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Research article

On ψ-convex functions and related inequalities

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Abstract: We introduce the class of ψ-convex functions $f : [0,\infty) \to \mathbb{R}$, where $\psi \in C([0,1])$ satisfies $\psi \geq 0$ and $\psi(0) \neq \psi(1)$. This class includes several types of convex functions introduced in previous works. We first study some properties of such functions. Next, we establish a double Hermite-Hadamard-type inequality involving ψ-convex functions and a Simpson-type inequality for functions $f \in C^1([0,\infty))$ such that $|f'|$ is ψ-convex. Our obtained results are new and recover several existing results from the literature.

Keywords: ψ-convex functions; Hermite-Hadamard-type inequalities; Simpson-type inequalities
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1. Introduction

The class of convex functions is widely used in many branches of pure and applied mathematics. Due to this reason, we find in the literature several studies related to convex functions, see e.g., [1–7]. One of the most famous inequalities involving convex functions is the double Hermite-Hadamard inequality [8, 9]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f : I \to \mathbb{R}$ and $a, b \in I$, where $a < b$ and I is an interval of $\mathbb{R}$. For more details about (1.1), see e.g., [10]. The double inequality (1.1) has been refined and extended to various classes of functions such as log-convex functions [11], s-logarithmically convex functions [12], hyperbolic p-convex functions [13], s-convex functions [14], convex functions on the co-ordinates [15], F-convex functions [16, 17], h-convex and harmonically h-convex interval-valued functions [18], m-convex functions [19], (m_1, m_2)-convex functions [20], (α, m)-convex functions [21], (α, m_1, m_2)-convex functions [22], etc. In particular, when $f : [0, \infty) \to \mathbb{R}$ is m-convex (see Definition 2.2), where $0 < m \leq 1$, Dragomir [19] proved that for all $a, b \geq 0$ with $0 \leq a < b$, we have

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{2}\min\left\{f(a) + mf\left(\frac{b}{m}\right), f(b) + mf\left(\frac{a}{m}\right)\right\}$$

and

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2}\,dx \leq \frac{m+1}{4}\left(\frac{f(a)+f(b)}{2} + m\frac{f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)}{2}\right).$$

Notice that, if f is convex (so $m = 1$, see Definition 2.2), the above double inequality reduces to (1.1).

Another important inequality, which is very useful in numerical integrations, is the Simpson's inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{(b-a)^4}{2880}\|f^{(4)}\|_\infty,$$

where $f \in C^4([a, b])$ and $\|f^{(4)}\|_\infty = \max_{a\leq x\leq b}|f^{(4)}(x)|$. The above inequality has been studied in several papers for different classes of functions, see e.g., [23–28]. For instance, Dragomir [25], proved that, if $f : [a, b] \to \mathbb{R}$ is a function of bounded variation, then

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{3}V_a^b,$$

where V_a^b denotes the total variation of f on $[a, b]$.

Notice that it is always interesting to extend the above important inequalities to other classes of functions. Such extensions will be useful for example in numerical integrations and many other applications. Motivated by this fact, we introduce in this paper the class of ψ-convex functions $f : [0, \infty) \to \mathbb{R}$, where $\psi \in C([0, 1])$ is a function satisfying certain conditions. This class includes several types of convex functions from the literature: m-convex functions, (m_1, m_2)-convex functions, (α, m)-convex functions and (α, m_1, m_2)-convex functions. Moreover, after studying some properties of this introduced class of functions, we establish a double Hermite-Hadamard-type inequality involving ψ-convex functions and a Simpson-type inequality for functions $f \in C^1([0, \infty))$, where $|f'|$ is ψ-convex. Our obtained results are new and recover several results from the literature.

Th rest of the paper is as follows. In Section 2, we introduce the class of ψ-convex functions and we study some properties of such functions. We also provide several examples of functions that belong to the introduced class. In Section 3, we establish Hermite-Hadamard-type inequalities involving ψ-convex functions. In Section 4, a Simpson-type inequality is proved for functions $f \in C^1([0, \infty))$, where $|f'|$ is ψ-convex.

2. ψ-convex functions

In this section, we introduce the class of ψ-convex functions. We first introduce the set

$$\Psi = \{\psi \in C([0,1]) : \psi \geq 0, \psi(0) \neq \psi(1)\}.$$

Definition 2.1. *Let $\psi \in \Psi$. A function $f : [0,\infty) \to \mathbb{R}$ is said to be ψ-convex, if*

$$f(\psi(0)tx + \psi(1)(1-t)y) \leq \psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}f(x) + \psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)}f(y) \tag{2.1}$$

for all $t \in [0,1]$ and $x, y \geq 0$.

We will show below that the introduced class of functions includes several classes of functions from the literature. Let us first start with some simple examples of ψ-convex functions.

Example 2.1. *Let us consider the function $\psi : [0,1] \to \mathbb{R}$ defined by*

$$\psi(t) = at, \quad 0 \leq t \leq 1,$$

where $0 < a \leq 1$ is a constant. Clearly, the function $\psi \in \Psi$ and $\psi(0) = 0 < a = \psi(1)$. On the other hand, for all $t \in [0,1]$, we have

$$\psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)} = 0$$

and

$$\psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)} = a(1-t).$$

Let $f : [0,\infty) \to \mathbb{R}$ be the function defined by

$$f(x) = Ax^2 + Bx, \quad x \geq 0,$$

where $A \geq 0$ and $B \in \mathbb{R}$ are constants. For all $t \in [0,1]$ and $x, y \geq 0$, we have

$$\begin{aligned}
&\psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}f(x) + \psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)}f(y) - f(\psi(0)tx + \psi(1)(1-t)y) \\
&= a(1-t)f(y) - f(a(1-t)y) \\
&= a(1-t)\left(Ay^2 + By\right) - A(a(1-t))^2 - B(a(1-t)) \\
&= aA(1-t)y^2(at + 1 - a) \geq 0,
\end{aligned}$$

which shows that f is ψ-convex.

Example 2.2. *We consider the function $\psi : [0,1] \to \mathbb{R}$ defined by*

$$\psi(t) = t^2, \quad 0 \leq t \leq 1.$$

Clearly, the function $\psi \in \Psi$. On the other hand, for all $t \in [0,1]$, we have

$$\psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)} = 0$$

and

$$\psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)} = 1-t^2.$$

Let $f:[0,\infty)\to\mathbb{R}$ be the function defined by

$$f(x) = x^3 - x^2 + x, \quad x \geq 0.$$

For all $t\in[0,1]$ and $x,y\geq 0$, we have

$$\begin{aligned}
&\psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}f(x) + \psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)}f(y) - f\left(\psi(0)tx + \psi(1)(1-t)y\right)\\
&= (1-t^2)(y^3-y^2+y) - [(1-t)y]^3 + [(1-t)y]^2 - [(1-t)y]\\
&= ty(1-t)P_t(y),
\end{aligned}$$

where $P_t(y)$ is the second order polynomial function (with respect to y) given by

$$P_t(y) = (3-t)y^2 - 2y + 1.$$

Observe that for all $t\in[0,1]$, the the discriminant of P_t is given by

$$\Delta = 4(t-2) < 0,$$

which implies (since $3-t>0$) that $P_t(y)\geq 0$. Consequently, for all $t\in[0,1]$ and $x,y\geq 0$,

$$\psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}f(x) + \psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)}f(y) - f\left(\psi(0)tx + \psi(1)(1-t)y\right) \geq 0,$$

which shows that f is ψ-convex.

We now recall the following notion introduced by Toader [29].

Definition 2.2. *Let $m\in[0,1]$ and $f:[0,\infty)\to\mathbb{R}$. The function f is said to be m-convex, if*

$$f(tx+m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $t\in[0,1]$ and $x,y\geq 0$.

Proposition 2.1. *If $f:[0,\infty)\to\mathbb{R}$ is m-convex, where $0\leq m<1$, then f is ψ-convex for some $\psi\in\Psi$.*

Proof. Let

$$\psi(t) = (m-1)t + 1, \quad 0\leq t\leq 1.$$

Clearly, ψ is a nonnegative and continuous function and $\psi(0)-\psi(1) = 1-m \neq 0$, which shows that $\psi\in\Psi$. We also have

$$\psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)} = \frac{(m-1)t}{m-1} = t$$

and

$$\psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)} = m\frac{(m-1)(1-t)}{m-1} = m(1-t).$$

Consequently, we get

$$\begin{aligned} f\left(\psi(0)tx+\psi(1)(1-t)y\right) &= f(tx+m(1-t)y) \\ &\le tf(x)+m(1-t)f(y) \\ &= \psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}f(x)+\psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)}f(y), \end{aligned}$$

which shows that f is ψ-convex. □

The following class of functions was introduced by Kadakal [20].

Definition 2.3. *Let $m_1, m_2 \in [0,1]$ and $f : [0,\infty) \to \mathbb{R}$. The function f is said to be (m_1, m_2)-convex, if*

$$f(m_1 tx + m_2(1-t)y) \le m_1 t f(x) + m_2(1-t) f(y)$$

for all $t \in [0,1]$ and $x, y \ge 0$.

Proposition 2.2. *If $f : [0,\infty) \to \mathbb{R}$ is (m_1, m_2)-convex, where $m_1, m_2 \in [0,1]$ with $m_1 \neq m_2$, then f is ψ-convex for some $\psi \in \Psi$.*

Proof. Let

$$\psi(t) = (m_2 - m_1)t + m_1, \quad 0 \le t \le 1.$$

Clearly, ψ is a nonnegative and continuous function and $\psi(0)-\psi(1) = m_1 - m_2 \neq 0$, which shows that $\psi \in \Psi$. We also have

$$\psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)} = m_1\frac{(m_2-m_1)t}{m_2-m_1} = m_1 t$$

and

$$\psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)} = m_2\frac{(m_2-m_1)(1-t)}{m_2-m_1} = m_2(1-t).$$

Consequently, we get

$$\begin{aligned} f\left(\psi(0)tx+\psi(1)(1-t)y\right) &= f(m_1tx+m_2(1-t)y) \\ &\le m_1tf(x)+m_2(1-t)f(y) \\ &= \psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}f(x)+\psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)}f(y), \end{aligned}$$

which shows that f is ψ-convex. □

Miheşan [30] was introduced the following concept.

Definition 2.4. *Let $\alpha, m \in [0,1]$ and $f : [0,\infty) \to \mathbb{R}$. The function f is said to be (α, m)-convex, if*

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha) f(y)$$

for all $t \in [0,1]$ and $x, y \ge 0$.

Proposition 2.3. *If $f : [0,\infty) \to \mathbb{R}$ is a (α, m)-convex function, where $\alpha \in (0,1]$ and $0 \le m < 1$, then f is ψ-convex for some $\psi \in \Psi$.*

Proof. Let

$$\psi(t) = (m-1)t^{\alpha} + 1, \quad 0 \le t \le 1.$$

It is clear that $\psi \in \Psi$, $\psi(0) = 1$ and $\psi(1) = m$. We also have

$$\psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)} = \frac{(m-1)t^{\alpha}}{m-1} = t^{\alpha}$$

and

$$\psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)} = m\frac{(m-1)(1-t^{\alpha})}{m-1} = m(1-t^{\alpha}).$$

Consequently, we get

$$\begin{aligned} f\left(\psi(0)tx + \psi(1)(1-t)y\right) &= f(tx + m(1-t)y) \\ &\le t^{\alpha} f(x) + m(1-t^{\alpha})f(y) \\ &= \psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}f(x) + \psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)}f(y), \end{aligned}$$

which shows that f is ψ-convex. □

In [22], Kadakal introduced the following notion.

Definition 2.5. *Let $\alpha, m_1, m_2 \in [0,1]$ and $f : [0,\infty) \to \mathbb{R}$. The function f is said to be (α, m_1, m_2)-convex, if*

$$f(m_1 tx + m_2(1-t)y) \le m_1 t^{\alpha} f(x) + m_2(1-t^{\alpha})f(y)$$

for all $t \in [0,1]$ and $x, y \ge 0$.

Proposition 2.4. *If $f : [0,\infty) \to \mathbb{R}$ is a (α, m_1, m_2)-convex function, where $\alpha \in (0,1]$ and $m_1, m_2 \in [0,1]$ with $m_1 \neq m_2$, then f is ψ-convex for some $\psi \in \Psi$.*

Proof. Let

$$\psi(t) = (m_2 - m_1)t^{\alpha} + m_1, \quad 0 \le t \le 1.$$

It is clear that $\psi \in \Psi$, $\psi(0) = m_1$ and $\psi(1) = m_2$. We also have

$$\psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)} = m_1 t^{\alpha}$$

and

$$\psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)} = m_2(1-t^{\alpha}).$$

Consequently, we get

$$\begin{aligned} f\left(\psi(0)tx + \psi(1)(1-t)y\right) &= f(m_1 tx + m_2(1-t)y) \\ &\le m_1 t^{\alpha} f(x) + m_2(1-t^{\alpha})f(y) \\ &= \psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}f(x) + \psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)}f(y), \end{aligned}$$

which shows that f is ψ-convex. □

Remark 2.1. *Let $f : [0,\infty) \to \mathbb{R}$ be (α, m_1, m_2)-convex, where $\alpha, m_1, m_2 \in [0,1]$.*

(i) If $\alpha = 1$, then f is (m_1, m_2)-convex.
(ii) If $m_1 = 1$ and $m_2 = m$, then f is (α, m)-convex.
(iii) If $\alpha = 1$, $m_1 = 1$ and $m_2 = m$, then f is m-convex.

We provide below some properties of ψ-convex functions.

Proposition 2.5.

(i) *Let $\sigma, \mu \geq 0$ and $\psi \in \Psi$. If $f, g : [0,\infty) \to \mathbb{R}$ are ψ-convex, then $\sigma f + \mu g$ is ψ-convex.*
(ii) *Let $\psi \in \Psi$. If $f : [0,\infty) \to \mathbb{R}$ is ψ-convex, then for all $x \geq 0$,*

$$\begin{aligned} f(\psi(0)x) &\leq \psi(0)f(x), \\ f(\psi(1)x) &\leq \psi(1)f(x), \\ \frac{f(\psi(0)x) + f(\psi(1)x)}{\psi(0) + \psi(1)} &\leq f(x). \end{aligned}$$

(iii) *Let $\psi \in \Psi$ and $f : [0,\infty) \to \mathbb{R}$ be ψ-convex.*
- *If $f(0) > 0$, then for all $t \in [0,1]$,*

$$\psi(t) \leq \psi(0) + \psi(1) - 1.$$

- *If $f(0) < 0$, then for all $t \in [0,1]$,*

$$\psi(t) \geq \psi(0) + \psi(1) - 1.$$

Proof. (i) Let $f, g : [0,\infty) \to \mathbb{R}$ be two ψ-convex functions. Let $h = \sigma f + \mu g$. For all $t \in [0,1]$ and $x, y \geq 0$, we have

$$f\left(\psi(0)tx + \psi(1)(1-t)y\right) \leq \psi(0)\frac{\psi(t) - \psi(0)}{\psi(1) - \psi(0)}f(x) + \psi(1)\frac{\psi(1) - \psi(t)}{\psi(1) - \psi(0)}f(y)$$

and

$$g\left(\psi(0)tx + \psi(1)(1-t)y\right) \leq \psi(0)\frac{\psi(t) - \psi(0)}{\psi(1) - \psi(0)}g(x) + \psi(1)\frac{\psi(1) - \psi(t)}{\psi(1) - \psi(0)}g(y).$$

Multiplying the first inequality (resp. second inequality) by $\sigma \geq 0$ (resp. $\mu \geq 0$), we obtain

$$\begin{aligned} & h\left(\psi(0)tx + \psi(1)(1-t)y\right) \\ & \leq \psi(0)\frac{\psi(t) - \psi(0)}{\psi(1) - \psi(0)}\sigma f(x) + \psi(1)\frac{\psi(1) - \psi(t)}{\psi(1) - \psi(0)}\sigma f(y) \\ & \quad + \psi(0)\frac{\psi(t) - \psi(0)}{\psi(1) - \psi(0)}\mu g(x) + \psi(1)\frac{\psi(1) - \psi(t)}{\psi(1) - \psi(0)}\mu g(y) \\ & = \psi(0)\frac{\psi(t) - \psi(0)}{\psi(1) - \psi(0)}h(x) + \psi(1)\frac{\psi(1) - \psi(t)}{\psi(1) - \psi(0)}h(y), \end{aligned}$$

which shows that h is ψ-convex.

(ii) Let $x \geq 0$. Taking $t = 1$ in (2.1), we get

$$f(\psi(0)x) \leq \psi(0)f(x). \tag{2.2}$$

Taking $t = 1$ in (2.1), we get

$$f(\psi(1)x) \leq \psi(1)f(x). \tag{2.3}$$

Summing (2.2) and (2.3), we obtain

$$f(\psi(0)x) + f(\psi(1)x) \leq (\psi(0) + \psi(1))f(x),$$

that is,

$$\frac{f(\psi(0)x) + f(\psi(1)x)}{\psi(0) + \psi(1)} \leq f(x).$$

(iii) Let $t \in [0, 1]$. Taking $x = y = 0$ in (2.1), we obtain

$$\begin{aligned} f(0) &\leq \frac{1}{\psi(1) - \psi(0)} \left(\psi(0)\psi(t) - \psi^2(0) + \psi^2(1) - \psi(1)\psi(t)\right) f(0) \\ &= (\psi(1) + \psi(0) - \psi(t))\, f(0). \end{aligned}$$

Hence, if $f(0) > 0$, the above inequality yields

$$\psi(t) \leq \psi(1) + \psi(0) - 1.$$

Similarly, if $f(0) < 0$, we get

$$\psi(t) \geq \psi(1) + \psi(0) - 1.$$

□

3. Hermite-Hadamard-type inequalities

In this section, we extend the Hermite-Hadamard double inequality (1.1) to the class of ψ-convex functions. We first fix some notations.

For all $\psi \in \psi$, let

$$\begin{aligned} A_\psi &= \frac{\psi(0)}{\psi(1) - \psi(0)} \int_0^1 (\psi(t) - \psi(0))\, dt, \\ B_\psi &= \frac{\psi(1)}{\psi(1) - \psi(0)} \int_0^1 (\psi(1) - \psi(t))\, dt, \\ E_\psi &= \frac{\psi(1)\left(\psi(1) - \psi\left(\frac{1}{2}\right)\right)}{\psi(1) - \psi(0)}, \\ F_\psi &= \frac{\psi(0)\left(\psi\left(\frac{1}{2}\right) - \psi(0)\right)}{\psi(1) - \psi(0)}. \end{aligned}$$

Theorem 3.1. *Let $\psi \in \Psi$ be such that $\psi(0)\psi(1) \neq 0$. If $f : [0, \infty) \to \mathbb{R}$ is a ψ-convex function, then for all $a, b \geq 0$ with $a < b$, we have*

$$\frac{1}{b - a} \int_a^b f(x)\, dx \leq \min\left\{ f\left(\frac{a}{\psi(0)}\right) A_\psi + f\left(\frac{b}{\psi(1)}\right) B_\psi, f\left(\frac{b}{\psi(0)}\right) A_\psi + f\left(\frac{a}{\psi(1)}\right) B_\psi \right\}. \tag{3.1}$$

Proof. Let $0 \leq a < b$. For all $t \in (0,1)$, we have

$$\begin{aligned} f(ta+(1-t)b) &= f\left(\psi(0)t\left[\frac{a}{\psi(0)}\right]+\psi(1)(1-t)\left[\frac{b}{\psi(1)}\right]\right) \\ &\leq \psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}f\left(\frac{a}{\psi(0)}\right)+\psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)}f\left(\frac{b}{\psi(1)}\right), \end{aligned}$$

which implies after integration over $t \in (0,1)$ that

$$\int_0^1 f(ta+(1-t)b)\,dt \leq f\left(\frac{a}{\psi(0)}\right)A_\psi + f\left(\frac{b}{\psi(1)}\right)B_\psi. \tag{3.2}$$

Similarly, we have

$$f(tb+(1-t)a) \leq \psi(0)\frac{\psi(t)-\psi(0)}{\psi(1)-\psi(0)}f\left(\frac{b}{\psi(0)}\right)+\psi(1)\frac{\psi(1)-\psi(t)}{\psi(1)-\psi(0)}f\left(\frac{a}{\psi(1)}\right),$$

which implies after integration over $t \in (0,1)$ that

$$\int_0^1 f(tb+(1-t)a)\,dt \leq f\left(\frac{b}{\psi(0)}\right)A_\psi + f\left(\frac{a}{\psi(1)}\right)B_\psi. \tag{3.3}$$

Finally, using (3.2), (3.3) and

$$\int_0^1 f(ta+(1-t)b)\,dt = \int_0^1 f(tb+(1-t)a)\,dt = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

we obtain (3.1). □

Remark 3.1. *Let us consider the case when $f : [0,\infty) \to \mathbb{R}$ is (α, m_1, m_2) convex, where $\alpha, m_1, m_1 \in (0,1]$ with $m_1 \neq m_2$. From Proposition 2.4, f is ψ-convex, where*

$$\psi(t) = (m_2 - m_1)t^\alpha + m_1, \quad 0 \leq t \leq 1.$$

In this case, elementary calculations give us that

$$\begin{aligned} A_\psi &= \frac{\psi(0)}{\psi(1)-\psi(0)}\int_0^1 (\psi(t)-\psi(0))\,dt \\ &= \frac{m_1}{m_2-m_1}\int_0^1 (m_2-m_1)t^\alpha\,dt \\ &= \frac{m_1}{\alpha+1} \end{aligned}$$

and

$$\begin{aligned} B_\psi &= \frac{\psi(1)}{\psi(1)-\psi(0)}\int_0^1 (\psi(1)-\psi(t))\,dt \\ &= \frac{m_2}{m_2-m_1}\int_0^1 (m_2-m_1)(1-t^\alpha)\,dt \\ &= \frac{m_2\alpha}{\alpha+1}. \end{aligned}$$

Hence, (3.1) *reduces to*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \min\left\{\frac{m_1}{\alpha+1}f\left(\frac{a}{m_1}\right)+\frac{m_2\alpha}{\alpha+1}f\left(\frac{b}{m_2}\right),\frac{m_1}{\alpha+1}f\left(\frac{b}{m_1}\right)+\frac{m_2\alpha}{\alpha+1}f\left(\frac{a}{m_2}\right)\right\} \tag{3.4}$$

and we recover the obtained inequality in [22].

Notice that,

- *if* $\alpha = 1$*, then* (3.4) *reduces to the right Hermite-Hadamard inequality for* (m_1, m_2)*-convex functions [20],*
- *if* $m_1 = 1$ *and* $m_2 = m$*, then* (3.4) *reduces to the right Hermite-Hadamard inequality for* (α, m)*-convex functions [21],*
- *if* $\alpha = 1$*,* $m_1 = 1$ *and* $m_2 = m$*, then* (3.4) *reduces to the right Hermite-Hadamard inequality for m-convex functions [19].*

Let us denote by $L^1_{\text{loc}}([0,\infty))$ the set of functions $f : [0,\infty) \to \mathbb{R}$ such that

$$\int_I |f(x)|\,dx < \infty$$

for every closed and bounded interval $I \subset [0,\infty)$.

Our second main result is the following.

Theorem 3.2. *Let* $\psi \in \Psi$ *be such that* $\psi(0)\psi(1) \neq 0$*. If* $f : [0,\infty) \to \mathbb{R}$ *is a* ψ*-convex function and* $f \in L^1_{\text{loc}}([0,\infty))$*, then for all* $a, b \ge 0$ *with* $a < b$*, we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{F_\psi}{b-a}\int_a^b f\left(\frac{x}{\psi(0)}\right)dx + \frac{E_\psi}{b-a}\int_a^b f\left(\frac{x}{\psi(1)}\right)dx. \tag{3.5}$$

Proof. For all $x, y \ge 0$, writing

$$\frac{x+y}{2} = \psi(0)\frac{1}{2}\left(\frac{x}{\psi(0)}\right) + \psi(1)\left(1-\frac{1}{2}\right)\left(\frac{y}{\psi(1)}\right)$$

and using the ψ-convexity of f, we obtain

$$f\left(\frac{x+y}{2}\right) \le \psi(0)\frac{\psi\left(\frac{1}{2}\right)-\psi(0)}{\psi(1)-\psi(0)}f\left(\frac{x}{\psi(0)}\right) + \psi(1)\frac{\psi(1)-\psi\left(\frac{1}{2}\right)}{\psi(1)-\psi(0)}f\left(\frac{y}{\psi(1)}\right),$$

that is,

$$f\left(\frac{x+y}{2}\right) \le F_\psi f\left(\frac{x}{\psi(0)}\right) + E_\psi f\left(\frac{y}{\psi(1)}\right). \tag{3.6}$$

In particular, for $x = ta + (1-t)b$ and $y = (1-t)a + tb$, where $t \in (0,1)$, (3.6) reduces to

$$f\left(\frac{a+b}{2}\right) \le F_\psi f\left(\frac{ta+(1-t)b}{\psi(0)}\right) + E_\psi f\left(\frac{1-t)a+tb}{\psi(1)}\right).$$

Integrating the above inequality over $t \in (0,1)$, we obtain

$$f\left(\frac{a+b}{2}\right) \le F_\psi \int_0^1 f\left(\frac{ta+(1-t)b}{\psi(0)}\right)dt + E_\psi \int_0^1 f\left(\frac{1-t)a+tb}{\psi(1)}\right)dt. \tag{3.7}$$

On the other hand, one has

$$\int_0^1 f\left(\frac{ta+(1-t)b}{\psi(0)}\right) dt = \frac{1}{b-a}\int_a^b f\left(\frac{z}{\psi(0)}\right) dz \tag{3.8}$$

and

$$\int_0^1 f\left(\frac{1-t)a+tb}{\psi(1)}\right) dt = \frac{1}{b-a}\int_a^b f\left(\frac{z}{\psi(1)}\right) dz. \tag{3.9}$$

Thus, (3.5) follows from (3.7)–(3.9). □

Remark 3.2. *Let us consider the case when $f:[0,\infty)\to\mathbb{R}$ is (α,m_1,m_2) convex, where $\alpha,m_1,m_1\in(0,1]$ with $m_1\neq m_2$. Then f is ψ-convex, where*

$$\psi(t)=(m_2-m_1)t^{\alpha}+m_1,\quad 0\le t\le 1.$$

In this case, elementary calculations give us that

$$\begin{aligned} E_{\psi} &= \frac{\psi(1)\left(\psi(1)-\psi\left(\frac{1}{2}\right)\right)}{\psi(1)-\psi(0)} \\ &= \frac{2^{\alpha}-1}{2^{\alpha}}m_2 \end{aligned}$$

and

$$\begin{aligned} F_{\psi} &= \frac{\psi(0)\left(\psi\left(\frac{1}{2}\right)-\psi(0)\right)}{\psi(1)-\psi(0)} \\ &= \frac{m_1}{2^{\alpha}}. \end{aligned}$$

Hence, (3.5) *reduces to*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\left(\frac{m_1}{2^{\alpha}}\int_a^b f\left(\frac{x}{m_1}\right)dx+\frac{(2^{\alpha}-1)m_2}{2^{\alpha}}\int_a^b f\left(\frac{x}{m_2}\right)dx\right)$$

and we recover the obtained inequality in [22].

4. Simpson-type inequalities

In this section, we establish Simpson-type inequalities for the class of functions $f\in C^1([0,\infty))$ such that $|f'|$ is ψ-convex. We first need the following lemma.

Lemma 4.1. *Let $\psi\in\Psi$ be such that $\psi(1)<\psi(0)$. If $f\in C^1([0,\infty))$, then for all $a,b\ge 0$ with $a<b$, we have*

$$\begin{aligned} &\frac{1}{6}\left[f(\psi(1)a)+4f\left(\frac{\psi(1)a+\psi(0)b}{2}\right)+f(\psi(0)b)\right]-\frac{1}{\psi(0)b-\psi(1)a}\int_{\psi(1)a}^{\psi(0)b}f(x)\,dx \\ &=(\psi(0)b-\psi(1)a)\int_0^1 H(t)f'(t\psi(0)b+(1-t)\psi(1)a)\,dt, \end{aligned} \tag{4.1}$$

where

$$H(t)=\begin{cases} t-\frac{1}{6} & \text{if } 0\le t\le\frac{1}{2},\\ t-\frac{5}{6} & \text{if } \frac{1}{2}<t\le 1.\end{cases}$$

Proof. By definition of H and using integrations by parts, we obtain

$$\begin{aligned}
&(\psi(0)b-\psi(1)a)\int_0^1 H(t)f'(t\psi(0)b+(1-t)\psi(1)a)\,dt\\
&=(\psi(0)b-\psi(1)a)\int_0^{\frac{1}{2}} H(t)f'(t\psi(0)b+(1-t)\psi(1)a)\,dt\\
&\quad+(\psi(0)b-\psi(1)a)\int_{\frac{1}{2}}^1 H(t)f'(t\psi(0)b+(1-t)\psi(1)a)\,dt\\
&=\left[H(t)f(t\psi(0)b+(1-t)\psi(1)a)\right]_{t=0}^{\frac{1}{2}}+\left[H(t)f(t\psi(0)b+(1-t)\psi(1)a)\right]_{t=\frac{1}{2}}^{1}\\
&\quad-\int_0^1 f(t\psi(0)b+(1-t)\psi(1)a)\,dt\\
&=\frac{1}{3}f\left(\frac{\psi(0)b+\psi(1)a}{2}\right)+\frac{1}{6}f\left(\psi(1)a\right)+\frac{1}{6}f\left(\psi(0)b\right)+\frac{1}{3}f\left(\frac{\psi(0)b+\psi(1)a}{2}\right)\\
&\quad-\frac{1}{\psi(0)b-\psi(1)a}\int_{\psi(1)a}^{\psi(0)b} f(x)\,dx,
\end{aligned}$$

which proves (4.1). □

We have the following Simpson-type inequality.

Theorem 4.1. *Let $\psi\in\Psi$ be a decreasing function. If $f\in C^1([0,\infty))$ and $|f'|$ is ψ-convex, then for all $a,b\geq 0$ with $a<b$, we have*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(\psi(1)a)+4f\left(\frac{\psi(1)a+\psi(0)b}{2}\right)+f(\psi(0)b)\right]-\frac{1}{\psi(0)b-\psi(1)a}\int_{\psi(1)a}^{\psi(0)b} f(x)\,dx\right|\\
&\leq\frac{\psi(0)b-\psi(1)a}{\psi(0)-\psi(1)}\left[\left(L_\psi-\frac{5}{36}\psi(1)\right)\psi(1)|f'(a)|+\left(\frac{5}{36}\psi(0)-L_\psi\right)\psi(0)|f'(b)|\right],
\end{aligned}\tag{4.2}$$

where

$$L_\psi=\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\psi(t)\,dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\psi(t)\,dt.$$

Proof. By Lemma 4.1, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(\psi(1)a)+4f\left(\frac{\psi(1)a+\psi(0)b}{2}\right)+f(\psi(0)b)\right]-\frac{1}{\psi(0)b-\psi(1)a}\int_{\psi(1)a}^{\psi(0)b} f(x)\,dx\right|\\
&\leq(\psi(0)b-\psi(1)a)\int_0^1 |H(t)|\,|f'(t\psi(0)b+(1-t)\psi(1)a)|\,dt.
\end{aligned}\tag{4.3}$$

On the other hand, due to the ψ-convexity of $|f'|$, we have

$$|f'(t\psi(0)b+(1-t)\psi(1)a)|\leq\psi(1)\frac{\psi(t)-\psi(1)}{\psi(0)-\psi(1)}|f'(a)|+\psi(0)\frac{\psi(0)-\psi(t)}{\psi(0)-\psi(1)}|f'(b)|$$

for all $t \in (0,1)$, which implies after integration over $t \in (0,1)$ that

$$\begin{aligned}
&(\psi(0)b - \psi(1)a)\int_0^1 |H(t)|\,|f'(t\psi(0)b + (1-t)\psi(1)a)|\,dt \\
&\leq \frac{\psi(0)b - \psi(1)a}{\psi(0) - \psi(1)} \\
&\quad \cdot \left[\psi(1)|f'(a)|\int_0^1 |H(t)|(\psi(t) - \psi(1))\,dt + \psi(0)|f'(b)|\int_0^1 |H(t)|(\psi(0) - \psi(t))\,dt\right].
\end{aligned} \tag{4.4}$$

On the other hand, by the definition of H, we have

$$\begin{aligned}
&\int_0^1 |H(t)|(\psi(t) - \psi(1))\,dt \\
&= \int_0^{\frac{1}{2}} \left|t - \frac{1}{6}\right| (\psi(t) - \psi(1))\,dt + \int_{\frac{1}{2}}^1 \left|t - \frac{5}{6}\right| (\psi(t) - \psi(1))\,dt \\
&= -\psi(1)\left(\int_0^{\frac{1}{2}} \left|t - \frac{1}{6}\right|\,dt + \int_{\frac{1}{2}}^1 \left|t - \frac{5}{6}\right|\,dt\right) + \int_0^{\frac{1}{2}} \left|t - \frac{1}{6}\right|\psi(t)\,dt + \int_{\frac{1}{2}}^1 \left|t - \frac{5}{6}\right|\psi(t)\,dt.
\end{aligned}$$

Notice that $\int_0^{\frac{1}{2}} \left|t - \frac{1}{6}\right|\,dt = \int_{\frac{1}{2}}^1 \left|t - \frac{5}{6}\right|\,dt = \frac{5}{72}$. Hence, we get

$$\int_0^1 |H(t)|(\psi(t) - \psi(1))\,dt = -\frac{5}{36}\psi(1) + \int_0^{\frac{1}{2}} \left|t - \frac{1}{6}\right|\psi(t)\,dt + \int_{\frac{1}{2}}^1 \left|t - \frac{5}{6}\right|\psi(t)\,dt. \tag{4.5}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 |H(t)|(\psi(0) - \psi(t))\,dt \\
&= \int_0^{\frac{1}{2}} \left|t - \frac{1}{6}\right| (\psi(0) - \psi(t))\,dt + \int_{\frac{1}{2}}^1 \left|t - \frac{5}{6}\right| (\psi(0) - \psi(t))\,dt \\
&= \psi(0)\left(\int_0^{\frac{1}{2}} \left|t - \frac{1}{6}\right|\,dt + \int_{\frac{1}{2}}^1 \left|t - \frac{5}{6}\right|\,dt\right) - \int_0^{\frac{1}{2}} \left|t - \frac{1}{6}\right|\psi(t)\,dt - \int_{\frac{1}{2}}^1 \left|t - \frac{5}{6}\right|\psi(t)\,dt,
\end{aligned}$$

that is,

$$\int_0^1 |H(t)|(\psi(0) - \psi(t))\,dt = \frac{5}{36}\psi(0) - \int_0^{\frac{1}{2}} \left|t - \frac{1}{6}\right|\psi(t)\,dt - \int_{\frac{1}{2}}^1 \left|t - \frac{5}{6}\right|\psi(t)\,dt. \tag{4.6}$$

Hence, it follows from (4.4)–(4.6) that

$$\begin{aligned}
&(\psi(0)b - \psi(1)a)\int_0^1 |H(t)|\,|f'(t\psi(0)b + (1-t)\psi(1)a)|\,dt \\
&\leq \frac{\psi(0)b - \psi(1)a}{\psi(0) - \psi(1)}\left[\left(L_\psi - \frac{5}{36}\psi(1)\right)\psi(1)|f'(a)| + \left(\frac{5}{36}\psi(0) - L_\psi\right)\psi(0)|f'(b)|\right].
\end{aligned}$$

Finally, (4.2) follows from (4.3) and the above estimate. □

Remark 4.1. *Assume that $|f'|$ is (α, m)-convex, where $\alpha \in (0,1]$ and $m \in [0,1)$. Then, by Proposition 2.3, $|f'|$ is ψ-convex, where*

$$\psi(t) = (m-1)t^{\alpha} + 1, \quad 0 \leq t \leq 1.$$

Clearly, ψ is a decreasing function. On the other hand, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(\psi(1)a) + 4f\left(\frac{\psi(1)a+\psi(0)b}{2}\right) + f(\psi(0)b)\right] - \frac{1}{\psi(0)b-\psi(1)a}\int_{\psi(1)a}^{\psi(0)b} f(x)\,dx\right| \\
&= \left|\frac{1}{6}\left[f(ma) + 4f\left(\frac{ma+b}{2}\right) + f(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx\right|.
\end{aligned}$$

Furthermore, elementary calculations show that

$$\begin{aligned}
L_{\psi} &= \int_0^{\frac{1}{2}} \left|t-\frac{1}{6}\right| \psi(t)\,dt + \int_{\frac{1}{2}}^1 \left|t-\frac{5}{6}\right| \psi(t)\,dt \\
&= (m-1)\left(\int_0^{\frac{1}{2}} t^{\alpha}\left|t-\frac{1}{6}\right|\,dt + \int_{\frac{1}{2}}^1 t^{\alpha}\left|t-\frac{5}{6}\right|\,dt\right) + \left(\int_0^{\frac{1}{2}} \left|t-\frac{1}{6}\right|\,dt + \int_{\frac{1}{2}}^1 \left|t-\frac{5}{6}\right|\,dt\right) \\
&= (m-1)\frac{6^{-\alpha} - 9\times 2^{-\alpha} + 5^{\alpha+2}\times 6^{-\alpha} + 3\alpha - 12}{18(\alpha+1)(\alpha+2)} + \frac{5}{36} \\
&:= (m-1)\nu_1 + \frac{5}{36}
\end{aligned}$$

which yields

$$\left(L_{\psi} - \frac{5}{36}\psi(1)\right)\psi(1) = \left(\frac{5}{36} - \nu_1\right)m(1-m) := \nu_2 m(1-m)$$

and

$$\left(\frac{5}{36}\psi(0) - L_{\psi}\right)\psi(0) = (1-m)\nu_1.$$

We also have

$$\frac{\psi(0)b-\psi(1)a}{\psi(0)-\psi(1)} = \frac{b-ma}{1-m}.$$

Consequently,

$$\begin{aligned}
&\frac{\psi(0)b-\psi(1)a}{\psi(0)-\psi(1)}\left[\left(L_{\psi} - \frac{5}{36}\psi(1)\right)\psi(1)|f'(a)| + \left(\frac{5}{36}\psi(0) - L_{\psi}\right)\psi(0)|f'(b)|\right] \\
&= (b-ma)\left(\nu_2 m|f'(a)| + \nu_1|f'(b)|\right).
\end{aligned}$$

Hence, (4.2) *reduces to*

$$\left|\frac{1}{6}\left[f(ma) + 4f\left(\frac{ma+b}{2}\right) + f(b)\right] - \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx\right| \leq (b-ma)\left(\nu_2 m|f'(a)| + \nu_1|f'(b)|\right)$$

and we recover the obtained inequality in [23].

5. Conclusions

We introduced the class of ψ-convex functions, where $\psi \in C([0,1])$ is nonnegative and satisfies $\psi(0) \neq \psi(1)$. The introduced class includes several classes of functions from the literature: m-convex functions, (m_1, m_2)-convex functions, (α, m)-convex functions and (α, m_1, m_2)-convex functions. After studying some properties of ψ-convex functions, some known inequalities are extended to this set of functions. Namely, when $\psi(0)\psi(1) \neq 0$ and f is ψ-convex, we obtained an upper bound of $\frac{1}{b-a}\int_a^b f(x)\,dx$ (see Theorem 3.1) and an upper bound of $f\left(\frac{a+b}{2}\right)$ (see Theorem 3.2). When ψ is nondecreasing and $|f'|$ is ψ-convex, we proved a Simpson-type inequality (see Theorem 4.1), which provides an estimate of

$$\left|\frac{1}{6}\left[f(\psi(1)a) + 4f\left(\frac{\psi(1)a + \psi(0)b}{2}\right) + f(\psi(0)b)\right] - \frac{1}{\psi(0)b - \psi(1)a}\int_{\psi(1)a}^{\psi(0)b} f(x)\,dx\right|.$$

It would be interesting to continue the study of ψ-convex functions in various directions. For instance, in [31], a sandwich like theorem was established for m-convex functions. Recall that any m-convex function is ψ-convex for some $\psi \in \Psi$ (see Proposition 2.1). A natural question is to ask whether it is possible to extend the sandwich like result in [31] to the class of ψ-convex functions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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