [0,\mathbb{G}]$ by utilizing of $F_\lambda(\mathbb{M}(\mathrm{t}))$ as in Eqs (3.2) and (3.3).*

$$\begin{aligned}\left\|F_\lambda(\mathbb{M}(\mathrm{t}))\right\|_{\mathrm{p}}^{\mathrm{p}} \leq & 2^{\mathrm{p}-1}\left\|\lambda\right\|_{\mathrm{p}}^{\mathrm{p}}+2^{2\mathrm{p}-2}\left\|\int_0^{\mathrm{t}} s^{\gamma-1}\Theta(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))ds\right\|_{\mathrm{p}}^{\mathrm{p}} \\ &+2^{2\mathrm{p}-2}\left\|\int_0^{\mathrm{t}} s^{\gamma-1}\Xi(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))d\mathcal{W}_{\mathrm{t}}\right\|_{\mathrm{p}}^{\mathrm{p}}.\end{aligned} \tag{3.4}$$

The Hölder inequality gives us the result that

$$\begin{aligned}\left\|\int_0^{\mathrm{t}} s^{\gamma-1}\Theta(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))ds\right\|_{\mathrm{p}}^{\mathrm{p}} &\leq \sum_{\iota=1}^{m} \mathrm{E}\Big(\int_0^{\mathrm{t}} s^{\gamma-1}\left|\Theta_\iota(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))\right|ds\Big)^{\mathrm{p}} \\ &\leq \sum_{\iota=1}^{m} \mathrm{E}\bigg(\Big(\int_0^{\mathrm{t}} s^{\frac{(\gamma-1)\mathrm{p}}{(\mathrm{p}-1)}}ds\Big)^{\mathrm{p}-1}\int_0^t\left|\Theta_\iota(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))\right|^{\mathrm{p}}ds\bigg) \\ &\leq \frac{\mathbb{G}^{(\mathrm{p}\gamma-1)}(\mathrm{p}-1)^{(\mathrm{p}-1)}}{(\mathrm{p}\gamma-1)^{(\mathrm{p}-1)}}\int_0^t\left\|\Theta(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))\right\|_{\mathrm{p}}^{\mathrm{p}}ds.\end{aligned} \tag{3.5}$$

According to ($\widetilde{\mathbb{H}}_1$), we acquire

$$\begin{aligned}\left\|\Theta(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))\right\|_{\mathrm{p}}^{\mathrm{p}} \leq & 2^{\mathrm{p}-1}\Big(\left\|\Theta(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))+\Theta(s,0,0)\right\|_{\mathrm{p}}^{\mathrm{p}}-\left\|\Theta(s,0,0)\right\|_{\mathrm{p}}^{\mathrm{p}}\Big) \\ \leq & 2^{\mathrm{p}-1}\Big(\mathbb{L}^{\mathrm{p}}\Big(\left\|\mathbb{M}(s)\right\|_{\mathrm{p}}^{\mathrm{p}}+\left\|\mathbb{M}(s-\upsilon)\right\|_{\mathrm{p}}^{\mathrm{p}}\Big)+\left\|\Theta(s,0,0)\right\|_{\mathrm{p}}^{\mathrm{p}}\Big).\end{aligned} \tag{3.6}$$

Therefore,

$$\begin{aligned}\int_o^{\mathrm{t}}\left\|\Theta(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))\right\|_{\mathrm{p}}^{\mathrm{p}}ds \leq & 2^{\mathrm{p}-1}\mathbb{L}^{\mathrm{p}}\bigg(\Big(\underset{s\in[0,\mathbb{G}]}{esssup}\|\mathbb{M}(s)\|_{\mathrm{p}}\Big)^{\mathrm{p}}+\Big(\underset{s\in[0,\mathbb{G}]}{esssup}\left\|\mathbb{M}(\mathrm{s}-\upsilon)\right\|_{\mathrm{p}}\Big)^{\mathrm{p}}\bigg) \\ & \int_0^{\mathrm{t}} 1ds+2^{\mathrm{p}-1}\int_0^{\mathrm{t}}\left\|\Theta(s,0,0)\right\|_{\mathrm{p}}^{\mathrm{p}}ds \\ \leq & 2^{\mathrm{p}-1}\mathbb{L}^{\mathrm{p}}\mathbb{G}\Big(\left\|\mathbb{M}(s)\right\|_{\widetilde{\mathcal{H}}^{\mathrm{p}}}^{\mathrm{p}}+\left\|\mathbb{M}(\mathrm{s}-\upsilon)\right\|_{\widetilde{\mathcal{H}}^{\mathrm{p}}}^{\mathrm{p}}\Big)2^{\mathrm{p}-1}\int_0^{\mathrm{t}}\left\|\Theta(s,0,0)\right\|_{\mathrm{p}}^{\mathrm{p}}ds.\end{aligned} \tag{3.7}$$

By Eqs (3.5) and (3.7) , we get

$$\begin{aligned}\left\|\int_0^{\mathrm{t}} s^{\gamma-1}\Theta(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))ds\right\|_{\mathrm{p}}^{\mathrm{p}} \leq & \frac{\mathbb{G}^{(\mathrm{p}\gamma-1)}(2\mathrm{p}-2)^{(\mathrm{p}-2)}}{(\mathrm{p}\gamma-1)^{(\mathrm{p}-1)}}\Big(\mathbb{L}^{\mathrm{p}}\mathbb{G}\Big(\left\|\mathbb{M}(s)\right\|_{\widetilde{\mathcal{H}}^{\mathrm{p}}}^{\mathrm{p}} \\ &+\left\|\mathbb{M}(s-\upsilon)\right\|_{\widetilde{\mathcal{H}}^{\mathrm{p}}}^{\mathrm{p}}\Big)+\int_0^{\mathrm{t}}\left\|\Theta(s,0,0)\right\|_{\mathrm{p}}^{\mathrm{p}}ds\Big).\end{aligned} \tag{3.8}$$

Now, applying the Burkholder-Davis-Gundy inequality and Hölder inequality, we get

$$
\begin{aligned}
\Big\| \int_0^{\mathrm{t}} s^{\gamma-1}\Xi(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))d\mathcal{W}_{\mathrm{t}}\Big\|_{\mathrm{p}}^{\mathrm{p}} &\le \sum_{\iota=1}^{m} \mathrm{E}\Big| \int_0^{\mathrm{t}} s^{\gamma-1}(\Xi_\iota(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))d\mathcal{W}_s\Big|^{\mathrm{p}} \\
&\le \sum_{\iota=1}^{m} C_{\mathrm{p}}\mathrm{E}\Big|\int_0^{\mathrm{t}} s^{2\gamma-2}\Big|\Xi_\iota(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))\Big|^2 ds\Big|^{\frac{\mathrm{p}}{2}} \\
&\le \sum_{\iota=1}^{m} C_{\mathrm{p}}\mathrm{E}\int_0^{\mathrm{t}} s^{2\gamma-2}\Big|\Xi_\iota(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))\Big|^{\mathrm{p}} ds \\
&\quad \Big(\int_0^{\mathrm{t}} s^{2\gamma-2}ds\Big)^{\frac{\mathrm{p}-2}{2}} \\
&\le C_{\mathrm{p}}\Big(\frac{\mathbb{G}^{2\gamma-1}}{2\gamma-1}\Big)^{\frac{\mathrm{p}-2}{2}}\int_0^{\mathrm{t}} s^{2\gamma-2}\Big\|\Xi(s,\mathbb{M}(s,\mathbb{M}(s-\upsilon))\Big\|_{\mathrm{p}}^{\mathrm{p}} ds, \qquad (3.9)
\end{aligned}
$$

where $C_{\mathrm{p}} = \left(\frac{\mathrm{p}^{\mathrm{p}+1}}{2(\mathrm{p}-1)^{\mathrm{p}-1}}\right)^{\frac{\mathrm{p}}{2}}$. *We have from* $(\widetilde{\mathbb{H}}_1)$ *and* $(\widetilde{\mathbb{H}}_2)$ *that*

$$
\begin{aligned}
\Big\|\Xi(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))\Big\|_{\mathrm{p}}^{\mathrm{p}} &\le 2^{\mathrm{p}-1}\mathbb{L}^{\mathrm{p}}\Big(\Big\|\mathbb{M}(s)\Big\|_{\mathrm{p}}^{\mathrm{p}} + \Big\|\mathbb{M}(s-\upsilon)\Big\|_{\mathrm{p}}^{\mathrm{p}}\Big) + 2^{\mathrm{p}-1}\Big\|\Xi(s,0,0)\Big\|_{\mathrm{p}}^{\mathrm{p}} \\
&\le 2^{\mathrm{p}-1}\mathbb{L}^{\mathrm{p}}\Big(\Big\|\mathbb{M}(s)\Big\|_{\mathrm{p}}^{\mathrm{p}} + \Big\|\mathbb{M}(s-\upsilon)\Big\|_{\mathrm{p}}^{\mathrm{p}}\Big) + 2^{\mathrm{p}-1}\mathcal{Z}^{\mathrm{p}}. \qquad (3.10)
\end{aligned}
$$

Thus, $\forall \mathrm{t} \in [0,\mathbb{G}]$. *We get the following:*

$$
\begin{aligned}
\int_0^{\mathrm{t}} s^{2\gamma-2}\Big\|\Xi(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon))\Big\|_{\mathrm{p}}^{\mathrm{p}} ds &\le 2^{\mathrm{p}-1}\mathbb{L}^{\mathrm{p}}\int_0^{\mathrm{t}} s^{2\gamma-2}\Big(\Big(\underset{s\in[0,\mathbb{G}]}{esssup}\Big\|\mathbb{M}(s)_{\mathrm{p}}\Big\|\Big)^{\mathrm{p}} \\
&\quad + \Big(\underset{s\in[0,\mathbb{G}]}{esssup}\Big\|\mathbb{M}(s-\upsilon)_{\mathrm{p}}\Big\|\Big)^{\mathrm{p}}\Big)ds + 2^{\mathrm{p}-1}\mathcal{Z}^{\mathrm{p}}\int_0^{\mathrm{t}} s^{2\gamma-2}ds \\
&\le \frac{2^{\mathrm{p}-1}\mathbb{G}^{2\gamma-1}}{2\gamma-1}\Big(\mathbb{L}^{\mathrm{p}}\Big(||\mathbb{M}(s)||_{\widetilde{\mathcal{H}}_{\mathrm{p}}}^{\mathrm{p}} + ||\mathbb{M}(s-\upsilon)||_{\widetilde{\mathcal{H}}_{\mathrm{p}}}^{\mathrm{p}}\Big) + \mathcal{Z}^{\mathrm{p}}\Big). \qquad (3.11)
\end{aligned}
$$

With Eqs (3.4) *and* (3.8) *and* $(\widetilde{\mathbb{H}}_2)$ *, we obtain that* $||F(\mathbb{M}(\mathrm{t}))||_{\widetilde{\mathcal{H}}_{\mathrm{p}}} < \infty$. *As a consequence, the map* F_λ *is well-defined.*

To prove EU, we have to prove the following lemma:

Lemma 3.2. *When* $\gamma > \frac{1}{2}$ *and* $\mathrm{t} > 0$, *the subsequent satisfies:*

$$\frac{\hbar}{\Gamma(2\gamma-1)}\int_0^{\mathrm{t}} s^{2\gamma-2}\mathbb{E}_{2\gamma-1}(\hbar s^{2\gamma-1})ds \le \mathbb{E}_{2\gamma-1}(\hbar \mathrm{t}^{2\gamma-1}), \qquad (3.12)$$

where $\mathbb{E}_{2\gamma-1}(.)$ *is a Mittag-Leffler function (MLF), which is defined as*

$$\mathbb{E}_{(2\gamma-1)}(\mathrm{t}) = \sum_{\iota=0}^{\infty}\frac{\mathrm{t}^\iota}{\Gamma((2\gamma-1)\iota+1)}. \qquad (3.13)$$

Proof. *Assume that $\hbar > 0$ is random. First, we substitute integral and sum, and then we apply the procedure that follows identity.*

$$\int_0^{\mathrm{t}} s^{2\gamma-2} s^{\iota(2\gamma-1)} ds = \mathrm{t}^{(\iota+1)(2\gamma-1)} \mathbb{B}(2\gamma-1, \iota(2\gamma-1)+1),\ \iota = 0,\ 1,\ 2, \cdots .$$

So, we get

$$\begin{aligned}\frac{\hbar}{\Gamma(2\gamma-1)} \int_0^{\mathrm{t}} s^{2\gamma-2} \mathbb{E}_{2\gamma-1}(\hbar s^{2\gamma-1}) ds =& \hbar \sum\nolimits_{\iota=0}^{\infty} \frac{\hbar^{\iota}}{\Gamma(\iota(2\gamma-1)+1)} \int_0^{\mathrm{t}} s^{2\gamma-2} s^{\iota(2\gamma-1)} ds \\ =& \sum\nolimits_{\iota=0}^{\infty} \frac{\hbar^{\iota+1} \mathrm{t}^{(\iota+1)(2\gamma-1)}}{\Gamma(2\gamma-1)\Gamma(\iota(2\gamma-1)+1)} \\ =& \sum\nolimits_{\iota=1}^{\infty} \frac{\hbar^{\iota} \mathrm{t}^{\iota(2\gamma-1)}}{\Gamma(\iota(2\gamma-1)+1)} \\ =& \mathbb{E}_{2\gamma-1}(\hbar \mathrm{t}^{2\gamma-1}) - 1 \\ \leq& \mathbb{E}_{2\gamma-1}(\hbar \mathrm{t}^{2\gamma-1}).\end{aligned}$$

In this case, a beta function is $\mathbb{B}$. Consequently, it finishes the proof.

To prove the EU of solutions, we will show that the operator F_λ is contractive under a suitably weighted norm ([28], Remark 2.1). The MLF $\mathbb{E}_{(2\gamma-1)}(\mathrm{t})$, as found in Eq (3.13), serves as the weight function in this instance.

Theorem 3.3. *If* $(\widetilde{\mathbb{H}}_1)$ *and* $(\widetilde{\mathbb{H}}_2)$ *are fulfilled, then the Eq* (1.1)*, when* $\mathbb{U}(0) = \lambda$*, posses a unique solution when* $[0, \mathbb{G}]$ *for any* $\lambda \in \widetilde{\mathcal{M}}_0^{\mathrm{p}}$.
Proof. We take $\hbar > 0$ *as follows:*

$$\hbar > \sigma 2^{\mathrm{p}-1} \Gamma(2\gamma-1), \tag{3.14}$$

where

$$\sigma = 2^{\mathrm{p}-1} \mathbb{L}^{\mathrm{p}} \Bigg((\mathbb{G}^{(\mathrm{p}-2)\gamma+1}) \frac{1}{(\frac{(\mathrm{p}-2)\gamma+1}{\mathrm{p}-1})^{\mathrm{p}-1}} + \Big(\frac{\mathbb{G}^{2\gamma-1}}{2\gamma-1} \Big)^{\frac{\mathrm{p}-2}{2}} \Big(\frac{\mathrm{p}^{\mathrm{p}+1}}{2(\mathrm{p}-1)^{\mathrm{p}-1}} \Big)^{\frac{\mathrm{p}}{2}} \Bigg). \tag{3.15}$$

Over the space $\widetilde{\mathcal{H}}^{\mathrm{p}}([o, \mathbb{G}])$*, we construct a weighted norm* $\|.\|_\hbar$ *to be*

$$\|\mathbb{U}(\mathrm{t})\|_\hbar = \underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup} \Big(\frac{\|\mathbb{U}(\mathrm{t})\|_{\mathrm{p}}^{\mathrm{p}}}{\mathbb{E}_{2\gamma-1}(\hbar \mathrm{t}^{2\gamma-1})} \Big)^{\frac{1}{\mathrm{p}}},\ \forall \mathbb{U}(\mathrm{t}) \in \widetilde{\mathcal{H}}^{\mathrm{p}}([0, \mathbb{G}]). \tag{3.16}$$

Two norms, $\|.\|_{\widetilde{\mathcal{H}}^{\mathrm{p}}}$ *and* $\|.\|_\hbar$*, are equivalent.* $(\widetilde{\mathcal{H}}^{\mathrm{p}}([0, \mathbb{G}]), \|.\|_\hbar)$ *is a Banach space as a result. Choose and fix* $\lambda \in \widetilde{\mathcal{M}}_0^{\mathrm{p}}$. *By virtue of Lemma 3.1, the operator* F_λ *is well-defined. We are going to now demonstrate the contractivity of the map* F_λ *concerning the norm* $\|.\|_\hbar$*. Let* $\mathbb{M}$ *and* $\widetilde{\mathbb{M}}$ *be arbitrary for this purpose. We acquire the subsequent* $\forall \mathrm{t} \in [0, \mathbb{G}]$ *from Eqs* (3.2) *and* (3.3)*:*

$$\begin{aligned}\|F_\lambda(\mathbb{M}(\mathrm{t})) - F_\lambda(\widetilde{\mathbb{M}}(\mathrm{t}))\|_{\mathrm{p}}^{\mathrm{p}} \leq& 2^{\mathrm{p}-1} \Big\| \int_0^{\mathrm{t}} s^{\gamma-1} \Big(\Theta(s, \mathbb{M}(s), \mathbb{M}(s-\upsilon)) - \Theta(s, \widetilde{\mathbb{M}}(s), \widetilde{\mathbb{M}}(s-\upsilon)) \Big) ds \Big\|_{\mathrm{p}}^{\mathrm{p}} \\ &+ 2^{\mathrm{p}-1} \Big\| \int_0^{\mathrm{t}} s^{\gamma-1} \Big(\Xi(s, \mathbb{M}(s), \mathbb{M}(s-\upsilon)) - \Xi(s, \widetilde{\mathbb{M}}(s), \widetilde{\mathbb{M}}(s-\upsilon)) \Big) d\mathcal{W}_s \Big\|_{\mathrm{p}}^{\mathrm{p}}.\end{aligned} \tag{3.17}$$

Using the Hölder inequality and ($\widetilde{\mathbb{H}}_1$)*, we obtain*

$$\begin{aligned}
&\Big\| \int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Theta(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon)) - \Theta(s,\widetilde{\mathbb{M}}(s),\widetilde{\mathbb{M}}(s-\upsilon))\Big)ds\Big\|_{\mathrm{p}}^{\mathrm{p}} \\
&\leq \sum_{\iota=1}^{m} \mathrm{E}\Big(\int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Theta_\iota(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon)) - \Theta_\iota(s,\widetilde{\mathbb{M}}(s),\widetilde{\mathbb{M}}(s-\upsilon))\Big)ds\Big)^{\mathrm{p}} \\
&\leq \sum_{\iota=1}^{m} \mathrm{E}\Big(\Big(\int_0^{\mathrm{t}} s^{\frac{(\gamma-1)(\mathrm{p}-2)}{\mathrm{p}-1}}ds\Big)^{\mathrm{p}-1}\Big(\int_0^{\mathrm{t}} s^{2\gamma-2}\Big|\Theta_\iota(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon)) - \Theta_\iota(s,\widetilde{\mathbb{M}}(s),\widetilde{\mathbb{M}}(s-\upsilon))\Big|\Big)\Big) \\
&\leq \frac{\mathbb{L}^{\mathrm{p}}\mathbb{G}^{\mathrm{p}\gamma-2\gamma+1}(\mathrm{p}-1)^{\mathrm{p}-1}}{(\mathrm{p}\gamma-2\gamma+1)^{\mathrm{p}-1}}\int_0^{\mathrm{t}} s^{2\gamma-2}\Big(\Big\|\mathbb{M}(s)-\widetilde{\mathbb{M}}(s))\Big\|_{\mathrm{p}}^{\mathrm{p}} + \Big\|\mathbb{M}(s-\upsilon)-\widetilde{\mathbb{M}}(s-\upsilon))\Big\|_{\mathrm{p}}^{\mathrm{p}}\Big)ds. \qquad (3.18)
\end{aligned}$$

However, using ($\widetilde{\mathbb{H}}_1$) *and the Burkholder-Davis-Gundy inequality, we have*

$$\begin{aligned}
&\Big\| \int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Xi(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon)) - \Xi(s,\widetilde{\mathbb{M}}(s),\widetilde{\mathbb{M}}(s-\upsilon))\Big)d\mathcal{W}_s\Big\|_{\mathrm{p}}^{\mathrm{p}} \\
&= \sum_{\iota=1}^{m} \mathrm{E}\Big|\int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Xi_\iota(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon)) - \Xi_\iota(s,\widetilde{\mathbb{M}}(s),\widetilde{\mathbb{M}}(s-\upsilon))\Big)d\mathcal{W}_s\Big|^{\mathrm{p}} \\
&\leq \sum_{\iota=1}^{m} C_{\mathrm{p}}\mathrm{E}\Big|\int_0^{\mathrm{t}} s^{2\gamma-2}\Big|\Xi_\iota(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon)) - \Xi_\iota(s,\widetilde{\mathbb{M}}(s),\widetilde{\mathbb{M}}(s-\upsilon))\Big|^2 ds\Big|^{\frac{\mathrm{p}}{2}} \\
&\leq \sum_{\iota=1}^{m} C_{\mathrm{p}}\mathrm{E}\int_0^{\mathrm{t}} s^{2\gamma-2}\Big|\Xi_\iota(s,\mathbb{M}(s),\mathbb{M}(s-\upsilon)) - \Xi_\iota(s,\widetilde{\mathbb{M}}(s),\widetilde{\mathbb{M}}(s-\upsilon))\Big|^{\mathrm{p}} ds\Big(\int_0^{\mathrm{t}} s^{2\gamma-2}ds\Big)^{\frac{\mathrm{p}-2}{2}} \\
&\leq \Big(\frac{\mathbb{G}^{2\gamma-1}}{2\gamma-1}\Big)^{\frac{\mathrm{p}-2}{2}}\mathbb{L}^{\mathrm{p}}C_{\mathrm{p}}\int_0^{\mathrm{t}} s^{2\gamma-2}\Big(\Big\|\mathbb{M}(s)-\widetilde{\mathbb{M}}(s)\|_{\mathrm{p}}^{\mathrm{p}} + \Big\|\mathbb{M}(s-\upsilon)-\widetilde{\mathbb{M}}(s-\upsilon)\|_{\mathrm{p}}^{\mathrm{p}}\Big)ds. \qquad (3.19)
\end{aligned}$$

Thus, $\forall \mathrm{t} \in [0,\mathbb{G}]$. *We have*

$$\|F_\lambda(\mathbb{M}(\mathrm{t})) - F_\lambda(\widetilde{\mathbb{M}}(\mathrm{t}))\Big\|_{\mathrm{p}}^{\mathrm{p}} \leq \sigma\int_0^{\mathrm{t}}\Big(\Big\|\mathbb{M}(s)-\widetilde{\mathbb{M}}(s)\|_{\mathrm{p}}^{\mathrm{p}} + \Big\|\mathbb{M}(s-\upsilon)-\widetilde{\mathbb{M}}(s-\upsilon)\|_{\mathrm{p}}^{\mathrm{p}}\Big)s^{2\gamma-2}ds, \qquad (3.20)$$

where σ *is specified in Eq* (3.15)*. The result suggests using the definition of* $\|.\|_\hbar$ *from Eq* (3.16)*:*

$$\begin{aligned}
\frac{\|F_\lambda\mathbb{M}(\mathrm{t}) - F_\lambda\widetilde{\mathbb{M}}(\mathrm{t})\|_{\mathrm{p}}^{\mathrm{p}}}{\mathbb{E}_{2\gamma-1}(\hbar\mathrm{t}^{2\gamma-1})} &\leq \frac{\sigma\int_0^{\mathrm{t}} s^{2\gamma-2}\frac{\left(\|\mathbb{M}(s)-\widetilde{\mathbb{M}}(s)\|_{\mathrm{p}}^{\mathrm{p}}+\|\mathbb{M}(s-\upsilon)-\widetilde{\mathbb{M}}(s-\upsilon)\|_{\mathrm{p}}^{\mathrm{p}}\right)}{\mathbb{E}_{2\gamma-1}(\hbar s^{2\gamma-1})}\mathbb{E}_{2\gamma-1}(\hbar s^{2\gamma-1})ds}{\mathbb{E}_{2\gamma-1}(\hbar\mathrm{t}^{2\gamma-1})} \\
&\leq \sigma\left(\underset{s\in[0,\mathbb{G}]}{esssup}\left(\frac{(\|\mathbb{M}(s)-\widetilde{\mathbb{M}}(s)\|_{\mathrm{p}}^{\mathrm{p}} + \|\mathbb{M}(s-\upsilon)-\widetilde{\mathbb{M}}(s-\upsilon)\|_{\mathrm{p}}^{\mathrm{p}})}{\mathbb{E}_{2\gamma-1}(\hbar\mathrm{t}^{2\gamma-1})}\right)^{\frac{1}{\mathrm{p}}}\right)^{\mathrm{p}} \\
&\quad \frac{\int_0^{\mathrm{t}} s^{2\gamma-2}\mathbb{E}_{2\gamma-1}(\hbar s^{2\gamma-1})ds}{\mathbb{E}_{2\gamma-1}(\hbar\mathrm{t}^{2\gamma-1})} \\
&\leq \frac{\sigma\Gamma(2\gamma-1)}{\hbar}\Big(\|\mathbb{M}(s)-\widetilde{\mathbb{M}}(s)\|_{\hbar}^{\mathrm{p}} + \|\mathbb{M}(s-\upsilon)-\widetilde{\mathbb{M}}(s-\upsilon)\|_{\hbar}^{\mathrm{p}}\Big). \qquad (3.21)
\end{aligned}$$

By utilizing Lemma 3.2, we get the required result.

$$\|F_\lambda(\mathbb{M}(\mathrm{t})) - F_\lambda(\widetilde{\mathbb{M}}(\mathrm{t}))\|_\hbar \leq \Big(\frac{\sigma\Gamma(2\gamma-1)}{\hbar}\Big)^{\frac{1}{p}}\Big(\|\mathbb{M}(s)-\widetilde{\mathbb{M}}(s)\|_\hbar + \|\mathbb{M}(s-\upsilon)-\widetilde{\mathbb{M}}(s-\upsilon)\|_\hbar\Big). \qquad (3.22)$$

From Eq (3.14)*, we get* $\frac{\sigma\Gamma(2\gamma-1)}{\hbar} < 1$*, and the operator* F_λ *on* $(\widetilde{\mathcal{H}}^p([0,\mathbb{G}]), \|.\|_\hbar)$ *is a contractive map. There is a single fixed point of this map in* $\widetilde{\mathcal{H}}^p([0,\mathbb{G}])$*, according to the Banach fixed point theorem. This fixed point is also the unique solution to Eq* (1.1) *with the In.C* $\mathbb{U}(0) = \lambda$*. We have demonstrated this theorem.*

We are going to demonstrate that the solution relies constantly on γ in the subsequent theorem.

Theorem 3.4. *The solution* $\mathbb{V}_\gamma(\mathrm{t},\lambda)$ *depends continuously on* γ*, i.e.,*

$$\lim_{\gamma\to\tilde{\gamma}} \underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup} \|\mathbb{V}_\gamma(\mathrm{t},\lambda) - \mathbb{V}_{\tilde{\gamma}}(\mathrm{t},\lambda)\|_{\mathrm{p}} = 0. \tag{3.23}$$

Proof. Suppose $\gamma, \tilde{\gamma} \in (\frac{1}{2}, 1)$ *and further take* $\lambda \in \widetilde{\mathcal{M}}_0^{\mathrm{p}}$*. As* $\mathbb{V}_\gamma(\lambda,\mathrm{t})$ *and* $\mathbb{V}_{\tilde{\gamma}}(\lambda,\mathrm{t})$ *are solutions to Eq* (1.1)*, we obtain the following:*

$$\begin{aligned}
\mathbb{V}_\gamma(\lambda,\mathrm{t}) - \mathbb{V}_{\tilde{\gamma}}(\lambda,\mathrm{t}) =& \int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Theta(s,\mathbb{V}_\gamma(s),\mathbb{V}_\gamma(s-\upsilon)) - \Theta(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))\Big)ds \\
&+ \int_0^{\mathrm{t}} (s^{\gamma-1} - s^{\tilde{\gamma}-1})\Theta(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))ds \\
&+ \int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Xi(s,\mathbb{V}_\gamma(s),\mathbb{V}_\gamma(s-\upsilon)) - \Xi(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))\Big)d\mathcal{W}_s \\
&+ \int_0^{\mathrm{t}} (s^{\gamma-1} - s^{\tilde{\gamma}-1})\Xi(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s))d\mathcal{W}_s.
\end{aligned} \tag{3.24}$$

Using Eq (3.3)*, we obtain the subsequent outcome using Eq* (3.24)*.*

$$\begin{aligned}
\Big\|\mathbb{V}_\gamma(\lambda,\mathrm{t}) - \mathbb{V}_{\tilde{\gamma}}(\lambda,\mathrm{t})\Big\|_{\mathrm{p}}^{\mathrm{p}} \leq& 2^{\mathrm{p}-1}\sigma\int_0^{\mathrm{t}} s^{2\gamma-2}\Big\|\mathbb{V}_\gamma(\lambda,\mathrm{t}) - \mathbb{V}_{\tilde{\gamma}}(\lambda,\mathrm{t})\Big\|_{\mathrm{p}}^{\mathrm{p}}ds \\
&+ 2^{2\mathrm{p}-2}\Big\|\int_0^{\mathrm{t}} (s^{\gamma-1} - s^{\tilde{\gamma}-1})\Theta(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))ds\Big\|_{\mathrm{p}}^{\mathrm{p}} \\
&+ \Big\|\int_0^{\mathrm{t}} (s^{\gamma-1} - s^{\tilde{\gamma}-1})\Xi(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))d\mathcal{W}_s\Big\|_{\mathrm{p}}^{\mathrm{p}}.
\end{aligned} \tag{3.25}$$

Suppose the following:

$$\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}) = \Big|s^{\gamma-1} - s^{\tilde{\gamma}-1}\Big|. \tag{3.26}$$

We are going to now simplify Eq (3.25)*. By using Eq* (3.3)*, the Hölder inequality,* $(\widetilde{\mathbb{H}}_1)$*, and* $(\widetilde{\mathbb{H}}_2)$*, we get the following result:*

$$\begin{aligned}
&\Big\|\int_0^{\mathrm{t}} (s^{\gamma-1} - s^{\tilde{\gamma}-1})\Theta(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))ds\Big\|_{\mathrm{p}}^{\mathrm{p}} \\
&\leq \sum_1^m \mathrm{E}\Big(\int_0^{\mathrm{t}} \mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma})\Big|\Theta_\iota(s,\mathbb{V}_{\tilde{\gamma}})\Big|ds\Big)^{\mathrm{p}} \\
&\leq \sum_1^m \mathrm{E}\Big(\Big(\int_0^{\mathrm{t}} (\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^{\frac{\mathrm{p}}{\mathrm{p}-1}}\Big)^{\mathrm{p}-1}\int_0^{\mathrm{t}}\Big|\Theta_\iota(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))\Big|^{\mathrm{p}}ds\Big)
\end{aligned} \tag{3.27}$$

$$\leq\Big(\int_0^{\mathrm{t}}(\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2\Big)^{\frac{\mathrm{p}}{2}}\Big(\int_0^{\mathrm{t}}1ds\Big)^{\frac{\mathrm{p}-2}{2}}\int_0^{\mathrm{t}}\Big\|\Theta(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))\Big\|_{\mathrm{p}}^{\mathrm{p}}ds$$
$$\leq\Big(\int_0^{\mathrm{t}}(\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2\Big)^{\frac{\mathrm{p}}{2}}\mathbb{G}^{\frac{\mathrm{p}-2}{2}}\int_0^{\mathrm{t}}2^{\mathrm{p}-1}\Big(\mathbb{L}^{\mathrm{p}}\|\mathbb{V}_{\tilde{\gamma}}(s,\lambda)\|_{\mathrm{p}}^{\mathrm{p}})+\|\Theta(s,0)\|_{\mathrm{p}}^{\mathrm{p}}\Big)$$
$$\leq\Big(\int_0^{\mathrm{t}}(\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2\Big)^{\frac{\mathrm{p}}{2}}\mathbb{G}^{\frac{\mathrm{p}}{2}}2^{\mathrm{p}-1}\Big(\mathbb{L}^{\mathrm{p}}\underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup}\|\mathbb{V}_{\tilde{\gamma}}(s,\lambda)\|_{\mathrm{p}}^{\mathrm{p}})+\mathcal{Z}^{\mathrm{p}}\Big). \tag{3.28}$$

We will now streamline the latter portion of Eq (3.25)*. For this, using the Burkholder-Davis-Gundy inequality, Eq* (3.26)*,* ($\widetilde{\mathbb{H}}_1$)*, and* ($\widetilde{\mathbb{H}}_2$)*, we have the result:*

$$\Big\|\int_0^{\mathrm{t}}(s^{\gamma-1}-s^{\tilde{\gamma}-1})\Xi(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))d\mathcal{W}_s\Big\|_{\mathrm{p}}^{\mathrm{p}} \tag{3.29}$$
$$\leq\sum\nolimits_1^m\mathrm{E}\Big|\int_0^{\mathrm{t}}\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma})\Xi_\iota(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))d\mathcal{W}_s\Big|^{\mathrm{p}}$$
$$\leq\sum\nolimits_1^m C_{\mathrm{p}}\mathrm{E}\Big|\int_0^{\mathrm{t}}\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma})^2\Big|\Xi_\iota(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))\Big|^2d\mathcal{W}_s\Big|^{\frac{\mathrm{p}}{2}}$$
$$\leq\sum\nolimits_1^m C_{\mathrm{p}}\mathrm{E}\Big[\Big(\int_0^{\mathrm{t}}\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma})^2\Big|\Xi_\iota(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))\Big|^{\mathrm{p}}ds\Big)^{\frac{2}{\mathrm{p}}}\Big(\int_0^{\mathrm{t}}(\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2ds\Big)^{\frac{\mathrm{p}-2}{\mathrm{p}}}\Big]^{\frac{\mathrm{p}}{2}}$$
$$=C_{\mathrm{p}}\int_0^{\mathrm{t}}\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma})^2\Big\|\Xi(s,\mathbb{V}_{\tilde{\gamma}}(s),\mathbb{V}_{\tilde{\gamma}}(s-\upsilon))\Big\|_{\mathrm{p}}^{\mathrm{p}}ds\Big(\int_0^{\mathrm{t}}(\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2ds\Big)^{\frac{\mathrm{p}-2}{\mathrm{p}}}$$
$$\leq C_{\mathrm{p}}\Big(\int_0^{\mathrm{t}}\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma})^2ds\Big)^{\frac{\mathrm{p}}{2}}2^{\mathrm{p}-1}\Big(\mathbb{L}^{\mathrm{p}}\underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup}\|\mathbb{V}_{\tilde{\gamma}}(s,\lambda)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{Z}^{\mathrm{p}}\Big). \tag{3.30}$$

Using $\|.\|_\hbar$ *and the previously obtained results, we obtain the following:*

$$\begin{aligned}\frac{\Big\|\mathbb{V}_\gamma(\lambda,\mathrm{t})-\mathbb{V}_{\tilde{\gamma}}(\lambda,\mathrm{t})\Big\|_{\mathrm{p}}^{\mathrm{p}}}{\mathbb{E}_{2\gamma-1}(\sigma\mathrm{t}^{2\gamma-1})}\leq&\frac{\sigma 2^{\mathrm{p}-1}\int_0^{\mathrm{t}}s^{2\gamma-2}\frac{\|\mathbb{V}_\gamma(s,\lambda)-\mathbb{V}_{\tilde{\gamma}}(s,\lambda)\|_{\mathrm{p}}^{\mathrm{p}}}{\mathbb{E}_{2\gamma-1}(\sigma\mathrm{t}^{2\gamma-1})}\mathbb{E}_{2\gamma-1}(\sigma\mathrm{t}^{2\gamma-1})}{\mathbb{E}_{2\gamma-1}(\sigma\mathrm{t}^{2\gamma-1})}\\&+2^{3\mathrm{p}-3}\Big(\mathbb{L}^{\mathrm{p}}\underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup}\|\mathbb{V}_{\tilde{\gamma}}(s,\lambda)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{Z}^{\mathrm{p}}\Big)\Big(\int_0^{\mathrm{t}}(\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2ds\Big)^{\frac{\mathrm{p}}{2}}\mathbb{G}^{\frac{\mathrm{p}}{2}}\\&+2^{3\mathrm{p}-3}\Big(\mathbb{L}^{\mathrm{p}}\underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup}\|\mathbb{V}_{\tilde{\gamma}}(s,\lambda)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{Z}^{\mathrm{p}}\Big)C_{\mathrm{p}}\Big(\int_0^{\mathrm{t}}(\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2ds\Big)^{\frac{\mathrm{p}}{2}}\\\leq&\frac{\sigma 2^{\mathrm{p}-1}\Gamma(2\gamma-1)}{\hbar}\Big\|\mathbb{V}_\gamma(\mathrm{t},\lambda)-\mathbb{V}_{\tilde{\gamma}}(\mathrm{t},\lambda)\Big\|_\hbar^{\mathrm{p}}\\&+2^{3\mathrm{p}-3}\Big(\mathbb{L}^{\mathrm{p}}\underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup}\|\mathbb{V}_{\tilde{\gamma}}(s,\lambda)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{Z}^{\mathrm{p}}\Big)\Big(\int_0^{\mathrm{t}}(\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2ds\Big)^{\frac{\mathrm{p}}{2}}\mathbb{G}^{\frac{\mathrm{p}}{2}}\\&+2^{3\mathrm{p}-3}\Big(\mathbb{L}^{\mathrm{p}}\underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup}\|\mathbb{V}_{\tilde{\gamma}}(s,\lambda)\|_{\mathrm{p}}^{\mathrm{p}}+\mathcal{Z}^{\mathrm{p}}\Big)C_{\mathrm{p}}\Big(\int_0^{\mathrm{t}}(\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2ds\Big)^{\frac{\mathrm{p}}{2}}.\end{aligned} \tag{3.31}$$

Ultimately, applying Lemma 3.2, we obtain

$$\Big(1-\frac{\sigma 2^{\mathrm{p}-1}\Gamma(2\gamma-1)}{\hbar}\Big)\Big\|\mathbb{V}_\gamma(\mathrm{t},\lambda)-\mathbb{V}_{\tilde{\gamma}}(\mathrm{t},\lambda)\Big\|_\hbar^{\mathrm{p}} \tag{3.32}$$

$$\leq 2^{3\mathrm{p}-3}\Big(\mathbb{L}^{\mathrm{p}} \underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup} \|\mathbb{V}_{\tilde{\gamma}}(s,\lambda)\|_{\mathrm{p}}^{\mathrm{p}} + \mathcal{Z}^{\mathrm{p}}\Big)\Big(\int_0^{\mathrm{t}} (\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2 ds\Big)^{\frac{\mathrm{p}}{2}} \mathbb{G}^{\frac{\mathrm{p}}{2}}$$

$$+ 2^{3\mathrm{p}-3}\Big(\mathbb{L}^{\mathrm{p}} \underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup} \|\mathbb{V}_{\tilde{\gamma}}(s,\lambda)\|_{\mathrm{p}}^{\mathrm{p}} + \mathcal{Z}^{\mathrm{p}}\Big) C_{\mathrm{p}}\Big(\int_0^{\mathrm{t}} (\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2 ds\Big)^{\frac{\mathrm{p}}{2}}.$$

Therefore, as a way to finish the proof, it is necessary to show the following using Eq (3.14) *and* $\mathrm{p} \geq 2$*:*

$$\underset{\tilde{\gamma}\to\gamma\mathrm{t}\in[0,\mathbb{G}]}{lim\ sup} \int_0^{\mathrm{t}} (\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2 ds = 0. \tag{3.33}$$

We have the following:

$$\begin{aligned}\int_0^{\mathrm{t}} (\mathbb{I}(\mathrm{t},s,\gamma,\tilde{\gamma}))^2 ds &= \int_0^{\mathrm{t}} s^{2\gamma-2} ds + \int_0^{\mathrm{t}} s^{2\tilde{\gamma}-2} ds - \int_0^{\mathrm{t}} s^{\gamma+\tilde{\gamma}-2} ds \\ &= \frac{s^{2\gamma-1}}{2\gamma-1} + \frac{s^{2\tilde{\gamma}-1}}{2\tilde{\gamma}-1} - \frac{2\mathrm{t}^{\gamma+\tilde{\gamma}-1}}{\gamma+\tilde{\gamma}-1}.\end{aligned} \tag{3.34}$$

As such, it demonstrated the needed outcome.

By comparing the two distinct solutions, we get the solutions for the Lipschitz continuity dependency for the initial values.

Theorem 3.5. *For any* $\lambda,\ \delta \in \widetilde{\mathcal{M}}_0^{\mathrm{p}}$*, the solution* $\mathbb{V}_\gamma(\mathrm{t},\lambda)$ *depends Lipschitz continuously on* λ*, i.e., there exists* $\mathbb{L}_1 > 0$ *such that*

$$\|\mathbb{V}_\gamma(\mathrm{t},\lambda) - \mathbb{V}_\gamma(\mathrm{t},\delta)|_{\mathrm{p}} \leq \mathbb{L}\|\lambda-\delta\|_{\mathrm{p}},\ \forall\ \mathrm{t}\in[0,\ \mathbb{G}]. \tag{3.35}$$

Proof. Take $\delta \in \widetilde{\mathcal{M}}_0^{\mathrm{p}}$*. Assume* $\lambda \in \widetilde{\mathcal{M}}_0^{\mathrm{p}}$ *randomly. As* $\mathbb{V}_\gamma(\mathrm{t},\lambda)$ *and* $\mathbb{V}_\gamma(\mathrm{t},\delta)$ *are solutions of Eq* (1.1)*, it implies that*

$$\begin{aligned}\mathbb{V}_\gamma(\mathrm{t},\lambda) - \mathbb{V}_\gamma(\mathrm{t},\delta) =& \lambda - \delta + \int_0^{\mathrm{t}} s^{\gamma-1}(\Theta(s,\mathbb{V}_\gamma(s,\lambda),\mathbb{V}_\gamma(s-\upsilon,\lambda)) \\ &- \Theta(s,\mathbb{V}_\gamma(s,\delta),\mathbb{V}_\gamma(s-\upsilon,\delta)))ds + \int_0^{\mathrm{t}} s^{\gamma-1}(\Xi(s,\mathbb{V}_\gamma(s,\lambda),\mathbb{V}_\gamma(s-\upsilon,\lambda)) \\ &- \Xi(s,\mathbb{V}_\gamma(s,\delta),\mathbb{V}_\gamma(s-\upsilon,\delta))d\mathcal{W}_s.\end{aligned} \tag{3.36}$$

Hence, using Eq (3.3)*:*

$$\begin{aligned}&\Big\|\mathbb{V}_\gamma(\mathrm{t},\lambda) - \mathbb{V}_\gamma(\mathrm{t},\delta)\Big\|_{\mathrm{p}}^{\mathrm{p}} \\ \leq& 2^{\mathrm{p}-1}\Big\|\int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Theta(s,\mathbb{V}_\gamma(s,\lambda),\mathbb{V}_\gamma(s-\upsilon,\lambda),)) - \Theta(s,\mathbb{V}_\gamma(s,\delta),\mathbb{V}_\gamma(s-\upsilon,\delta))\Big)ds\Big\|_{\mathrm{p}}^{\mathrm{p}} \\ &+ 2^{\mathrm{p}-1}\Big\|\int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Xi(s,\mathbb{V}_\gamma(s,\lambda),\mathbb{V}_\gamma(s-\upsilon,\lambda)) - \Xi(s,\mathbb{V}_\gamma(s,\lambda),\mathbb{V}_\gamma(s-\upsilon,\delta))\Big)d\mathcal{W}_s\Big\|_{\mathrm{p}}^{\mathrm{p}}.\end{aligned} \tag{3.37}$$

Now, using Eq (3.37)*, Eq* (3.3)*, the Hölder inequality, and* ($\widetilde{\mathbb{H}}_1$)*, we get the following result:*

$$\Big\|\int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Theta(s,\mathbb{V}_\gamma(s,\lambda),\mathbb{V}_\gamma(s-\upsilon,\lambda)) - \Theta(s,\mathbb{V}_\gamma(s,\delta),\mathbb{V}_\gamma(s-\upsilon,\delta))\Big)ds\Big\|_{\mathrm{p}}^{\mathrm{p}}$$

$$\leq \sum_{\imath=1}^{m} \mathrm{E}\Big(\int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Theta_\imath(s, \mathbb{V}_\gamma(s,\lambda), \mathbb{V}_\gamma(s-\upsilon,\lambda)) - \Theta_\imath(s, \mathbb{V}_\gamma(s,\delta), \mathbb{V}_\gamma(s-\upsilon,\delta))\Big)ds\Big)^{\mathrm{p}}$$
$$\leq \sum_{\imath=1}^{m} \mathrm{E}\Big(\Big(\int_0^{\mathrm{t}} s^{\frac{(\gamma-1)(\mathrm{p}-2)}{\mathrm{p}-1}} ds\Big)^{\mathrm{p}-1}$$
$$\Big(\int_0^{\mathrm{t}} s^{2\gamma-2}\Big|\Theta_\imath(s, \mathbb{V}_\gamma(s,\lambda), \mathbb{V}_\gamma(s-\upsilon,\lambda)) - \Theta_\imath(s, \mathbb{V}_\gamma(s,\delta), \mathbb{V}_\gamma(s-\upsilon,\delta))\Big|\Big)\Big)$$
$$\leq \Big(\frac{\mathbb{L}^{\mathrm{p}}\mathbb{G}^{\mathrm{p}\gamma-2\gamma+1}(\mathrm{p}-1)^{\mathrm{p}-1}}{(\mathrm{p}\gamma-2\gamma+1)^{\mathrm{p}-1}}\Big)$$
$$\int_0^{\mathrm{t}} s^{2\gamma-2}\Big(\Big\|\mathbb{V}_\gamma(s,\lambda) - \mathbb{V}_\gamma(s,\delta)\Big\|_{\mathrm{p}}^{\mathrm{p}} + \Big\|\mathbb{V}_\gamma(s-\upsilon,\lambda) - \mathbb{V}_\gamma(s-\upsilon,\delta)\Big\|_{\mathrm{p}}^{\mathrm{p}}\Big)ds. \tag{3.38}$$

Now using Eq (3.3)*, the Hölder inequality, Burkholder-Davis-Gundy inequality, and* ($\widetilde{\mathbb{H}}_1$)*, we get*

$$\Big\|\int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Xi(s, \mathbb{V}_\gamma(s,\lambda), \mathbb{V}_\gamma(s-\upsilon,\lambda)) - \Xi(s, \mathbb{V}_\gamma(s,\delta), \mathbb{V}_\gamma(s-\upsilon,\delta))\Big)d\mathcal{W}_s\Big\|_{\mathrm{p}}^{\mathrm{p}}$$
$$= \sum_{\imath=1}^{m} \mathrm{E}\Big|\int_0^{\mathrm{t}} s^{\gamma-1}\Big(\Xi_\imath(s, \mathbb{V}_\gamma(s,\lambda), \mathbb{V}_\gamma(s-\upsilon,\lambda)) - \Xi_\imath(s, \mathbb{V}_\gamma(s,\delta), \mathbb{V}_\gamma(s-\upsilon,\delta))\Big)d\mathcal{W}_s\Big|^{\mathrm{p}}$$
$$\leq \sum_{\imath=1}^{m} C_{\mathrm{p}}\mathrm{E}\Big|\int_0^{\mathrm{t}} s^{2\gamma-2}\Big|\Xi_\imath(s, \mathbb{V}_\gamma(s,\lambda)\mathbb{V}_\gamma(s-\upsilon,\lambda)) - \Xi_\imath(s, \mathbb{V}_\gamma(s,\delta), \mathbb{V}_\gamma(s-\upsilon,\delta))\Big|^2 ds\Big|^{\frac{\mathrm{p}}{2}}$$
$$\leq \sum_{\imath=1}^{m} C_{\mathrm{p}}\mathrm{E}\int_0^{\mathrm{t}} s^{2\gamma-2}\Big|\Xi_\imath(s, \mathbb{V}_\gamma(s,\lambda), \mathbb{V}_\gamma(s-\upsilon,\lambda)) - \Xi_\imath(s, \mathbb{V}_\gamma(s,\delta), \mathbb{V}_\gamma(s-\upsilon,\delta))\Big|^{\mathrm{p}} ds$$
$$\Big(\int_0^{\mathrm{t}} s^{2\gamma-2} ds\Big)^{\frac{\mathrm{p}-2}{2}}$$
$$\leq \mathbb{L}^{\mathrm{p}} C_{\mathrm{p}}\Big(\frac{\mathbb{G}^{2\gamma-1}}{2\gamma-1}\Big)^{\frac{\mathrm{p}-2}{2}} \int_0^{\mathrm{t}} s^{2\gamma-2}\Big(\Big\|\mathbb{V}_\gamma(s,\lambda) - \mathbb{V}_\gamma(s,\delta)\Big\|_{\mathrm{p}}^{\mathrm{p}}$$
$$+ \Big\|\mathbb{V}_\gamma(s-\upsilon,\lambda) - \mathbb{V}_\gamma(s-\upsilon,\delta)\Big\|_{\mathrm{p}}^{\mathrm{p}}\Big)ds. \tag{3.39}$$

Utilizing Eqs (3.38) *and* (3.39)*, we can therefore extract the following from Eq* (3.37).

$$\Big\|\mathbb{V}_\gamma(\mathrm{t},\lambda) - \mathbb{V}_\gamma(\mathrm{t},\delta)\Big\|_{\mathrm{p}}^{\mathrm{p}} \leq 2^{\mathrm{p}-1}\Big\|\lambda-\delta\Big\|_{\mathrm{p}}^{\mathrm{p}} + 2^{\mathrm{p}-1}\sigma\int_0^{\mathrm{t}} s^{2\gamma-2}\Big(\Big\|\mathbb{V}_\gamma(s,\lambda) - \mathbb{V}_\gamma(s,\delta)\Big\|_{\mathrm{p}}^{\mathrm{p}}$$
$$+ \Big\|\mathbb{V}_\gamma(s-\upsilon,\lambda) - \mathbb{V}_\gamma(s-\upsilon,\delta)\Big\|_{\mathrm{p}}^{\mathrm{p}}\Big)ds. \tag{3.40}$$

Taking the Gronwall inequality into account, we arrive at the following ([29]*, Lemma 7.1.1*)*:*

$$\Big\|\mathbb{V}_\gamma(\mathrm{t}-\upsilon,\lambda) - \mathbb{V}_\gamma(\mathrm{t}-\upsilon,\delta)\Big\|_{\mathrm{p}}^{\mathrm{p}} \leq 2^{\mathrm{p}-1}\mathbb{E}_{2\gamma-1}(2^{\mathrm{p}-1}\sigma\Gamma(2\gamma-1)\mathrm{t}^{2\gamma-1})\Big\|\lambda-\delta\Big\|_{\mathrm{p}}^{\mathrm{p}}. \tag{3.41}$$

Hence, the proof is complete.

3.2. Regularity for CFrSDDEs

We are going to demonstrate the regularity of CFrSDDEs solutions in this portion.

Proof of Theorem 2.3.

$$2^{2-2\mathrm{p}}\Big\|\mathbb{V}_\gamma(\mathrm{t},\lambda)-\mathbb{V}_\gamma(u,\lambda)\Big\|_{\mathrm{p}}^{\mathrm{p}}\leq\Big\|\int_u^{\mathrm{t}}s^{\gamma-1}\Theta(s,\mathbb{V}_\gamma(s,\lambda))ds\Big\|_{\mathrm{p}}^{\mathrm{p}}+\Big\|\int_u^{\mathrm{t}}s^{\gamma-1}\Xi(s,\mathbb{V}_\gamma(s,\lambda))d\mathcal{W}_s\Big\|_{\mathrm{p}}^{\mathrm{p}},$$

applying inequality

$$\begin{aligned}2^{2-2\mathrm{p}}\Big\|\mathbb{V}_\gamma(\mathrm{t},\lambda)-\mathbb{V}_\gamma(\mathrm{t},\lambda)\Big\|_{\mathrm{p}}^{\mathrm{p}}\leq&\frac{(\mathrm{p}-1)^{\mathrm{p}-1}}{(\mathrm{p}\gamma-1)^{\mathrm{p}-1}}(\mathrm{t}^{\frac{\mathrm{p}\gamma-1}{\mathrm{p}-1}}-u^{\frac{\mathrm{p}\gamma-1}{\mathrm{p}-1}})^{\mathrm{p}-1}\int_u^{\mathrm{t}}\Big\|\Theta(s,\mathbb{V}_\gamma(s,\lambda))\Big\|_{\mathrm{p}}^{\mathrm{p}}ds\\&+C_{\mathrm{p}}\int_u^{\mathrm{t}}\frac{\Big\|\Xi(s,\mathbb{V}_\gamma(s,\lambda))\Big\|_{\mathrm{p}}^{\mathrm{p}}}{s^{2-2\gamma}}ds\Big(\int_u^{\mathrm{t}}\frac{1}{s^{2-2\gamma}}ds\Big)^{\frac{\mathrm{p}-2}{2}}.\end{aligned}$$

So, $\mathcal{Z}_1>0$ exists when $\underset{\mathrm{t}\in[0,\mathbb{G}]}{esssup}\|\mathbb{V}_\gamma(\mathrm{t},\lambda)\|_{\mathrm{p}}^{\mathrm{p}}\leq\mathcal{Z}_1$ because $\mathbb{V}_\gamma(s,\lambda)\in\widetilde{\mathcal{H}}^{\mathrm{p}}([0,\mathbb{G}])$. Along with $(\widetilde{\mathbb{H}}_1)$ and $(\widetilde{\mathbb{H}}_2)$, this implies

$$\Big\|\Theta(s,\mathbb{V}_\gamma(s,\lambda))\Big\|_{\mathrm{p}}^{\mathrm{p}}\leq2^{\mathrm{p}-1}(\mathbb{L}^{\mathrm{p}}\Big\|\mathbb{V}_\gamma(s,\lambda))\Big\|_{\mathrm{p}}^{\mathrm{p}}+\|\Theta(s,0)\|_{\mathrm{p}}^{\mathrm{p}})\leq2^{\mathrm{p}-1}(\mathbb{L}^{\mathrm{p}}\mathcal{Z}_1+\mathcal{Z}^{\mathrm{p}}).$$

$$\Big\|\Xi(s,\mathbb{V}_\gamma(s,\lambda))\Big\|_{\mathrm{p}}^{\mathrm{p}}\leq2^{\mathrm{p}-1}(\mathbb{L}^{\mathrm{p}}\Big\|\mathbb{V}_\gamma(s,\lambda))\Big\|_{\mathrm{p}}^{\mathrm{p}}+\|\Xi(s,0)\|_{\mathrm{p}}^{\mathrm{p}})\leq2^{\mathrm{p}-1}(\mathbb{L}^{\mathrm{p}}\mathcal{Z}_1+\mathcal{Z}^{\mathrm{p}}).$$

From above, we get the following:

$$\begin{aligned}2^{2-2\mathrm{p}}\Big\|\mathbb{V}_\gamma(\mathrm{t},\lambda)-\mathbb{V}_\gamma(\mathrm{t},\lambda)\Big\|_{\mathrm{p}}^{\mathrm{p}}\leq&\frac{(2\mathrm{p}-2)^{\mathrm{p}-1}}{(\mathrm{p}\gamma-1)^{\mathrm{p}-1}}(\mathrm{t}-\mathrm{f})^{\frac{(2\gamma-1)\mathrm{p}}{2}}(\mathbb{L}^{\mathrm{p}}\mathcal{Z}_1+\mathcal{Z}^{\mathrm{p}})\mathbb{G}^{\frac{\mathrm{p}}{2}}\\&+\frac{1}{(2\gamma-1)^{\frac{\mathrm{p}}{2}}}(\mathrm{t}-\mathrm{f})^{\frac{(2\gamma-1)\mathrm{p}}{2}}(\mathbb{L}^{\mathrm{p}}\mathcal{Z}_1+\mathcal{Z}^{\mathrm{p}})2^{\mathrm{p}-1}C_{\mathrm{p}}.\end{aligned}$$

Hence, we get

$$\Big\|\mathbb{V}_\gamma(\mathrm{t},\lambda)-\mathbb{V}_\gamma(\mathrm{t},\lambda)\Big\|_{\mathrm{p}}\leq\mathcal{J}(\mathrm{t}-\ \mathrm{f})^{\gamma-\frac{1}{2}},$$

where

$$\mathcal{J}^{\mathrm{p}}=2^{2\mathrm{p}-2}\Big(\frac{(2\mathrm{p}-2)^{\mathrm{p}-1}}{(\mathrm{p}\gamma-1)^{\mathrm{p}-1}}(\mathbb{L}^{\mathrm{p}}\mathcal{Z}_1+\mathcal{Z}^{\mathrm{p}})\mathbb{G}^{\frac{\mathrm{p}}{2}}+\frac{1}{(2\gamma-1)^{\frac{\mathrm{p}}{2}}}(\mathbb{L}^{\mathrm{p}}\mathcal{Z}_1+\mathcal{Z}^{\mathrm{p}})2^{\mathrm{p}-1}C_{\mathrm{p}}\Big).$$

We provided two examples in the part that follows to demonstrate the usefulness of our demonstrated outcome.

4. Examples

Example 4.1. Consider the following problem:

$$\mathcal{T}_{\mathrm{t}}^{\gamma}\mathbb{U}(\mathrm{t})=-a\mathbb{U}(\mathrm{t}-\upsilon)+b\frac{d\mathcal{W}_s}{d\mathrm{t}},\ \frac{1}{2}<\gamma<1,\ 0<\mathrm{t}<\mathbb{G},\tag{4.1}$$

$$\mathbb{U}(\mathrm{t})=1,\ \ -\upsilon\leq\mathrm{t}\leq0.\tag{4.2}$$

In this case, the delay time is υ, the drift and diffusion terms are $\Theta=-a\mathbb{U}(\mathrm{t}-\upsilon)$ and $\Xi=b$ accordingly, and the constants a and b are positive. In the aforementioned model, which was studied in [30, 31],

$\gamma = 1$ characterized the statistical physics of motorists. Assume the following: $\upsilon = 0.1, a = 0.5, b = 1$. The terms $-a\mathbb{U}(\mathrm{t}-\upsilon)$ and b satisfied the require condition. Therefore, with $\mathbb{U} = 1$, $\upsilon \leq \mathrm{t} \leq 0$ in $-\upsilon \leq \mathrm{t} \leq \mathbb{G}$, the Theorem 3.3 indicates that a solution exists and is unique.

Example 4.2. Examine the following problem:

$$\mathcal{T}_{\mathrm{t}}^{\gamma}\mathbb{U}(\mathrm{t}) = \frac{c\mathbb{U}(\mathrm{t}-\upsilon)}{1+\mathbb{U}^{10}(\mathrm{t}-\upsilon)} - a\mathbb{U}(\mathrm{t}) + b\mathbb{U}\frac{d\mathcal{W}_s}{d\mathrm{t}},\ \frac{1}{2} < \gamma < 1,\ 0 < \mathrm{t} < \mathbb{G}, \tag{4.3}$$

$$\mathbb{U}(\mathrm{t}) = 0.5,\ \ -\upsilon \leq \mathrm{t} \leq 0. \tag{4.4}$$

In this case, the delay time is υ, the drift and diffusion terms are $\Theta = \frac{c\mathbb{U}(\mathrm{t}-\upsilon)}{1+\mathbb{U}^{10}(\mathrm{t}-\upsilon)}$ and $\Xi = -a\mathbb{U}(\mathrm{t}) + b\mathbb{U}$ accordingly, and the constants a, b, and c are positive. In the aforementioned model, in [29] it was examined using $\gamma = 1$ to elucidate the random rise in blood vessel concentration. Assume the following: $\upsilon = 5, a = 1, b = 2, c = 2$. The required condition of Theorem 3.3 is satsfied by $\frac{c\mathbb{U}(\mathrm{t}-\upsilon)}{1+\mathbb{U}^{10}(\mathrm{t}-\upsilon)}$ and $-a\mathbb{U}(\mathrm{t}) + b\mathbb{U}$. With $\mathbb{U} = 0.5$, $\upsilon \leq \mathrm{t} \leq 0$ in $-\upsilon \leq \mathrm{t} \leq \mathbb{G}$, Theorem 3.3 states that there is a unique solution that exists.

5. Conclusions

The existence and uniqueness are important because they guarantee that the differential equation has a meaningful and reliable solution that can be used to analyze or predict the behavior of the model or system. A differential equation without existence and uniqueness may have no solution or several contradictory solutions that rely on arbitrary presumptions. In this research work, we have proven the well-posedness and regularity of solutions to CFrSDDEs. After proving the existence and uniqueness of the solutions, we showed how the solutions continuously depend on the fractional exponent γ as well as the initial values. The regularity of time is the subject of the second section. Finally, two instances are given to illustrate our results.

In this article, we paid attention to the well-posedness and regularity of the solutions of a class of CFrSDDEs. Due to the importance of this topic in the future, we will apply the numerical method to solve various problems that exist in various domains.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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